

Reflexive Injective Oriented Colourings

by

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B.Sc., University of the Fraser Valley, 2007

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ABSTRACT

We define a variation of injective oriented colouring as reflexive injective oriented colouring, or rio-colouring for short, which requires an oriented colouring to be injective on the neighbourhoods of the underlying graph, without requiring the colouring to be proper. An analysis of the complexity of the problem of deciding whether an oriented graph has a k -rio-colouring is considered, and we find a dichotomy for the values of k below 3 and above, being in **P** or **NP**-complete, respectively. Characterizations are given for the oriented graphs resulting from the polynomial-time solvable cases, and bounds are given for the rio-chromatic number in terms of maximum in-degree and out-degree, in general, and for oriented trees. Also, a polynomial-time algorithm is developed to aid in the rio-colouring of oriented trees.

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*The value of mathematics is that it is fun and amazing and brings us great joy. To say that mathematics is important because it is useful is like saying that children are important because we can train them to do spiritually meaningless labor in order to increase corporate profits. Or is that in fact what we **are** saying?*

Paul Lockhart, A Mathematician's Lament

DEDICATION

To my parents. I love you!

Introduction

1.1 Application to Business Organization

We open this paper with a brief explanation of where our results have potential for use within business. A new business that starts off with a few employees may work extremely well, where all employees share all different kinds of tasks. As this hypothetical business grows, and new employees are hired, it becomes necessary to define clear cut job descriptions at some point, in order to maintain efficiency and accountability. Obviously, this business will not want to make drastic changes in the way work is currently being performed, since they are successful and growing, but how do they further their proven employee organization, without resorting to potentially unhelpful restructuring?

There are restrictions on using our results to answer this question, where a business must first have a clear representation of how their employees currently communicate, or which employee oversees another, which we call a work-flow graph. Treat each employee as a vertex, and place an arc from one employee to another if the

former is responsible for the latter, or the former communicates with the latter.

The organization graph is a set of vertices which represent the positions you wish your business to have, together with the arcs between them, represent the way in which one position is to interact with another, or which position oversees another. A loop at any position could, therefore, allow for the training of an employee who will hold the same position as their trainer. Our results suppose that every position is trainable. It may well be that a business is most efficient when it has the least number of positions possible, and our results show that this will most likely not be a simple problem, which should be expected given the amount of time and energy businesses already put forward to obtaining optimal efficiency in their employee structure, and partially explains why we would mostly see work-flow graphs in a tree hierarchy as the paradigm today (see Chapter 6).

A rio-colouring, which we define in Chapter 2, is a way of assigning employees with positions such that all the other employees they interact with must hold different positions, except maybe the position they hold (in the case for which they are training another employee), such that any two employees that work together must then be assigned positions that will always interact the same way (the arc between any two employees assigned with these positions must always appear oriented in the same direction). To be specific, the relationship between positions in the organization graph is a target digraph for an injective homomorphism, also defined in Chapter 2, but it can be adjusted to suit the best needs of the work flow graph, so that we really view an assignment of positions as a rio-colouring. The main idea is to keep the current employee work flow already in place, at the local level around each employee so that it appears no changes have been made to any individual employee's role, and describe it with the most simple organization graph possible (the fewest number of positions), while new employees hired may continue the already established success of the business with a clear understanding of how to work in the business' team.

However, it would be the work of the owner to give said positions their name and description, which may not fit with traditional roles given their employees may not currently work in a tree hierarchy.

1.2 Preceding Results of Colourings

The theory of graph colouring first arises in most introductory texts on Graph Theory and gained initial widespread interest with the popularizing of the famous Four Colour Conjecture, which is of course, now a theorem.

Prior to the study of oriented colourings, to colour a digraph the orientation was ignored and one studied the colouring of the underlying graph. This is not to say that a colouring of the underlying graph gives no information about its digraph, for example, a theorem proven by Gallai (1968) and Roy (1967) (see [1]) states: *Every digraph D contains a directed path of length $\chi - 1$.* An immediate corollary of this theorem when applied to tournaments shows that every tournament has a directed Hamilton path, a result of Redei's (1934) proven earlier.

Adding a requirement of a colouring to further satisfy some property involving the orientation of the arcs in a digraph first arose in a paper by Courcelle (1994) [5]. There we find what is termed a *good and semi-strong colouring*, where 'good' is synonymous with 'oriented' and 'semi-strong' is synonymous with 'injective'. Many authors have developed the theory of oriented colourings, such as Raspaud, Kostochka, Boiron, Borodin, Sopena, Hell, and Nešetřil (see [21] for a survey). Their efforts have mainly focused on bounds for the oriented chromatic number of differing classes of digraphs: oriented outerplanar graphs have oriented chromatic number at most 7 [20], while the oriented chromatic number of an oriented planar graph is at most 80 [19]; however, no example of an oriented planar graph with oriented chromatic number larger than 17 has ever been provided. Finally, results by Maurer, Sudborough and Welzl (1980) [17] show that, for $k = 1, 2, 3$, the problem of deciding whether an oriented graph has

an oriented k -colouring is in \mathbf{P} , and so algorithms exist that either find an oriented k -colouring of the desired oriented graph, or return an obstruction, a minimal subgraph which deadlocks the possibility of such a colouring. Further work by Klostermeyer and MacGillivray showed for each fixed integer $k \geq 4$, the problem of deciding whether an oriented graph has an oriented k -colouring is \mathbf{NP} -complete [14].

In Courcelle's work *On The Monadic Second Order Logic of Graphs*, the result that necessitated the oriented property to be defined showed that every directed planar graph with in-degree at most three has a proper injective oriented k -colouring, where $k = 64 \cdot 3^{63}$. Work by Raspaud and Sopena improved this result to $k = 5 \cdot 2^6$ [19]. The theory of injective oriented colourings, both proper (irreflexive) and improper (reflexive), has been advanced by MacGillivray, Raspaud and Swartz [16]. The more finely tuned problem of the existence of a homomorphism, with the added property of being injective on in-neighbourhoods, from a given oriented graph to a fixed target digraph was explored by Swartz in his thesis [22].

Colourings of undirected graphs that are injective have been studied after their emergence in Courcelle's work [5], but there exists only a handful of papers up to the present. We make use of results by Hahn, Kratochvíl, Širáň, and Sotteau (2002) [9] on the injective chromatic number χ_i of a graph that relates it to the chromatic number, $\chi(G) \leq \chi_i(G)$, when G is different from K_2 . Note that they do not require the injective colouring to be proper. We also use their bounds on the injective chromatic number in terms of the maximum degree, $\chi_i(G) \leq (\Delta(\Delta - 1) + 1)$, and a bound on the number of vertices of an r -clique K , $\chi_i(K) = |V(K)| \leq (\Delta^2 - \Delta)$. Another result they obtained which gives our work greater context is that the problem of deciding whether a graph has an injective k -colouring, for fixed $k \geq 3$, is \mathbf{NP} -complete. There is recent work by Hahn, Raspaud, and Wang (2006) [10], the main result being an upper bound of the injective chromatic number with respect to G a K_4 -minor free graph, $\chi_i(G) \leq \lceil \frac{3}{2}\Delta \rceil$. Bu, Chen, Raspaud, and Wang (2008) [2], independently

from Lužar, Škrekovski, and Tancer (2008) [15] have all proven various bounds on the injective chromatic number with respect to planar graphs, their maximum degree, and their girth. Kranston, Kim and Yu in their works titled *Injective colourings of graphs with low average degree*, and *Injective colourings of sparse graphs*, both available freely on the internet (there is no reference to any journal - use Google Scholar) prove that if $\Delta \geq 4$ and the maximum average degree of a graph G is $mad(G) < \frac{14}{5}$, then $\chi_i(G) \leq (\Delta + 2)$, with various other similar results in both short papers. Hell, Raspaud, and Stacho (2008) [12] show the problem of whether a chordal graph has an injective k -colouring, for fixed positive integer k , to be polynomial-time solvable, however, computing the injective chromatic number of such graphs is **NP**-hard.

The concept of injective colourings of oriented graphs provides a plethora of ways to define the injective property, allowing for a rich variety of results. Of these, one may stipulate that the oriented colourings must be injective on in-neighbourhoods and out-neighbourhoods separately, with further variation for either proper or improper condition to be satisfied [4], or another variation considers the injective property to hold on an underlying graph's neighbourhoods, again with further variation for either proper or improper conditions. Curiously, irreflexive (proper) oriented colourings of oriented graphs that satisfy the injective property on the underlying graph's neighbourhoods results in the same theory as those colourings with the injective property on the in-neighbourhoods and out-neighbourhoods separately. There remains one case left to be studied, the improper (reflexive) oriented colourings of oriented graphs which are injective on in-neighbourhoods and out-neighbourhoods simultaneously, which is discussed throughout this thesis.

1.3 Summary of Thesis

Needed definitions clarifying our terminology used throughout this thesis are given in Chapter 2. There, the second section sets up some detailed ground work for the

context of work in Chapter 4.

In Chapter 3, we establish the dichotomy that exists for the problem of deciding whether an oriented graph has a k -rio-colouring, for fixed k . The discriminating value of k echoes results for injective colouring, and only need be a low value of 3 to force the problem to be **NP**-complete.

When we have $k = 2$, the complexity results suggest there is an easy characterization for the oriented 2-rio-colourable graphs, which we describe in Chapter 4. To hint at the difficulty for attempting the same work when $k = 3$, we work through a partial characterization of the oriented cycles, which naturally leads into the subject of bounding the rio-chromatic number.

Chapter 5 starts off with a preliminary discussion where we show the rio-chromatic number must be exponential in terms of maximum in-degree and maximum out-degree, in general, and contrast this with the injective chromatic number being polynomial in terms of maximum degree. This is followed by a section that finishes our discussion of rio-colouring the oriented cycles, to show that, at most, only four colours are ever needed. Of course, any discussion on the bounds of the rio-chromatic number would not be complete if we left out a characterization of rio-cliques, oriented graphs which have order equal to their rio-chromatic number, and coincidentally, are the same as such graphs for injective colouring. These enable us to get a better idea of what bounds we can expect for other families of oriented graphs. We then move on to less trivial work by finding an explicit upper bound for the rio-chromatic number in terms of maximum in-degree and out-degree, followed by an analysis of how much this bound can improve when restricting ourselves to the family of oriented trees.

We finish off with Chapter 6 containing an algorithm developed to aid in determining the existence of an injective homomorphism of oriented trees to a specific target digraph, by keeping two types of lists for each vertex of the tree desired. These two lists, respectively, contain target vertices as candidates for a potential injective

homomorphism's images and systems of distinct representatives to link candidates together in a way that satisfies the injective property.

Preliminaries

2.1 Definitions and Basic Results

For foundational concepts in graph theory, we utilize Chartrand and Lesniak's text [3]. In this section we state the definitions, results and notations that can not be found in an introductory graph theory text.

We use directed graphs that never have multiple arcs, and are asymmetric, but sometimes may have loops. An *irreflexive* digraph has no vertex incident with a loop, and a *reflexive* digraph has every vertex incident with a loop.

An *oriented graph* is an asymmetric directed graph as above. That is, a directed graph such that for every pair of vertices x and y , at most one of the arcs xy or yx exists. Intuitively, the same is obtained by assigning a direction to each edge of a simple graph.

For digraphs the open in-neighbourhood and open out-neighbourhood of a vertex x are denoted $N^-(x)$ and $N^+(x)$, respectively, and when we wish to combine these sets and discuss the neighbourhood of the underlying undirected graph, we simply

use $N(x)$. We deal with closed neighbourhoods, $N^-[\cdot]$, $N^+[\cdot]$, $N[\cdot]$, analogously.

There are many varieties of families of graphs and digraphs. For example, a *star* graph S_k , is a tree with k leaves which are all adjacent to a central vertex, and an *in-star* is an oriented star that has every arc adjacent from the leaves to the center. Also, T_k is generally used to refer to a tournament on k vertices. We even mention obscure types such as *book* graphs, which can be described as a complete bipartite graph where one of the partite sets has exactly two vertices x and y such that the edge xy is included. A *wheel* graph is a cycle such that a new vertex is added and connected to every vertex in the cycle.

A *triangle* of a graph G is a subgraph of G isomorphic to K_3 , the complete graph on three vertices.

The *eccentricity* of a vertex v in a graph G is $\max_{x \in V(G)} \{d(v, x)\}$, the maximum distance of any vertex from v . The *diameter* of a graph G , $\text{diam}(G)$, is the maximum eccentricity among the vertices of G . We will use these, and other terms, for oriented graphs, and when we do so it should be understood to apply to the underlying graph.

Let G and H be directed graphs. A *homomorphism of G to H* is a function $f : V(G) \rightarrow V(H)$ such that $f(x)f(y)$ is an arc of H whenever xy is an arc of G , and therefore induces a mapping of the arcs of G to the arcs of H . We denote the existence of a homomorphism between G and H by $G \rightarrow H$. A homomorphism of G to H is *injective* if, for each vertex x of G , no two neighbours of x have the same image. The existence of an injective homomorphism between G and H is also denoted $G \xrightarrow{\text{inj}} H$. We make use of the fact that composing two injective homomorphisms is an injective homomorphism in Section 2.2 of the Preliminaries and Chapter 4 to help analyze the structure of characterizations. Because homomorphisms generalize colourings, the problem of deciding if a given digraph has a homomorphism to a fixed digraph H has been called *H -colouring*. The book by Hell and Nešetřil [11] has been recommended as an excellent introduction to the theory of homomorphisms of graphs and digraphs.

Let D be an oriented graph and k be a positive integer. An *oriented k -colouring* of D is a proper k -colouring of D such that if some arc of D joins a vertex of colour i to a vertex of colour j , for $1 \leq i \neq j \leq k$, then no arc of D joins a vertex of colour j to a vertex of colour i . The *oriented chromatic number of D* is the smallest k for which there is an oriented k -colouring of D . Equivalently, an oriented k -colouring of D is a homomorphism to an oriented graph T on k vertices, and the oriented chromatic number of D is the least k for which such a homomorphism exists. Since arcs can be added joining non-adjacent vertices of T without destroying the existence of a homomorphism of D to T , the oriented graph T can be taken as a tournament.

An *injective oriented k -colouring* of an oriented graph D is an oriented k -colouring of D such that, for each vertex x of D , no two neighbours of x are assigned the same colour. Not surprisingly, such can be seen as an injective homomorphism to some digraph T on k vertices. Again, the digraph T can always be taken as a tournament, for adding arcs to non-adjacent vertices of T does nothing to an already existing mapping. If we did not stipulate that an oriented colouring be proper, we could get away with a trivial monochromatic colouring every time, but in contrast, the injective condition allows a non-trivial relaxation of injective oriented colourings from being proper, and since we study this throughout our work, *the target digraphs of the homomorphism model of the colourings we discuss are always reflexive*. Altogether, we shorten the adjectival phrase ‘reflexive injective oriented’ to the prefix *rio* for shorthand.

The *rio-chromatic number* of an oriented graph D , denoted $\chi_!(D)$, is the smallest k for which there is a k -rio-colouring of D . Since injective homomorphisms compose, if $D_1 \xrightarrow{\text{inj}} D_2$, then $\chi_!(D_1) \leq \chi_!(D_2)$.

We often talk of H -colouring being injective, which means that we are looking for an injective homomorphism from an oriented graph G to some reflexive tournament H . If H has k vertices, then an injective homomorphism can be thought of as a k -rio-

colouring of G with the vertices of H being distinct colours. Also, a k -rio-colouring must be an injective homomorphism of G to some reflexive tournament on k vertices. For the kinesthetic learner who desires an interactive exploration of these definitions, we recommend trying to rio-colour an oriented cycle of their choice with the minimum number of colours, and check it against Lemma 5.1 and Figure 2.1.

To avoid any confusion, while the target tournament T is always reflexive, the digraph we seek to colour is always irreflexive unless otherwise stated.

We make use of two target tournaments enough to merit special mention. One is a reflexive transitive triple T_3 , and the other is a reflexive directed 3-cycle C_3 . Both are shown in Figure 2.1, and in the sequel we will refer to the vertex labels that are used.

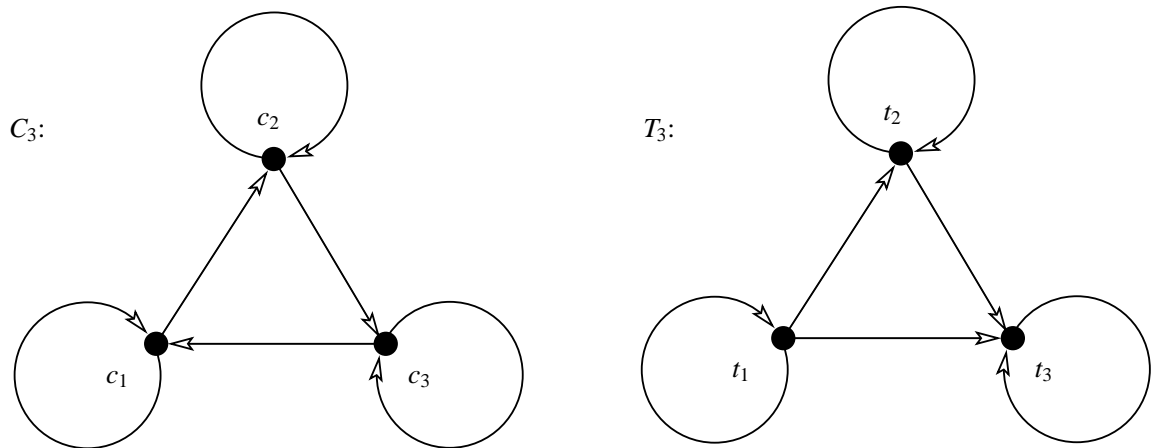


Figure 2.1: Target tournaments

In an injective homomorphism $f : V(D) \rightarrow V(H)$ we must have $d_D^-(x) \leq d_H^-(f(x))$ and $d_D^+(x) \leq d_H^+(f(x))$ for every vertex x of D . In the reflexive tournament C_3 , every vertex has in-degree two and out-degree two. In the tournament T_3 , the vertex t_1 has in-degree one and out-degree three, the vertex t_2 has in-degree two and out-degree two, and the vertex t_3 has in-degree three and out-degree one. Therefore, for oriented graphs that we wish to injective C_3 -colour, their in-degrees cannot exceed two, nor their out-degrees. As well, any injective T_3 -coloured oriented graph must map vertices

of out-degree three to t_1 and vertices of in-degree three to t_3 .

We finish off this section with a basic result before we embark on any proofs of theorems. There is a very well behaved family of oriented cycles which have properties that we will revisit when necessary. Consider the family of oriented cycles \mathcal{C} such that each $C \in \mathcal{C}$ is comprised of two disjoint perfect matchings both oriented in opposite directions from each other. That is, C of order $n = 2k$ has vertices $V = \{v_1, v_2, \dots, v_{2k}\}$ with arc set $A = (\{v_1v_2, v_3v_4, \dots, v_{2k-1}v_{2k}\} \cup \{v_1v_{2k}, v_3v_2, \dots, v_{2k-1}v_{2k-2}\})$. Clearly, every oriented cycle in \mathcal{C} has even order.

Lemma 2.1. *Let C be an oriented cycle from the family \mathcal{C} with order n .*

(i) $n \equiv 0 \pmod{6}$ if and only if C has an injective C_3 -colouring.

(ii) $n \equiv 0 \pmod{4}$ if and only if C has an injective T_3 -colouring.

The proof of Lemma 2.1 is immediate from our later characterizations of oriented cycles (see Lemmas 4.2 and 4.3). From another viewpoint, mapping the vertices of C in some order, while trying to maintain the rules of our colouring, to the desired target, forces a specific pattern of images to be used repeatedly.

2.2 Obstructions

An oriented graph G is said to be *critically k -rio-colourable* if $\chi_!(G) = k$, but for every element e (vertex or arc) of G , $G - e$ is $(k - 1)$ -rio-colourable. Any critically k -rio-colourable oriented graph is also said to be an *obstruction* to any of the $(k - 1)$ -rio-colourable oriented graphs.

A more intuitive way of seeing obstructions lies in the viewpoint of a partial order on oriented graphs. We use $G \leq H$, for oriented graphs G and H , when G is a subgraph of H to set up a partial order on \mathcal{G} , and partition the set of all oriented graphs \mathcal{G} into subsets \mathcal{G}_i consisting of oriented graphs with rio-chromatic number i . Then the obstructions to the oriented graphs of \mathcal{G}_i are the minimal elements of \mathcal{G}_{i+1} ,

for we cannot have the rio-chromatic number of an oriented graph G less than $(i + 1)$ if G contains one of these minimal elements of \mathcal{G}_{i+1} as a subgraph.

Here we begin a rather lengthy discussion contrasting the previous partial order against a much more interesting, elaborate and descriptive kind, that also yields more concise results.

Suppose we have two oriented graphs D_1 and D_2 . Then we wish to turn the relation $D_1 \preccurlyeq D_2$ if and only if $D_1 \xrightarrow{\text{inj}} D_2$, into a partial order, for on its own, anti-symmetry of \preccurlyeq does not hold (i.e., take D_1 a directed three cycle, and D_2 the union of two directed three cycles and vice versa). The choice of this partial order is motivated by the fact that if $D_1 \xrightarrow{\text{inj}} D_2$, then $\chi!(D_1) \leq \chi!(D_2)$.

With this in mind, note that we can partition the set of oriented graphs using the equivalence relation $G \sim H$ if and only if $G \xrightarrow{\text{inj}} H$ and $H \xrightarrow{\text{inj}} G$, so that we have left only to find a convenient representative from each equivalence class.

Definition 2.2. Let D be an oriented graph. An *injective core* of D is a subgraph D' of D such that $D \xrightarrow{\text{inj}} D'$ and yet no proper subgraph B of D' allows $D' \xrightarrow{\text{inj}} B$.

Proposition 2.3. *Every oriented graph has a unique injective core (up to isomorphism).*

Proof. Suppose D_1 and D_2 were both injective cores of an oriented graph D . Let $f : D \xrightarrow{\text{inj}} D_1$ and $g : D \xrightarrow{\text{inj}} D_2$. Since D_1 and D_2 are subgraphs of D , we also have $i_1 : D_1 \xrightarrow{\text{inj}} D$ and $i_2 : D_2 \xrightarrow{\text{inj}} D$ as identity maps (obviously injective).

Then we can compose the following injective homomorphisms, $\alpha = (g \circ i_1)$ and $\beta = (f \circ i_2)$ (where \circ denotes rightmost function applied first). Further, note that $(\beta \circ \alpha)$ is an injective homomorphism by composition of the same, but that it must be an automorphism of D_1 , else D_1 would have an injective homomorphism to some proper subgraph as a contradiction to D_1 being an injective core. Then $(\beta \circ \alpha)$ is also a bijection, and hence, α must be one-to-one and β must be onto. However, the

same argument applies to $(\alpha \circ \beta)$ with D_2 . Hence, both α and β are isomorphisms, so that $D_1 \cong D_2$. \square

Since Proposition 2.3 holds, we denote *the* injective core of an oriented graph D as $D^!$. Note that D may contain multiple copies of $D^!$ as separate components.

Proposition 2.4. *If for oriented graphs D_1 and D_2 we have $D_1 \xrightarrow{\text{inj}} D_2$ and $D_2 \xrightarrow{\text{inj}} D_1$, then D_1 and D_2 have isomorphic injective cores.*

Proof. Suppose $f : D_1 \xrightarrow{\text{inj}} D_1^!$ and $g : D_2 \xrightarrow{\text{inj}} D_2^!$. As with the proof of Proposition 2.3, we have identity maps $i_1 : D_1^! \xrightarrow{\text{inj}} D_1$ and $i_2 : D_2^! \xrightarrow{\text{inj}} D_2$. We also have injective homomorphisms $q : D_1 \xrightarrow{\text{inj}} D_2$ and $r : D_2 \xrightarrow{\text{inj}} D_1$.

Then by composition of injective homomorphisms, we have $(g \circ q \circ i_1) = \alpha : D_1^! \xrightarrow{\text{inj}} D_2^!$ and $(f \circ r \circ i_2) = \beta : D_2^! \xrightarrow{\text{inj}} D_1^!$. Further, $(\alpha \circ \beta)$ is also an injective homomorphism, which must be an automorphism of $D_1^!$. The rest of the proof unfolds exactly as Proposition 2.3. \square

As a consequence of Proposition 2.4, we have our convenient representative, the injective core, of oriented graphs that are equivalent with respect to injective homomorphism. Let \mathcal{G}' be the set of all injective cores, so that (\mathcal{G}', \preceq) is a poset.

We can further refine discussions surrounding injective cores. We make use of a homomorphism r between a labelled oriented graph G and a subgraph H of G as a *retraction* if $r(x) = x$ for all $x \in V(H)$.

Proposition 2.5. *If D is an oriented and connected graph, D is its own injective core.*

Proof. Let $\phi : D \xrightarrow{\text{inj}} D^!$. Of course, $\phi|_{D^!}$ must be an automorphism, otherwise $D^!$ is not an injective core. Then $(\phi|_{D^!})$ has an inverse, which we simply call σ , so that $(\sigma \circ \phi)$ is a retraction of D to $D^!$. Also note that σ is an automorphism and therefore

satisfies the properties of being an injective homomorphism. Hence, $(\sigma \circ \phi)$ is also an injective homomorphism by composition.

The statement obviously holds for $D = K_1$, but also see that for any other connected D , it is not possible for $D^! = K_1$, for then $(\sigma \circ \phi)$ would not respect the homomorphism property, as there are no loops on any single vertex of an oriented graph.

Suppose $D^!$ were a proper non-trivial subgraph of D . There must then exist vertices $x \in (V(D) \setminus V(D^!))$ and $y \in V(D^!)$ such that x and y are adjacent, because D is connected. However, without loss of generality, x being an out-neighbour of y , must map via $(\sigma \circ \phi)$ to an out-neighbour of y that lies in $D^!$, say u . This leads to $(\sigma \circ \phi)(x) = (\sigma \circ \phi)(u)$, which contradicts the injective property of $(\sigma \circ \phi)$ on the neighbourhood of y . \square

Proposition 2.6. *A disjoint union of oriented graphs $G = (D_1 \cup D_2 \cup \dots \cup D_k)$ is its own injective core if and only if for all $1 \leq i \neq j \leq k$ we have $D_j \xrightarrow{\text{inj}} D_i$.*

Proof. If G has only one component, we are done by Proposition 2.5. For G containing more components, we prove the contrapositive. Without loss of generality, say $D_2 \xrightarrow{\text{inj}} D_1$. Then, obviously, $G \xrightarrow{\text{inj}} (G - D_2)$, so that G cannot be its own injective core.

Conversely, suppose $D_i \not\xrightarrow{\text{inj}} D_j$ for all $i \neq j$. Hence, for $f : G \xrightarrow{\text{inj}} G^!$, f must map every component of G to itself. Since each D_i is connected, $f|_{D_i}$ must be an automorphism, else we contradict Proposition 2.5. \square

Thus far, this allows us to generate injective cores from the disjoint union of connected oriented graphs as long as no injective homomorphism exists between any two of them.

If we partition the set of injective cores \mathcal{G}' into sets \mathcal{G}'_i of injective cores of riochromatic number i , for any positive integer i , and apply our partial order \preceq to \mathcal{G}'_i , we then have minimal elements of \mathcal{G}'_{i+1} as obstructions to the i -rio-colourable oriented

graphs.

The last question we would like to resolve is if all obstructions are properly represented by minimal elements of \mathcal{G}'_i for some i with respect to \preceq , as opposed to \mathcal{G}_i with respect to \leq , and it turns out that every obstruction is a minimal element of \mathcal{G}_i with respect to \leq , but not all obstructions are minimal elements of \mathcal{G}'_i when \preceq is used instead. However, we can press the set of obstructions into a more concise collection with \preceq , if we choose to use this partial order's minimal elements of \mathcal{G}'_i instead. We refer to the minimal elements of \mathcal{G}'_i with respect to \preceq as *rio-obstructions*, and continue to reserve the regular notion of obstruction for the subgraph partial order.

If some obstruction G were not the minimal element of \mathcal{G}_i for some positive integer i , with respect to \leq , then we would have some proper subgraph G' of G with rio-chromatic number i . This implies for some element $e \in (V(G) \cup E(G)) \setminus (V(G') \cup E(G'))$ we have $\chi_!(G - e) = \chi_!(G)$, a contradiction. Clearly, the converse holds.

Suppose G was a rio-obstruction. Then every proper subgraph G' of G must have smaller rio-chromatic number, else $G' \xrightarrow{\text{inj}} G$ (using the identity to map) with $G' \in \mathcal{G}'_i$ leaves us in contradiction. Hence, rio-obstructions are always obstructions. On the other hand, obstructions G are always injective cores, for otherwise we would have a contradiction by $\chi_!(G) \leq \chi_!(G')$, for G cannot have the proper subgraph G' with smaller rio-chromatic number. Finally, for any obstruction G that is not a rio-obstruction, if there is a minimal element $D \preceq G$, then D must also be an obstruction simply by minimality. To sum up, when some oriented graph H contains G or D as a subgraph, then D by itself would be enough to describe a deadlock to rio-colouring H with less than i colours, because $D \xrightarrow{\text{inj}} G \xrightarrow{\text{inj}} T_{i-1}$ for any reflexive tournament T_{i-1} .

Lastly, we always need the existence of some rio-obstruction less than any given obstruction. Note that for any oriented graphs D and G , we must always have $d_D^\pm(x) \leq d_G^\pm(f(x))$ for all $x \in V(D)$ when $f : D \xrightarrow{\text{inj}} G$, so that 'smaller' elements of \mathcal{G}'_i

have non-increasing maximum in/out-degree from G . It is not the case that minimal elements of \mathcal{G}'_i are acyclic, in general, for any i , but this is the case when $i = 2$, which we finalize in Chapter 4.

For convention, when we discuss any set \mathcal{F} of oriented graphs we can make use of the set $Forb(\mathcal{F}) = \{D \mid \text{there is no } F \in \mathcal{F} \text{ such that } F \xrightarrow{\text{inj}} D\}$. We wish to describe \mathcal{F} such that $Forb(\mathcal{F})$ is the set of 2-rio-colourable oriented graphs. Suppose we had used $F \leq D$ in the setup of $Forb(\mathcal{F})$ instead; however, as we have established that the set of rio-obstructions is a subset of obstructions, we reduce \mathcal{F} into a more concise set when using $F \xrightarrow{\text{inj}} D$, for if we have an injective core G of \mathcal{G}'_3 that is not a rio-obstruction, while G' is a minimal element of \mathcal{G}'_3 , then $G \xrightarrow{\text{inj}} D$ implies $G' \xrightarrow{\text{inj}} D$, so that $G' \in \mathcal{F}$ is enough to describe both G' and G as forbidden. Thus, we need only the set of minimal elements of \mathcal{G}'_3 , or the rio-obstructions, to describe \mathcal{F} completely.

If any vertex v of an oriented graph G has degree three in the underlying graph, then obviously the subgraph star centered on v has rio-chromatic number three, and hence, any orientation of the star graph on four vertices is in \mathcal{F} . Then every other oriented graph in \mathcal{F} must have maximum degree two in its underlying graph. Since there is only one target tournament for 2-rio-colourings, namely the reflexive tournament T_2 on two vertices, any rio-obstruction $F \in \mathcal{F}$ cannot have more than one component, for say $F = F_1 \cup F_2$, then removal of any element $e \in (V(F_1) \cup E(F_1))$ would yield $((F_1 - e) \cup F_2) \xrightarrow{\text{inj}} T_2$ by minimality, with a 2-rio-colouring of F_2 , but the same would also hold for removal of an element in F_2 and imply $F_1 \xrightarrow{\text{inj}} T_2$, however, then we must have that $F \xrightarrow{\text{inj}} T_2$, which is a contradiction. Then every rio-obstruction to 2-rio-colouring in F , besides the oriented stars on four vertices, must consist of only one component, namely a path or a cycle.

Continuing on to Chapter 4, we will arrive at the conclusion that for $Forb(\mathcal{F})$ as the set of 2-rio-colourable graphs, \mathcal{F} consists of paths with the exception of the oriented stars on four vertices.

Complexity

We make use of the details of the theory of **NP**-completeness established by Garey and Johnson [7].

Few colours are needed to invoke complex decisions for the rio-colouring of digraphs. Analogous to the other generalizations of injective oriented colouring [16, 22, 4], we show that, for a fixed integer $k \geq 2$ the problem of deciding if a given oriented graph is k -rio-colourable is in **P** if $k \leq 2$ and in **NP**-complete if $k \geq 3$. We also characterize the oriented graphs which are 2-rio-colourable, and explore characteristics of oriented cycles, which both further display the dichotomy of the problem of rio-colouring the oriented graphs. The 1-rio-colourable oriented graphs obviously consist of unions of disjoint arcs.

To prove that deciding whether an oriented graph has a 2-rio-colouring is polynomial-time solvable, we perform a reduction of its conditions to 2-SAT, which is well known to be polynomial-time solvable [6]. We need the following machinery to break down this reduction into a simple proof:

- (•) The two colours we use are 0 and 1.

- (•) There is a variable x corresponding to each vertex of the oriented graph G to be checked for 2-rio-colourability.
- (a) For any two vertices x and y with a common neighbour in the underlying graph of G , add the clause: $x \text{ xor } y$.
- (b) For every arc xy of G , add the clause: $x \rightarrow y$.

Proposition 3.1. *An oriented graph G is 2-rio-colourable if and only if the set of clauses generated from (a) and (b) are able to be satisfied simultaneously.*

Proof. Suppose we have an oriented graph G with a 2-rio-colouring $f : V(G) \rightarrow \{0,1\}$. For any two common neighbours x and y in the underlying graph of G , $f(x) \neq f(y)$ because f is injective on common neighbours, so that we may assign the variable x true when $f(x) = 1$, false otherwise, and the variable y is assigned similarly. Hence, any clause generated from (a) is satisfied.

For every arc xy of G , without loss of generality, we must have $f(x) = 0$ and $f(y) = 0$ or 1 , or $f(x) = 1$ and $f(y) = 1$, because f is an oriented colouring that allows adjacent vertices to have the same colour, so that our truth assignment of the variables x and y always satisfy their clause generated from (b). Altogether, the clauses generated from (a) and (b) are satisfied.

Conversely, suppose all the clauses generated from (a) and (b) were satisfied simultaneously. Let f be the colouring of the vertices of G corresponding to each variable x with $f(x) = 0$ when x is false, and $f(x) = 1$ when x is true. Then for any two common neighbours x and y of the underlying graph of G must have $f(x) \neq f(y)$, else their clause generated from (a) would be false. Thus, f must be an injective colouring.

Considering the arcs xy of G , we must have truth assignment of their clause generated from (b) as x false and y true or false, or x true and y true. This implies, in the first case, $f(x) = 0$ and $f(y) = 0$ or 1 , and in the second case $f(x) = 1$ and

$f(y) = 1$, so that in all cases the arcs of G have been coloured to satisfy the oriented property while allowing adjacent vertices to be coloured the same. Altogether, f satisfies the reflexive, injective, and oriented properties and is a 2-rio-colouring of G . \square

Theorem 3.2. *Deciding whether an oriented graph has an injective homomorphism to a reflexive directed 3-cycle is NP-complete.*

Proof. The problem is clearly in NP, since, given a colouring, we can check the injective and oriented properties are satisfied in polynomial time.

We make use of an oriented cycle on 18 vertices $C_{18} \in \mathcal{C}$. List the vertices around the cycle as $V = \{v_0, v_1, \dots, v_{17}\}$, with v_0 to be of out-degree two. By Lemma 2.1, C_{18} cannot have an injective homomorphism to T_3 in Figure 2.1.

Lemma 2.1 also allows us to focus on an injective homomorphism of C_{18} to C_3 . Note that for any injective homomorphism f of C_{18} to C_3 we have $f(v_i) = f(v_j)$ for $i \equiv j \pmod{6}$, essentially because every vertex must have one of its neighbours assigned the same image as itself. Also observe that if we have no other vertices assigned an image other than v_0, v_6, v_{12} , which must be mapped the same, then this pre-assignment can be extended to an injective homomorphism of C_{18} to C_3 .

We show that 3-colouring a 3-regular graph polynomially transforms to an injective homomorphism of an oriented graph to a reflexive C_3 . Say we have a 3-regular graph G . We will show how to transform G into an oriented graph G' as we complete this paragraph. Replace every vertex $x \in V(G)$ with an instance X of C_{18} . Then for any edge $xy \in E(G)$, being the i^{th} edge with respect to x and j^{th} edge with respect to y , identify $x_{6i} \in V(X)$ with the terminal vertex of one copy of a directed path of length two, do the same for $y_{6j} \in V(Y)$ with another copy, identify the two copies' initial vertices together, calling the new vertex t , and finally, call the resulting oriented path P . The transformation replaces each of the n vertices of G with 36 elements and each of the m edges of G with 7 elements, while taking $2m$ steps to

connect the replacement paths to the replaced vertex structures. The transformation can therefore be carried out in polynomial time.

We claim $u = x_{6i}$ and $v = y_{6j}$ cannot map to the same image, nor any of their copies in G' , for if so, u in X and v in Y are both forced to map their respective neighbourhoods the same, contradicting injectivity at the vertex t . On the other hand, when u and v are mapped to different vertices of C_3 , say $f(u) = c_1$, $f(v) = c_2$, the neighbours of u in X are mapped to c_1, c_2 , while the neighbours of v in Y are mapped to c_2, c_3 , and so the neighbours of u and v on P must be mapped to c_3 and c_1 , respectively, so that t may be mapped to c_3 . We can extend any pre-assignment because the other possible combinations map similarly by symmetry of C_3 .

We can now justify our claim that G is 3-colourable if and only if G' has an injective homomorphism to C_3 . Suppose G has a specific 3-colouring, $C : V(G) \rightarrow \{c_1, c_2, c_3\}$ (the colours of C are the vertices of C_3). Then adjacent vertices x, y in G must receive different colours, $c(x), c(y)$, so that we assign the vertices x_0, x_6, x_{12} of X to map to $c(x)$ while y_0, y_6, y_{12} of Y will map to $c(y)$, and further, we have shown this can be extended to an injective homomorphism of G' which must map to C_3 , because C_{18} is a subgraph of G' .

On the other hand, suppose G' has a fixed injective homomorphism to C_3 . Then for a vertex x in G , we assign its colour to be the image of $x_0, x_6, x_{12} \in V(X)$, which must be different for adjacent vertices x, y since the path P ensures its end vertices receive a different image.

Therefore, 3-colouring of 3-regular graphs polynomially transforms to injective C_3 -colouring. However, the former is NP-complete [8], and therefore, so is the latter. It must then follow that deciding if an oriented graph to be injective C_3 -colourable is NP-complete. \square

Theorem 3.3. *Deciding whether an oriented graph has an injective homomorphism to a reflexive transitive tournament on three vertices is NP-complete.*

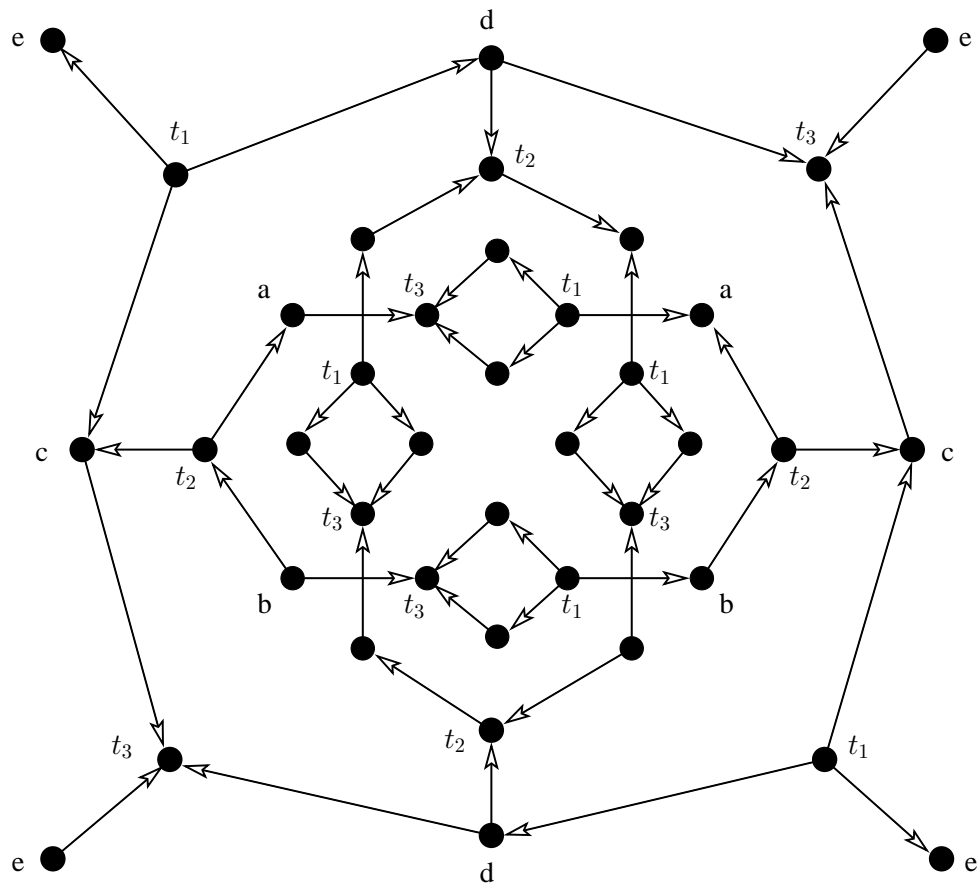


Figure 3.1: The Neighbour-Juggling Graph

Proof. As above, the problem is clearly in NP.

In Figure 3.1, we have what we call the *Neighbour-Juggling Graph*, but for immediate simplicity refer to the Neighbour-Juggling Graph as H . The first notable property is that there is no injective homomorphism of H to C_3 in Figure 2.1, because H has vertices of in-degree three.

Now consider any injective homomorphism of H to T_3 . Observe Figure 3.1 and see that we have labelled some of the vertices of H with their only possible images, while others are labelled with unindexed letters a, b, c, d , and e to facilitate our discussion. Labelling vertices with the image t_1 or t_3 is due to their respective degrees, while labelling vertices with t_2 is due to the need to satisfy the injective property within the neighbourhoods of the vertices labelled c and d . Further, vertices labelled with the same unindexed letter must map to the same image. Now, vertices labelled c , having in-degree two, may map to t_2 or t_3 , and we claim that either is possible. This is because there is a vertex of in-degree two labelled a , so that all vertices with that label may map to anything but t_1 . Similarly, all vertices labelled b may map to anything but t_3 . Hence, the neighbourhoods of vertices labelled a, b, c may always map injective, and the claim is true. By a parallel argument, the vertices labelled d may map to t_1 or t_2 . Altogether, it is possible for the vertices labelled e to have *any* image by mapping the neighbourhoods labelled with unindexed letters appropriately, effectively 'juggling' neighbourhoods. To be perfectly clear, those vertices labelled e cannot map to different images without violating injectivity in some neighbourhood within H .

By the juggling argument above, if we have a pre-assignment of the vertices in H labelled e , it can be extended to an injective homomorphism of H to T_3 .

As before, we wish to show that 3-edge-colourability of 3-regular graphs polynomially transforms to injective T_3 -colouring. Suppose we have a 3-regular graph G . We construct an oriented graph G' from G such that G' has an injective T_3 -colouring.

Replace every vertex x of G with a copy X of an in-star with three leaves, giving the leaves themselves and their incident edges an order $\{1, 2, 3\}$. For each edge xy of G , being the i^{th} edge incident with x , and the j^{th} edge incident with y , replace it with a copy of H , and identify the two vertices labelled e of in-degree one, the first with the i^{th} leaf of X and the second with the j^{th} leaf of Y . The transformation replaces each of the n vertices of G with 7 elements and each of the m edges of G with 88 elements, while taking $2m$ steps to connect replaced structures. The transformation may therefore be accomplished in polynomial time.

We claim that G is 3-edge-colourable if and only if G' has an injective homomorphism to T_3 . Suppose G is equipped with a 3-edge-colouring, $f : E(G) \rightarrow \{t_1, t_2, t_3\}$ (the vertices of T_3). Then for any edge xy of G , assign the vertices labelled e in the copy of H associated with xy to map to $f(xy)$. For the center of each in-star of G' , map these vertices to their only possible image, t_3 . We have already shown that each copy of H in G' can have the current pre-assignment extended to an injective homomorphism of G' , which must map to T_3 because H is a subgraph of G' .

Conversely, suppose G' was endowed with an injective homomorphism g to T_3 . For each edge xy of G , its corresponding structure H in G' must have the same image at the connecting leaves of in-stars X and Y , so we use this for the colour of xy . In addition, the edges incident with any vertex x of G must have different colours because the leaves of X in G' must map differently to satisfy the injective property.

Therefore, 3-edge-colourability of 3-regular graphs polynomially transforms to injective T_3 -colourability. However, the former problem is NP-complete [13], so that the latter must be so as well. Then deciding whether any oriented graph has an injective T_3 -colouring must also be NP-complete. \square

Corollary 3.4. *Deciding whether an oriented graph is k -rio-colourable for fixed $k \geq 3$ is NP-complete.*

Proof. The problem is clearly in NP, as with the previous theorems.

We wish to show that the problem of injective C_3 -colouring of oriented graphs polynomially transforms to injective T_k -colouring, where T_k is a reflexive tournament consisting of a C_3 and a reflexive transitive tournament T_{k-3} on $(k-3)$ vertices such that every vertex is adjacent to the vertices of C_3 . We form an oriented graph G' from some given oriented graph G with the following adjustment: attach to every vertex x of G a copy of T_{k-3} by adding arcs from the vertices of T_{k-3} to make them adjacent to the vertex x , and finally, for every vertex t of each copy of T_{k-3} , add three vertices, t_a, t_b, t_c , making t adjacent to them.

We claim that G has an injective C_3 -colouring if and only if G' has an injective T_k -colouring. Suppose that G did have an injective homomorphism f of G to C_3 , so that we simply extend f to an injective T_k -colouring, as each copy of T_{k-3} has all of its vertices in the neighbourhood of some vertex x of G , and hence, we can simply map each T_{k-3} directly to the transitive subtournament of T_k . Note that an injective homomorphism of each T_{k-3} to T_k is unique, since the vertices of T_{k-3} have exactly the same in/out-degree as the vertices of the transitive subtournament of T_k because of the arcs from t to x, t_a, t_b, t_c for each vertex t in each copy of T_{k-3} . To finish, we map two of t_a, t_b, t_c to the two vertices of $(V(C_3) - \{f(x)\})$, and whichever vertex is leftover we map to t in T_k . Hence, G' has an injective T_k -colouring.

Conversely, suppose G' had an injective T_k -colouring, call it g . We know that none of the vertices t of each copy of T_{k-3} in G' can have $g(t) \in V(C_3)$, for the out-degree of the vertices of C_3 in T_k is two, while the out-degree of t is at least four, and we have already shown that if t maps to a vertex of the reflexive transitive subtournament of T_k , it does so uniquely because of the vertices x, t_a, t_b, t_c , where again, x is the vertex of G associated with this particular instance of T_{k-3} . Hence, every copy of T_{k-3} by its only possible mapping, forces $g(x) \in V(C_3)$ for all $x \in V(G)$. Since all vertices of G must map to $V(C_3)$, and g is a rio-colouring, we then have an injective C_3 -colouring of G .

Therefore, injective C_3 -colouring of oriented graphs polynomially transforms to injective T_k -colouring. Yet the former problem is NP-complete by Theorem 3.2, so that the latter must be as well.

Altogether, since a k -rio-colouring can be seen as an injective T -colouring for some reflexive tournament T on k vertices, it must be that k -rio-colouring of oriented graphs is also NP-complete. \square

Characterization

The low threshold of NP-completeness leaves us one remaining non-trivial case, the 2-rio-colourable graphs, which we can completely describe.

Unmistakably, any orientation of a graph with a vertex of degree three cannot be 2-rio-coloured, so that any orientation of the star graph S_3 is an obstruction; call the set of these \mathcal{S} .

Let \mathcal{P} be the family of oriented paths P of odd order $n = 2k + 1 \geq 5$ with vertex set $\{v_0, v_1, \dots, v_{2k}\}$ and arc set of two disjoint maximum matchings, M_1, M_2 , such that M_1 has arcs incident on v_0, v_1 and v_2, v_3 oriented the same direction, with arcs of $(P - \{v_0, v_1\}) \cap M_1$ alternating in orientation along P , and M_2 has arcs incident on v_{2k}, v_{2k-1} and v_{2k-2}, v_{2k-3} oriented the same direction, with arcs of $(P - \{v_{2k}, v_{2k-1}\}) \cap M_2$ alternating in orientation along P . Also, no oriented path $P \in \mathcal{P}$ can be rio-coloured with two colours because the natural colouring of heads and tails of arcs in M_i always violates the injective property at one end of P , depending on our choice of $i = 1$ or $i = 2$; otherwise a 2-colouring violates the oriented property. We show in 5.1 that any oriented path has an injective homomorphism to C_3 , so that any $P \in \mathcal{P}$

has a 3-*rio-colouring*.

Combine the above into $\mathcal{F} = \mathcal{S} \cup \mathcal{P}$. Then the elements of \mathcal{F} are, of course, critically 3-*rio-colourable*. Hence, they are obstructions of oriented graphs with *rio-chromatic number two*, and further, are *rio-obstructions* such that $Forb(\mathcal{F})$ is the set of 2-*rio-colourable* graphs, established below.

Theorem 4.1. *Let D be a digraph. The following are equivalent:*

- (i) *D has a 2-*rio-colouring*.*
- (ii) *No graph from \mathcal{F} has an injective homomorphism to D .*
- (iii) *D is a disjoint union of: paths of order $n < 5$, paths having a maximal matching M that saturates every internal vertex of the path, while all arcs of M alternate in orientation, and cycles of order $n = 4k$, $k \geq 1$, with such a matching that instead saturates every vertex.*

Proof. ((i) \Rightarrow (ii)) Suppose $D \xrightarrow{\text{inj}} T_2$. If for $F \in \mathcal{F}$ we have $F \xrightarrow{\text{inj}} D$, then by composition of injective homomorphisms $F \xrightarrow{\text{inj}} T_2$, yet $\chi!(F) = 3$.

((ii) \Rightarrow (iii)) We prove the contrapositive. If D has a component that is not a path nor cycle, the underlying undirected graph of this component must have a vertex of degree three, and so for some $S \in \mathcal{S}$ we have $S \xrightarrow{\text{inj}} D$. Hence, we may further assume $\Delta(D) \leq 2$.

If a component D' of D is an oriented cycle of odd length, then by parity it is not possible to have every *other* arc alternating in orientation around the cycle. Then without loss of generality, let the vertex set of D' be $V = \{v_0, v_1, \dots, v_{2k}\}$ where the arcs incident on v_0, v_1 and v_2, v_3 are oriented the same direction so that we may form a path $P : v_0 v_1 \dots v_{2k} v'_0 v'_1 v'_2 v'_3$ where the prime notation denotes copies of vertices, and where the arc set of P is induced from D' , copied vertices inducing arcs from their original vertices. Note that P is of odd order, having two copies of the vertices

v_0, v_1, v_2 and v_3 . Take two disjoint maximum matchings of P , call them M_1, M_2 . Let two consecutive arcs of M_1 oriented in the same direction be denoted a and b , which must exist as we may have $a = v_0v_1, b = v_2v_3$. Let two consecutive arcs of M_2 be oriented in the same direction such that they are situated as close as possible to a and b , call them c and d , which must be possible, since, at worst, $c = v'_0v'_1, d = v'_2v'_3$. These arcs may be so close together that we have the oriented path $acbd$, but $acbd \in \mathcal{P}$. In fact, any distance between a, b, c, d forms a path starting with a and ending with d from \mathcal{P} , since the arcs between them must alternate as described, else a, b, c, d would not be of minimum distance away from each other. Note that any path starting with a and ending with d must be of odd order, since a and d belong to different maximum matchings.

If some component D' of D is an even oriented cycle with length not divisible by four, then take two disjoint maximum matchings, M_1, M_2 , and note that each $|M_i|$ is odd. Because the M_i are arcs in a cycle, by parity, there is no way to have arcs of M_i alternating in orientation as we traverse D' , so that we again have arcs a, b, c, d as described previously, and the argument is the same for some path of \mathcal{P} being a subgraph of D' as before.

Finally, suppose D has some component D' an oriented cycle of length divisible by four or a path of order $n \geq 5$, both without a maximal matching that has arcs alternating in orientation. Let M_1, M_2 be the two disjoint maximal matchings of D . Since neither M_1 nor M_2 satisfy said properties, each must contain an arc that breaks the alternation of orientation pattern. Then again, we have arcs a, b, c, d as described previously, so that some path in \mathcal{P} is a subgraph of D .

((iii) \Rightarrow (i)) It is easy to see that such graphs are 2-rio-colourable. □

Unless $P = NP$, it is unlikely that, for $k \geq 3$, the k -rio-colourable graphs admit a “yes, if and only if no theorem”. An example of such a theorem is “a graph G is 2-colourable if and only if it has no odd cycle.” A succinct certificate that a graph

is 2-colourable is the 2-colouring itself; it is easy to check whether the description passes as a 2-colouring of the given graph. A succinct certificate that a graph is not 2-colourable is an odd cycle. It is easy to check whether what is presented is in fact an odd cycle in the graph. Decision problems that admit such “yes if and only if no” theorems are usually solvable by a polynomial-time algorithm. An NP-complete problem is solvable by a polynomial-time algorithm if and only if $P = NP$. Since most people believe that $P \neq NP$, a “yes if and only if no” characterization is unlikely when the corresponding decision problem is NP-complete.

The characterizations below of the oriented cycles are examples of certificates that are not succinct, which we should expect from our results with the complexity of 3-rio-colourings in Chapter 3.

We make use of the term *factor* of an oriented graph to mean a spanning subgraph which has no isolated vertices, as well as a *traversal* as some fixed enumeration of the vertices of an oriented cycle, so that we *traverse* each vertex by considering the vertices in the order they appear in the enumeration. Also, the *net length* of a factor is the number of forward arcs minus the number of backward arcs with respect to a fixed orientation of the cycle it belongs to.

Lemma 4.2. *An oriented cycle C is C_3 -colourable if and only if there exists a factor in C consisting of directed paths whose net length is congruent to 0 modulo 3.*

Proof. Consider any such factor of C , say M , of net length $3k + i$ where k is some nonzero integer and $i = 0, 1, 2$. Label the vertices of C as v_1, v_2, \dots, v_n starting at some end vertex of a directed path P of M , following along the edges of P and continuing to the end vertex of the directed path immediately preceding P ; thus, the arc incident with v_1 and v_n is not in M . Let f be a homomorphism from C to C_3 defined by the following rules: map end vertices of M so that arcs of the matching $(E(C) - M)$ map to the reflexive arcs of C_3 and map all internal vertices of directed paths of M so that their arcs map to irreflexive arcs of C_3 . Note that any injective

C_3 -colouring can be described as f , because the subdigraph

$$\langle g^{-1}(c_1) \rangle \cup \langle g^{-1}(c_2) \rangle \cup \langle g^{-1}(c_3) \rangle$$

is a matching, for any injective homomorphism g to C_3 .

Now suppose $i = 0$ for some such M ; then we have M mapping around C_3 exactly k times, or in other words $f(v_1) = f(v_n)$, so that C is C_3 -colourable.

On the other hand, suppose $i = 1$ or 2 for every such M . When $i = 1$, suppose we may choose our labelling of vertices of C so that the arc $v_{n-1}v_n$ is possible. Then after mapping the vertices of C in their labelled order ensuring that we agree with the target C_3 , we inevitably arrive at $f(v_1) = f(v_{n-1})$, which contradicts injective property of f on the neighbourhood of v_n .

For the case when $i = 2$, simply traverse M in the opposite direction to reverse the sign of k , so that we have $i = 1$ and proceed as in the previous case.

The only case left is when $i = 2$ and all arcs of M are oriented in the positive direction, or equivalently, $i = 1$ and all the arcs of M are oriented in the negative direction, but without loss of generality, we deal with it in the former description. In this case, note that we cannot have the arc v_nv_1 , otherwise we would be able to include this arc in M to form a factor of directed paths that have net length 0 modulo 3. Then say we have $f(v_1) = c$, where c is any vertex of C_3 , and let c^- be the in-neighbour of c different from c . Then after mapping the vertices of C in their labelled order to agree with C_3 as the target, we have $f(v_n) = c^-$, however the arc v_1v_n is then mapped to cc^- , which contradicts the oriented property of the homomorphism f .

Thus, if there is no such matching M with $i = 0$ then there is no C_3 -colouring of C . Altogether, this establishes the lemma. \square

To characterize the oriented cycles that have a homomorphism to T_3 , we need a

special oriented two-way infinite path we call P_∞ with vertex and arc set

$$\{\dots v_{-2}, v_{-1}, v_0, v_1, v_2 \dots\} \quad \{\dots v_0v_1, v_1v_2, v_2v_3 \dots\}$$

such that the same pattern of three arcs is repeated throughout the path.

We also need to define the *signed sum of lengths* of oriented paths which are subgraphs of P_∞ in a factor M of an oriented cycle C as the sum of their underlying path lengths with positive length if an oriented path maps to T_3 in a clockwise fashion and negative if it maps counterclockwise.

Lemma 4.3. *An oriented cycle C has a T_3 -colouring if and only if there exists a factor M of C the components of which are oriented paths isomorphic to some subgraph of P_∞ such that the signed sum of lengths of the underlying undirected paths is congruent to 0 modulo 3.*

Proof. Suppose C has a T_3 -colouring obtained from an injective homomorphism f . Then the set of all arcs uv of C with $f(u) = f(v)$ is a matching B , so let $M = C - B$ be our factor. Note that no arc in M maps to a loop. Notice that if we traverse T_3 without using loops, i.e.: traversing clockwise, $t_1, t_2, t_3, t_1, t_2, t_3, \dots$, we follow the same arc pattern as P_∞ , so that all components of M must be subdigraphs of P_∞ . Thus, the length of each component gives us the number of edges used to traverse the irreflexive triangle in T_3 , considering a clockwise traversal as being a positive length, and a counterclockwise traversal as being negative. Choose any end vertex v as our origin for counting the lengths of the components in M . Then because the end vertex u in M adjacent to v in C both have the same image under f , the signed sum of lengths of the components in M must be congruent to 0 modulo 3.

Conversely, suppose we have our factor M of C . Starting with some end vertex of M and following its adjacent internal vertices we define a homomorphism f of C to T_3 as we traverse C with the following rules: f induces a mapping of the arcs in

M to no loop of T_3 , while the arcs in the matching $E(C) - M$ are mapped to the loops of T_3 . As discussed previously, the components of M traverse T_3 in a clockwise or counterclockwise fashion and their lengths keep track of which image the end vertices are assigned. Upon reaching the last vertex after assigning images, because the components of M have signed sum of lengths congruent to 0 modulo 3, it must be that the first and last vertex we assigned have the same image in T_3 . Therefore, C is T_3 -colourable. \square

Naturally, one would wonder if all oriented cycles are 3-rio-colourable, and if a counterexample does exist, the above lemmas hint that it must be fairly well behaved, since an oriented cycle C that contains either a directed path of length five or any combination of two (not necessarily distinct) oriented paths in Figure 4.1, automatically has a factor that ensures a mapping to C_3 .

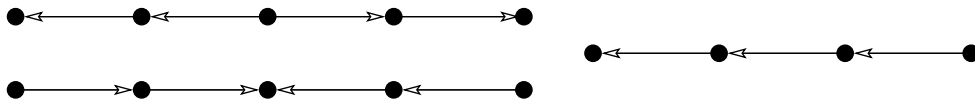


Figure 4.1: Oriented paths of note as subdigraphs of oriented cycles

Further, recall Lemma 2.1 to notice that we already have an infinite set of graphs that neither have an injective homomorphism to C_3 nor T_3 : simply take any $C \in \mathcal{C}$ of order congruent to ± 2 modulo 12. Unfortunately, we do not characterize oriented cycles completely, but we will show, in Lemma 5.1, that their rio-chromatic number is never more than four.

Lower and Upper Bound of $\chi!$ on Various Oriented Graphs

The goal of this chapter is to determine lower and upper bounds to the rio-chromatic number of oriented graphs within certain families, in terms of the maximum oriented degrees, Δ^- , Δ^+ . Satisfying the injective property immediately implies $\Delta^- + \Delta^+ \leq \chi!$, so to follow the usual trend, we would like to use the maximum oriented degrees to formulate an upper bound as well. Of course, no upper bound exists in terms of just one of Δ^- or Δ^+ , because it is always possible to make one parameter arbitrarily larger than the other for some oriented graph and satisfying the injective property for the larger parameter disproves any bounds exclusively using the smaller.

As in the case for oriented colourings satisfying the injective property only on in-neighbourhoods, as well as all other cases, formulating an upper bound on $\chi!(G)$ using $\Delta^-(G)$ and $\Delta^+(G)$ for general oriented graphs G is possible, but it will always be exponential. We use the same oriented graph D as in [16] to show this, the disjoint union of tournaments on k vertices by adding a dominating (or dominated, it makes no difference for our case) vertex to each one. Then every k -tournament T in D is

in the neighbourhood of some vertex, so that T must appear as a subdigraph of the target, and therefore the target must be a k -universal tournament with order at least $2^{(k-1)/2}$ [18], where a k -universal tournament is a tournament that contains every tournament on k vertices as a subgraph. Then, explicitly, $\chi_!(D) \geq 2^{(k-1)/2}$.

Contrast the upper bound with the injective chromatic number's upper bound using Δ as the maximum degree, $\chi_i \leq \Delta(\Delta - 1) + 1$, found in [9]. The cause for the upper bound of the rio-chromatic number being exponential is clearly due to the oriented property.

For any oriented graph G , we have the underlying graph of G , which we denote as G' . The injective chromatic number of G' , $\chi_i(G')$, returning to [9], is related to the original chromatic number of G' , where $\chi(G') \leq \chi_i(G')$ is established in their Lemma 2. Taking an injective colouring of G' immediately satisfies the same injective property of a rio-colouring of G , but it does not generally satisfy the oriented property, so that more colours are sometimes necessary to do so. Hence, we have $\chi(G') \leq \chi_i(G') \leq \chi_!(G)$, with the only exception being an oriented K_2 . Also, we have the irreflexive injective oriented chromatic number, $\chi_!^{irr}(G) \leq 2\chi_!(G)$, which is established by taking a rio-colouring and splitting every colour class (a matching) into two new colour classes, one for the head of each arc, and one for the tails.

5.1 Oriented Cycles

We finalize our discussion of oriented cycles from the previous section. Take C_3 , add a vertex t that is adjacent to every vertex of C_3 , and add a loop to t ; call the resulting tournament on four vertices T . Of course, T also contains T_3 as a subtournament.

Lemma 5.1. *Every oriented path has an injective homomorphism to C_3 , and every oriented cycle has an injective homomorphism to T .*

Proof. We prove the statement for paths during the proof of the statement for cycles.

The case of directed cycles is easy, since they clearly map to C_3 .

The case of C of order n from \mathcal{C} is characterized for $n \equiv 0, 4, 6, 8 \pmod{12}$, so suppose $n \equiv \pm 2 \pmod{12}$. Then $n \equiv 2, 4 \pmod{6}$. Select a vertex v of out-degree two from C with out-neighbours u and w . Consider the path $C - v$ of length $6k + i$, for $i = 2, 4$, which has only two disjoint factors of directed paths, each a matching of net length $3k + j$, for $j = 1, 2$. Then following the logic of the proof of Lemma 4.2 it must be that for any injective homomorphism of $C - v$ to C_3 , u and w both map to different vertices of C_3 . But then we can always map v to t in T , and thus establish an injective homomorphism of C to T .

Then suppose we have an oriented cycle C that is neither a directed cycle nor from the family \mathcal{C} . Consider the largest directed subpath of C , so that there must exist an oriented subpath $uxvw$ in C with arcs $\{xu, xv, vw\}$. Take any injective homomorphism f of $C - x$ to C_3 , which must exist, because there is always a factor of $C - x$ consisting of directed paths of any net length modulo 3.

If $f(u) \neq f(v)$, then simply map x to t in T and we are done. Otherwise, modify f to map both x and v to t in T . In either case, f is then an injective homomorphism of C to T . □

Corollary 5.2. *For all oriented cycles C , we have $\chi!(C) \leq 4$.*

5.2 Rio-cliques

We define the oriented graphs for which the rio-chromatic number equals the order as *rio-cliques*, and analogously the same concept arises as io-cliques in [16]. They are worthy of analysis, for the largest subdigraph in an oriented graph yields a lower bound on its rio-chromatic number.

In characterizing rio-cliques, we are amused by the fact that they do not depend in any way on the orientation of arcs. It seems that when we demand the injective property to include the entire neighbourhood of any vertex, as opposed to just in-neighbourhoods, it is enough to reduce our discussion to the edges of a graph and

the undirected distance between its vertices.

Theorem 5.3. *Let G be an oriented graph with G' the underlying simple graph of G . Then G is a rio-clique if and only if every arc of G lies on a triangle and $\text{diam}(G') \leq 2$.*

Proof. Suppose we have an oriented graph G with every arc on a triangle and $\text{diam}(G') \leq 2$. Then for any two vertices x and y of G , whether they are adjacent or not, they share a neighbour, and hence, by injectivity, must be coloured distinctly.

Inversely, suppose first that $\text{diam}(G') \geq 3$. Then there exists $x, y \in V(G)$ such that $d(x, y) \geq 3$. Consider the rio-colouring where every vertex of G receives a different colour except x and y . Then it is an injective colouring because x and y do not share a neighbour, and it is oriented because x and y both have disjoint neighbourhoods each coloured completely distinct. Thus, G is not a rio-clique.

Now suppose G were of diameter two or less, but flaunted an arc xy not on a triangle. Now either $N(x) - \{y\} = \emptyset$ or $N(y) - \{x\} = \emptyset$, else $\text{diam}(G') > 2$. Consider again the rio-colouring where x and y are the only vertices to receive the same colour. It is trivial to see that this is an oriented colouring. Further, it is injective, since x and y induce the only non-trivial monochromatic subgraph of G . \square

An expert in this area will immediately recognize the same result for research that has been done on injective colourings [9], although they do not give such graphs a name, so we call a simple graph that has its reflexive injective chromatic number equal to its order an *ri-clique*, for obvious reasons. We formalize this equivalence with the following corollary, the truth of which is trivial, and so we omit the proof.

Corollary 5.4. *Let G be an oriented graph. Then G is an rio-clique if and only if the underlying graph of G is an ri-clique. Also, any orientation of an ri-clique is an rio-clique.*

Examples of rio-cliques that are not tournaments include orientations of: complete tripartite graphs, wheel graphs, book graphs, two complete graphs identified at a vertex; this is not meant to be exhaustive, and the reader could possibly list others.

Note that if we wish to obtain an even better lower bound on the rio-chromatic number of some oriented graph we may obtain one by paying attention to the collection of all its rio-cliques because together they may increase the lower bound to its rio-chromatic number, which follows the same asymptotic argument given for the general lower bound of the rio-chromatic number already given.

An immediate consequence of Theorem 5.3 is that these rio-cliques give us the ability to automatically disprove overly optimistic upper bounds of certain families of graphs. For instance, the upper bound on the rio-chromatic number of an oriented series parallel graph, in general, cannot be better than $6 \cdot \max\{\Delta^-, \Delta^+\} - 3$ due to the following construction of a series parallel rio-clique K : take a triangle $\{a, b, c\}$ and with each edge e of the triangle, connect the incident vertices of e to a set of $2k$ isolated vertices - orient the triangle as a directed cycle, orient half of the $2k$ 6-cycles in one direction, and the other half in the opposite direction. Then see that K has every arc in a triangle, $\text{diam}(K) = 2$, and K has no K_4 minor [1], since K has only three vertices with degree larger than three, and we cannot increase the degree of any vertex by contracting edges other than those on abc , which reduces the number of vertices with large degree. We have oriented the edges of K to minimize $\max\{\Delta^-, \Delta^+\} = k + 1$ in order to obtain the largest ratio between $\max\{\Delta^-, \Delta^+\}$ and the order n of K . Here, we have $n = 6k + 3$, and hence, the upper bound on the rio-chromatic number of series-parallel graphs, in general, cannot be lower than $6 \cdot \max\{\Delta^-, \Delta^+\} - 3$.

5.3 Analogue of Brooks' Theorem

The fundamental theorem of Brooks' [3] bounds the chromatic number of a graph, excepting complete graphs and odd cycles, by its maximum degree while characterizing the exceptions with $\chi = \Delta + 1$. In analogy with this result, we seek a bound on χ_i in terms of its maximum in- and out-degrees.

To establish an upper bound in terms of the maximum oriented degrees, we consider an injective colouring of an oriented graph G . With the work of Hahn et al. [9] we use their proof of Lemma 5 to obtain the neighbour graph $(G')^{(2)}$, with vertex set $V(G)$ and x adjacent with y when there exists a path of length two between x and y in G' , implying $\chi_i(G') \leq k^2 - k + 1$, where $k = \Delta^- + \Delta^+$. Then taking an injective colouring C of G' using χ_i colours, we need only adjust this colouring to satisfy the oriented property to rio-colour G . We do this formally below.

Theorem 5.5. *For all oriented graphs G we have $\chi(G) \leq 2^{t+1} - 4$, where $t = \chi_i(G) \leq (k^2 - k + 1)$, and $k = (\Delta^- + \Delta^+)$.*

Proof. Suppose we have an oriented graph G . Now any vertex x of G has at most k neighbours that each potentially have $(k - 1)$ neighbours other than x themselves, so that the maximum degree of a vertex in $(G')^{(2)}$ is at worst $k(k - 1)$. Since we have $\chi((G')^{(2)}) = \chi_i(G)$, using bounds on the chromatic number yields $\chi_i(G) \leq (k(k - 1) + 1)$. Then take C an injective t -colouring of G , for $t = \chi_i(G)$, although the oriented property will be met with quite a bit of adjustment of this colouring.

Here, we make use of the concept of the *signature* of a vertex v in G with respect to C , as defined in [16], which is a shorthand method of keeping track of the adjacency of v to/from each colour class of C . Suppose v is in colour class i , for $1 \leq i, j \leq t$, then the signature is an ordered $(k^2 + 1)$ -tuple, with a '+' in the j^{th} position if v is adjacent to a vertex in colour class j , or a '-' in the j^{th} position if v is adjacent from a vertex in colour class j . Note that C need not be proper, so that we need $j = i$ as

well.

Looking at colour class 2, and noticing that not all arcs between colour class 1 and 2 are oriented the same direction, we must then split colour class 2 into two colours, depending on adjacency with colour class 1. Although, this by itself is not sufficient, since colour class 2 contains disjoint arcs and we may break the oriented property if we introduce only one new colour here. Therefore, colour class 2 must be partitioned into four sets, each a different colour class, depending on the first two coordinates of the signature of each vertex.

Notice we have the same situation between colour class 3 and 1, as well as between colour class 3 and 2. Partitioning the same way, we must use the first three coordinates of signatures for vertices in colour class 3, which means this set then splits into eight sets.

Considering any vertex in colour class i of C , and the signatures which differ on the first i coordinates, there are 2^i such different signatures, so that the vertices of each type of signature must receive a unique colour to satisfy the oriented property of a rio-colouring of G . Therefore, for each colour class i , we must partition it into 2^i colours. Altogether, we accumulate $\sum_{i=2}^t 2^i = 2^{t+1} - 4$ colours for a rio-colouring of G . \square

It is ironic that we can get a much better bound for rio-cliques, say when we are working without knowledge of their order, but again, this is due to the fact that the oriented property is automatically satisfied when every vertex has a different colour. Applying Proposition 6 from [9], we have $\chi_1(G) = |V(G)| \leq (k^2 - k)$, for all non-complete rio-cliques G . Of course, complete rio-cliques only have $\Delta + 1$ vertices.

5.4 Oriented Trees

We now turn our attention to obtaining a better bound on the rio-chromatic number of oriented trees in terms of the maximum in-degree or maximum out-degree. Since

no arc of an oriented tree lies on a triangle, rio-cliques do not give us any further insight.

We wish to reduce our argument in what follows to only involve one of the two maximum oriented degrees, viz. Δ^- . To accomplish this, if we have an oriented tree T with $\Delta^+(T) \geq \Delta^-(T)$ that is already rio-coloured, or more precisely, T has an injective homomorphism f to a reflexive tournament H , then simply orient all arcs of both T and H the opposite direction to obtain T' , H' and note that f can be used as an injective homomorphism of T' to H' . Thus, T and T' can both be rio-coloured with the same number of colours.

Lemma 5.6. *Let T be any oriented tree. Then $\chi_!(T) \leq 2 \cdot \max\{\Delta^-, \Delta^+\} + 1$.*

Proof. Let k be some positive integer, and H a reflexive $(2k+1)$ -regular tournament. We show that any oriented tree T with $\Delta^+ \leq \Delta^- \leq k$ has an injective homomorphism to H via induction on $n = |V(T)|$.

For the basis step, start with a directed in-star with k leaves, which obviously has an injective homomorphism to H .

Now suppose any tree T with $\Delta^- \leq k$ and order $n \geq (k+1)$ has an injective homomorphism to H .

Say we had a tree T with $\Delta^- \leq k$ of order $(n+1)$. Remove a leaf v from T whose neighbour we call u and apply the inductive hypothesis to $(T-v)$. Since

$$d_{T-v}^-(u) \leq (\Delta^- - 1) \quad \text{or} \quad d_{T-v}^+(u) \leq (\Delta^+ - 1)$$

depending on whether v was an in-neighbour or an out-neighbour of u , there is always an extra vertex in H left for the image of v , which guarantees the injective property is satisfied, so that our lemma holds true. \square

Showing the upper bound was quite trite, seeing as how we never even needed to use a loop of H on an injective homomorphism. However, we assure the bound is

sharp and the proof of this cruxes on some vertex of the target graph being forced to fill two conflicting roles when the target is too small.

Theorem 5.7. *The bound of Lemma 5.6 is sharp.*

Proof. We consider the tree T with height two, and $\Delta^- = \Delta^+ = k$, where the root v of T has the property that for all $w \in N[v]$, $d^-(w) = d^+(w) = k$. Suppose, to the contrary, that T could have an injective homomorphism f to a reflexive tournament H on $2k$ vertices. Now v has $2k$ neighbours, so that we must have $u \in N^\pm(v)$ such that $f(v) = f(u)$ in H . Without loss of generality, say $u \in N^+(v)$, so that $f(v)f(u)$ must be counted in the out-degree of $f(v)$, and $f(v)$ must have k in-neighbours distinct from itself. However, from the viewpoint of $f(u)$, we must have $f(v)f(u)$ counted in the in-degree of $f(u)$, so that $f(u)$ has k out-neighbours distinct from itself. But then we have $2k$ neighbours of $f(v) = f(u)$, plus $f(v)$ itself, contradicting the digraph target H having only $2k$ vertices. \square

An Algorithm for Rio-colourings of Trees

Suppose we have some oriented tree T together with a reflexive digraph H . What we would like to know is whether T has an injective homomorphism to H or not. We present an algorithm that determines this, and further finds such a homomorphism if one exists.

Start by assigning a list of possible images $L(v) \subseteq V(H)$ to each vertex v of T , excluding $w \in V(H)$ from $L(v)$ only when the in or out degree of w is less than that of v . Formally, $L(v) = \{w \in V(H) \mid d^\pm(w) \geq d^\pm(v)\}$. We call the elements of $L(v)$ the *candidates* of v .

Dissecting the tree, we take some vertex x of T to be the root, and partition the vertices of T into level sets determined by their distance from x in the underlying graph T' of T , $V_i = \{v \in V(T) \mid d_{T'}(x, v) = i\}$, for $0 \leq i \leq \ell$, where ℓ is the eccentricity of x , and we call V_i the i^{th} *level* of T . Further, V_0 is $\{x\}$ which we call the *top* of T and V_ℓ is contained in the set of leaves of T which we refer to as the *bottom*.

Next, for all the vertices of T , starting at the bottom and working towards the

top, we must step through each one's candidates and ensure that three conditions are satisfied:

- (i) candidates extend; that is, for any candidate t of $v \in V_i$, the in-neighbours (resp. out-neighbours) of v in V_{i+1} have in-neighbours (resp. out-neighbours) of t as candidates. This also ensures arcs are preserved when mapped from T to H .
- (ii) for vertices v such that $|(N^-[v] \cup N^+[v])| \geq 3$ there is, for each $t \in L(v)$, an injective mapping to $(N^-[t] \cup N^+[t])$.
- (iii) a candidate t of a vertex $v \in V_i$ does not stalemate the ability to satisfy the injective property of some closed neighbourhood of a vertex w in $(V_{i+1} \cap (N^-(v) \cup N^+(v)))$, so that $N[w]$ for all such w are able to satisfy the injective property while simultaneously mapping v to t . There is a recursive check we must also perform, but we leave its description for later.

If any of the above conditions are broken for a candidate, then it is no longer considered to be a candidate and is removed from its list. Suppose that all candidates have been checked up to the i^{th} level. Take away the levels that have not been checked yet, so that we are left with a forest of trees, $(T - (\cup_{j=0}^{(i-1)} V_j))$, and let T_v be the tree rooted at $v \in V_i$ in this forest. Notice that when we iterate to v , if $L(v) \neq \emptyset$, we can define an injective homomorphism of T_v rooted at v to H selecting appropriate candidates as we backtrack our steps, which is the main motivation for condition (i), and we emphasize again that it is not solely for the express purpose of preserving arcs.

A candidate t of a vertex v , with $N^-(v) = \{v_1^-, v_2^-, \dots, v_k^-\}$ and $N^+(v) = \{v_1^+, v_2^+, \dots, v_j^+\}$, $k + j \geq 2$, satisfies condition (ii) if and only if there is a system of

distinct representatives (SDR) for

$$L(v_1^-) \cap N^-[t], L(v_2^-) \cap N^-[t], \dots, L(v_k^-) \cap N^-[t],$$

$$L(v_1^+) \cap N^+[t], L(v_2^+) \cap N^+[t], \dots, L(v_j^+) \cap N^+[t],$$

which collection of sets we refer to more simply as \mathbb{V}^t . This condition also prompts the creation of the set $A = \{v \in V(T) \mid |N^-[v] \cup N^+[v]| \geq 3\}$ because there is no need to check for an SDR of a collection involving one set in the case that v only has a single neighbour.

Condition (iii) is necessary because if we retrace our steps back down T , assigning candidates as we go, we must guarantee that a choice of candidate at level i can extend to an injective H -colouring of all previous levels back down to the bottom. We need only process $L(v)$ for some vertex $v \in V_i$ if $(V_{i+1} \cap (N^-(v) \cup N^+(v)))$ is non-empty, and since there is no need to check for SDR's at the bottom, there is no need to check condition (iii) holds at level $(\ell - 1)$; hence, let the set of all such vertices throughout $(T - (V_{\ell-1} \cup V_\ell))$ be A^* .

Before we stipulate the details of condition (iii), we need to emphasize the recursive property of $SDR(\cdot, \cdot)$, which is left undefined until after its appearance below. Let $v \in (A^* \cap V_i)$, so that v is an in-neighbour of vertices $\{v_1, v_2, \dots, v_k\} = (V_{i+1} \cap N^+(v) \cap A)$ and an out-neighbour of vertices $\{v_{k+1}, v_{k+2}, \dots, v_{k+j}\} = (V_{i+1} \cap N^-(v) \cap A)$. Then condition (iii) is satisfied for some $t \in L(v)$ (and subsequently remains in $L(v)$) if and only if there exists an SDR for \mathbb{V}^t , for which we denote the selection of candidates with $\mathbb{V}^t(\cdot) : N[v] \rightarrow V(H)$, such that for every $1 \leq p \leq (k+j)$, there is an SDR from $SDR(\mathbb{V}^t(v_p), v_p)$, say $\mathbb{V}^t(v_p) = z$, for

$$L(v_{p_1}^-) \cap N^-[z], L(v_{p_2}^-) \cap N^-[z], \dots, L(v_{p_q}^-) \cap N^-[z],$$

$$L(v_{p_1}^+) \cap N^+[z], L(v_{p_2}^+) \cap N^+[z], \dots, L(v_{p_r}^+) \cap N^+[z], \{t\} \cap N^\pm[z],$$

for $(N^-(v_p) - \{v\}) = \{v_{p_1}^-, v_{p_2}^-, \dots, v_{p_q}^-\}$, $(N^+(v_p) - \{v\}) = \{v_{p_1}^+, v_{p_2}^+, \dots, v_{p_r}^+\}$, and where $N^\pm(z)$ uses “-” for when $p \leq k$ or “+” otherwise. Again, we condense any future discussion of the above collection of sets by referring to it as \mathbb{V}_p^t , and denote the choice of candidates for its SDR with the function $\mathbb{V}_p^t(\cdot) : N[v_p] \rightarrow V(H)$ when needed. Note the intersection of A above, since we need only ensure there are SDR’s for neighbourhoods that are sufficiently large. Of course, the set of all such SDR’s involving t chosen from $L(v)$ will now be clearly understood as $SDR(t, v)$, and note that the definition is not complete until we set $SDR(t, v)$ equal to the set of all SDR’s of \mathbb{V}^t when v has only leaves for neighbours in the level below v .

We collect our discussion into polished form as Algorithm 6.1 and dub its title as *the H-sieve*. Note that line 19 of the *H-sieve* exits all loops and terminates the program.

Algorithm 6.1: (The H -sieve) removing vertex images from use in possible injective H -colourings

Input:

An oriented tree T with root vertex x .
 Level sets $V_0 = \{x\}$, $V_1 = N(x)$, \dots , V_ℓ .
 Sets A and A^* as previously defined.
 A target digraph H .

Output:

Candidates $L(v)$ for each vertex v of T as previously defined.
 Lists $SDR(t, v)$ of SDR's, for all $v \in V(T)$ and all $t \in L(v)$.

```

1 Assign lists to  $v \in V(T)$  as follows:  $L(v) = \{t \in V(H) \mid d^\pm(t) \geq d^\pm(v)\}$ ;
2 for  $i = \ell - 1$  to 0 do
3   foreach  $v \in V_i$  do
4     foreach  $t \in L(v)$  do
5       foreach arc  $vu$  ( $uv$ ) with  $u \in V_{i+1}$  do
6         if no  $w \in L(u)$  has  $tw$  ( $wt$ ) as an arc of  $H$  then
7           remove  $t$  from  $L(v)$ ;
8       if  $v \in A$  then
9         if the collection  $\mathbb{V}^t$  has no SDR then remove  $t$  from  $L(v)$ ;
10        else put SDR's into list  $SDR(t, v)$ ;
11       if  $v \in A^*$  then
12         foreach SDR  $f$  of  $SDR(t, v)$  do
13           find  $k = |V_{i+1} \cap N^+(v) \cap A|$  and  $j = |V_{i+1} \cap N^-(v) \cap A|$ ;
14           for  $p = 1$  to  $(k + j)$  do
15             if  $\mathbb{V}_p^t$  has no SDR from  $SDR(f(v_p), v_p)$  then
16               remove  $f$  from  $SDR(t, v)$ ;
17               set  $p = k + j$ ;
18           if  $SDR(t, v) = \emptyset$  then remove  $t$  from  $L(v)$ ;
19   if  $L(v) = \emptyset$  then break;
```

Note that Algorithm 6.1 has linear running time, since the target digraph H remains fixed, and therefore, so do the number of candidates t and the number of SDR's for \mathbb{V}^t of any vertex v of any tree T as input.

Theorem 6.1. *Let T be an oriented tree with root vertex x , and H a digraph. If the H -sieve Algorithm terminates without executing line 19, then for any $v \in V(T)$ and*

$t \in L(v)$ with SDR g for \mathbb{V}^t in $SDR(t, v)$, there exists an injective homomorphism f of T_v to H with $f(w) = g(w)$ for all $w \in (N[v] \cap V(T_v))$.

Proof. Any oriented tree T rooted at x with maximum height one (which are just oriented stars) clearly satisfy the statement, since any SDR of \mathbb{X}^t with $t \in L(x)$ is an injective H -colouring of T .

Let \mathcal{B} be the set of oriented trees that terminate the H -sieve with a nonempty set of candidates at their root, yet have a vertex v such that any $t \in L(v)$ with SDR g for \mathbb{V}^t from $SDR(t, v)$, and no injective homomorphism f of T_v to H has $f(w) = g(w)$ for all $w \in (N[v] \cap V(T_v))$.

We know that for any $T \in \mathcal{B}$, the order of T must be at least four, so suppose we take a minimal such T with root x . However, for all $v \in (V(T) - \{x\})$ we must have $T_v \notin \mathcal{B}$ by minimal choice of T .

Upon reconsideration of the H -sieve, $z \in L(x)$ must satisfy the three conditions discussed, and therefore, there is an SDR g for \mathbb{X}^z in $SDR(z, x)$ such that for each $v_p \in N(x) \cap A$ we have an SDR g_p for \mathbb{V}_p^t in $SDR(g(v_p), v_p)$. But we know $T_{v_p} \notin \mathcal{B}$ for all p , so each has an injective H -colouring f_p such that $f_p(w) = g_p(w)$ for all $w \in (N[v_p] \cap V(T_{v_p}))$. Since this holds for any $g \in SDR(z, x)$, it must be that $T \notin \mathcal{B}$, which implies that $\mathcal{B} = \emptyset$, else we have this contradiction. \square

Corollary 6.2. *Let H be a digraph and T an oriented tree with root vertex x . Then there exists an injective H -colouring of T if and only if the H -sieve terminates with $L(x) \neq \emptyset$.*

Proof. If $f : T \xrightarrow{\text{inj}} H$, then $d_H^\pm(f(v)) \geq d_T^\pm(v)$ for all $v \in V(T)$ and so $f(v)$ becomes a candidate for v .

Running the H -sieve on T , when we arrive at the vertex v , and check that $f(v)$ satisfies property (ii), we see that $f(N[v])$ is an SDR for $\mathbb{V}^{f(v)}$ and is placed in $SDR(f(v), v)$.

To check that property (iii) is satisfied for candidate $f(v)$, for when $v \in V_s$, we perform strong induction. We know (iii) is satisfied automatically for vertices in $V_{\ell-1}$, as we have SDR's for each such v from the previous paragraph. Let the neighbours v_p of v in $(V_{s+1} \cap A)$ be as described previously. Then suppose $f(v)$ satisfies (iii) for all $v \in \left(\bigcup_{i=1}^{\ell-(s-1)} V_{\ell-i}\right)$ such that $f(N[v_p])$ is in $SDR(f(v_p), v_p)$. Then immediately, we have that $f(N[v])$ is not removed from $SDR(f(v), v)$ by the inductive hypothesis on each v_p of v .

Therefore, the H -sieve completes without executing line 19 and $L(x) \neq \emptyset$.

The converse follows from Theorem 6.1, as $L(x) \neq \emptyset$ implies that line 19 is not executed. □

Corollary 6.3. *Let k be any fixed positive integer and T an oriented tree. Then there exists a linear-time algorithm that determines if T has a k -rio-colouring, and finds one if it exists.*

Proof. We know T has a k -rio-colouring if and only if T has an injective homomorphism to a reflexive tournament on k vertices. For fixed k , the number of such tournaments is a function of k alone and is in no way affected by the order of T . The existence of an injective homomorphism to each possible target tournament can be tested in linear time by Algorithm 6.1. □

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