NEW CHARACTERIZATIONS OF CERTAIN STARLIKE AND CONVEX GENERALIZED HYPERGEOMETRIC FUNCTIONS

By

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Applying various properties of a certain class of linear integral operators, the authors prove a number of theorems which provide interesting characterizations of starlike and convex generalized hypergeometric functions. Several useful corollaries are also deduced.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \qquad (a_1 = 1), \tag{1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. We denote by $\mathcal{S}^*$ and $\mathcal{K}$ the subclasses of $\mathcal{A}$ consisting of all starlike functions in $\mathcal{U}$ and of all convex functions in $\mathcal{U}$, respectively. Note that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$.

Let α_j $(j = 1,\ldots,p)$ and β_j $(j = 1,\ldots,q)$ be complex numbers with

$$\beta_j \neq 0, -1, -2, \ldots; \quad j = 1, \ldots, q.$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by (see, e.g., [6], p. 19)

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \qquad (p \leq q + 1), \end{aligned} \tag{2}$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ is the Pochhammer symbol.

In order to prove our characterization theorems for these generalized hypergeometric functions, we need the following lemmas due to Owa and Srivastava [4].

LEMMA 1. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (2) *satisfy the condition:*

$$\left|\frac{z\ {}_pF_q'(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z)}{{}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z)}\right| < 1 \tag{3}$$

for $z \in \mathcal{U}$. *Then*

$$z\ {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z) \in \mathcal{S}^*.$$

LEMMA 2. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (2) *satisfy the condition* (3) *for* $z \in \mathcal{U}$. *Then*

$$z\ {}_{p+1}F_{q+1}(\alpha_1,\ldots,\alpha_p,1;\ \beta_1,\ldots,\beta_q,2;\ z) \in \mathcal{K}.$$

LEMMA 3. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (2) *satisfy the condition:*

$$\left|\frac{z\ {}_pF_q''(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z)}{{}_pF_q'(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z)}\right| < 1 \tag{4}$$

for

$$z \in \mathcal{U} \text{ and } \prod_{j=1}^{p} \alpha_j \neq 0.$$

Then

$$z\ {}_{p+1}F_{q+1}(\alpha_1+1,\ldots,\alpha_p+1,1;\ \beta_1+1,\ldots,\beta_q+1,2;\ z) \in \mathcal{S}^*.$$

2. PROPERTIES OF A CERTAIN LINEAR INTEGRAL OPERATOR

Let $\mathcal{J}_\gamma(f)$ be a linear integral operator defined by

$$\mathcal{J}_\gamma(f) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)\,dt \qquad (\gamma > -1) \qquad (5)$$

for $f(z) \in \mathcal{A}$. The operator $\mathcal{J}_\gamma(f)$, when $\gamma \in \mathcal{N} = \{1,2,3,\ldots\}$ was studied by Bernardi [1]. In particular, the operator $\mathcal{J}_1(f)$ was considered earlier by Libera [2] and Livingston [3].

We require the following properties of $\mathcal{J}_\gamma(f)$ obtained by Pascu [5].

LEMMA 4. *If* $f(z) \in \mathcal{S}^*$, *then* $\mathcal{J}_\gamma(f) \in \mathcal{S}^*$ *for* $0 \leqq \gamma \leqq 1$.

LEMMA 5. *If* $f(z) \in \mathcal{K}$, *then* $\mathcal{J}_\gamma(f) \in \mathcal{K}$ *for* $0 \leqq \gamma \leqq 1$.

3. APPLICATIONS TO GENERALIZED HYPERGEOMETRIC FUNCTIONS

Our characterization theorem for a class of starlike generalized hypergeometric functions is contained in

THEOREM 1. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (2) *satisfy the condition* (3) *for* $z \in \mathcal{U}$. *Then*

$$z\ {}_{p+1}F_{q+1}(\alpha_1,\ldots,\alpha_p,\gamma+1;\ \beta_1,\ldots,\beta_q,\gamma+2;\ z) \in \mathcal{S}^*$$

for $0 \leqq \gamma \leqq 1$.

PROOF. First of all, we note that

$$z\ {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z) \in \mathcal{S}^*$$

by Lemma 1. Next, applying Lemma 4 to

$$z\ {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ z),$$

we have

$$\begin{aligned}\mathcal{J}_\gamma(z\ {}_pF_q) &= \frac{\gamma+1}{z^\gamma}\int_0^z t^\gamma\ {}_pF_q(\alpha_1,\ldots,\alpha_p;\ \beta_1,\ldots,\beta_q;\ t)\,dt \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \cdot \frac{(\gamma+1)_n}{(\gamma+2)_n} \cdot \frac{z^{n+1}}{n!} \\ &= z\ {}_{p+1}F_{q+1}(\alpha_1,\ldots,\alpha_p,\gamma+1;\ \beta_1,\ldots,\beta_q,\gamma+2;\ z) \in \mathcal{S}^*.\end{aligned}$$

which completes the proof of Theorem 1.

For $\gamma = 0$, Theorem 1 evidently yields

COROLLARY 1. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (2) satisfy the condition (3) for $z \in \mathcal{U}$. Then

$$z\ {}_{p+1}F_{q+1}(\alpha_1,\ldots,\alpha_p,1;\ \beta_1,\ldots,\beta_q,2;\ z) \in \mathcal{S}^*.$$

By using Lemma 2 and Lemma 5, we similarly derive

THEOREM 2. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (2) satisfy the condition (3) for $z \in \mathcal{U}$. Then

$$z\ {}_{p+2}F_{q+2}(\alpha_1,\ldots,\alpha_p,1,\gamma+1;\ \beta_1,\ldots,\beta_q,2,\gamma+2;\ z) \in \mathcal{K}$$

for $0 \leqq \gamma \leqq 1$.

By setting $\gamma = 1$ in Theorem 2, we obtain

COROLLARY 2. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (2) *satisfy the condition* (3) *for* $z \in \mathcal{U}$. *Then*

$$z \; {}_{p+1}F_{q+1}(\alpha_1,\ldots,\alpha_p,1;\ \beta_1,\ldots,\beta_q,3;\ z) \in \mathcal{K}.$$

Finally, by applying Lemma 3 and Lemma 4, we have

THEOREM 3. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (2) *satisfy the condition* (4) *for*

$$z \in \mathcal{U} \quad \text{and} \quad \prod_{j=1}^{p} \alpha_j \neq 0.$$

Then

$$z \; {}_{p+2}F_{q+2}(\alpha_1+1,\ldots,\alpha_p+1,1,\gamma+1;\ \beta_1+1,\ldots,\beta_q+1,2,\gamma+2;\ z) \in \mathcal{S}^*$$

for $0 \leqq \gamma \leqq 1$.

The special case of Theorem 3 when $\gamma = 1$ is given by

COROLLARY 3. *Let the generalized hypergeometric function* ${}_pF_q(z)$ *defined by* (2) *satisfy the condition* (4) *for*

$$z \in \mathcal{U} \quad \text{and} \quad \prod_{j=1}^{p} \alpha_j \neq 0.$$

Then

$$z \; {}_{p+1}F_{q+1}(\alpha_1+1,\ldots,\alpha_p+1,1;\ \beta_1+1,\ldots,\beta_q+1,3;\ z) \in \mathcal{S}^*.$$

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