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New Numerical Results on Existence of Volterra–Fredholm Integral Equation of Nonlinear Boundary Integro-Differential Type

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Abstract: Symmetry is presented in many works involving differential and integral equations. Whenever a human is involved in the design of an integral equation, they naturally tend to opt for symmetric features. The most common examples are the Green functions and linguistic kernels that are often designed symmetrically and regularly distributed over the universe of discourse. In the current study, the authors report a study on boundary value problem (BVP) for a nonlinear integro Volterra–Fredholm integral equation with variable coefficients and show the existence of solution by applying some fixed-point theorems. The authors employ various numerical common approaches as the homotopy analysis methodology established by Liao and the modified Adomain decomposition technique to produce a numerical approximate solution, then graphical depiction reveals that both methods are most effective and convenient. In this regard, the authors address the requirements that ensure the existence and uniqueness of the solution for various variations of nonlinearity power. The authors also show numerical examples of how to apply our primary theorems and test the convergence and validity of our suggested approach.

Keywords: boundary conditions; nonlinear integro-differential equations; Krasnoselskii fixed point theorem; Arzela–Ascoli theorem

MSC: 34A45; 34B15; 65L10; 45J05

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In the setting of integral equations, boundary value problems play a significant role in the theory of applied differential calculus. Differential equation of ordinary, partial, integro, and stochastic types are some examples (see [1–4]). Numerous mathematical formulations of mathematical phenomena incorporate integro-differential equations, which appear in many domains such as in physics, material sciences, fractional calculus theory, number theory, ecology, and epidemiology (see [5–9]). They often occur in approximation models of the real-world problems and this motivates the in-depth study of these types of integral models aiming to prove existence or/and uniqueness of their solutions.

The Fredholm, Volterra, and integro-differential equations have important features and are used widely in mathematics. Many mathematicians have explored generating functions and combinatorial sums of specific polynomials and integro-differential equations in particular. Because integral questions of this sort exist in many mathematical models, computer algorithms, engineering difficulties, physics, and fractional calculus theory (cf. other articles [10–16]).

On the other hand, the Adomian decomposition and its modifications are widely utilized in many fields of applied mathematics, particularly in integral equation theory. As a result, numerous researchers, including Wazwaz and his students, have researched these numerical methods in order to solve difficult problems and get accurate outcomes. In previous articles, these approaches were also applied to the numerical solution of Abel's integral equations, the Bagley–Torvik equations, the Fredholm and Volterra integral equations, the integro equations that play a significant role in mathematics to obtain meaningful relations and representations; see [17–24] and closed references therein.

Furthermore, the exploration and solution of integro differential equation of nonlinear Volterra and Fredholm types have attracted more and more attention by using homotopy analysis methods. Over the years, this method has been proposed to find solution of linear and nonlinear integral equations, for example, see [25–29].

Among the previous obtained results in study of the BVPs including the construction of an integro-differential solution are those obtained in previous studies. For instance, in this paper, we will consider a nonlinear integro-differential equation of the form and solving it by using modified Adomian and homotopy analysis methods,

$$\begin{aligned} \mho\varphi''(\mathtt{t}) + \mathcal{A}(\mathtt{t})\varphi'(\mathtt{t}) + \mathcal{B}(\mathtt{t})\varphi(\mathtt{t}) = \mathtt{f}(\mathtt{t}) + \lambda_1 \int_{b_0}^{\mathtt{t}} \Psi(\mathtt{t},\mathtt{y})[\varphi(\mathtt{y})]^p \mathtt{dy} \\ + \lambda_2 \int_{b_0}^{b_1} \Psi(\mathtt{t},\mathtt{y})[\varphi(\mathtt{y})]^q \mathtt{dy}, \quad \text{for } \mathtt{t} \in J, \end{aligned} \quad (1)$$

with the boundary conditions

$$\varphi(b_0) = \ell_1, \quad \varphi(b_1) = \ell_2, \quad \ell_1, \ell_2 \in \mathbb{R}, \quad (2)$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathtt{f}$ and the kernel Ψ are known functions under fulfilling the requirements to be provided in the following section. The parameters λ_1, λ_2 and $\mho$ are nonzero real parameters and p, q are finite natural numbers. While $\varphi(\mathtt{t})$ is an unknown function that must be discovered in the space $\mathcal{C}^2(J, \mathbb{R})$ and $J = [b_0, b_1]$.

On the other hand, the main advantage of this problem is the application of boundary value problem on integral equation, which enable us to convert and analyse the problem to an ordinary differential equation. In addition, the large number of studies on differential equations in neutral, delay, and KdV with different boundary conditions have been investigated by researchers; see for example, [30–32].

The rest of our study is arranged as follows: In Section 1, we recall the main concepts, and existence and uniqueness of the solution. Section 2 describes the methods of solution of (1) by the algorithms proposed in this article in detail in Sections 2.1 and 2.2, respectively. Section 3 describes the numerical results and analysis. Finally, Section 4 gives the conclusion of our study.

1. Basic Tools and Existence of Solutions

In this section we briefly review some basic elements of the Volterra–Fredholm integral equations and integro-differential equations. For a comprehensive study on these topics, we refer the interested reader to [33–37].

Definition 1 (See [33]). *Let (X, d) be a metric space. Then, we say a function $\mathtt{f}{:}X \to X$ is a contraction mapping, if there is a non-negative real number $0 \le k < 1$ such that*

$$\mathtt{d}(\mathtt{f}(\mathtt{x}), \mathtt{f}(\mathtt{y})) \le k\mathtt{d}(\mathtt{x}, \mathtt{y}), \quad \text{for all} \quad \mathtt{x}, \mathtt{y} \in X.$$

Theorem 1 (See [37]). *Suppose that $g(\mathtt{x}) = lim_{n\to\infty} g_n(\mathtt{x})$ on J, where $g, g_1, g_2, \ldots$ are all Riemann integrable functions on I. If $\{g_n(x)\}_{n=1}^{\infty}$ is uniformly bounded on J, then $\int_{b_0}^{b_1} g(\mathtt{t})\mathtt{dt} = lim_{n\to\infty} \int_{b_0}^{b_1} g_n(\mathtt{t})\mathtt{dt}$ and $lim_{n\to\infty} \int_{b_0}^{b_1} |g_n(\mathtt{t}) - g(\mathtt{t})|\mathtt{dt} = 0$*

Theorem 2 (See [34]). *Let (X, d) be a metric space, then for each contraction mapping $\tau : X \to X$ has a unique fixed point of τ in X.*

Theorem 3 (See [35]). *Let $(X, ||.||)$ be a Banach space over $\mathbb{R}$ and $\mathcal{K}$ be a nonempty closed, convex and bounded subset of X. Any Compact operator $b_0 : \mathcal{K} \to \mathcal{K}$ has at least one fixed point.*

Our two theorems are considering the Arzela–Ascoli theorem and Krasnoselskii fixed point theorem, respectively.

Theorem 4 (See [33]). *Every bounded and equicontinuous sequence in the closed and bounded interval $[a, b]$ has a uniformly convergent subsequence.*

Theorem 5 (See [36]). *Let X be a Banach space and μ be a closed and convex nonempty subset of X, then the functions $\mathcal{H}, \mathcal{K} : \mu \to X$ with the following properties:*

1. *$\mathcal{H}$ is a contraction mapping,*
2. *$\mathcal{K}$ is compact and continuous,*
3. *for all $\mathtt{x}, \mathtt{y} \in \mu$, such that $\mathcal{H}\mathtt{x} + \mathcal{K}\mathtt{y} \in \mu$.*

   Then, there is $\mathtt{y}$ in μ such that $\mathcal{H}\mathtt{y} + \mathcal{K}\mathtt{y} = \mathtt{y}$.

Let us briefly recall the following concepts that will be involved in proving the next theorem of existence and uniqueness of the solutions.

Main postulates:

We suppose the following Hypotheses to prove all theorems.

Hypothesis 1 (H1). The functions $\mathcal{A}$ and $\mathcal{B}$ belong to $\mathcal{C}(J, \mathbb{R})$.

Hypothesis 2 (H2). The known free function $\mathtt{f}$ is a member of the $\mathcal{C}^2(J, \mathbb{R})$.

Hypothesis 3 (H3). For any $\mathtt{y} \in J$, the known kernel $(\mathtt{t}, \mathtt{y}) \mapsto \Psi(\mathtt{t}, \mathtt{y})$ is continuous in $\mathtt{t}$, for all $\mathtt{t} \in \mathbb{R}$.

$$\left(\int_{b_0}^{b_1} (\Psi(\mathtt{t}, \mathtt{y}))^2 dy\right)^{\frac{1}{2}} \le \gamma, \ \text{for all } \mathtt{t} \in J, \ \gamma > 0,$$

Hypothesis 4 (H4).

$$(\xi + k_1|\lambda_1|\varepsilon_1(1) + |\lambda_2|\varepsilon_2(1)) \le |\mho|,$$

where

$$\xi = \Big((b_1 - b_0)\|\mathcal{A}\|_\infty + (b_1 - b_0)^2\|\mathcal{B}\|_\infty\Big),$$

$$\varepsilon_1(l) = \binom{p}{l} \frac{\gamma(b_1 - b_0)^{2l+\frac{1}{2}}\sqrt{\Upsilon_1(l)}}{\sqrt{2p - 2l + 1}},$$

$$\varepsilon_2(l) = \binom{q}{l} \frac{\gamma(b_1 - b_0)^{2l+\frac{1}{2}}\sqrt{\Upsilon_2(l)}}{\sqrt{2q - 2l + 1}},$$

$$\mathrm{Y}_1(l) = \{\ell_2^{2p-2l} + \ell_2^{2p-2l-1}\ell_1 + \cdots + \ell_1^{2p-2l}\},$$

and

$$\mathrm{Y}_2(l) = \{\ell_2^{2q-2l} + \ell_2^{2q-2l-1}\ell_1 + \cdots + \ell_1^{2q-2l}\}.$$

Hypothesis 5 (H5).

$$(\xi + |\lambda_1|\Lambda_1 + |\lambda_2|\Lambda_2) \leq |\mho|,$$

where

$$\Lambda_1 = \sum_{l=1}^{p} \frac{\rho_1(l)k_1^l\varepsilon_1(l)}{(b_1-b_0)^{3l-3}},$$

and

$$\Lambda_2 = \sum_{l=1}^{q} \frac{\rho_2(l)\varepsilon_2(l)}{(b_1-b_0)^{3l-3}},$$

where $\rho_1(l), \rho_2(l)$ such that $\rho_1(1) = 1 = \rho_2(1)$ are finite positive constants depend on l and k_1 is a positive real numbers.

Theorem 6. *If the conditions* (**H1**)–(**H3**) are applied, then (1)–(2) *are reduce to the as follows Equation (3) is nonlinear Volterra–Fredholm integral equations (NVFIE).*

$$\begin{aligned}
&\mho\sigma(\mathtt{t}) + \int_{b_0}^{b_1}[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}} \mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y} \\
&\quad - \lambda_2\int_{b_0}^{b_1} \mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y}]\sigma(\mathtt{x})d\mathtt{x} \\
&\quad = \mathcal{F}(\mathtt{t}) + \lambda_1\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y} \\
&\quad + \lambda_2\int_{b_0}^{b_1}\sum_{l=2}^{q}\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y},
\end{aligned} \tag{3}$$

where

$$\sigma(\mathtt{t}) := \varphi''(\mathtt{t}), \tag{4}$$

$$\mathcal{H}(\mathtt{t},\mathtt{x}) := \frac{1}{(b_1-b_0)}\begin{cases} (\mathtt{x}-b_0)(\mathcal{A}(\mathtt{t}) - (b_1-\mathtt{t})\mathcal{B}(\mathtt{t})) & b_0 \leq \mathtt{x} \leq \mathtt{t} \\ (\mathtt{x}-b_1)(\mathcal{A}(\mathtt{t}) - (b_0-\mathtt{t})\mathcal{B}(\mathtt{t})) & \mathtt{t} \leq \mathtt{x} \leq b_1 \end{cases}, \tag{5}$$

$$\mathcal{S}_p(\mathtt{t},\mathtt{y};l) := \binom{p}{l}\frac{\Psi(\mathtt{t},\mathtt{y})}{(b_1-b_0)^p}[\ell_1(b_1-\mathtt{y}) + \ell_2(\mathtt{y}-b_0)]^{p-l}, \tag{6}$$

$$\mathcal{S}_q(\mathtt{t},\mathtt{y};l) := \binom{q}{l}\frac{\Psi(\mathtt{t},\mathtt{y})}{(b_1-b_0)^q}[\ell_1(b_1-\mathtt{y}) + \ell_2(\mathtt{y}-b_0)]^{q-l}, \tag{7}$$

$$\mathscr{M}_2(\mathtt{y},\mathtt{x}) := \begin{cases} (\mathtt{x}-b_0)(\mathtt{y}-b_1) & b_0 \leq \mathtt{x} \leq \mathtt{y} \\ (\mathtt{x}-b_1)(\mathtt{y}-b_0) & \mathtt{y} \leq \mathtt{x} \leq b_1 \end{cases}, \tag{8}$$

$$\mu(\mathtt{t}) := \frac{(-\mathcal{A}(\mathtt{t}) + (b_1-\mathtt{t})\mathcal{B}(\mathtt{t}))\ell_1 + (\mathcal{A}(\mathtt{t}) + (\mathtt{t}-b_0)\mathcal{B}(\mathtt{t}))\ell_2}{(b_1-b_0)}, \tag{9}$$

$$\mathcal{F}(\mathtt{t}) := \mathtt{f}(\mathtt{t}) + \lambda_1\int_{b_0}^{\mathtt{t}}\mathcal{S}_p(\mathtt{t},\mathtt{y};0)d\mathtt{y} + \lambda_2\int_{b_0}^{b_1}\mathcal{S}_q(\mathtt{t},\mathtt{y};0)d\mathtt{y} - \mu(\mathtt{t}). \tag{10}$$

Proof. If a function $\mathtt{t} \mapsto \sigma(\mathtt{t})$ belongs $C(J,\mathbb{R})$, then:

$$\text{Let} \quad \varphi^{''}(\mathtt{t}) = \sigma(\mathtt{t}), \tag{11}$$

$$\varphi^{'}(\mathtt{t}) = \int_{b_0}^{\mathtt{t}} \sigma(\mathtt{x})\mathrm{d}\mathtt{x} + \varphi^{'}(b_0), \tag{12}$$

and

$$\varphi(\mathtt{t}) = \int_{b_0}^{\mathtt{t}} (\mathtt{t}-\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x} + \ell_1 + (\mathtt{t}-b_0)\varphi^{'}(b_0). \tag{13}$$

It follows from Equation (13) with $\mathtt{t} = b_1$ that

$$\varphi^{'}(b_0) = \frac{1}{(b_1-b_0)}\left[(\ell_2-\ell_1) + \int_{b_0}^{b_1}(x-b_0)\sigma(x)dx\right] \tag{14}$$

and by putting Equation (14) into Equation (12), we obtain

$$\varphi^{'}(\mathtt{t}) = \frac{1}{(b_1-b_0)}\left[(\ell_2-\ell_1) + \int_{b_0}^{b_1}\mathcal{H}_1(\mathtt{t},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right]. \tag{15}$$

By using the result in Equations (12) and (11), we can obtain

$$\varphi(\mathtt{t}) = \frac{1}{(b_1-b_0)}\left[(\mathtt{t}-b_0)\ell_2 + (b_1-\mathtt{x})\ell_1 + \int_{b_0}^{b_1}\mathcal{H}_2(\mathtt{t},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right], \tag{16}$$

where

$$\mathscr{M}_1(\mathtt{t},\mathtt{x}) := \begin{cases} (b_0-\mathtt{x}) & b_0 \le \mathtt{x} \le \mathtt{t} \\ (b_1-\mathtt{x}) & \mathtt{t} \le \mathtt{x} \le b_1 \end{cases}, \tag{17}$$

$$\mathscr{M}_2(\mathtt{t},\mathtt{x}) := \begin{cases} (b_1-\mathtt{t})(b_0-\mathtt{x}) & b_0 \le \mathtt{x} \le \mathtt{x} \\ (b_0-\mathtt{t})(b_1-\mathtt{x}) & \mathtt{t} \le \mathtt{x} \le b_1 \end{cases}, \tag{18}$$

and

$$[\varphi(\mathtt{t})]^p = \frac{1}{(b_1-b_0)^p}\sum_{l=0}^{p}\binom{p}{l}[(\mathtt{t}-b_0)\ell_2 + (b_1-\mathtt{t})\ell_1]^{p-l}\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{t},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right)^l, \tag{19}$$

$$[\varphi(\mathtt{t})]^q = \frac{1}{(b_1-b_0)^q}\sum_{l=0}^{q}\binom{q}{l}[\ell_1(b_1-\mathtt{t}) + \ell_2(\mathtt{t}-b_0)]^{q-l}\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{t},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right)^l. \tag{20}$$

Substitution Equations (11), (15), (16), (19) and (20) into Equation (1) to get

$$\mho\sigma(\mathtt{t}) + \frac{\mathcal{A}(\mathtt{t})}{b_1 - b_0}(\ell_2 - \ell_1) + \frac{1}{b_1 - b_0}\int_{b_0}^{b_1}[\mathcal{A}(\mathtt{t})M_1(t,x) + \mathcal{B}(\mathtt{t})\mathscr{M}_2(\mathtt{t},\mathtt{x})]\sigma(\mathtt{x})\mathrm{d}\mathtt{x}$$
$$+ \frac{\mathcal{B}(\mathtt{t})}{b_1 - b_0}((b_1 - \mathtt{t})\ell_1 + (\mathtt{t} - b_0)\ell_2)$$
$$= \mathcal{F}(\mathtt{t}) + \frac{\lambda_1}{(b_1 - b_0)^p}\int_{b_0}^{\mathtt{t}}\sum_{l=0}^{p}\binom{p}{l}\Psi(\mathtt{t},\mathtt{y})[(b_1 - \mathtt{t})\ell_1 + (\mathtt{t} - b_0)\ell_2]^{p-l}\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right)^l \mathrm{d}\mathtt{y}$$
$$+ \frac{\lambda_2}{(b_1 - b_0)^q}\int_{b_0}^{b_1}\sum_{l=0}^{q}\binom{q}{l}\Psi(\mathtt{t},\mathtt{y})[(b_1 - \mathtt{t})\ell_1 + (\mathtt{t} - b_0)\ell_2]^{q-l}\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right)^l \mathrm{d}\mathtt{y},$$

$$\begin{aligned}\mho\sigma(\mathtt{t}) + \int_{b_0}^{b_1}[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}}\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})\mathrm{d}\mathtt{y} - \lambda_2\int_{b_0}^{b_1}\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})\mathrm{d}\mathtt{y}]\sigma(\mathtt{x})\mathrm{d}\mathtt{x} \\ = \mathcal{F}(\mathtt{t}) + \lambda_1\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right)^l\mathrm{d}\mathtt{y} \\ + \lambda_2\int_{b_0}^{b_1}\sum_{l=2}^{q}\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right)^l\mathrm{d}\mathtt{y},\end{aligned} \tag{21}$$

where $\mathcal{F}(\mathtt{t}), \mathcal{H}(\mathtt{t},\mathtt{x}), \mathcal{S}_p(\mathtt{t},\mathtt{y};l), \mathcal{S}_q(\mathtt{t},\mathtt{y};l)$ and $\mu(\mathtt{t})$ are determined in Equations (5)–(7), (9) and (10) above, respectively.

The converse is straight forward and thereby it is omitted. □

The following theorem states that if the NVFIE (3) has a continuous solution if it meet the requirements (**H1**)–(**H4**).

Theorem 7. *If conditions* (**H1**)–(**H4**) *hold, then an NVFIE* (3) has a continuous solution.

Proof. Let $\nabla_\varsigma = \{\|\sigma\|_\infty = \sup_{\mathtt{t}\in J}|\sigma(\mathtt{t})| \leq \varsigma, \quad \sigma \in \mathcal{C}(J,\mathbb{R})\}$. Where the positive, finite solution of the Equation (22) is denoted by the symbol ς.

$$|\lambda_1|\sum_{l=1}^{p}(k_1\varsigma)^l\varepsilon_1(l) + |\lambda_2|\sum_{l=1}^{q}\varsigma^l\varepsilon_2(l) + (\xi - |\mho|)\varsigma + \|\mathcal{F}\|_\infty = 0, \tag{22}$$

and k_1 is an upper bound of $|\mathscr{M}_2(\mathtt{t},\mathtt{x})|$. Considering (3) and setting the following two operators

$$\begin{aligned}(\mathscr{P}_1\sigma_1)(\mathtt{t}) = \frac{1}{\mho}\mathcal{F}(\mathtt{t}) - \frac{1}{\mho}\int_{b_0}^{b_1}\Bigg[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}}\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})\mathrm{d}\mathtt{y} \\ - \lambda_2\int_{b_0}^{b_1}\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})\mathrm{d}\mathtt{y}\Bigg]\sigma(\mathtt{x})\mathrm{d}\mathtt{x},\end{aligned}$$

$$\begin{aligned}(\mathscr{P}_2\sigma_2)(\mathtt{t}) = \frac{\lambda_1}{\mho}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x})\right)^l\mathrm{d}\mathtt{y} \\ + \frac{\lambda_2}{\mho}\int_{b_0}^{b_1}\sum_{l=2}^{q}\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})\mathrm{d}\mathtt{x}\right)^l\mathrm{d}\mathtt{y},\end{aligned}$$

where σ_1, σ_2 are two arbitrary functions in the set ∇_ς. Now,

$$\begin{aligned}|(\mathscr{P}_1\sigma_1)(\mathtt{t})| &\leq \frac{1}{|\mho|}|\mathcal{F}(\mathtt{t})| + \frac{\varsigma}{|\mho|}\int_{b_0}^{b_1}|\mathcal{H}(\mathtt{t},\mathtt{x})|d\mathtt{x} + \frac{|\lambda_1|\varsigma}{|\mho|}\int_{b_0}^{\mathtt{t}}\int_{b_0}^{b_1}|\mathcal{S}_p(\mathtt{t},\mathtt{y};1)||\mathscr{M}_2(\mathtt{y},\mathtt{x})|d\mathtt{x}d\mathtt{y}\\ &+\frac{|\lambda_2|\varsigma}{|\mho|}\int_{b_0}^{b_1}\int_{b_0}^{b_1}|\mathcal{S}_q(\mathtt{t},\mathtt{y};1)||\mathscr{M}_2(\mathtt{y},\mathtt{x})|d\mathtt{x}d\mathtt{y}\\ &\leq \frac{1}{|\mho|}|\mathcal{F}(\mathtt{t})| + \frac{\xi\varsigma}{|\mho|} + \frac{k_1|\lambda_1|p\varsigma}{|\mho|(b_1-b_0)^{p-3}}\int_{b_0}^{b_1}\frac{|\Psi(\mathtt{t},\mathtt{y})|}{|(\ell_1-\ell_2)\mathtt{y}+(\ell_2 b_1-\ell_1 b_0)|^{1-p}}d\mathtt{y}\\ &+\frac{|\lambda_2|q\varsigma}{|\mho|(b_1-b_0)^{q-3}}\int_{b_0}^{b_1}\frac{|\Psi(\mathtt{t},\mathtt{y})|}{|(\ell_1-\ell_2)\mathtt{y}+(\ell_2 b_1-\ell_1 b_0)|^{1-q}}d\mathtt{y}\\ &\leq \frac{1}{|\mho|}|\mathcal{F}(\mathtt{t})| + \frac{\xi\varsigma}{|\mho|} + k_1|\lambda_1|\frac{p(b_1-b_0)^{\frac{5}{2}}(\Upsilon_1(1))^{\frac{1}{2}}\varsigma}{|\mho|(2p-1)^{\frac{1}{2}}}\Big(\int_{b_0}^{b_1}(\Psi(\mathtt{x},\mathtt{y}))^2d\mathtt{y}\Big)^{\frac{1}{2}}\\ &+|\lambda_2|\frac{q(b_1-b_0)^{\frac{5}{2}}(\Upsilon_2(1))^{\frac{1}{2}}\varsigma}{|\mho|(2q-1)^{\frac{1}{2}}}\Big(\int_{b_0}^{b_1}(\Psi(\mathtt{t},\mathtt{y}))^2d\mathtt{y}\Big)^{\frac{1}{2}}\\ &\leq \frac{1}{|\mho|}\|\mathcal{F}(\mathtt{t})\|_\infty + \frac{1}{|\mho|}(\xi + (k_1|\lambda_1|\varepsilon_1(1) + |\lambda_2|\varepsilon_2(1))\varsigma. \end{aligned} \tag{23}$$

By using the same arguments as above, we can deduce

$$\begin{aligned}|(\mathscr{P}_2\sigma_2)(\mathtt{t})| &\leq \frac{|\lambda_1|}{|\mho|}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)|\left(\int_{b_0}^{b_1}|\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})|d\mathtt{x}\right)^l d\mathtt{y}\\ &+\frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\sum_{l=2}^{q}|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)|\left(\int_{b_0}^{b_1}|\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})|d\mathtt{x}\right)^l d\mathtt{y}\\ &\leq |\lambda_1|\sum_{l=2}^{p}\binom{p}{l}\frac{(b_1-b_0)^{2l+\frac{1}{2}}(\Upsilon_1(l))^{\frac{1}{2}}(k_1\varsigma)^l}{|\mho|(2p-2l+1)^{\frac{1}{2}}}\left(\int_{b_0}^{b_1}(\Psi(\mathtt{t},\mathtt{y}))^2d\mathtt{y}\right)^{\frac{1}{2}}\\ &+ |\lambda_2|\sum_{l=2}^{q}\binom{q}{l}\frac{(b_1-b_0)^{2l+\frac{1}{2}}(\Upsilon_2(l))^{\frac{1}{2}}\varsigma^l}{|\mho|(2q-2l+1)^{\frac{1}{2}}}\left(\int_{b_0}^{b_1}(\Psi(\mathtt{t},\mathtt{y}))^2d\mathtt{y}\right)^{\frac{1}{2}}\\ &\leq \frac{1}{|\mho|}\left(|\lambda_1|\sum_{l=2}^{p}(k_1\varsigma)^l\varepsilon_1(l) + |\lambda_2|\sum_{l=2}^{q}\varsigma^l\varepsilon_2(l)\right). \end{aligned} \tag{24}$$

From Equations (23) and (24), it follows that

$$\begin{aligned}\|\mathscr{P}_1(\sigma_1)+\mathscr{P}_2(\sigma_2)\|_\infty &\leq \|\mathscr{P}_1(\sigma_1)\|_\infty + \|\mathscr{P}_2(\sigma_2)\|_\infty\\ &\leq \frac{1}{|\mho|}\|\mathcal{F}(\mathtt{t})\|_\infty + \frac{\varsigma}{|\mho|}(\xi + (k_1|\lambda_1|\varepsilon_1(1) + |\lambda_2|\varepsilon_2(1))\\ &+\frac{1}{|\mho|}\left(|\lambda_1|\sum_{l=2}^{p}\varepsilon_1(l)(k_1\varsigma)^l + |\lambda_2|\sum_{l=2}^{q}\varepsilon_2(l)\varsigma^l\right) = \varsigma. \end{aligned} \tag{25}$$

Therefore,

$$\mathscr{P}_1(\sigma_1) + \mathscr{P}_2(\sigma_2) \in \nabla_\varsigma, \forall \sigma_1, \sigma_2 \in \nabla_\varsigma.$$

Now, if $\mathtt{t}_1, \mathtt{t}_2$ are two elements in J, without loss generality $\mathtt{t}_1 < \mathtt{t}_2$. Applying the conditions (**H1**)–(**H3**) and using the continuous functions $\mathcal{F}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathtt{t}$, we have

$$
\begin{aligned}
|(\mathscr{P}_1\sigma_1)(\mathtt{t}_2)-(\mathscr{P}_1\sigma_1)(\mathtt{t}_1)| \leq{}& \frac{1}{|\mho|}|\mathcal{F}(\mathtt{t}_2)-\mathcal{F}(\mathtt{t}_1)| + \frac{\varsigma}{|\mho|(b_1-b_0)}\int_{b_0}^{\mathtt{t}_1}|\mathcal{H}_1(\mathtt{t}_2,\mathtt{x})-\mathcal{H}_1(\mathtt{t}_1,\mathtt{x})|\mathtt{dx} \\
&+ \frac{\varsigma}{|\mho|(b_1-b_0)}\int_{b_0}^{\mathtt{t}_1}|\mathcal{H}_2(\mathtt{t}_2,\mathtt{x})-\mathcal{H}_2(\mathtt{t}_1,\mathtt{x})|\mathtt{dx} + \int_{b_0}^{\mathtt{t}_1}|\mathcal{H}_1(\mathtt{t}_2,\mathtt{t})-\mathcal{H}_2(\mathtt{t}_1,\mathtt{x})|\mathtt{dx} \\
&+ \frac{p\varsigma k_1|\lambda_1|}{(b_1-b_0)^{p-3}|\mho|}\times\int_{b_0}^{b_1}\frac{|\Psi(\mathtt{t}_2,\mathtt{y})-\Psi(\mathtt{t}_1,\mathtt{y})|}{|(\ell_1-\ell_2)\mathtt{y}+(\ell_2 b_1-\ell_1 b_0)|^{1-p}}\mathtt{dy} \\
&+ \frac{|\lambda_2|q\varsigma}{|\mho|(b_1-b_0)^{q-3}}\times\int_{b_0}^{b_1}\frac{|\Psi(\mathtt{t}_2,\mathtt{y})-\Psi(\mathtt{t}_1,\mathtt{y})|}{|(\ell_1-\ell_2)\mathtt{y}+(\ell_2 b_1-\ell_1 b_0)|^{1-q}}\mathtt{dy}.
\end{aligned} \tag{26}
$$

We can conclude that the right-hand side of Equation (26) is independent of $\sigma \in \nabla_\varsigma$. In addition, it tends zero when $\mathtt{t}_2 - \mathtt{t}_1$ tends zero. Therefore, this leads to $|(\mathscr{P}_1\sigma_1)(\mathtt{t}_2) - (\mathscr{P}_1\sigma_1)(\mathtt{t}_1)|$ approaches zero.

Also, we have

$$
\begin{aligned}
|(\mathscr{P}_2\sigma_2)(\mathtt{t}_2)-(\mathscr{P}_2\sigma_2)(\mathtt{t}_1)| \leq{}& \frac{|\lambda_1|}{|\mho|}\sum_{l=2}^{p}\binom{p}{l}(b_1-b_0)^{3l-p}(k_1\varsigma)^l \\
&\times\int_{b_0}^{b_1}\frac{|\Psi(\mathtt{t}_2,\mathtt{y})-\Psi(\mathtt{t}_1,\mathtt{y})|}{|(\ell_1-\ell_2)\mathtt{y}+(\ell_2 b_1-\ell_1 b_0)|^{2l-2p}}\mathtt{dy} \\
&+ \frac{|\lambda_2|}{|\mho|}\sum_{l=2}^{q}\binom{q}{l}(b_1-b_0)^{3l-q}\varsigma^l\times\int_{b_0}^{b_1}\frac{|\Psi(\mathtt{t}_2,\mathtt{y})-\Psi(\mathtt{t}_1,\mathtt{y})|}{|(\ell_1-\ell_2)\mathtt{y}+(\ell_2 b_1-\ell_1 b_0)|^{2l-2q}}\mathtt{dy}.
\end{aligned} \tag{27}
$$

Considering Equation (27), if $\mathtt{t}_2 - \mathtt{t}_1$ approaches zero, then $\mathtt{dy}$ tends zero. Hence, the set $(\mathscr{P}_1 + \mathscr{P}_2)\nabla_\varsigma$ is equicontinuous. Also, we have $\mathscr{P}_1\sigma_1$ and $\mathscr{P}_2\sigma_2$ are two elements in $(J, \mathbb{R})$. As a result, considering $\nabla\varsigma$, $\mathscr{P}_1 + \mathscr{P}_2$ is a self-operator. If σ and $\sigma*$ are any two functions, then they belong to $\mathscr{P}_1\varsigma$. Therefore,

$$
\|\mathscr{P}_1(\sigma) - \mathscr{P}_1(\sigma^*)\|_\infty \leq \frac{1}{|\mho|}(\xi + k_1|\lambda_1|\varepsilon_1(1) + |\lambda_2|\varepsilon_2(1))\|\sigma - \sigma^*\|_\infty, \tag{28}
$$

and according to the condition (5), $\mathscr{P}_1$ is a contraction operator on ∇_ς.

$$
\|\mathscr{P}_1(\sigma) - \mathscr{P}_1(\sigma^*)\|_\infty \leq \|\sigma - \sigma^*\|_\infty.
$$

Let $\{\sigma_n\}_{n\in\mathbb{N}}$ with $\sigma_n \in \nabla_\varsigma$ be a sequence such that σ_n approaches σ whereas n tends to ∞. Then, for any two elements σ_n, σ which contains in ∇_ς and $\forall \mathtt{t} \in J$, we have

$$
\begin{aligned}
|(\mathscr{P}_2\sigma_n)(\mathtt{t})-(\mathscr{P}_2\sigma)(\mathtt{t})| &\leq \frac{|\lambda_1|}{|\mho|}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)| \\
&\times\left[\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma_n(\mathtt{x})d\mathtt{x}\right)^l-\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l\right]d\mathtt{y} \\
&+\frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\sum_{l=2}^{q}|\mathcal{S}_q(\mathtt{t},\mathtt{y};l)|\left[\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma_n(\mathtt{x})d\mathtt{x}\right)^l-\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l\right]d\mathtt{y} \\
&\leq \frac{|\lambda_1|}{|\mho|}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)|\rho_1(l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})|\sigma_n(\mathtt{x})-\sigma(\mathtt{x})|d\mathtt{x}\right)d\mathtt{y} \\
&+\frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\sum_{l=2}^{q}|\mathcal{S}_q(\mathtt{t},\mathtt{y};l)|\rho_2(l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})|\sigma_n(\mathtt{x})-\sigma(\mathtt{x})|d\mathtt{x}\right)d\mathtt{y}.
\end{aligned}
$$

By applying (1), it follows that

$$
\begin{aligned}
&\lim_{n\to\infty}|(\mathscr{P}_2\sigma_n)(\mathtt{t})-(\mathscr{P}_2\sigma)(\mathtt{t})| \\
&\leq \frac{|\lambda_1|}{|\mho|}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)|\rho_1(l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\lim_{n\to\infty}|\sigma_n(\mathtt{x})-\sigma(\mathtt{x})|d\mathtt{x}\right)d\mathtt{y} \\
&+\frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\sum_{l=2}^{q}|\mathcal{S}_q(\mathtt{t},\mathtt{y};l)|\rho_2(l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\lim_{n\to\infty}|\sigma_n(\mathtt{x})-\sigma(\mathtt{x})|d\mathtt{x}\right)d\mathtt{y}=0,
\end{aligned}
$$

where $\rho_1(l)$ and $\rho_2(l)$ are finite positive real numbers that depend on l. As a result, the operator $\mathscr{P}_2$ is a sequentially continuous operator on ∇_ς.

The sequence $\mathscr{P}_2(\sigma_n)$ is uniformly bounded on J since

$$|\mathscr{P}_2\sigma(\mathtt{t})| \leq \frac{1}{|\mho|}\left(|\lambda_1|\sum_{l=2}^{p}(k_1\varsigma)^l\varepsilon_1(l)+|\lambda_2|\sum_{l=2}^{q}\varsigma^l\varepsilon_2(l)\right).$$

Furthermore, the sequence $\mathscr{P}_2(\sigma_n)$ is equicontinuous because

$$|\mathscr{P}_2(\sigma_n)(\mathtt{t}_2)-\mathscr{P}_2(\sigma_n)(\mathtt{t}_1)|<\epsilon,\text{ as }|\mathtt{t}_2-\mathtt{t}_1|<\xi,\forall n\in\mathbb{N}.$$

The sequence $\mathscr{P}_2(\sigma_n)$ has a subsequence $\mathscr{P}_2\{\sigma_{nk}\}_{k\in\mathbb{N}}$ which uniformly converges according to the Arzela–Ascoli theorem (4). Furthermore, the operator $\mathscr{P}_2$ is completely continuous and the collection $\mathscr{P}_2\nabla_\varsigma$ is compact. After satisfying all of the conditions of the Krasnosel'skii theorem (5), the operator $\mathscr{P}_1+\mathscr{P}_2$ has at least one fixed point in ∇_ς, which is a solution to the NVFIE (3). □

Theorem 8. *The NVFIE* (21) *has a unique solution, whenever the conditions* (**H1**)–(**H3**) *and* (**H5**) *are satisfied.*

Proof. It is obvious that the operator $\mathscr{P}_1+\mathscr{P}_2$ is a self-adjoint operator on ∇_ς. By using the same method used in Equation (28), we have

$$\|\mathscr{P}_1(\sigma)-\mathscr{P}_1(\sigma^*)\|_\infty \leq \frac{1}{|\mho|}(\xi+k_1|\lambda_1|\varepsilon_1(1)+|\lambda_2|\varepsilon_2(1))\|\sigma-\sigma^*\|_\infty.$$

Also, $\forall\sigma, \sigma^* \in \nabla_\varsigma$, we have

$$\|\mathscr{P}_2(\sigma) - \mathscr{P}_2(\sigma^*)\|_\infty \leq \frac{1}{|\mho|}\left(|\lambda_1|\sum_{l=2}^{p}\frac{\rho_1(l)k_1^l\varepsilon_1(l)}{(b_1-b_0)^{3l-3}} + |\lambda_2|\sum_{l=2}^{q}\frac{\rho_2(l)\varepsilon_2(l)}{(b_1-b_0)^{3l-3}}\right)\|\sigma-\sigma^*\|_\infty. \tag{29}$$

By using Equation (28) with $\rho_1(1) = 1 = \rho_2(1)$ and (29), it follows that

$$\begin{aligned}
&\|(\mathscr{P}_1+\mathscr{P}_2)(\sigma) - (\mathscr{P}_1+\mathscr{P}_2)(\sigma^*)\|_\infty \\
&\leq \|(\mathscr{P}_1)(\sigma) - (\mathscr{P}_1)(\sigma^*)\|_\infty + \|(\mathscr{P}_2)(\sigma) - (\mathscr{P}_2)(\sigma^*)\|_\infty \\
&\leq \frac{1}{|\mho|}(\xi + |\lambda_1|k_1\varepsilon_1(1) + |\lambda_2|\varepsilon_2(1))\|\sigma-\sigma^*\|_\infty \\
&+ \frac{1}{|\mho|}\left(|\lambda_1|\sum_{l=2}^{p}\frac{\rho_1(l)k_1^l\varepsilon_1(l)}{(b_1-b_0)^{3l-3}} + |\lambda_2|\sum_{l=2}^{q}\frac{\rho_2(l)\varepsilon_2(l)}{(b_1-b_0)^{3l-3}}\right)\|\sigma-\sigma^*\|_\infty \\
&\leq \frac{1}{|\mho|}(\xi + |\lambda_1|\Lambda_1 + |\lambda_2|\Lambda_2)\|\sigma-\sigma^*\|_\infty.
\end{aligned}$$

Hence, one can have

$$\|(\mathscr{P}_1+\mathscr{P}_2)(\sigma) - (\mathscr{P}_1+\mathscr{P}_2)(\sigma^*)\|_\infty \leq \|\sigma-\sigma^*\|_\infty. \tag{30}$$

We conclude that the operator is contraction on ∇_ς according to the Banach contraction principal (3) and the condition (**H5**). As a result, the NVFIE (21) has a unique continuous solution in ∇_ς. □

2. Methods of Solutions

Our main section is divided into two subsections that deal with the solution technique for the given nonlinear problem including the modified Adomain decomposition and homotopy analysis methods.

2.1. The Modified Adomain Decomposition Method Solution

If the criteria of Theorem 8 hold, then the next section will explain how can we apply the MADM to get an approximate solution to the NVFIE (21). Assume that the formula can be used to estimate the unknown function $\sigma(\mathtt{t})$ of the Equation (21)

$$\sigma(\mathtt{t}) = \sum_{n=0}^{\infty}\sigma_n(\mathtt{t}). \tag{31}$$

If $\mathcal{F}(\mathtt{t}) = \mathcal{F}_1(\mathtt{t}) + \mathcal{F}_2(\mathtt{t})$, then we have

$$\sigma_0(\mathtt{t}) = \frac{1}{\mho}\mathcal{F}_1(\mathtt{t}), \tag{32}$$

$$\begin{aligned}
\sigma_1(\mathtt{t}) &= \frac{1}{\mho}\mathcal{F}_2(\mathtt{t}) \\
&- \frac{1}{\mho}\left\{\int_{b_0}^{b_1}\left[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}}\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y} - \lambda_2\int_{b_0}^{b_1}\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y}\right]\sigma_0(\mathtt{x})d\mathtt{x}\right\} \\
&+ \frac{\lambda_1}{\mho}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\mathsf{A}_0(\mathtt{y},\mathtt{x})d\mathtt{y} + \frac{\lambda_2}{\mho}\int_{b_0}^{b_1}\sum_{l=2}^{q}\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\mathsf{A}_0(\mathtt{y},\mathtt{x})d\mathtt{y},
\end{aligned} \tag{33}$$

$$\sigma_n(\mathtt{t}) = -\frac{1}{\mho}\Bigg\{\int_{b_0}^{b_1}\Big[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}}\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y} - \lambda_2\int_{b_0}^{b_1}\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathcal{H}_2(\mathtt{y},\mathtt{x})d\mathtt{y}\Big]\sigma_{n-1}(\mathtt{x})d\mathtt{x}\Bigg\}$$

$$+\frac{\lambda_1}{\mho}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\mathsf{A}_{n-1}(\mathtt{y},\mathtt{x})d\mathtt{y} + \frac{\lambda_2}{\mho}\int_{b_0}^{b_1}\sum_{l=2}^{q}\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\mathsf{A}_{n-1}(\mathtt{y},\mathtt{x})d\mathtt{y}, \quad \forall n \geq 2, \tag{34}$$

$$\mathsf{A}_n(\sigma_n(\mathtt{t}),\mathtt{y};l) = \frac{1}{n!}\left(\frac{d^n}{d\nu^n}\left[\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sum_{i=0}^{\infty}\nu^i\sigma_i(\mathtt{x})d\mathtt{x}\right]^l\right)\Bigg|_{\nu=0}, \tag{35}$$

where A_n is an Adomain's polynomial, for $n = 0, 1, 2, \ldots$.

The following theorem holds if the conditions of Theorem 8 are satisfied:

Theorem 9. *The* NVFIE (21) *converges to the exact solution* $\sigma(\mathtt{t})$*, whenever the approximate solution established by Equations* (32)–(34)*.*

Proof. Define a sequence of partial sum $\{S_k(\mathtt{t})\}$ as follows:

$$S_k(\mathtt{t}) = \sum_{i=0}^{k}\sigma_i(\mathtt{t}).$$

For each pair of positive integers n, m with $n > m$ and $m \geq 1$, we have

$$\begin{aligned}
&\|S_n(\mathtt{t}) - S_m(\mathtt{t})\|_\infty \\
&= |\sum_{i=m+1}^{n}\sigma_i(\mathtt{t})| \\
&\leq \frac{1}{|\mho|}\int_{b_0}^{b_1}|\mathcal{H}(\mathtt{t},\mathtt{x})\sum_{i=m}^{n-1}\sigma_i(\mathtt{x})|d\mathtt{x} + \frac{|\lambda_1|}{|\mho|}\int_{b_0}^{b_1}\int_{b_0}^{\mathtt{t}}|\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})\sum_{i=m}^{n-1}\sigma_i(\mathtt{x})|d\mathtt{y}d\mathtt{x} \\
&+ \frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\int_{b_0}^{b_1}|\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})\sum_{i=m}^{n-1}\sigma_i(\mathtt{x})|d\mathtt{y}d\mathtt{x} \\
&+ \frac{|\lambda_1|}{|\mho|}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\sum_{i=m}^{n-1}\mathsf{A}_i(\mathtt{y},\mathtt{x})|d\mathtt{y}d\mathtt{x} + \frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\sum_{l=2}^{q}|\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\sum_{i=m}^{n-1}\mathsf{A}_i(\mathtt{y},\mathtt{x})|d\mathtt{y}d\mathtt{x}
\end{aligned}$$

$$\leq \frac{\xi}{|\mho|}\|S_{n-1}-S_{m-1}\|_\infty + \frac{k_1|\lambda_1|}{|\mho|}(b_1-b_0)^3\int_{b_0}^{b_1}|\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\sum_{i=m}^{n-1}\sigma_i(\mathtt{x})|\mathtt{dx}$$
$$+\frac{|\lambda_2|}{|\mho|}(b_1-b_0)^3\int_{b_0}^{b_1}|\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\sum_{i=m}^{n-1}\sigma_i(\mathtt{x})|\mathtt{dx}$$
$$+\frac{|\lambda_1|}{|\mho|}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\sum_{i=m}^{n-1}\sigma_i(\mathtt{x})\mathtt{dx}\right)^l|\mathtt{dy}+\frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\sum_{l=2}^{q}|\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\sum_{i=m}^{n-1}\sigma_i(\mathtt{x})\mathtt{dx}\right)^l|\mathtt{dy}$$
$$\leq \frac{1}{|\mho|}(\xi+(k_1|\lambda_1|\varepsilon_1(1)+|\lambda_2|\varepsilon_2(1)))\|S_{n-1}-S_{m-1}\|_\infty$$
$$+\frac{|\lambda_1|}{|\mho|}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}(b_1-b_0)^3\rho_1(l)\big|\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\big|\mathtt{dy}+\frac{|\lambda_2|}{|\mho|}\int_{b_0}^{b_1}\sum_{l=2}^{q}(b_1-b_0)^3\rho_2(l)\big|\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\big|\mathtt{dy}$$
$$\leq \frac{1}{|\mho|}\left(\xi+k_1|\lambda_1|\varepsilon_1(1)+|\lambda_2|\varepsilon_2(1)+|\lambda_1|\sum_{l=2}^{p}\frac{\rho_1(l)k_1^l\varepsilon_1(l)}{(b_1-b_0)^{3l-3}}\right)\|S_{n-1}-S_{m-1}\|_\infty$$
$$+\frac{1}{|\mho|}\left(|\lambda_2|\sum_{l=2}^{q}\frac{\rho_2(l)\varepsilon_2(l)}{(b_1-b_0)^{3l-3}}\right)\|S_{n-1}-S_{m-1}\|_\infty.$$

By setting $\rho_1(1),\rho_2(1)=1$, we have

$$\begin{aligned}&\|S_n(\mathtt{t})-S_m(\mathtt{t})\|_\infty\\&\leq\frac{1}{|\mho|}\left(\xi+|\lambda_1|\sum_{l=1}^{p}\frac{\rho_1(l)k_1^l\varepsilon_1(l)}{(b_1-b_0)^{3l-3}}+|\lambda_2|\sum_{l=1}^{q}\frac{\rho_2(l)\varepsilon_2(l)}{(b_1-b_0)^{3l-3}}\right)\|S_{n-1}-S_{m-1}\|_\infty\\&=\frac{1}{|\mho|}(\xi+|\lambda_1|\Lambda_1+|\lambda_2|\Lambda_2)\|S_{n-1}-S_{m-1}\|_\infty\\&=\vartheta\|S_{n-1}(\mathtt{t})-S_{m-1}(\mathtt{t})\|_\infty,\end{aligned}\tag{36}$$

where $\vartheta=\dfrac{(\mathcal{A}+|\lambda_1|\Lambda_1+|\lambda_2|\Lambda_2)}{|\mho|}$, and $\vartheta<1$. Take $n=m+1$ to get

$$\begin{aligned}\|S_{m+1}-S_m\|_\infty&\leq\vartheta\|S_m(\mathtt{t})-S_{m-1}(\mathtt{t})\|_\infty\\&\leq\vartheta^2\|S_{m-1}(\mathtt{t})-S_{m-2}(\mathtt{t})\|_\infty\\&\leq\cdots\leq\vartheta^m\|S_1(\mathtt{t})-S_0(\mathtt{t})\|_\infty\\&=\vartheta^m\|\sigma_1\|_\infty.\end{aligned}\tag{37}$$

Substituting the inequality (37) into the inequality (36), setting $n>m>N\in\mathbb{N}$, and applying the triangle inequality, we get

$$\|S_n-S_m\|_\infty\leq\frac{\vartheta^n}{1-\vartheta}\|\sigma_1\|_\infty=\epsilon,$$

where

$$\lim_{n\to\infty}\vartheta^n=0.$$

Therefore,

$$\forall n,m\in\mathbb{N},\quad \|S_n-S_m\|_\infty<\epsilon.$$

As a result, the sequence $\{S_k(\mathtt{t})\}$ is a Cauchy in the Banach space $\mathcal{C}(J,\mathbb{R})$, and hence,

$$\lim_{n\to\infty}S_n(\mathtt{t})=\sigma(\mathtt{t}).$$

□

2.2. The Homotopy Analysis Method Solution

In this section, we analyse for the NVFIE (21) under the conditions of Theorem 8 by applying the HAM (see [27]) to (2) as follows: If the criteria of Theorem 8 hold, then the HAM will be used to obtain an approximate solution to the NVFIE (21) in the section that follows. Equation (2) provides that

$$\sigma(\mathtt{t}) + \frac{1}{\mho}\left(\int_{b_0}^{b_1}\left[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}}\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y} - \lambda_2\int_{b_0}^{b_1}\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y}\right]\sigma(\mathtt{x})d\mathtt{x}\right) - \frac{1}{\mho}\mathcal{F}(\mathtt{t}) + \frac{\lambda_1}{\mho}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y} - \frac{\lambda_2}{\mho}\int_{b_0}^{b_1}\sum_{l=2}^{q}\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y} = 0. \quad (38)$$

We define the nonlinear operator $\mathcal{N}$ by

$$\mathcal{N}[\sigma(\mathtt{t})] = \sigma(\mathtt{t}) + \frac{1}{\mho}\left(\int_{b_0}^{b_1}\left[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}}\mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y} - \lambda_2\int_{b_0}^{b_1}\mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y}\right]\sigma(\mathtt{x})d\mathtt{x}\right) - \frac{1}{\mho}\mathcal{F}(\mathtt{t}) - \frac{\lambda_1}{\mho}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y} - \frac{\lambda_2}{\mho}\int_{b_0}^{b_1}\sum_{l=2}^{q}R_q(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathscr{M}_2(\mathtt{y},\mathtt{x})\sigma(\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y}. \quad (39)$$

From Equations (38) and (39), we get

$$\mathcal{N}[\sigma(\mathtt{t})] = 0, \qquad \text{for all} \quad \mathtt{t} \in J. \quad (40)$$

The following explanation for the homotopy of the unknown function $\sigma(\mathtt{t})$ can be considered

$$\mathcal{F}^*[\mathcal{M}(\mathtt{t};h,\jmath)] = (1-\jmath)\mathcal{L}(\mathcal{M}(\mathtt{t};h,\jmath) - \sigma_0(\mathtt{t})) - \jmath h N[\mathcal{M}(\mathtt{t};h,\jmath)]. \quad (41)$$

1. The function $\sigma_0(\mathtt{t})$ is the initial approximate solution of the unknown function $\sigma(\mathtt{t})$;
2. The rate of convergence parameter $h \in \mathbb{R} - \{0\}$ is used to the method suggested;
3. Equation (41) embeds the homotopy parameter $\jmath \in [0,1]$.
4. The operator $\mathcal{L}$ is known as an auxiliary linear operator if $\jmath(\mathtt{t}) = 0$ when $\mathcal{L}[\jmath(\mathtt{t})] = 0$;
5. If the Equation (39) is denoted by the operator $\mathcal{N}$, then we obtain

$$\mathcal{N}[\mathcal{M}(\mathtt{t};h,\jmath)] = \mathcal{M}(\mathtt{t};h,\jmath)$$
$$+\frac{1}{\mho}\left\{\int_{b_0}^{b_1}\left[\mathcal{H}(\mathtt{t},\mathtt{x}) - \lambda_1\int_{b_0}^{\mathtt{t}} \mathcal{S}_p(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y} - \lambda_2\int_{b_0}^{b_1} \mathcal{S}_q(\mathtt{t},\mathtt{y};1)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{y}\right]\mathcal{M}(\mathtt{x};h,\jmath)d\mathtt{x}\right\}$$
$$-\frac{1}{\mho}\mathcal{F}(\mathtt{t}) - \frac{\lambda_1}{\mho}\int_{b_0}^{\mathtt{t}}\sum_{l=2}^{p}\mathcal{S}_p(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathcal{M}(\mathtt{x};h,\jmath)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y}$$
$$-\frac{\lambda_2}{\mho}\int_{b_0}^{b_1}\sum_{l=2}^{q}\mathcal{S}_q(\mathtt{t},\mathtt{y};l)\left(\int_{b_0}^{b_1}\mathcal{M}(\mathtt{x};h,\jmath)\mathscr{M}_2(\mathtt{y},\mathtt{x})d\mathtt{x}\right)^l d\mathtt{y}, \quad (42)$$

and

$$\mathcal{F}^*[\mathcal{M}(\mathtt{t};h,\jmath)] = 0. \quad (43)$$

Solving Equation (43) yields

$$(1-\jmath)\mathcal{L}[\mathcal{M}(\mathtt{t};h,\jmath) - \sigma_0(\mathtt{t})] = \jmath h\mathcal{N}[\mathcal{M}(\mathtt{t};h,\jmath)]$$

$$\sigma(\mathtt{t}) = \sum_{n=0}^{\infty}\sigma_n(\mathtt{t}) = \sigma_0(\mathtt{t}) + \sum_{n=1}^{\infty}\sigma_n(\mathtt{t}), \quad (44)$$

where

$$\sigma_n(\mathtt{t}) = \frac{1}{n!}\frac{\partial^n}{\partial\jmath^n}\mathcal{M}(\mathtt{t};h,\jmath)\Big|_{\jmath=0};$$
$$\sigma_1(\mathtt{t}) = h\mathcal{R}_{\mathbf{1}}[\sigma_0(\mathtt{t})];$$
$$\sigma_n(\mathtt{t}) = \sigma_{n-1}(t) + h\mathcal{R}_{\mathbf{n}}\sigma_{n-1}(\mathtt{t}), \quad \forall n \geq 2;$$
$$\sigma_{n-1}(\mathtt{t}) = (\sigma_0(\mathtt{t}), \sigma_1(\mathtt{t}), \ldots, \sigma_{n-1}(\mathtt{t}));$$
$$\mathcal{R}_{\mathbf{n}}[\sigma_{n-1}(\mathtt{t})] = \frac{1}{(n-1)!}\left[\frac{\partial^{n-1}}{\partial\jmath^{n-1}}\mathcal{N}\left(\sum_{i=0}^{\infty}\sigma_i(\mathtt{t})\jmath^i\right)\Big|_{\jmath=0}\right].$$

3. Numerical Results

As an application of the construction of the above algorithms in Theorems 7 and 8, we can now present some numerical examples. Data calculations and graphs are implemented by MATLAB 2022a.

Example 1. *Our first example considers the boundary value problem*

$$\mho\phi''(\mathtt{t}) + 2\phi'(\mathtt{t}) = \mathtt{f}(\mathtt{t}) + \lambda_1\int_0^t \mathtt{t}(3s^2-2)\phi^3(s)ds + \lambda_2\int_0^1 \mathtt{t}(3s^2-2)\phi^2(s)ds \quad (45)$$

where $\mathtt{f}(\mathtt{t}) = 6\mho\mathtt{t} + 6\mathtt{t}^2 - 4 - \lambda_1(\frac{\mathtt{t}(\mathtt{t}^3-2\mathtt{t}+1)^4}{4} - \frac{\mathtt{t}}{4}) + \lambda_2\frac{\mathtt{t}}{3}$ *and the exact solution* $\phi(\mathtt{t}) = \mathtt{t}^3 - 2\mathtt{t} + 1, \mathtt{t} \in [0,1]$ *with boundary conditions* $\phi(0) = 1, \phi(1) = 0$ *and* $\mho = 5\times10^3$, $\lambda_1 = \frac{1}{300}$, *and* $\lambda_2 = \frac{1}{400}$. *Considering the postulate* (**H3**),

$$\left[\int_0^1\left(\mathtt{t}(3\mathtt{y}^2-2)\right)^2 dy\right]^{\frac{1}{2}} \leq \frac{3}{\sqrt{5}},$$

postulate (**H4**),

$$\begin{aligned}(\xi + k_1|\lambda_1|\varepsilon_1(1) + |\lambda_2|\varepsilon_2(1)) = 2.0099 \leq |\mho|,\\ \text{where} \quad \xi = 2, k_1 = 1,\\ \lambda_1 = \frac{1}{300}, \varepsilon_1(1) = \frac{9}{5},\\ \lambda_2 = \frac{1}{400}, \varepsilon_2(1) = \frac{6}{\sqrt{15}},\end{aligned}$$

and postulate (**H5**),

$$\begin{aligned}(\xi + |\lambda_1|\Lambda_1 + |\lambda_2|\Lambda_2) = 2.0254 \leq |\mho|,\\ \text{where} \quad \xi = 2, \lambda_1 = \frac{1}{300}, \lambda_2 = \frac{1}{400},\\ \Lambda_1 = 5.4654, \Lambda_2 = 2.8908,\end{aligned}$$

respectively. These confirm the convergence of the problem.

Repeating the above process as in Section 1 by setting $\sigma(\mathtt{t}) = \phi''(\mathtt{t})$, *we can deduce a nonlinear Volterra–Fredholm integral equation in the form of (3). Moreover, (45) can satisfy the condition postulate* (**H5**)*it has a unique solution. Thus, Theorem 8 confirms the uniqueness of solution of this problem. Finally, we tabulate the numerical results in Table 1 with* $h = -0.3333064738985$ *for the proposed methods and their absolute errors between them with the exact value. Moreover, we have drawn it graphically in Figure 1 for the same value of h.*

Table 1. Numerical solutions for Example 1 solved by the MADM (σ_{MADM}) and HAM (σ_{HAM}).

t	σ_{exact}	σ_{MADM}	σ_{HAM}	$\|\sigma_{exact} - \sigma_{MADM}\|$	$\|\sigma_{exact} - \sigma_{HAM}\|$
0	0	0.000000000018898	−0.000266659995017	0.000000000018898	0.000266659995017
0.200000000000000	1.200000000000000	1.200000011921477	1.199658611206104	0.000000011921477	0.000341388793896
0.400000000000000	2.400000000000000	2.400000086393681	2.399647950502708	0.000000086393681	0.000352049497292
0.600000000000000	3.600000000000000	3.600000175841783	3.599701327382030	0.000000175841783	0.000298672617970
0.800000000000000	4.800000000000001	4.800000247296660	4.799818687831255	0.000000247296660	0.000181312168746
1.000000000000000	6.000000000000000	6.000000295194053	5.999999999994251	0.000000295194053	0.000000000005749

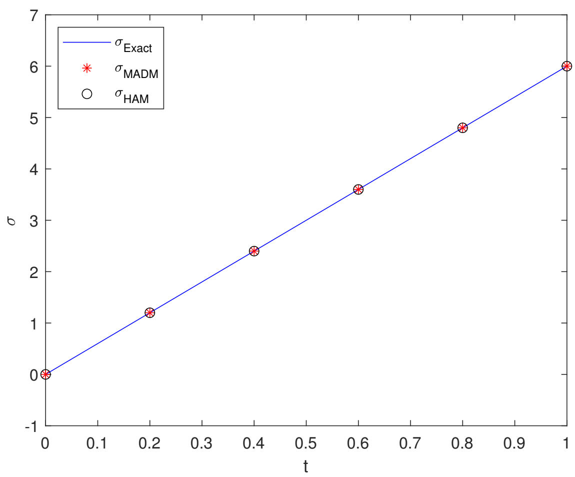

Figure 1. Plot of the proposed methods compared with the exact solution of Example 1.

For several values of $h = [-0.333306473899000 \ \ -0.333306473898500 \ \ -0.333306473898000]$, *the best approximations of* σ_{HAM} *can be deduced as tabulated in Table 2.*

Table 2. Different values of σ_{HAM} with respect to the values of h.

$h = -0.333306473899$	$h = -0.3333064738985$	$h = -0.333306473898$
−0.000266659995017	−0.000266659995017	−0.000266659995017
1.199658611207904	1.199658611206104	1.199658611204305
2.399647950506308	2.399647950502708	2.399647950499109
3.599701327387429	3.599701327382030	3.599701327376630
4.799818687838455	4.799818687831255	4.799818687824056
6.000000000003250	5.999999999994251	5.999999999985251

It is worth mentioning that the h values can confirm the convergence of the approximate solution, which are demonstrated in Figure 2.

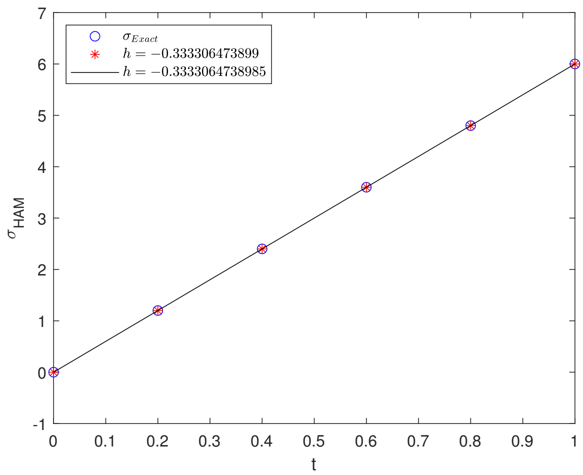

Figure 2. σ_{HAM} solutions for some values of h in Example 1.

Example 2. *Consider the following boundary value problem:*

$$\mho\phi''(\mathtt{t}) = \mathtt{f}(\mathtt{t}) + \lambda_1 \int_0^{\mathtt{t}} \sin(\mathtt{t}-s)\phi^2(s)ds + \lambda_2 \int_0^{\frac{\pi}{2}} \sin(\mathtt{t}-s)\phi(s)ds \tag{46}$$

with boundary conditions

$$\phi(0) = 1 \qquad and \qquad \phi(\frac{\pi}{2}) = 0,$$

where $\mathtt{f}(\mathtt{t}) = -\mho\cos(\mathtt{t}) - \lambda_1\Big(\frac{\sin^2(\mathtt{t})-\cos(\mathtt{t})+1}{3}\Big) - \lambda_2\Big(\frac{\pi\sin(\mathtt{t})-2\cos(\mathtt{t})}{4}\Big)$, $\phi(\mathtt{t}) = \cos(\mathtt{t})$ *is exact solution, for all* $\mathtt{t} \in [0, \frac{\pi}{2}]$, $\mho = 4\times 10^3$, $\lambda_1 = \frac{1}{400}$ *and* $\lambda_2 = \frac{1}{1200}$.

Observe that the postulate (**H3**) :

$$\left[\int_0^{\frac{\pi}{2}} ((\sin(\mathtt{t}-\mathtt{y}))^2 d\mathtt{y}\right]^{\frac{1}{2}} \leq \frac{\sqrt{\pi}}{2},$$

postulate (**H4**) :

$$\begin{aligned}
&(\xi + k_1|\lambda_1|\varepsilon_1(1) + |\lambda_2|\varepsilon_2(1)) = 0.0102 \leq |\mho|,\\
&where \quad \xi = 0, k_1 = 1,\\
&\lambda_1 = \frac{1}{400}, \varepsilon_1(1) = 3.1646,\\
&\lambda_2 = \frac{1}{1200}, \varepsilon_2(1) = 2.7406,
\end{aligned}$$

and postulate (**H5**) :

$$(\xi + |\lambda_1|\Lambda_1 + |\lambda_2|\Lambda_2) = 0.0260 \leq |\mho|,$$
$$\text{where} \quad \xi = 0, \lambda_1 = \frac{1}{400}, \lambda_2 = \frac{1}{1200},$$
$$\Lambda_1 = 4.9093, \Lambda_2 = 2.7406,$$

respectively. These confirm the convergence of the problem.

By repeating the approach described in Section 1 and setting $\sigma(\mathtt{t}) := \phi''(\mathtt{t})$, *we may get a nonlinear Volterra–Fredholm integral equation in the form of (3). Furthermore, (46) meets the requirement postulate* (**H5**). *Thus, Theorem 8 confirms the uniqueness of this problem's solution. Finally, we arrange the numerical results for the suggested approaches and their absolute errors with the exact value in Table 3 with* $h = -0.3335010$. *Furthermore, we have illustrated it graphically in Figure 3 for the same value of* h. *Table 2 illustrates the average absolute infinity norm errors between the exact and approximate solutions (MADM and HAM) with* $h = -0.3335010$.

Table 3. Numerical solutions for Example 1 solved by the MADM (σ_{MADM}) and HAM ($\sigma_H AM$).

$\mathtt{t}$	σ_{exact}	σ_{MADM}	σ_{HAM}	$\|\sigma_{Exact} - \sigma_{MADM}\|$	$\|\sigma_{Exact} - \sigma_{HAM}\|$
0	−1.000000000000000	–1.000000843016519	−1.000503767496914	0.000000843016519	0.000503767496914
0.314159265358979	−0.951056516295154	–0.951056973274357	−0.951535349824139	0.000000456979203	0.000478833528985
0.628318530717959	−0.809016994374947	–0.809017188656542	−0.809424201820296	0.000000194281594	0.000407207445348
0.942477796076938	−0.587785252292473	–0.587785364820836	−0.588081184954149	0.000000112528363	0.000295932661675
1.256637061435917	−0.309016994374947	–0.309017207000090	−0.309172883226273	0.000000212625142	0.000155888851325
1.570796326794897	−0.000000000000000	–0.000000440291301	−0.000000736172456	0.000000440291300	0.000000736172456

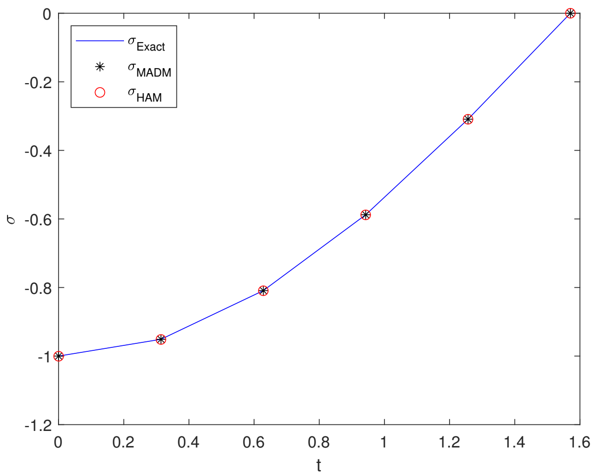

Figure 3. Plot of the proposed methods compared with the exact solution of Example 2.

In addition, Figure 4 illustrates the absolute errors of infinity of the σ_{MADM}, (σ_{HAM}, *with* $h = -0.33330490$) *and the exact solution at the same points used in Table 2.*

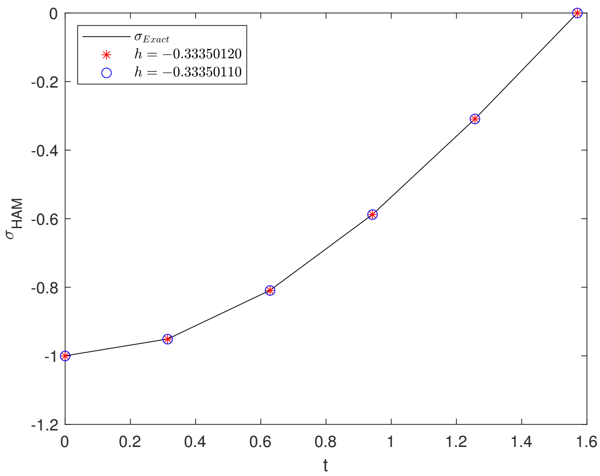

Figure 4. σ_{HAM} solutions for some values of h in Example 2.

Again, there are many values of $h = [-0.33350120 \;\; -0.33350110 \;\; -0.3335010]$ *which give the best approximations of* σ_{HAM} *as shown in Table 4.*

Table 4. Different values of σ_{HAM} with respect to the values of h.

$h = -0.33350120$	$h = -0.33350110$	$h = -0.3335010$
−1.000504367497397	−1.000504067497155	−1.000503767496914
−0.951535920505433	−0.951535635164786	−0.951535349824139
−0.809424687324857	−0.809424444572576	−0.809424201820296
−0.588081537766754	−0.588081361360451	−0.588081184954149
−0.309173068825120	−0.309172976025696	−0.309172883226273
−0.000000736408373	−0.000000736290414	−0.000000736172456

4. Conclusions and Future Directions

The main theme of this research is focused on the solution and analysis of a nonlinear boundary value problems for a Volterra–Fredholm integro equation with specific boundary conditions. We explicitly build the existence and uniqueness of an auxiliary problem with the simplified right-hand side using Arzela–Ascoli and Krasnoselskii fixed point theorems. Furthermore, using the theory of the Banach contraction principle index, we demonstrate the existence of at least one continuous solution to the original issue, as stated in Theorem 7. We have included some numerical talks and clear graphical representations for Volterra–Fredholm integro issues for several eigenvalues and homotopy parameters to help you grasp the resultant boundary models. Many solutions have been found and are depicted in Figures 1–3. In addition, efficiency of the proposed schemes is also presented in tables by calculating absolute errors.

The Volterra–Fredholm integro fractional differential problems have a bright future as a type of highly integrated boundary value problem in integrated fractional operators; however, there is still room for improvement in transmission efficiency and numerical solutions, which is also the future direction of our work.

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