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Sustainable Inventory Managements for Non-Instantaneous Deteriorating Items: Preservation Technology and Green Technology Approaches with Advanced Purchase Discounts and Joint Emission Regulations

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Abstract: The present article aims to determine the green economic policies of an inventory model for non-instantaneous deteriorating items under practical scenarios. These scenarios involve specific maximum lifetimes for items with deteriorations controllable through investments in preservation technologies, which can affect the period without deterioration. Additionally, carbon is emitted due to energy-related costs, prompting retailers to invest in green technology investments to reduce carbon emissions concurrently under the carbon tax policy and the carbon cap-and-trade policy simultaneously. Meanwhile, when a retailer is required to make a prepayment, the purchase discount policy is contingent on the number of installments offered. This means that the retailer prepays off the entire purchasing cost with a single installment, thereby receiving a maximum percentage of price discount. Otherwise, the retailer prepays a certain fraction of the purchasing cost with multiple installments, and the percentage of the price discount will be contingent on the number of n identical installments. In this context, we present theoretical results for optimal solutions, and a salient algorithm is presented, which is derived from these theoretical findings within a sustainable inventory system. To better illustrate the proposed mathematical problems, several numerical examples are presented, followed by sensitivity analysis for different scenarios.

Keywords: non-instantaneous deteriorating items; carbon regulations; green technology; preservation technology; prepayment installments; discount

1. Introduction

Over the century, global warming and its destructive effect on the human lifestyle has become a significant concern, such as a report of climate data at the World Resources Institute (WRI) in 2014, which revealed that GHG emissions are rising at an alarming rate around the world and were around 150 times more in 2011 than they were in 1850. Additionally, the European Commission (EC), an official regulatory agency, has assessed that 2011–2020 was the hottest recorded decade and that global warming is currently rising at a rate of 0.2 °C per decade: this means that the change in weather patterns, the rapid melting of glaciers, global warming, etc. are alarming issues directly related to carbon emissions. The EC also estimates that carbon emissions from human activities are the single largest contributor to global warming. For this, an international treaty was signed between 37 counties and European Union (EU) members in 1992 by the name Kyoto Protocol under the committee called the United Nation Framework Convention on Climate Change (UNFCCC). As mentioned above, many governments have considered environmental disasters and increased public awareness about environmental issues, have undertaken strict environmental policies, and have instituted regulations to ensure that industries and businesses do not excessively pollute the air and water of the planet. For instance, governments impose a tax known as a "carbon tax" for each unit of carbon emissions, it is based on the amount of carbon dioxide emitted from the enterprises' operations. Another carbon mechanism is a cap-and-trade system: it is set by a government to limit on the CO_2 emissions of enterprises and regulates emission allowances trading between enterprise. Meanwhile, a carbon limit/cap system strictly sets an upper limit of emissions that enterprises may not exceed. From the perspective of practical viewpoints, as the effect of GHG emissions becomes more and more severe, enterprises are also under increasing pressure from their stakeholders to reduce carbon emissions. According to the above explanation, this article will jointly consider the two most frequently utilized strategies, carbon tax, and cap-and-trade, to reduce carbon emissions generated through different operational activities in an inventory system.

In the real world, the deterioration of items is a natural phenomenon, because these items may deteriorate due to decay, damage, dryness, obsolescence, evaporation, etc. over time when they are stored in warehouses. Furthermore, Ghare and Schrader [1] investigated how deterioration affects inventory and introduced an Economic Order Quantity (EOQ) model, under the assumption of exponential inventory decay over time. From then on, a great deal of research efforts has been devoted to inventory models of deteriorating items such as how Raafat [2] surveyed the existing literature concerning inventory models characterized by continuous deterioration. Later, Goyal and Giri [3] conducted an extensive survey of inventory models designed for items affected by both deterioration and random lifetimes. Bakker et al. [4] wrote a review of inventory systems with deterioration since 2001. Janssen et al. [5] wrote a survey on the recent trends in the modeling of deteriorating inventory. Meanwhile, from a practical viewpoint, some items like fruits, fish, vegetables, top-quality food items, etc., lose their originality after some time, which are caused financial and physical loss. Wu et al. [6] pioneered the introduction of deterioration occurring after a non-deterioration period and formulated an optimal replenishment policy for items with partial backlogged shortages. In addition, in the case of every deteriorating item, the rate of deterioration is contingent upon their maximum lifetimes. These items stay in a usable condition throughout their lifetimes, but once they exceed their expiration dates, they are no longer suitable for sale. Accordingly, the inevitable degradation of items that do not deteriorate instantly is an unavoidable process. These processes can be influenced by multiple factors, including enzyme activity, autoxidation, oxidation, and the involvement of microorganisms. These deterioration mechanisms are influenced by environmental conditions such as temperature, humidity, and air pressure. In this connection, to mitigate the risk of deterioration in items prone to degradation, the deterioration processes can be curtailed through specific methods or the acquisition of specialized equipment. Examples of such measures include refrigeration, freezing, cooling, salting, pickling, dehydrating, and other related techniques. Additionally, an empirical observation on deteriorating items (The Profit Experts, 2011) revealed that a store's total profit could increase by 33% if it successfully reduces its perishable waste by 20%, so it indicates that preservation technology presents economic benefits through the reduction of inventory waste. As previously mentioned, the use of preservation technology is crucial for safeguarding deteriorating items. Therefore, this article will examine how investments in preservation technology affect the deterioration rate and the duration of the non-deterioration period for items that do not degrade immediately, especially within their defined maximum lifetimes. Unfortunately, while investments in preservation technology can extend the lifespan of items, the efforts to maintain freshness often lead to the emission of harmful gases into the atmosphere, contributing to global warming and the greenhouse effect. In response to this environmental concern, the adoption of green technology becomes imperative as a measure to protect the environment and mitigate the risk of global warming. For instance, carbon emissions can be minimized through the utilization of advanced technology equipment or increased investment in greener technologies such as solar panels, renewable energies, and energy-efficient machinery. Accordingly, it is clear that green technology has a significant positive impact on the reduction of environmental carbon emissions, and it also is a sustainable mechanism that considers the short- and long-term impacts on the environment. It's a win–win for the environment!

Nevertheless, investing in both preservation technology and green technology can cut down on deterioration and emission rates. However, these investments may hike up the enterprise's expenses, prompting enterprises to be very conscious about their operations concerning carbon emissions and preservation technology investments. This article will discuss strategies for reducing emissions using green technology and the importance of controlling or decreasing the deterioration rate of items through preservation technology for enterprises to achieve optimal operational efficiency and cost reduction. Finally, effectively managing inventory is a crucial task for any business organization. Developing an accurate and realistic inventory model requires the consideration of various practical business schemes, including prepayment, trade credit, and price discounts. Among them, an advance payment scheme offered by the supplier can assist traders in completing payments by paying a portion of the total purchase price across a few installments. The benefit for the supplier is the elimination of the risk of order cancellation, helping to build confidence with customers. In this continuation, the prepayment mechanism reduces order cancellations significantly, enabling them to offer price discounts as an incentive to attract more potential customers using the prepayment strategy. However, to the best of the authors' knowledge, no prior work has considered discounts linked to multiple prepayments. As a result of this, this article will explore a scenario where the supplier offers a discount facility if the retailer prepays off the entire purchasing cost with a single installment, and they gain a maximum percentage of the price discount. Conversely, if the retailer prepays a certain fraction of the purchasing cost with more installments, the percentage of the price discount depends on the number of installments.

Taking all the aforementioned factors into account, we investigate various scenarios to examine the impacts of preservation technology investment and green technology investment for a non-instantaneous deteriorating item with maximum lifetimes under concurrent carbon regulations. Moreover, our inventory models incorporate the influence of the discount policy on installments. Based upon above discussion, the main contributions of this article are as follows:

(1) The article identifies key factors, namely preservation technology, green technology investment, and the discount policy depending on the number of installments, that influence the total cost in EOQ models for non-instantaneous deteriorating items with maximum lifetimes subject to carbon constraints.
(2) This article presents various decision models that enable retailers to respond to the purchase discount policy, which depends on the number of installments, alongside government-imposed carbon regulations.
(3) This article proposes theorems to derive optimal replenishment cycle time, preservation technology, and green technology investment under the discount policy and concurrent carbon emission regulations. These optimal decisions are necessary for enterprises to control their operational costs.

The structure of this article is as follows: Section 2 reviews the relevant literature on carbon emission regulations, investments in preservation technology, green technology, and prepayment schemes. Section 3 discusses the analysis of existing research gaps. Section 4 introduces mathematical models that integrate preservation technology, green technology, and joint carbon regulations within the framework of prepayment discount policies. Section 5 presents lemmas, theorems, and an algorithm to optimize the total cost of these inventory models. Section 6 includes several numerical examples to demonstrate the theoretical findings. Section 7 carries out a sensitivity analysis to discuss managerial observations. Section 8 offers managerial insights. Finally, Section 9 provides conclusions and suggests directions for future research.

2. Literature Review

The growing awareness of environmental damage caused by greenhouse gas (GHG) emissions has gained significant attention in the last decade. Numerous studies have emphasized the long-term national objective to reduce greenhouse gas emissions as follows: a common goal is to achieve a substantial reduction, with targets such as reducing emissions to less than 50% of the levels measured in 2005 by the year 2050. In response to these objectives, many countries have initiated efforts to mitigate greenhouse gas emissions stemming from supply chain activities. An illustrative example is provided by Hoffman (2007), who demonstrated the environmental impact of Walmart's initiatives. Walmart successfully achieved a significant reduction of 667,000 metric tons of emissions by collaborating with over 60,000 suppliers. This reduction was accomplished through a 5% decrease in packaging, which was a change implemented in response to Walmart's environmental request. Such examples highlight the role of collaboration between large corporations and their suppliers in achieving significant reductions in greenhouse gas emissions within the supply chain. Moreover, much research has identified different regulations on carbon emissions in inventory system practices like carbon tax and cap-and-trade, which are the two most frequently utilized strategies to reduce carbon emissions generated through different mechanisms of an enterprise. Hua et al. [7] delved into the impact of carbon caps-and-trade mechanisms on an enterprise's cost by formulating an EOQ model. They assumed that carbon emissions stem from product ordering and holding activities. Later, Chen et al. [8] considered purchased items as another factor contributing to carbon emissions. They explored the ordering policy of the inventory model within the context of a carbon cap policy. He et al. [9] investigated sustainable inventory management while examining the implications of applying carbon tax and cap-and-trade policies. Daryanto et al. [10] proposed a three-level model considering transportation emissions, storage emissions, and disposal activities. Lee [11] investigated the optimal order policy of an EOQ inventory model within the framework of cap-and-price regulation.

On the other hand, certain developing countries have numerous industries, like ready-made garment manufacturing, that release emissions into the environment without established policies to curb these emissions. This trend may lead to a gradual rise in greenhouse gases and waste disposal. Addressing this concern in this regard, Toptal et al. [12] first proposed the idea that green technology investments could mitigate emissions. They investigated investment decisions for carbon emissions reduction within an inventory model. Particularly, under carbon cap-and-trade and carbon tax regulations, they took into account scenarios where investments in green technology are permissible. Datta [13] and Lin [14] investigated the effect of green technology investment under the context of the carbon tax. Bhattacharyya and Sana [15] focused on optimizing green technology within an eco-friendly manufacturing system. In a related study, Huang et al. [16] incorporated the influence of green technology into an inventory system considering various carbon regulations. Mishra et al. [17] explored an inventory model that takes into account controllable carbon emissions arising from warehouse operations, and they discussed the impact of shortages within their model. Meanwhile, Shi et al. [18] investigated an inventory model for deteriorating items with expiration dates, examining carbon emissions associated with warehousing, purchasing, transportation, and manufacturing operations. Yang and Lin [19] indicated the significance of green innovation performance in fostering a sustainable supply chain. Hasan et al. [20] conducted a comparative analysis of three distinct carbon emission reduction strategies—carbon tax, carbon cap, and carbon trading—within the context of sustainable inventory decision making. Mishra et al. [21] examined sustainable inventory management within a greenhouse farm. Their study explored the possibilities of controlling carbon emissions through the adoption of green technologies. Numerous literature sources have examined the possibility of reducing carbon emissions through green technology investment such as Haoxuan et al. [22], Xu et al. [23], Xia et al. [24], Taleizadeh et al. [25], Gao et al. [26], Lu et al. [27], and Yu et al. [28–32].

In essence, the observation reveals that most of the deteriorating items come with expiration dates. As these items near their maximum lifetimes, the deterioration rate tends to increase gradually, ultimately reaching 100%. Hsu et al. [33] were pioneers in proposing a deterioration model that depends on the expiration rate. Building on this concept, Wu et al. [34] highlighted item freshness as a vital aspect of stocking policies, integrating the expiration rate and the freshness of items into the inventory system. Wu et al. [35] explored inventory models for deteriorating items with a maximum lifetime, analyzing scenarios that involve offering partial trade credits to credit-risk customers using Discounted Cash Flow (DCF) analysis. On a related note, Chung et al. [36] accounted for lot size decisions for deterioration items with a maximum lifetime under an advance-cash-credit payment scheme. Feng et al. [37] introduced the idea of incorporating product freshness into stocking policies, by accounting for the expiration rate of inventory items within the system. In a different context, Tiwari et al. [38] developed a profit-maximizing inventory model for a deteriorating item with price-dependent demand and a time-varying deterioration rate, considering the item's maximum lifetime. Liao et al. [39] explored scenarios where a retailer sells non-instantaneous deteriorating items that deteriorate completely near their expiration date and display imperfect quality. On a similar note, Mahato and Mahata [40] discussed deteriorating items with a maximum lifetime and dynamic demand, adding another dimension to the considerations in inventory management. Furthermore, substantial research has been conducted on this topic, including recent works by Hou et al. [41,42], Liao et al. [43,44], Shah et al. [45], Vandana and Srivastava [46], Khan et al. [47], and Srivastava et al. [48], who formulated inventory models for items subject to deterioration with specified expiration dates.

Additionally, it is important to acknowledge the nature of items being deteriorating and recognize that while the deterioration process may be inevitable, it can be managed and controlled to some extent. This control is often aimed at slowing down the speed of deterioration. As an example, the use of cold storage can effectively reduce the deterioration of perishable items such as sea and seasonal products. Additionally, the active application of preservation technologies serves as an additional means to provide extra protection for deteriorating items, contributing to the overall management of inventory and product quality. Hsu et al. [49] were pioneers in studying the impact of preservation technology investment on deteriorating items. Their research indicated that as the deterioration rate increased, a higher level of investment was required. Building on this perspective, Singh et al. [50] developed an EOQ model that includes stock-dependent demand, a trade credit facility from the retailer's perspective, and the integration of preservation technology to address the effects of deterioration. Shah et al. [45] conducted an inventory model considering items' expiration date and deterioration. Their study incorporated two-level trade credit and explored the impact of preservation technology investment, employing a DCF approach. On a related note, Mishra et al. [51] studied the characteristics of preservation technology investment for deteriorating items with demand dependent on the item's price. Li et al. [52] investigated an inventory model incorporating preservation technology for non-instantaneous deteriorating items. They presented two distinct models, showing how preservation technology can be instrumental in optimizing profits. On a related note, Bardhan et al. [53] investigated optimal replenishment policies and preservation technology investments for items with non-instantaneous deterioration and non-linear demand that varies with inventory levels. They compared outcomes with and without preservation investments to assess the impact of such technology on the overall system. Iqbal and Sarkar [54] also revealed the capability of certain preservation technologies to extend the freshness lifespan of products. In a different context, Yu et al. [55] developed an inventory model for deteriorating items over a finite time horizon, taking into account preservation technology within carbon regulations. Similarly, Mashud et al. [56] investigated a sustainable inventory model aimed at maximizing supply chain profit through investments in both preservation and green technologies. Mishra et al. [21] emphasized the potential for controlling both carbon emissions and deterioration by employing green and preservation technologies, respectively. The literature on preservation technology is vast; the following research works are worth mentioning: Liu et al. [57], Chaudhari et al. [58], Mishra et al. [59], and Priyamvada et al. [60].

Finally, the imposition of a prepayment strategy by suppliers can serve as a strategic approach to control order cancellations or suspensions. In this strategy, the supplier offers a prepayment strategy constituting a percentage of the purchasing cost. This prepayment can be arranged in single or multiple identical installments. Additionally, suppliers may request an advance payment from the enterprise before the delivery of products. The implementation of such a prepayment system is designed to mitigate default risks, including the cancellation of orders and inaccuracies in demand forecasting for the supplier. By securing a portion of the payment upfront, the supplier can enhance financial stability and minimize uncertainties associated with order fulfillment, contributing to a more reliable and efficient supply chain. Gupta et al. [61] were the pioneers in proposing the advance payment scheme. Teng et al. [62] investigated an inventory model that considered maximum lifetime-dependent deterioration rates alongside the incorporation of an advance payment scheme. In a related context, Li et al. [63] explored an inventory system dealing with deteriorating items, focusing on a scenario where a seller demands an advance–cash–delay payment scheme to derive optimal solutions for inventory management. Taleizadeh [64] anticipated an inventory model that incorporated advance payment pricing, disruption in supply, and allowed partial shortages. In a related vein, Diabat et al. [65] examined an inventory system for deteriorating items, taking into account partial upstream prepayment and partial downstream delayed payment. Khan et al. [66] explored optimal pricing and inventory decisions under capacity constraints, incorporating an advance payment strategy for the retailer, with considerations for the expiration dates of deteriorating items. Additionally, Khan et al. [67] considered a model for deteriorating items with expiration dates, allowing for item advertisement and partial backorder. Chung et al. [68] designed an inventory model for deteriorating items, taking into consideration factors such as cash discounts and trade credit. Khan et al. [69] explored a flexible prepayment regulation that aligns with the purchased quantity, requiring the retailer to prepay a small percentage of the acquisition cost for a higher purchased quantity. Duary et al. [70] explored an inventory system for deteriorating items considering both advance and delay in payment as key components. Additionally, Khan et al. [71] outlined an inventory management system for non-instantaneous deteriorating items, incorporating a hybrid cash advance payment business strategy. These studies contribute to the understanding of complex inventory management scenarios encompassing various factors such as various payment strategies and business strategies in the context of deteriorating items. These topics have been investigated by a large number of researchers such as Lashgari et al. [72], Khan et al. [73], and their references.

3. Research Gap Analysis

The survey findings reveal that the current literature is deficient in comprehensive inventory models that account for investments in preservation technology, green technology, and carbon regulations, especially in the context of a purchase discount policy. Specifically, preservation technology plays a crucial role in extending the non-deteriorating period of products and can significantly contribute to improving costs in the supply chain. By employing effective preservation techniques, businesses can delay the onset of deterioration, maintaining the quality and freshness of products for a longer duration. This extension of the non-deteriorating period has several positive implications for the supply chain. Additionally, the purchase discount policy contingent on the number of installments aligns with government-imposed carbon regulations. This study seeks to bridge these gaps by creating an all-encompassing supply chain model. The research gaps identified are as follows:

(1) Limited research has been conducted on the effects of preservation technology on supply chain models for items that deteriorate gradually. This study seeks to address this gap by analyzing how preservation technology can slow the deterioration process. The main goal is to explore how preservation technology can prolong the period during which items remain non-deteriorated.
(2) Existing inventory systems have not yet considered scenarios where suppliers offer discounts on purchasing costs based on installment decisions, especially in the context of government-imposed carbon regulations. This study aims to bridge this gap by introducing and analyzing purchase discount policies within the framework of carbon regulations. This research explores how such discount strategies impact both the financial aspects of inventory management and the environmental implications related to carbon emissions.
(3) No existing supply chain or inventory model has simultaneously taken into account the interrelated aspects of preservation technology, green technology, carbon emissions, and the purchase discount policy for non-instantaneous deteriorating items. This study constructs a comprehensive model that incorporates all these elements. Our study aims to evaluate how these factors interact with each other, providing insights into their collective influence on supply chain sustainability for non-instantaneous deteriorating items.

By addressing these research gaps, this study will contribute significantly to advancing sustainable supply chain management practices, providing valuable insights for practitioners in optimizing their decision-making processes.

4. Mathematical Model Formulation

Giving the notation and assumptions in Appendix A, as $t = 0$, Q units of the products arrive at the inventory system. During the period $(0, t_d(\xi)]$, the inventory level of the product $I_1(t)$ deteriorates with the demand rate. After time $t = t_d(\xi)$, the inventory level starts to deteriorate and be disposed at a rate of $(1 - m(\xi))\theta(t - t_d(\xi))$ for a given preservation investment ξ. Then, during the period $[t_d(\xi), T]$, the inventory level $I_2(t)$ decreases due to the combined effects of the demand rate and deterioration rate. So, the subsequent are representing the situation as follows:

$$\frac{dI_1(t)}{d(t)} = -D, \quad t \in (0, t_d(\xi)] \tag{1}$$

and

$$\frac{dI_2(t)}{d(t)} = -D - (1 - m(\xi))\theta(t - t_d(\xi))I_2(t), \quad t \in (t_d(\xi), T] \tag{2}$$

with boundary conditions $I_1(0) = Q$ and $I_2(T) = 0$.

The solution of the differential equations given in Equations (1) and (2) yields

$$I_1(t) = Q - Dt \tag{3}$$

and

$$I_2(t) = \frac{D}{m(\xi)}\{(1 + \lambda - t + t_d(\xi)) - (1 + \lambda - t + t_d(\xi))^{1-m(\xi)}(1 + \lambda - T + t_d(\xi))^{m(\xi)}\} \tag{4}$$

Additionally, since $I_1(t) = I_2(t)$ at $t = t_d(\xi)$, we have

$$Q - Dt_d(\xi) = \frac{D}{m(\xi)}\{(1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\}$$

Furthermore, the quantity of the product received is

$$Q = \frac{D}{m(\xi)}\{m(\xi)t_d(\xi) + (1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\} \quad (5)$$

Now, to substitute Equation (5) into Equation (3), we have

$$I_1(t) = \frac{D}{m(\xi)}\{m(\xi)(t_d(\xi)-t) + (1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\} \quad (6)$$

Specifically, the discount policies of prepayment are defined as follows:

Policy 1: The supplier offers a β% price discount as a benefit if the retailer decides to pay the full purchasing cost in advance with a single installment.

Policy 2: The supplier offers a $(\frac{\beta}{n})$% price discount if the retailer decides to pay the partial purchasing cost in advance with more installments. For this, the discount rate is dependent on the number of installments.

Based upon the above discussion, the inventory-associated costs are listed as follows:

Policy 1: This is with full advance payment in a single installment:

(1) The annual cost is $\frac{A}{T}$.

(2) The annual purchasing cost is

$$\frac{(1-\beta)c_pQ}{T} = \frac{(1-\beta)c_pD}{m(\xi)T}\{m(\xi)t_d(\xi) + (1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\}$$

(3) The annual stock holding cost is

$$\frac{c_h}{T}[\int_0^{t_d(\xi)} I_1(t)dt + \int_{t_d(\xi)}^{T} I_2(t)dt]$$
$$= \frac{c_hD}{m(\xi)T}\{\frac{m(\xi)}{2}t_d^2(\xi) + (1+\lambda)t_d(\xi) - (1+\lambda)^{1-m(\xi)}t_d(\xi)(1+\lambda-T+t_d(\xi))^{m(\xi)}$$
$$+\frac{(1+\lambda)^2}{2} + \frac{m(\xi)}{2(2-m(\xi))}(1+\lambda-T+t_d(\xi))^2 - \frac{(1+\lambda)^{2-m(\xi)}}{2-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\}$$

(4) The annual instalment capital cost is

$$\frac{\phi}{T}(1-\beta)c_pt_0Q = \frac{\phi(1-\beta)c_pD}{m(\xi)T}t_0\{m(\xi)t_d(\xi) + (1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\}$$

(5) The annual preservation technology for controlling the deterioration rate is $\frac{\xi}{T}$.

(6) The annual green technology for controlling the emission rate is $\frac{G}{T}$.

(7) Taking into account that the sales revenue in the delay time is less than the loan used for prepayment or not, the interest earned can be discussed in two parts: one is $(1-\beta)c_pQ \le sDt_1$, and the other is $(1-\beta)c_pQ > sDt_1$. Herein, we find that the above inequalities seem too complicated and tedious to derive explicit inequalities in T. For this, we approximate the order quantity using the truncated Taylor series expansion as follows:

$$(1+x)^a \approx 1 + ax$$

So, the approximated order quantity would be $Q \approx DT$. Using this approximation is common in the relevant literature such as Chen and Chang [74]. As discussed above, we used the approximated order quantity in the inequalities and then $(1-\beta)c_pDT \le sDt_1$ if and only if $T \le \frac{st_1}{(1-\beta)c_p}$. Hereafter, the interest earned is discussed as follows:

Case 1-1: $T \leq \dfrac{st_1}{(1-\beta)c_p}$

In this sub-case, the sales revenue in the delay time is larger than the loan used for full advance payment. Therefore, the retailer pays off the purchasing cost before the delay time, so there is no interest earned.

Case 1-2: $T > \dfrac{st_1}{(1-\beta)c_p}$

In this sub-case, the sales revenue in the delay time is less than the loan used for prepayment. So, there is no interest earned as well.

(8) The interest charges can be discussed as follows:

Case 1-1: $T \leq \dfrac{st_1}{(1-\beta)c_p}$

In this sub-case, the sales begin, and the revenue from sales is continuously used to pay the loan from time 0, so we have the following:

$$\text{The annual interest charges are } \frac{\phi_L D}{2sT}[(1-\beta)c_p T]^2$$

Case 1-2: $T > \dfrac{st_1}{(1-\beta)c_p}$

In this sub-case, the amount of the outstanding loan does not pay off at time t_1, and the balance of the outstanding loan will pay off at time $(1-\beta)(\frac{c_p}{s})T$, so the interest charges consist of the following elements:

(i) The cost of interest charges during time $(0, t_1]$ is

$$\phi_L[(1-\beta)c_pT + (1-\beta)c_pT - sDt_1]t_1/2 = \frac{\phi_L D}{2}t_1[2(1-\beta)c_pT - st_1]$$

(ii) The loan amounts at time t_1 are $(1-\beta)c_pDT - sDt_1$, so the cost of interest charges during time $[t_1, (1-\beta)(\frac{c_p}{s})T]$ is

$$\phi_L[(1-\beta)(\frac{c_p}{s})T - t_1]Ds[(1-\beta)(\frac{c_p}{s})T - t_1]/2 = \frac{\phi_L D}{2s}[(1-\beta)c_pT - st_1]^2$$

Combining above, we have the following:

$$\text{The annual interest charges are } \frac{\phi_L D}{2T}t_1[2(1-\beta)c_pT - st_1] + \frac{\phi_L D}{2sT}[(1-\beta)c_pT - st_1]^2$$

(9) The cost of carbon emissions under the carbon tax regulation is discussed as follows:

In considering carbon dioxide that is emitted from the processes of ordering purchasing cost and storage processes, they are calculated as follows:

(i) The fixed carbon emissions associated with placing an order is $\hat{A}$.

(ii) The variable amount of carbon emissions associated with purchasing process is

$$\hat{c}_pQ = \frac{\hat{c}_pD}{m(\xi)}\{m(\xi)t_d(\xi) + (1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\}$$

(iii) The variable amount of carbon emissions associated with the storage process is

$$\frac{\hat{c}_hD}{m(\xi)}\{\frac{m(\xi)}{2}t_d^2(\xi) + (1+\lambda)t_d(\xi) - (1+\lambda)^{1-m(\xi)}t_d(\xi)(1+\lambda-T+t_d(\xi))^{m(\xi)}$$
$$+\frac{(1+\lambda)^2}{2} + \frac{m(\xi)}{2(2-m(\xi))}(1+\lambda-T+t_d(\xi))^2 - \frac{(1+\lambda)^{2-m(\xi)}}{2-m(\xi)}$$
$$\times(1+\lambda-T+t_d(\xi))^{m(\xi)}\}$$

Based on above the total carbon emission (CE) without carbon reduction investment, we have the following:

$$\begin{aligned}CE = \{&\hat{A} + \frac{\hat{c}_p D}{m(\xi)}[m(\xi)t_d(\xi) + (1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}] \\ &+ \frac{\hat{c}_h D}{m(\xi)}[\frac{m(\xi)}{2}t_d^2(\xi) + (1+\lambda)t_d(\xi) - (1+\lambda)^{1-m(\xi)}t_d(\xi)(1+\lambda-T+t_d(\xi))^{m(\xi)} \\ &+ \frac{(1+\lambda)^2}{2} + \frac{m(\xi)}{2(2-m(\xi))}(1+\lambda-T+t_d(\xi))^2 - \frac{(1+\lambda)^{2-m(\xi)}}{2-m(\xi)} \\ &\times(1+\lambda-T+t_d(\xi))^{m(\xi)}]\}\end{aligned} \tag{7}$$

Next, the retailer will invest in green technologies to reduce carbon emissions when the government applies carbon regulations. So, the function of emissions reduction is

$$R(G) = aG - bG^2, \quad G < \frac{a}{b} \tag{8}$$

As a result of above, the total carbon emission carbon reduction investment is $CE - R(G)$. Furthermore, the carbon emission cost under the tax regulation is

$$\frac{\delta_1}{T}[CE - R(G)]$$

(10) The cost of carbon emissions under the cap-and-trade regulation is discussed as follows:

In considering that the government sets a limit Z for the total allowable carbon emissions, the retailer will buy more allowances from other institutions or invest in green technology when his/her carbon emission exceeds the carbon cap Z. Covertly, the retailer can sell the surplus at a rate of δ_2 when his/her carbon emissions do not exceed the carbon cap Z. Based on above, the carbon emission cost under cap-and-trade regulation is

$$\frac{\delta_2}{T}[CE - R(G) - Z]$$

Based upon above discussion, the annual total cost function can be expressed as

$$\begin{gathered}TC(T,\xi,G) = \text{ordering cost} + \text{purchasing cost} + \text{stock-holding cost} + \text{capital cost} + \text{preservation technology} \\ \text{cost} + \text{green technologycost} + \text{carbon emission cost for carbon taxation} + \text{carbon emission cost for carbon} \\ \text{cap-and-trade} + \text{interest charges} - \text{interest earned}\end{gathered}$$

Thereafter, the annual total cost function under Policy 1 is

$$TC_1(T,\xi,G) = \begin{cases} TC_{11}(T,\xi,G) & if \quad 0 < T < \dfrac{st_1}{(1-\beta)c_p}, \quad 0 < \xi, \quad 0 < G \\ TC_{12}(T,\xi,G) & if \quad \dfrac{st_1}{(1-\beta)c_p} \le T, \quad 0 < \xi, \quad 0 < G \end{cases} \tag{9}$$

where

$$\begin{aligned}TC_{11}(T,\xi,G) \quad &= \frac{1}{T}\{A + \frac{(1-\beta)c_p D}{m(\xi)}H_{10}(T,\xi) + \frac{c_h D}{m(\xi)}H_{20}(T,\xi) \\ &+ \frac{\phi(1-\beta)c_p D}{m(\xi)}t_0 H_{10}(T,\xi) + \xi + G + \delta_1[\hat{A} + \frac{\hat{c}_p D}{m(\xi)}H_{10}(T,\xi) \\ &+ \frac{\hat{c}_h D}{m(\xi)}H_{20}(T,\xi) - R(G)] + \delta_2[\hat{A} + \frac{\hat{c}_p D}{m(\xi)}H_{10}(T,\xi) \\ &+ \frac{\hat{c}_h D}{m(\xi)}H_{20}(T,\xi) - R(G) - Z] + \frac{\phi_L D}{2s}[(1-\beta)c_p T]^2\}\end{aligned} \tag{10}$$

and

$$\begin{aligned} TC_{12}(T,\xi,G) &= \frac{1}{T}\{A + \frac{(1-\beta)c_pD}{m(\xi)}H_{10}(T,\xi) + \frac{c_hD}{m(\xi)}H_{20}(T,\xi) \\ &+ \frac{\phi(1-\beta)c_pD}{m(\xi)}t_0H_{10}(T,\xi) + \xi + G + \delta_1[\hat{A} + \frac{\hat{c}_pD}{m(\xi)}H_{10}(T,\xi) \\ &+ \frac{\hat{c}_hD}{m(\xi)}H_{20}(T,\xi) - R(G)] + \delta_2[\hat{A} + \frac{\hat{c}_pD}{m(\xi)}H_{10}(T,\xi) \\ &+ \frac{\hat{c}_hD}{m(\xi)}H_{20}(T,\xi) - R(G) - Z] + \frac{\phi_LD}{2}t_1[2(1-\beta)c_pT - st_1] \\ &+ \frac{\phi_LD}{2s}[(1-\beta)c_pT - st_1]^2\} \end{aligned} \tag{11}$$

Herein,

$$H_{10}(T,\xi) = m(\xi)t_d(\xi) + 1 + \lambda - (1+\lambda)^{1-m(\xi)}(1+\lambda - T + t_d(\xi))^{m(\xi)} \tag{12}$$

and

$$\begin{aligned} H_{20}(T,\xi) &= \frac{t_d^2(\xi)}{2}m(\xi) + (1+\lambda)t_d(\xi) - (1+\lambda)^{1-m(\xi)}t_d(\xi)(1+\lambda - T + t_d(\xi))^{m(\xi)} \\ &+ \frac{(1+\lambda)^2}{2} + \frac{m(\xi)}{2(2-m(\xi))}(1+\lambda - T + t_d(\xi))^2 \\ &- \frac{(1+\lambda)^{2-m(\xi)}}{2-m(\xi)}(1+\lambda - T + t_d(\xi))^{m(\xi)} \end{aligned} \tag{13}$$

Equations (10) and (11) imply that

$$TC_{11}(\frac{st_1}{(1-\beta)c_p},\xi,G) = TC_{12}(\frac{st_1}{(1-\beta)c_p},\xi,G) \tag{14}$$

Consequently, $TC_1(T,\xi,G)$ is continuous on its domain.

Policy 2: This is with partial advance payment in $n(n \geq 2)$ installments.

The inventory-associated costs in this policy are listed as follows:

(1) The annual purchasing cost is

$$\frac{c_p}{T}(1-\frac{\beta}{n})Q = \frac{c_pD}{m(\xi)T}(1-\frac{\beta}{n})\{m(\xi)t_d(\xi) + (1+\lambda) - (1+\lambda)^{1-m(\xi)}(1+\lambda - T + t_d(\xi))^{m(\xi)}\}$$

(2) The annual instalment capital cost is

$$\begin{aligned} \frac{\phi}{T}(1-\frac{\beta}{n})c_pt_0\alpha Q(\frac{n+1}{2n}) &= \frac{\phi c_pD}{m(\xi)T}(1-\frac{\beta}{n})t_0\alpha(\frac{n+1}{2n})\{m(\xi)t_d(\xi) + (1+\lambda) \\ &-(1+\lambda)^{1-m(\xi)}(1+\lambda - T + t_d(\xi))^{m(\xi)}\} \end{aligned}$$

(3) The interest earned can be discussed as follows:

When the retailer prepays the partial purchasing cost in n installments, where $n \geq 2$, the supplier offers a $(\frac{\beta}{n})\%$ $(n \geq 2)$ discount. Likewise, there are two cases in this policy: one is $T \leq \frac{st_1}{\alpha(1-\frac{\beta}{n})c_p}$, and the other is $T > \frac{st_1}{\alpha(1-\frac{\beta}{n})c_p}$.

Case 2-1: $T \leq \frac{st_1}{(1-\frac{\beta}{n})c_p}$.

In the sub-case, the sales revenue in the delay time is larger than the loan used for partial advance payment. Therefore, the retailer can earn interest from time $\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T$ to the time t_1. So, the annual interest earned is

$$\frac{I_d}{T}[t_1-\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]Ds[t_1-\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]/2=\frac{I_dD}{2sT}[st_1-\alpha(1-\frac{\beta}{n})c_pT]^2$$

Case 2-2: $T>\frac{st_1}{(1-\frac{\beta}{n})c_p}$

In this sub-case, the sales revenue in the delay time is less than the loan used for prepayment. So, there is no interest earned.

(4) The interest charges can be divided as follows:

Case 2-1: $T\le\frac{st_1}{\alpha(1-\frac{\beta}{n})c_p}$.

In this sub-case, the interest charges are discussed as follows:

(i) The cost of interest charges during time (0, $(0,\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]$ is

$$\phi_L[\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T][\alpha(1-\frac{\beta}{n})sDT]/2=\frac{\phi_LD}{2s}[\alpha(1-\frac{\beta}{n})c_pT]^2$$

(ii) The sales revenue and interest earned during time $[\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T,t_1]$ are

$$Ds[t_1-\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]+I_d[t_1-\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]Ds[t_1-\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]/2$$
$$=Ds[t_1-\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]+\frac{I_dD}{2s}[st_1-\alpha(1-\frac{\beta}{n})c_pT]^2$$

(iii) At time t_1, the loan amounts are

$$(1-\alpha)(1-\frac{\beta}{n})c_pDT-Ds[t_1-\alpha(1-\frac{\beta}{n})(\frac{c_p}{s})T]=(1-\frac{\beta}{n})c_pDT-sDt_1$$

So, the cost of interest charges during time $[t_1,(1-\frac{\beta}{n})(\frac{c_p}{s})T]$ is

$$\phi_L[(1-\frac{\beta}{n})(\frac{c_p}{s})T][(1-\frac{\beta}{n})c_pDT-Dst_1]/2=\frac{\phi_LD}{2s}[(1-\frac{\beta}{n})c_pT-st_1]^2$$

As discussed above, we have the following:

The annual interest charges are $\frac{\phi_LD}{2sT}[\alpha(1-\frac{\beta}{n})c_pT]^2+\frac{\phi_LD}{2sT}[(1-\frac{\beta}{n})c_pT-st_1]^2$

Case 2-2: $T>\frac{st_1}{\alpha(1-\frac{\beta}{n})c_p}$

In this sub-case, the interest charges are discussed as follows:

(i) The cost of interest charges during time $(0,t_1]$ is

$$\phi_L[\alpha(1-\frac{\beta}{n})c_pDT+\alpha(1-\frac{\beta}{n})c_pDT-sDt_1]t_1/2=\frac{\phi_LD}{2}t_1[2\alpha(1-\frac{\beta}{n})c_pT-st_1]$$

(ii) At time t_1 the loan amounts are

$$(1-\alpha)(1-\frac{\beta}{n})c_pDT+\alpha(1-\frac{\beta}{n})c_pDT-Dst_1=(1-\frac{\beta}{n})c_pDT-sDt_1$$

So, the cost of interest charges during time $[t_1,(1-\frac{\beta}{n})(\frac{c_p}{s})T]$ is

$$\phi_L[(1-\frac{\beta}{n})(\frac{c_p}{s})T-t_1][(1-\frac{\beta}{n})c_pDT-sDt_1]/2=\frac{\phi_L D}{2s}[(1-\frac{\beta}{n})c_pT-st_1]^2$$

As discussed above, we have the following:

The annual interest charges are $\frac{\phi_L D}{2T}t_1[2\alpha(1-\frac{\beta}{n})c_pT-st_1]+\frac{\phi_L D}{2sT}[(1-\frac{\beta}{n})c_pT-st_1]^2$

Similarly, the annual total cost function under Policy 2 is

$$TC_2(T,\xi,G)=\begin{cases} TC_{21}(T,\xi,G) \quad if \quad 0<T<\dfrac{st_1}{\alpha(1-\frac{\beta}{n})c_p}, \quad 0<\xi, \quad 0<G \\ TC_{22}(T,\xi,G) \quad if \quad \dfrac{st_1}{\alpha(1-\frac{\beta}{n})c_p}\le T, \quad 0<\xi, \quad 0<G \end{cases} \tag{15}$$

where

$$\begin{aligned} TC_{21}(T,\xi,G) &= \frac{1}{T}\{A+(\frac{c_pD}{m(\xi)})(1-\frac{\beta}{n})H_{10}(T,\xi)+\frac{c_hD}{m(\xi)}H_{20}(T,\xi) \\ &+(\frac{n+1}{2n})\alpha(\frac{\phi c_pD}{m(\xi)})(1-\frac{\beta}{n})t_0H_0(T,\xi)+\xi+G \\ &+\delta_1[\hat{A}+\frac{\hat{c}_pD}{m(\xi)}H_{10}(T,\xi)+\frac{\hat{c}_hD}{m(\xi)}H_{20}(T,\xi)-R(G)] \\ &+\delta_2[\hat{A}+\frac{\hat{c}_pD}{m(\xi)}H_{10}(T,\xi)+\frac{\hat{c}_hD}{m(\xi)}H_{20}(T,\xi)-R(G)-Z] \\ &+\frac{\phi_LD}{2s}[\alpha(1-\frac{\beta}{n})c_pT]^2+\frac{\phi_LD}{2s}[(1-\frac{\beta}{n})c_pT-st_1]^2 \\ &-\frac{I_dD}{2s}[st_1-\alpha(1-\frac{\beta}{n})c_pT]^2\} \end{aligned} \tag{16}$$

and

$$\begin{aligned} TC_{22}(T,\xi,G) &= \frac{1}{T}\{A+(\frac{c_pD}{m(\xi)})(1-\frac{\beta}{n})H_{10}(T,\xi)+\frac{c_hD}{m(\xi)}H_{20}(T,\xi) \\ &+(\frac{n+1}{2n})\alpha(\frac{\phi c_pD}{m(\xi)})(1-\frac{\beta}{n})t_0H_{10}(T,\xi)+\xi+G \\ &+\delta_1[\hat{A}+\frac{\hat{c}_pD}{m(\xi)}H_{10}(T,\xi)+\frac{\hat{c}_hD}{m(\xi)}H_{20}(T,\xi)-R(G)] \\ &+\delta_2[\hat{A}+\frac{\hat{c}_pD}{m(\xi)}H_{10}(T,\xi)+\frac{\hat{c}_hD}{m(\xi)}H_{20}(T,\xi)-R(G)-Z] \\ &+\frac{\phi_LD}{2s}t_1[2\alpha(1-\frac{\beta}{n})c_pT-st_1]+\frac{\phi_LD}{2s}[(1-\frac{\beta}{n})c_pT-st_1]^2\} \end{aligned} \tag{17}$$

Equations (16) and (17) imply that

$$TC_{21}(\frac{st_1}{\alpha(1-\frac{\beta}{n})c_p},\xi,G)=TC_{22}(\frac{st_1}{\alpha(1-\frac{\beta}{n})c_p},\xi,G) \tag{18}$$

Furthermore, $TC_2(T,\xi,G)$ is continuous on its domain as well.

5. Solution Methodology

After the above discussion, the annual total cost function can be expressed as

$$TC(T,\xi,G)=\begin{cases} TC_1(T,\xi,G) & \text{if the retailer adopts Policy 1} \\ TC_2(T,\xi,G) & \text{if the retailer adopts Policy 2} \end{cases} \tag{19}$$

where

$$TC_1(T,\xi,G)=\begin{cases} TC_{11}(T,\xi,G) & if \quad 0<T<\dfrac{st_1}{(1-\beta)c_p}, \quad 0<\xi, \quad 0<G \\ TC_{12}(T,\xi,G) & if \quad \dfrac{st_1}{(1-\beta)c_p}\le T, \quad 0<\xi, \quad 0<G \end{cases}$$

and

$$TC_2(T,\xi,G)=\begin{cases} TC_{21}(T,\xi,G) & if \quad 0<T<\dfrac{st_1}{\alpha(1-\dfrac{\beta}{n})c_p}, \quad 0<\xi, \quad 0<G \\ TC_{22}(T,\xi,G) & if \quad \dfrac{st_1}{\alpha(1-\dfrac{\beta}{n})c_p}\le T, \quad 0<\xi, \quad 0<G \end{cases}$$

In differential calculus, the theory of optimizing functions of three variables is used to minimize the total cost function. However, due to the complexity of the model, it is difficult to demonstrate that $TC_1(T,\xi,G)$ and $TC_2(T,\xi,G)$ are joint convex in T, ξ, and G. Moreover, it is difficult to find the close forms of T, ξ, and G. Theoretically, by applying certain conditions, we can demonstrate that (1) for any given set of variables, the convexity property of the total cost per unit of time can be mathematically proven, and (2) there is a unique solution for any given set of conditions. Finally, we will illustrate the optimization of the total cost per unit of time through the use of a search algorithm.

Now, taking the first- and second-order partial derivatives of $TC_{ij}(T,\xi,G)$ ($i=1,2$ and $j=1,2$) with respect to the decision variable G for any given ξ and T, they are defined as follows:

$$\frac{\partial TC_{ij}(T,\xi,G)}{\partial G}=\frac{1}{T}\{1-(\delta_1+\delta_2)(a-2bG)\} \quad (i=1,2 \text{ and } j=1,2) \tag{20}$$

and

$$\frac{\partial^2 TC_{ij}(T,\xi,G)}{\partial G^2}=\frac{2b}{T}(\delta_1+\delta_2)>0 \quad (i=1,2 \text{ and } j=1,2) \tag{21}$$

As the second derivative gives positive values for $G>0$, the total cost function $TC_{ij}(T,\xi,G)$ ($i=1,2$ and $j=1,2$) is convex with respect to G for any given ξ and T. Later, putting the value of $G=\dfrac{a}{b}$ into Equation (20), we have

$$\frac{\partial TC_{ij}(T,\xi,G)}{\partial G}\Big|_{G=\frac{a}{b}}=\frac{1+(\delta_1+\delta_2)a}{T}>0 \tag{22}$$

By applying the convexity and monotonicity properties, we obtain that

$$\frac{\partial TC_{ij}(T,\xi,G)}{\partial G}\Big|_{G=0}=\frac{1-(\delta_1+\delta_2)a}{T}<0 \tag{23}$$

As discussed above, Equation (23) implies that $a>\dfrac{1}{\delta_1+\delta_2}$. Moreover, the optimal solution G^* of $TC_{ij}(T,\xi,G)$ ($i=1,2$ and $j=1,2$) for any given ξ and T is

$$G^*=\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)} \tag{24}$$

By substituting Equation (24) into Equations (10), (11), (16), and (17), the total cost per unit time can be reduced to

$$\overline{TC}_1(T,\xi) = \begin{cases} \overline{TC}_{11}(T,\xi) & if \quad 0 < T < \dfrac{st_1}{(1-\beta)c_p}, \quad 0 < \xi \\ \overline{TC}_{12}(T,\xi) & if \quad \dfrac{st_1}{(1-\beta)c_p} \leq T, \quad 0 < \xi \end{cases} \tag{25}$$

and

$$\overline{TC}_2(T,\xi) = \begin{cases} \overline{TC}_{21}(T,\xi) & if \quad 0 < T < \dfrac{st_1}{\alpha(1-\dfrac{\beta}{n})c_p}, \quad 0 < \xi \\ \overline{TC}_{22}(T,\xi) & if \quad \dfrac{st_1}{\alpha(1-\dfrac{\beta}{n})c_p} \leq T, \quad 0 < \xi \end{cases} \tag{26}$$

where

$$\begin{aligned} \overline{TC}_{11}(T,\xi) &= \frac{1}{T}\{A+(\frac{(1-\beta)c_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{c_hD}{m(\xi)})H_{20}(T,\xi) \\ &+(\frac{\phi(1-\beta)c_pD}{m(\xi)})t_0H_{10}(T,\xi)+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)} \\ &+\delta_1[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)-R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})] \\ &+\delta_2[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)- \\ &R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})-Z]+\frac{\phi_LD}{2s}[(1-\beta)c_pT]^2\} \end{aligned} \tag{27}$$

$$\begin{aligned} \overline{TC}_{12}(T,\xi) &= \frac{1}{T}\{A+(\frac{(1-\beta)c_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{c_hD}{m(\xi)})H_{20}(T,\xi) \\ &+(\frac{\phi(1-\beta)c_pD}{m(\xi)})t_0H_{10}(T,\xi)+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)} \\ &+\delta_1[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)-R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})] \\ &+\delta_2[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)-R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})-Z] \\ &+\frac{\phi_LD}{2}t_1[2(1-\beta)c_pT-st_1]+\frac{\phi_LD}{2s}[(1-\beta)c_pT-st_1]^2\} \end{aligned} \tag{28}$$

$$\begin{aligned} \overline{TC}_{21}(T,\xi) &= \frac{1}{T}\{A+(\frac{c_pD}{m(\xi)})(1-\frac{\beta}{n})H_{10}(T,\xi)+(\frac{c_hD}{m(\xi)})H_{20}(T,\xi) \\ &+(\frac{n+1}{2n})\alpha(\frac{\phi c_pD}{m(\xi)})(1-\frac{\beta}{n})t_0\times H_{10}(T,\xi)+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)} \\ &+\delta_1[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)-R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})] \\ &+\delta_2[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)-R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}) \\ &-Z]+\frac{\phi_LD}{2s}[\alpha(1-\frac{\beta}{n})c_pT]^2+\frac{\phi_LD}{2s}[(1-\frac{\beta}{n})c_pT-st_1]^2 \\ &-\frac{I_dD}{2s}[st_1-\alpha(1-\frac{\beta}{n})c_pT]^2\} \end{aligned} \tag{29}$$

and

$$\begin{aligned}\overline{TC}_{22}(T,\xi) &= \frac{1}{T}\{A+(\frac{c_pD}{m(\xi)})(1-\frac{\beta}{n})H_{10}(T,\xi)+(\frac{c_hD}{m(\xi)})H_{20}(T,\xi)\\ &+(\frac{n+1}{2n})\alpha(\frac{\phi c_pD}{m(\xi)})(1-\frac{\beta}{n})t_0\times H_{10}(T,\xi)+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}\\ &+\delta_1[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)-R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})]\\ &+\delta_2[\hat{A}+(\frac{\hat{c}_pD}{m(\xi)})H_{10}(T,\xi)+(\frac{\hat{c}_hD}{m(\xi)})H_{20}(T,\xi)-R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})\\ &-Z]+\frac{\phi_LD}{2}t_1[2\alpha(1-\frac{\beta}{n})c_pT-st_1]+\frac{\phi_LD}{2s}[(1-\frac{\beta}{n})c_pT-st_1]^2\}\end{aligned} \quad (30)$$

Similarly, for any given ξ, taking the first- and second-order partial derivatives of $\overline{TC}_{ij}(T,\xi)$ $(i=1,2$ and $j=1,2)$ with respect to the decision variable T yields

$$\begin{aligned}\frac{\partial TC_{11}(T,\xi)}{\partial T} &= \frac{1}{T^2}\{-[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]\\ &+(\frac{D}{m(\xi)})[(1-\beta)c_p+(\delta_1+\delta_2)\hat{c}_p+\phi(1-\beta)c_pt_0]H_{11}(T,\xi)\\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(T,\xi)+(\delta_1+\delta_2)\\ &\times R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2Z+\frac{\phi_LD}{2s}(1-\beta)^2c_p^2T^2\}\end{aligned} \quad (31)$$

$$\begin{aligned}\frac{\partial\overline{TC}_{12}(T,\xi)}{\partial T} &= \frac{1}{T^2}\{-[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]\\ &+(\frac{D}{m(\xi)})(1-\beta)[c_p+(\delta_1+\delta_2)\hat{c}_p+\phi(1-\beta)c_pt_0]H_{11}(T,\xi)\\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(T,\xi)+(\delta_1+\delta_2)\times R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})\\ &+\delta_2Z+\frac{\phi_LD}{2s}(1-\beta)^2c_p^2T^2\}\end{aligned} \quad (32)$$

$$\begin{aligned}\frac{\partial\overline{TC}_{21}(T,\xi)}{\partial T} &= \frac{1}{T^2}\{-[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]\\ &+(\frac{D}{m(\xi)})[(1-\frac{\beta}{n})c_p+(\delta_1+\delta_2)\hat{c}_p+(\frac{n+1}{2n})\alpha\phi c_p(1-\frac{\beta}{n})t_0]H_{11}(T,\xi)\\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(T,\xi)+(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})\\ &+\delta_2Z+[\frac{\phi_LD}{2s}(1+\alpha^2)(1-\frac{\beta}{n})^2c_p^2-\frac{I_dD}{2s}\alpha^2(1-\frac{\beta}{n})^2c_p^2]T^2\\ &-\frac{D}{2}st_1^2(\phi_L-I_d)\}\end{aligned} \quad (33)$$

$$\begin{aligned}\frac{\partial TC_{22}(T,\xi)}{\partial T} &= \frac{1}{T^2}\{-[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]\\ &+(\frac{D}{m(\xi)})[(1-\frac{\beta}{n})c_p+(\delta_1+\delta_2)\hat{c}_p+(\frac{n+1}{2n})\alpha\phi c_p(1-\frac{\beta}{n})t_0]H_{11}(T,\xi)\\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(T,\xi)+(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})\\ &+\delta_2Z+\frac{\phi_LD}{2s}(1-\frac{\beta}{n})^2c_p^2T^2\}\end{aligned} \quad (34)$$

$$\begin{aligned}\frac{\partial^2 \overline{TC}_{11}(T,\xi)}{\partial T^2} &= \frac{1}{T^3}\{2[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}] \\ &+(\frac{D}{m(\xi)})[(1-\beta)c_p+(\delta_1+\delta_2)\hat{c}_p+\phi(1-\beta)c_p t_0]H_{12}(T,\xi) \\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{22}(T,\xi)-2[(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}) \\ &+\delta_2 Z]\}\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{\partial^2 \overline{TC}_{12}(T,\xi)}{\partial T^2} &= \frac{1}{T^3}\{2[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}] \\ &+(\frac{D}{m(\xi)})[(1-\beta)c_p+(\delta_1+\delta_2)\hat{c}_p+\phi(1-\beta)c_p t_0]H_{12}(T,\xi) \\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{22}(T,\xi)-2[(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}) \\ &+\delta_2 Z]\}\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial^2 \overline{TC}_{21}(T,\xi)}{\partial T^2} &= \frac{1}{T^3}\{2[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}] \\ &+(\frac{D}{m(\xi)})[(1-\frac{\beta}{n})c_p+(\delta_1+\delta_2)\hat{c}_p+(\frac{n+1}{2n})\alpha\phi(1-\frac{\beta}{n})c_p t_0]H_{12}(T,\xi) \\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{22}(T,\xi)+Dst_1^2(\phi_L-I_d) \\ &-[2(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z]\}\end{aligned} \tag{37}$$

and

$$\begin{aligned}\frac{\partial^2 \overline{TC}_{22}(T,\xi)}{\partial T^2} &= \frac{1}{T^3}\{2[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]+(\frac{D}{m(\xi)})[(1-\frac{\beta}{n})c_p \\ &+(\delta_1+\delta_2)\hat{c}_p+(\frac{n+1}{2n})\alpha\phi(1-\frac{\beta}{n})c_p t_0]H_{12}(T,\xi) \\ &+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{22}(T,\xi) \\ &-[2(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z]\}\end{aligned} \tag{38}$$

where

$$\begin{aligned}H_{11}(T,\xi) &= (1+\lambda)^{1-m(\xi)}m(\xi)T(1+\lambda-T+t_d(\xi))^{m(\xi)-1}-m(\xi)t_d(\xi) \\ &-1-\lambda+(1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\end{aligned} \tag{39}$$

$$\begin{aligned}H_{21}(T,\xi) &= (1+\lambda)^{1-m(\xi)}m(\xi)Tt_d(\xi)(1+\lambda-T+t_d(\xi))^{m(\xi)-1} \\ &-\frac{m(\xi)}{2-m(\xi)}T(1+\lambda-T+t_d(\xi))+\frac{m(\xi)}{2-m(\xi)}(1+\lambda)^{2-m(\xi)}T \\ &\times(1+\lambda-T+t_d(\xi))^{m(\xi)-1}-\frac{m(\xi)}{2}t_d^2-(1+\lambda)t_d(\xi)+(1+\lambda)^{1-m(\xi)}t_d(\xi) \\ &\times(1+\lambda-T+t_d(\xi))^{m(\xi)}-\frac{(1+\lambda)^2}{2}-\frac{m(\xi)}{2(2-m(\xi))}(1+\lambda-T+t_d(\xi))^2 \\ &+\frac{(1+\lambda)^{2-m(\xi)}}{2-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\end{aligned} \tag{40}$$

$$\begin{aligned}H_{12}(T,\xi) &= (1+\lambda)^{1-m(\xi)}m(\xi)(1-m(\xi))T^2(1+\lambda-T+t_d(\xi))^{m(\xi)-2} \\ &-2(1+\lambda)^{1-m(\xi)}m(\xi)T(1+\lambda-T+t_d(\xi))^{m(\xi)-1} \\ &+2[m(\xi)t_d(\xi)+1+\lambda]-2(1+\lambda)^{1-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)}\end{aligned} \tag{41}$$

and

$$\begin{aligned} H_{22}(T,\xi) &= (1+\lambda)^{1-m(\xi)} m(\xi) t_d(\xi)[1-m(\xi)]T^2(1+\lambda-T+t_d(\xi))^{m(\xi)-2} \\ &+ \frac{m(\xi)}{2-m(\xi)}T^2 + \frac{m(\xi)}{2-m(\xi)}(1+\lambda)^{2-m(\xi)}[1-m(\xi)]T^2(1+\lambda-T+t_d(\xi))^{m(\xi)-2} \\ &-2(1+\lambda)^{1-m(\xi)} m(\xi) t_d(\xi) T(1+\lambda-T+t_d(\xi))^{m(\xi)-1} + \frac{2m(\xi)}{2-m(\xi)}T(1+\lambda-T+t_d(\xi)) \\ &-\frac{2m(\xi)}{2-m(\xi)}(1+\lambda)^{2-m(\xi)}T(1+\lambda-T+t_d(\xi))^{m(\xi)-1} + m(\xi)t_d^2(\xi) + 2(1+\lambda)t_d(\xi) \\ &-2(1+\lambda)^{1-m(\xi)} t_d(\xi)(1+\lambda-T+t_d(\xi))^{m(\xi)} + (1+\lambda)^2 + \frac{m(\xi)}{2-m(\xi)}(1+\lambda-T+t_d(\xi))^2 \\ &-\frac{2(1+\lambda)^{2-m(\xi)}}{2-m(\xi)}(1+\lambda-T+t_d(\xi))^{m(\xi)} \end{aligned} \tag{42}$$

Additionally, here are the results we have obtained:

Lemma 1. *For any given ξ, we have*

(1) $H_{i1}(T,\xi) > 0$ $(i=1,2)$ $T \geq t_d(\xi)$.
(2) $H_{j2}(T,\xi) > 0$ $(i=1,2)$ $T \geq t_d(\xi)$.

Proof. The complete proof of Lemma 1 is provided in Appendix B. □

Based on above properties of $H_{12}(T,\xi)$ and $H_{22}(T,\xi)$ for any given ξ, we have the following results.

Theorem 1. *For any given ξ,*

(1) *If* $A + (\delta_1+\delta_2)\hat{A} + \xi + \frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)} \geq (\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}) + \delta_2 Z$, *then* $\overline{TC}_{ij}(T,\xi)$ $(i=1,2$ *and* $j=1,2)$ *is convex with respect to* $T \geq t_d(\xi)$, *respectively.*
(2) *If* $A + (\delta_1+\delta_2)\hat{A} + \xi + \frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)} < (\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}) + \delta_2 Z$, *then* $\overline{TC}_{ij}(T,\xi)$ $(i=1,2$ *and* $j=1,2)$ *is increasing on* $T \geq t_d(\xi)$, *respectively.*

Proof. The proof is omitted based on the above analysis. □

Given these findings, we will investigate the optimal solutions for each policy. For this, let T_{ij}^* denote the solution to $\frac{\partial \overline{TC}_{ij}(T,\xi)}{\partial T}$.
$(i=1,2$ *and* $j=1,2)$ for any given ξ. Meanwhile, let T_i^* $(i=1,2)$ be the optimal replenishment cycle time for each policy.
Policy I: This is with full advance payment in a single installment.
For convenience, let $\Psi_1 = \frac{st_1}{(1-\beta)c_p}$; then, for any given ξ, Equations (31) and (32) yield

$$\frac{\partial \overline{TC}_{11}(T,\xi)}{\partial T}|_{T=t_d(\xi)} = \frac{\Omega_0}{t_d^2(\xi)} \tag{43}$$

$$\frac{\partial \overline{TC}_{11}(T,\xi)}{\partial T}|_{T=\Psi_1} = \frac{\partial \overline{TC}_{12}(T,\xi)}{\partial T}|_{T=\Psi_1} = \frac{\Omega_1}{\Psi_1^2} \tag{44}$$

and

$$\frac{\partial \overline{TC}_{12}(T,\xi)}{\partial T}|_{T=\lambda} = \frac{\Omega_2}{\lambda^2} \tag{45}$$

where

$$\Omega_0 = -[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]+(\frac{D}{m(\xi)})[(1-\beta)c_p+(\delta_1+\delta_2)\hat{c}_p +\phi(1-\beta)c_p t_0]H_{11}(t_d(\xi),\xi)+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(t_d(\xi),\xi) +(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z+\frac{\phi_L D}{2s}(1-\beta)^2 c_p^2 t_d^2(\xi) \tag{46}$$

$$\Omega_1 = -[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]+(\frac{D}{m(\xi)})[(1-\beta)c_p+(\delta_1+\delta_2)\hat{c}_p +\phi(1-\beta)c_p t_0]H_{11}(\Psi_1,\xi)+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(\Psi_1,\xi) +(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z+\frac{\phi_L D}{2s}(1-\beta)^2 c_p^2 \Psi_1^2 \tag{47}$$

and

$$\Omega_2 = -[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]+(\frac{D}{m(\xi)})[(1-\beta)c_p +(\delta_1+\delta_2)\hat{c}_p+\phi(1-\beta)c_p t_0]H_{11}(\lambda,\xi)+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(\lambda,\xi) +(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z+\frac{\phi_L D}{2s}(1-\beta)^2 c_p^2 \lambda^2 \tag{48}$$

Based upon the above discussions, we have the following Theorems where the retailer adopts Policy 1. Meanwhile, for simplicity, let

$$\varepsilon_1 = A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}$$

and

$$\varepsilon_2 = (\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z.$$

Theorem 2. *Suppose that $t_d(\xi) < \Psi_1$ for any given ξ :*

(1) When $\varepsilon_1 \geq \varepsilon_2$, we have the following results.

- *(1-1) If $\Omega_2 \leq 0$, then $\overline{TC}_1(T|\xi)$ is decreasing on $[t_d(\xi),\lambda]$. Furthermore, $T_1^* = T_{12}^* = \lambda$.*
- *(1-2) If $\Omega_0 \leq 0$, $\Omega_1 \leq 0$, and $\Omega_2 > 0$, then there exists a unique point $T_{12}^* \in [\Psi_1,\lambda]$ satisfying $\overline{TC'}_{12}(T_{12}^*|\xi) = 0$. Furthermore, $T_1^* = T_{12}^*$.*
- *(1-3) If $\Omega_0 \leq 0$, $\Omega_1 > 0$, and $\Omega_2 > 0$, then there exists a unique point satisfying $\overline{TC'}_{11}(T_{11}^*|\xi) = 0$. Furthermore, $T_1^* = T_{11}^*$.*
- *(1-4) If $\Omega_0 > 0$, then $\overline{TC}_1(T|\xi)$ is increasing on $[t_d(\xi),\lambda]$. Furthermore, $T_1^* = T_{11}^* = t_d(\xi)$.*

(2) When $\varepsilon_1 < \varepsilon_2$, we have the following:

- *(2-1) If $\Omega_0 > 0$, then $\overline{TC}_1(T|\xi)$ is increasing on $[t_d(\xi),\lambda]$. Furthermore, $T_1^* = T_{11}^* = t_d(\xi)$.*

Proof. The complete proof of Theorem 2 is provided in Appendix C. □

Theorem 3. *Suppose that $t_d(\xi) \geq \Psi_1$ for any given ξ:*

(1) When $\varepsilon_1 \geq \varepsilon_2$, we have the following results.

- *(1-1) If $\Omega_2 \leq 0$, then $\overline{TC}_1(T|\xi)$ is decreasing on $[t_d(\xi),\lambda]$. Furthermore, $T_1^* = T_{12}^* = \lambda$.*
- *(1-2) If $\Omega_0 \leq 0$ and $\Omega_2 > 0$, then there exists a unique point $T_{12}^* \in [t_d(\xi),\lambda]$ satisfying $\overline{TC'}_{12}(T_{12}^*|\xi) = 0$. Furthermore, $T_1^* = T_{12}^*$.*

(1-3) If $\Omega_0 > 0$, then $\overline{TC}_1(T|\xi)$ is increasing on $[t_d(\xi), \lambda]$. Furthermore, $T_1^* = T_{12}^* = t_d(\xi)$.

(2) When $\varepsilon_1 < \varepsilon_2$, we have the following:

(2-1) If $\Omega_0 > 0$, then $\overline{TC}_1(T|\xi)$ is increasing on $[t_d(\xi), \lambda]$. Furthermore, $T_1^* = T_{12}^* = t_d(\xi)$.

Proof. The proof is analogous to the one provided for Theorem 2. □

Policy II: This is with partial advance payment in n ($n \geq 2$) installments.

For any given ξ, let $\Psi_2 = \dfrac{st_1}{\alpha(1-\frac{\beta}{n})c_p}$; then, Equations (33) and (34) yield

$$\frac{\partial \overline{TC}_{21}(T,\xi)}{\partial T}|_{T=t_d(\xi)} = \frac{\Omega_3}{t_d^2(\xi)} \tag{49}$$

$$\frac{\partial \overline{TC}_{21}(T,\xi)}{\partial T}|_{T=\Psi_2} = \frac{\partial \overline{TC}_{22}(T,\xi)}{\partial T}|_{T=\Psi_2} = \frac{\Omega_4}{\Psi_2^2} \tag{50}$$

and

$$\frac{\partial \overline{TC}_{22}(T,\xi)}{\partial T}|_{T=\lambda} = \frac{\Omega_5}{\lambda^2} \tag{51}$$

where

$$\begin{aligned}\Omega_3 \;=\;& -[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]+(\frac{D}{m(\xi)})[(1-\frac{\beta}{n})c_p+(\delta_1+\delta_2)\hat{c}_p\\ &+(\frac{n+1}{2n})\alpha\phi c_p(1-\frac{\beta}{n})t_0]H_{11}(t_d(\xi),\xi)+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(t_d(\xi),\xi)\\ &+(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z+[\frac{\phi_L D}{2s}(1+\alpha^2)(1-\frac{\beta}{n})^2c_p^2\\ &-\frac{I_d D}{2s}\alpha^2(1-\frac{\beta}{n})^2c_p^2]t_d^2(\xi)-\frac{D}{2}st_1^2(\phi_L-I_d)\end{aligned} \tag{52}$$

$$\begin{aligned}\Omega_4 \;=\;& -[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]+(\frac{D}{m(\xi)})[(1-\frac{\beta}{n})c_p+(\delta_1+\delta_2)\hat{c}_p\\ &+(\frac{n+1}{2n})\alpha\phi c_p(1-\frac{\beta}{n})t_0]H_{11}(\Psi_2,\xi)+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(\Psi_2,\xi)\\ &+(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z+\frac{\phi_L D}{2\alpha^2}st_1^2\end{aligned} \tag{53}$$

and

$$\begin{aligned}\Omega_5 \;=\;& -[A+(\delta_1+\delta_2)\hat{A}+\xi+\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)}]+(\frac{D}{m(\xi)})[(1-\frac{\beta}{n})c_p+(\delta_1+\delta_2)\hat{c}_p\\ &+(\frac{n+1}{2n})\alpha\phi c_p(1-\frac{\beta}{n})t_0]H_{11}(\lambda,\xi)+(\frac{D}{m(\xi)})[c_h+(\delta_1+\delta_2)\hat{c}_h]H_{21}(\lambda,\xi)\\ &+(\delta_1+\delta_2)R(\frac{a(\delta_1+\delta_2)-1}{2b(\delta_1+\delta_2)})+\delta_2 Z+\frac{\phi_L D}{2s}(1-\frac{\beta}{n})^2c_p^2\lambda^2\end{aligned} \tag{54}$$

Similarly, based upon the above discussions, we have the following Theorems when the retailer adopts Policy 2.

Theorem 4. *Suppose that $t_d(\xi) < \Psi_2$ for any given ξ:*

(1) When $\varepsilon_1 \geq \varepsilon_2$, we have the following results.

(1-1) If $\Omega_5 \leq 0$, then $\overline{TC}_2(T|\xi)$ is decreasing on $[t_d(\xi), \lambda]$. Furthermore, $T_2^* = T_{22}^* = \lambda$.

(1-2) If $\Omega_3 \leq 0$, $\Omega_4 \leq 0$, and $\Omega_5 > 0$, then there exists a unique point $T_{22}^* \in [\Psi_2, \lambda]$ satisfying $\overline{TC'}_{22}(T_{22}^*|\xi) = 0$. Furthermore, $T_2^* = T_{22}^*$.

(1-3) *If* $\Omega_3 \leq 0, \Omega_4 > 0$, *and* $\Omega_5 > 0$, *then there exists a unique point* $T_{21}^* \in [t_d(\xi), \Psi_2]$ *satisfying* $\overline{TC'}_{21}(T_{21}^*|\xi) = 0$. *Furthermore,* $T_2^* = T_{21}^*$.

(1-4) *If* $\Omega_3 > 0$, *then* $\overline{TC}_2(T|\xi)$ *is increasing on* $[t_d(\xi), \lambda]$. *Furthermore,* $T_2^* = T_{21}^* = t_d(\xi)$.

(2) *When* $\varepsilon_1 < \varepsilon_2$, *we have the following:*

(2-1) *If* $\Omega_3 > 0$, *then* $\overline{TC}_2(T|\xi)$ *is increasing on* $[t_d(\xi), \lambda]$. *Furthermore,* $T_2^* = T_{21}^* = t_d(\xi)$.

Proof. The proof is analogous to the one provided for Theorem 2. □

Theorem 5. *Suppose that* $t_d(\xi) \geq \Psi_2$ *for any given* ξ:

(1) *When* $\varepsilon_1 \geq \varepsilon_2$, *we have the following results.*

(1-1) *If* $\Omega_5 \leq 0$, *then* $\overline{TC}_2(T|\xi)$ *is decreasing on* $[t_d(\xi), \lambda]$. *Furthermore,* $T_2^* = T_{22}^* = \lambda$.

(1-2) *If* $\Omega_3 \leq 0$ *and* $\Omega_5 > 0$, *then there exists a unique point* $T_{22}^* \in [t_d(\xi), \lambda]$ *satisfying* $\overline{TC'}_{22}(T_{22}^*|\xi) = 0$. *Furthermore,* $T_2^* = T_{22}^*$.

(1-3) *If* $\Omega_3 > 0$, *then* $\overline{TC}_2(T|\xi)$ *is increasing on* $[t_d(\xi), \lambda]$. *Furthermore,* $T_2^* = T_{22}^* = t_d(\xi)$.

(2) *When* $\varepsilon_1 < \varepsilon_2$, *we have the following:*

(2-1) *If* $\Omega_3 > 0$, *then* $\overline{TC}_2(T|\xi)$ *is increasing on* $[t_d(\xi), \lambda]$. *Furthermore,* $T_2^* = T_{22}^* = t_d(\xi)$.

Proof. The proof is analogous to the one provided for Theorem 2. □

Moreover, this article considers $t_d(\xi) \leq T$, which implies that $\xi \in [0, t_d^{-1}(T)]$ for a given T, where $t_d^{-1}(\cdot)$ represents the inverse function of the primitive function $t_d(\cdot)$. As a result of this, for a given T, the total cost function is given by

$$\overline{TC}_1(\xi|T) = \begin{cases} \overline{TC}_{11}(\xi|T) & \text{when } 0 < \xi < t_d^{-1}(T) \text{ and } 0 < T < \dfrac{st_1}{(1-\beta)c_p} \\ \overline{TC}_{12}(\xi|T) & \text{when } 0 < \xi < t_d^{-1}(T) \text{ and } \dfrac{st_1}{(1-\beta)c_p} \leq T \end{cases}$$

and

$$\overline{TC}_2(\xi|T) = \begin{cases} \overline{TC}_{21}(\xi|T) & \text{when } 0 < \xi < t_d^{-1}(T) \text{ and } 0 < T < \dfrac{st_1}{(1-\dfrac{\beta}{n})c_p} \\ \overline{TC}_{22}(\xi|T) & \text{when } 0 < \xi < t_d^{-1}(T) \text{ and } \dfrac{st_1}{(1-\dfrac{\beta}{n})c_p} \leq T \end{cases}$$

Now, differentiating Equations (27)–(30) with respect to ξ for any given T is as follows:

$$\begin{aligned} \frac{\partial TC_{11}(\xi|T)}{\partial \xi} &= \frac{\partial TC_{12}(\xi|T)}{\partial \xi} = \frac{D}{T}[(1-\beta)c_p + \phi(1-\beta)c_p t_0 + (\delta_1+\delta_2)\hat{c}_p] \\ &\times\{\frac{\partial}{\partial \xi}(\frac{H_{10}(\xi|T)}{m(\xi)})\} + 1 + \frac{D}{T}[c_h + (\delta_1+\delta_2)\hat{c}_h]\{\frac{\partial}{\partial \xi}(\frac{H_{20}(\xi|T)}{m(\xi)})\} \end{aligned} \tag{55}$$

and

$$\begin{aligned} \frac{\partial TC_{21}(\xi|T)}{\partial \xi} &= \frac{\partial TC_{22}(\xi|T)}{\partial \xi} = \frac{D}{T}[(1-\frac{\beta}{n})c_p + (\frac{n+1}{2n})\alpha(1-\frac{\beta}{n})\phi c_p t_0 \\ &+(\delta_1+\delta_2)\hat{c}_p]\{\frac{\partial}{\partial \xi}(\frac{H_{10}(\xi|T)}{m(\xi)})\} + 1 + \frac{D}{T}[c_h + (\delta_1+\delta_2)\hat{c}_h] \\ &\times\{\frac{\partial}{\partial \xi}(\frac{H_{20}(\xi|T)}{m(\xi)})\} \end{aligned} \tag{56}$$

As expected, it is quite challenging to analytically prove that, for any given T, the function $\overline{TC}_{ij}(\xi|T)$ $(i = 1,2$ and $j = 1,2)$ is convex with respect to ξ. As a result of this, we will investigate the mathematical properties of the function $\overline{TC}_{ij}(\xi|T)$ $(i = 1,2$ and $j = 1,2)$ for any given T to aid in developing an algorithm to search for the optimal solution. To prevent redundancy, we will omit the analogous calculations.

In other words, for any given T, let ξ_i^* denote the solution to $\frac{\partial \overline{TC}_{ij}(\xi|T)}{\partial \xi} = 0$ $(i = 1,2$ and $j = 1,2)$. Meanwhile, let

$$\Omega_{11} = \frac{\partial \overline{TC}_{11}(\xi|T)}{\partial \xi}|_{\xi=0} = \frac{\partial \overline{TC}_{12}(\xi|T)}{\partial \xi}|_{\xi=0} \tag{57}$$

$$\Omega_{12} = \frac{\partial \overline{TC}_{11}(\xi|T)}{\partial \xi}|_{\xi=t_d^{-1}(T)} = \frac{\partial \overline{TC}_{12}(\xi|T)}{\partial \xi}|_{\xi=t_d^{-1}(T)} \tag{58}$$

$$\Omega_{21} = \frac{\partial \overline{TC}_{21}(\xi|T)}{\partial \xi}|_{\xi=0} = \frac{\partial \overline{TC}_{22}(\xi|T)}{\partial \xi}|_{\xi=0} \tag{59}$$

and

$$\Omega_{22} = \frac{\partial \overline{TC}_{21}(\xi|T)}{\partial \xi}|_{\xi=t_d^{-1}(T)} = \frac{\partial \overline{TC}_{22}(\xi|T)}{\partial \xi}|_{\xi=t_d^{-1}(T)} \tag{60}$$

Finally, let ξ^* represent the optimal preservation investment decision; then, we arrive at the following results.

Lemma 2. *For any given T, when* $\frac{\partial \overline{TC}_{1i}^2(\xi|T)}{\partial \xi^2} > 0$ *(i = 1,2), we have the following results.*

(A) *If* $\Omega_{11} \le 0$, *then* $\xi^* = t_d^{-1}(T)$.
(B) *If* $\Omega_{11} > 0$ *and* $\Omega_{12} \le 0$, *then* $\xi^* = 0$ *or* $t_d^{-1}(T)$ *is associated with the least cost.*
(C) *If* $\Omega_{12} > 0$, *then* $\xi^* = 0$.

Proof. The complete proof of Lemma 2 is provided in Appendix D. □

Lemma 3. *For any given T, when* $\frac{\partial \overline{TC}_{1i}^2(\xi|T)}{\partial \xi^2} > 0$ *(i = 1,2), we have the following results.*

(A) *If* $\Omega_{11} > 0$, *then* $\xi^* = 0$.
(B) *If* $\Omega_{11} \le 0$ *and* $\Omega_{12} > 0$, *then* $\xi^* = \xi_1^*$.
(C) *If* $\Omega_{12} \le 0$, *then* $\xi^* = t_d^{-1}(T)$.

Proof. The complete proof of Lemma 3 is provided in Appendix E. □

Lemma 4. *For any given T, when* $\frac{\partial \overline{TC}_{2j}^2(\xi|T)}{\partial \xi^2} \le 0$ *(j = 1,2), we have the following results.*

(A) *If* $\Omega_{21} \le 0$, *then* $\xi^* = t_d^{-1}(T)$.
(B) *If* $\Omega_{21} > 0$ *and* $\Omega_{22} \le 0$, *then* $\xi^* = 0$ *or* $t_d^{-1}(T)$ *associated with the least cost.*
(C) *If* $\Omega_{22} > 0$, *then* $\xi^* = 0$.

Proof. The proof is analogous to the one provided for Lemma 2. □

Lemma 5. *For any given T, when* $\frac{\partial \overline{TC}_{2j}^2(\xi|T)}{\partial \xi^2} > 0$ *(j = 1,2), we have the following results.*

(A) *If* $\Omega_{21} > 0$, *then* $\xi^* = 0$.
(B) *If* $\Omega_{21} \le 0$ *and* $\Omega_{22} > 0$, *then* $\xi^* = \xi_2^*$.
(C) *If* $\Omega_{22} \le 0$, *then* $\xi^* = t_d^{-1}(T)$.

Proof. The proof is analogous to the one provided for Lemma 3. □

Indeed, an iterative algorithm (Algorithm 1) has been constructed based on the theoretical results to obtain the optimal solution for T and ξ (Lemmas and Theorems).

Algorithm 1. Solving procedures under various policies

Step 1: Adopting Policy (I), go to Step 2. Otherwise, go to Step 8.
Step 2: For $i = 1$
Let the initialize the value of $\xi = \xi^{(\mathrm{i})} \geq 0$.
Step 3: If $t_d(\xi) < \Psi_1$ and $\varepsilon_1 \geq \varepsilon_2$, according to Theorem 2, execute one of the following sub-steps. Otherwise, go to Step 4.
Step 3-1: If $\Omega_2 \leq 0$, then we get T_1^* is $T_{12}^* = \lambda$. Substituting T_1^* into $\overline{TC}_{12}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 3-2: If $\Omega_0 \leq 0$, $\Omega_1 > 0$, and $\Omega_2 > 0$, then we get T_1^* is T_{11}^*. Substituting T_1^* into $\overline{TC}_{11}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 3-3: If $\Omega_0 > 0$, then we get T_1^* is $T_{11}^* = t_d(\xi)$. Substituting T_1^* into $\overline{TC}_{11}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 4: If $t_d(\xi) < \Psi_1$ and $\varepsilon_1 < \varepsilon_2$, according to Theorem 2, execute one of the following sub-steps. Otherwise, go to Step 5.
Step 4-1: If $\Omega_0 > 0$, then we get T_1^* is $T_{11}^* = t_d(\xi)$. Substituting T_1^* into $\overline{TC}_{11}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 5: If $t_d(\xi) \geq \Psi_1$ and $\varepsilon_1 \geq \varepsilon_2$, according to Theorem 3, execute one of the following sub-steps. Otherwise, go to Step 6.
Step 5-1: If $\Omega_2 \leq 0$, then we get T_1^* is $T_{12}^* = \lambda$. Substituting T_1^* into $\overline{TC}_{12}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 5-2: If $\Omega_0 \leq 0$, and $\Omega_2 > 0$, then we get T_1^* is T_{12}^*. Substituting T_1^* into $\overline{TC}_{12}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 5-3: If $\Omega_0 > 0$, then we get T_1^* is $T_{12}^* = t_d(\xi)$. Substituting T_1^* into $\overline{TC}_{12}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 6: If $t_d(\xi) \geq \Psi_1$ and $\varepsilon_1 < \varepsilon_2$, according to Theorem 3, execute one of the following sub-steps.
Step 6-1: If $\Omega_0 > 0$, then we get T_1^* is $T_{12}^* = t_d(\xi)$. Substituting T_1^* into $\overline{TC}_{12}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 7.
Step 7: If the difference between $\xi^{(i)}$ and $\xi^{(i+1)}$ is small enough, such as $\left|\xi^{(i+1)} - \xi^{(i)}\right| \leq 10^{-4}$, set $\xi^* = \xi^{(i+1)}$, then (T^*, ξ^*) is the optimal solution and stop. Otherwise, set $i = i + 1$ and go back to Step 2.
Step 8: For $i = 1$
Let the initialize the value of $\xi = \xi^{(\mathrm{i})} \geq 0$.
Step 9: If $t_d(\xi) < \Psi_2$ and $\varepsilon_1 \geq \varepsilon_2$, according to Theorem 4, execute one of the following sub-steps. Otherwise, go to Step 10.
Step 9-1: If $\Omega_5 \leq 0$, then we get T_2^* is $T_{22}^* = \lambda$. Substituting T_2^* into $\overline{TC}_{22}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 9-2: If $\Omega_3 \leq 0$, $\Omega_4 > 0$, and $\Omega_5 > 0$, then we get T_2^* is T_{21}^*. Substituting T_2^* into $\overline{TC}_{21}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 9-3: If $\Omega_3 > 0$, then we get T_2^* is $T_{21}^* = t_d(\xi)$. Substituting T_2^* into $\overline{TC}_{21}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 10: If $t_d(\xi) < \Psi_1$ and $\varepsilon_1 < \varepsilon_2$, according to Theorem 4, execute one of the following sub-steps. Otherwise, go to Step 11.
Step 10-1: If $\Omega_3 > 0$, then we get T_2^* is $T_{21}^* = t_d(\xi)$. Substituting T_2^* into $\overline{TC}_{21}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 11: If $t_d(\xi) \geq \Psi_1$ and $\varepsilon_1 \geq \varepsilon_2$, according to Theorem 5, execute one of the following sub-steps. Otherwise, go to Step 12.
Step 11-1: If $\Omega_5 \leq 0$, then we get T_2^* is $T_{22}^* = \lambda$. Substituting T_2^* into $\overline{TC}_{22}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 11-2: If $T_{21}^* = t_d(\xi)$, and $\Omega_5 > 0$, then we get T_2^* is T_{22}^*. Substituting T_2^* into $\overline{TC}_{22}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 11-3: If $\Omega_3 > 0$, then we get T_2^* is $T_{22}^* = t_d(\xi)$. Substituting T_2^* into $\overline{TC}_{22}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 12: If $t_d(\xi) \geq \Psi_1$ and $\varepsilon_1 < \varepsilon_2$, according to Theorem 5, execute one of the following sub-steps.
Step 12-1: If $\Omega_3 > 0$, then we get T_2^* is $T_{22}^* = t_d(\xi)$. Substituting T_2^* into $\overline{TC}_{22}(\xi|T)$, we get $\xi^{(\mathrm{i}+1)} = \xi^*$ following the results of Lemmas 2, 3, 4 and 5, go to Step 13.
Step 13: If the difference between $\xi^{(i)}$ and $\xi^{(i+1)}$ is small enough, such as $\left|\xi^{(i+1)} - \xi^{(i)}\right| \leq 10^{-4}$, set $\xi^* = \xi^{(i+1)}$, then (T^*, ξ^*) is the optimal solution and stop. Otherwise, set $i = i + 1$ and go back to Step 8.

6. Numerical Analysis

To demonstrate the model introduced in the previous section and to showcase the efficacy of the suggested optimal solution method for this inventory system, we present numerical examples. The parameters in this section were drawn from information provided in earlier research papers. The functions, parameters, and initial solution settings used in the model are detailed in the initial values for Examples 1 through 6. Finally, the proposed model and sensitivity analysis can be solved using Algorithm.

To express the numerical verification, the following data were used:

Example 1. *Let* D= 50/*per year*, c_p= $10/*unit*, $\hat{c}_p$ = 5 *ton* CO_2/*unit*, s= $15/*unit*, c_h= $4/*unit*, $\hat{c}_h$ = 3 *ton* CO_2/*unit*, $n = 3$, α= 0.4, β= 0.2, λ= 1.2 *years*, t_o= 0.5 *years*, t_1= 0.6 *year*, ϕ= 0.25/*year*, ϕ_L= 0.2/*year*, I_d= 0.8/*year*, δ_1= 0.1, δ_2= 0.12, Z= 100, ξ= 0.5, *and* $t_d(\xi)$= 0.2 *year*. *The optimal solutions of Theorem 2 are shown in Table 1.*

Table 1. Optimal solutions for Theorem 2.

Theorem 2 ($t_d(\xi) < \Psi_1 = 1.125$)									
	ε_1	ε_2	A	$\hat{A}$	Ω_0	Ω_1	Ω_2	T_1^*	$\overline{TC}_1^*(T^*\|\xi)$
(1-1)	243.4545	20.7273	200	150	≤ 0	≤ 0	≤ 0	$T_{12}^* = \lambda$ =1.2	$\overline{TC}_{12}(\lambda\|\xi)$ = 865.0453
(1-2)	223.4545	20.7273	180	150	≤ 0	≤ 0	> 0	T_{12}^* = 1.1370	$\overline{TC}_{12}(T_{12}^*\|\xi)$ = 847.8829
(1-3)	193.4545	20.7273	150	150	≤ 0	> 0	> 0	T_{11}^* =1.048	$\overline{TC}_{11}(T_{11}^*\|\xi)$ = 820.4244
(1-4)	23.7545	20.7273	10	15	> 0	> 0	> 0	$T_{11}^* = t_d(\xi)$ = 0.2	$\overline{TC}_{11}(t_d(\xi)\|\xi)$ = 505.8848
(2-1)	19.5545	20.7273	8	5	> 0	> 0	> 0	$T_1^* = t_d(\xi)$ = 0.2	$\overline{TC}_{11}(t_d(\xi)\|\xi)$ = 484.8848

Example 2. *The data are similar to Example 1: let* $t_d(\xi)$= 1.16 *years; then, the optimal solutions of Theorem 3 are shown in Table 2.*

Table 2. Optimal solutions for Theorem 3.

Theorem 3 ($t_d(\xi) \geq \Psi_1 = 1.125$)								
	ε_1	ε_2	A	$\hat{A}$	Ω_0 Ω_1	Ω_2	T_1^*	$\overline{TC}_1^*(T^*\|\xi)$
(1-1)	243.4545	20.7273	200	150	≤ 0 ≤ 0	≤ 0	$T_{12}^* = \lambda = 1.2$	$\overline{TC}_{12}(\lambda\|\xi)$ = 857.7604
(1-2)	213.4545	20.7273	170	150	≤ 0 ≤ 0	> 0	$T_{12}^* = 1.19$	$\overline{TC}_{12}(T_{12}^*\|\xi)$ = 832.4729
(1-3)	193.4545	20.7273	150	150	> 0 > 0	> 0	$T_{12}^* = t_d(\xi) = 1.16$	$\overline{TC}_{12}(t_d(\xi)\|\xi)$ = 815.2315
(2-1)	19.5545	20.7273	8	5	> 0 > 0	> 0	$T_{12}^* = t_d(\xi) = 1.16$	$\overline{TC}_{12}(t_d(\xi)\|\xi)$ = 665.3177

Example 3. *The data are familiar to Example 1: let* c_p= $4/*unit*, s= $5/*unit*, $n = 8$, λ= 1.95 *years, and* $t_d(\xi) = 1.8$ *years. The optimal solutions of Theorem 4 are shown in Table 3.*

Table 3. Optimal solutions for Theorem 4.

Theorem 4 ($t_d(\xi) < \Psi_2 = 1.9231$)									
	ε_1	ε_2	A	$\hat{A}$	Ω_3	Ω_4	Ω_5	T_2^*	$\overline{TC}_2^*(T^*\|\xi)$
(1-1)	548.4545	20.7273	450	400	≤ 0	≤ 0	≤ 0	$T_{22}^* = \lambda = 1.95$	$\overline{TC}_{22}(\lambda\|\xi)$ = 774.0699
(1-2)	518.4545	20.7273	420	400	≤ 0	≤ 0	> 0	T_{22}^* = 1.9331	$\overline{TC}_{22}(T_{22}^*\|\xi)$ = 758.6138

Table 3. *Cont.*

Theorem 4 ($t_d(\xi) < \Psi_2 = 1.9231$)									
(1-3)	487.4545	20.7273	400	350	≤ 0	> 0	> 0	T^*_{21} =1.8550	$\overline{TC}_{21}(T^*_{21}\|\xi)$ = 742.1813
(1-4)	426.4545	20.7273	350	300	> 0	> 0	> 0	$T^*_{21} = t_d(\xi) = 1.8$	$\overline{TC}_{21}(t_d(\xi)\|\xi)$ = 708.5114
(2-1)	19.5545	20.7273	8	5	> 0	> 0	> 0	$T^*_{21} = t_d(\xi) = 1.8$	$\overline{TC}_{21}(t_d(\xi)\|\xi)$ = 482.4559

Example 4. *The data are familiar to Example 3: let $t_d(\xi)$= 1.95 years, and let λ= 1.98 years. The optimal solutions of Theorem 5 are shown in Table 4.*

Table 4. Optimal solutions for Theorem 5.

Theorem 5 ($t_d(\xi) < \Psi_2 = 1.9231$)								
	ε_1	ε_2	A	$\hat{A}$	Ω_3 Ω_1	Ω_5	T^*_2	$\overline{TC}^*_2(T^*\|\xi)$
(1-1)	548.4545	20.7273	450	400	≤ 0 ≤ 0	≤ 0	$T^*_{22} = \lambda = 1.98$	$\overline{TC}_{22}(\lambda\|\xi)$ = 773.9363
(1-2)	532.8545	20.7273	430	420	≤ 0 ≤ 0	> 0	$T^*_{22} = 1.9516$	$\overline{TC}_{22}(T^*_{22}\|\xi)$ = 766.0013
(1-3)	518.4545	20.7273	420	400	> 0 > 0	> 0	$T^*_{22} = t_d(\xi) = 1.95$	$\overline{TC}_{22}(t_d(\xi)\|\xi)$ = 758.6169
(2-1)	19.5545	20.7273	8	5	> 0 > 0	> 0	$T^*_{22} = t_d(\xi) = 1.95$	$\overline{TC}_{22}(t_d(\xi)\|\xi)$ = 502.7707

Example 5. *The data are familiar to Example 1: let A= 200/order, $\hat{A}$= 150 ton CO_2/unit, and $n = 8$. Now, based on the algorithm, the optimal solutions are shown in Table 5.*

Table 5. Optimal solutions for Policy 1 (Ψ_1 =1.125).

ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}^*_1(\xi^*\|T^*)$	Iterations
0.5000	0.8318	≤ 0	> 0	> 0	T^*_{11} = 1.0536	0.8999	858.0599	4
0.8999	1.0435	≤ 0	≤ 0	> 0	T^*_{12} = 1.1553	1.1173	854.5642	
1.1173	1.1420	≤ 0	x	≤ 0	T^*_{12} = 1.2000	1.2200	853.9867	
1.2200	1.1850	≤ 0	x	≤ 0	T^*_{12} = 1.2000	1.2200	853.9867	

Example 6. *The data are familiar to Example 5: let λ= 3 years; then, based on the algorithm, the optimal solutions are shown in Table 6.*

Table 6. Optimal solutions for Policy 2 (Ψ_2 =2.3077).

ξ_0	$t_d(\xi)$	Ω_3	Ω_4	Ω_5	T^*	$\xi^*\|T^*$	$\overline{TC}^*_2(\xi^*\|T^*)$	Iterations
0.5000	0.8318	≤ 0	> 0	> 0	T^*_{21} = 0.9379	0.6670	862.1282	7
0.6670	0.9254	≤ 0	> 0	> 0	T^*_{21} = 0.9706	0.7270	861.8222	
0.7270	0.9571	≤ 0	> 0	> 0	T^*_{21} = 0.9813	0.7470	861.7907	
0.7470	0.9675	≤ 0	> 0	> 0	T^*_{21} = 0.9848	0.7530	861.7874	
0.7530	0.9706	≤ 0	> 0	> 0	T^*_{21} = 0.9859	0.7550	861.7872	
0.7550	0.9716	≤ 0	> 0	> 0	T^*_{21} = 0.9862	0.7560	861.7870	
0.7560	0.9722	≤ 0	> 0	> 0	T^*_{21} = 0.9865	0.7560	861.7870	

7. Sensitivity Analysis

This section examines how the optimal solution and the cost per unit of time fluctuate in response to alterations in the initial values of key parameters. Tables 7–14 illustrate the sensitivity analyses. Following this, we analyze how different important parameters influence the optimal solution.

Example 7. *The data are similar to Example 1: let* $t_o = 0.15$*; then, based on the algorithm, the optimal solutions are shown in Table 7.*

Table 7. Effect of ordering cost (A).

A	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
100	0.5000	0.4818	≤ 0	> 0	> 0	$T_{11}^* = 0.7355$	0.9705	730.4591	9
	0.9705	0.7267	≤ 0	> 0	> 0	$T_{11}^* = 0.8352$	1.1912	726.2048	
	1.1912	0.8232	≤ 0	> 0	> 0	$T_{11}^* = 0.8731$	1.2812	725.6859	
	1.2812	0.8595	≤ 0	> 0	> 0	$T_{11}^* = 0.8868$	1.3146	725.6221	
	1.3146	0.8726	≤ 0	> 0	> 0	$T_{11}^* = 0.8917$	1.3266	725.6143	
	1.3266	0.8773	≤ 0	> 0	> 0	$T_{11}^* = 0.8934$	1.3308	725.6133	
	1.3308	0.8789	≤ 0	> 0	> 0	$T_{11}^* = 0.8940$	1.3323	725.6132	
	1.3323	0.8795	≤ 0	> 0	> 0	$T_{11}^* = 0.8942$	1.3328	725.6132	
	1.3328	0.8797	≤ 0	> 0	> 0	$T_{11}^* = 0.8942$	1.3328	725.6132	
300	0.5000	0.4818	≤ 0	> 0	> 0	$T_{11}^* = 1.0823$	1.8493	915.0527	3
	1.8493	1.0550	≤ 0	≤ 0	≤ 0	$T_{12}^* = \lambda = 1.2$	2.2299	903.3934	
	2.2299	1.1581	≤ 0	x	≤ 0	$T_{12}^* = \lambda = 1.2$	2.2299	903.3934	

Table 8. Effect of the carbon emissions due to ordering cost ($\hat{A}$) when A = 200.

$\hat{A}$	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
100	0.5000	0.4818	≤ 0	> 0	> 0	$T_{11}^* = 0.9172$	1.3905	822.7290	5
	1.3905	0.9016	≤ 0	> 0	> 0	$T_{11}^* = 1.1008$	1.9060	811.9789	
	1.9060	1.0716	≤ 0	≤ 0	> 0	$T_{12}^* = 1.1660$	2.1152	810.9700	
	2.1152	1.1291	≤ 0	x	≤ 0	$T_{12}^* = \lambda = 1.2$	2.2299	810.8934	
	2.2299	1.1581	≤ 0	x	≤ 0	$T_{12}^* = \lambda = 1.2$	2.2299	810.8934	
200	0.5000	0.4818	≤ 0	> 0	> 0	$T_{11}^* = 0.9540$	1.4857	842.4961	4
	1.4857	0.9364	≤ 0	≤ 0	> 0	$T_{12}^* = 1.1571$	2.0858	830.0516	
	2.0858	1.1214	≤ 0	≤ 0	≤ 0	$T_{12}^* = \lambda = 1.2$	2.2299	829.2268	
	2.2299	1.1581	≤ 0	x	≤ 0	$T_{12}^* = \lambda = 1.2$	2.2299	829.2268	

Table 9. Effect of carbon emissions due to holding cost ($\hat{c}_h$) when A = 200 and $\hat{A}$ =150.

$\hat{c}_h$	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
6	0.5000	0.4818	≤ 0	> 0	> 0	$T_{11}^* = 0.9073$	1.3671	850.8282	9
	1.3671	0.8928	≤ 0	> 0	> 0	$T_{11}^* = 1.0781$	1.8403	840.6191	
	1.8403	1.0523	≤ 0	≤ 0	> 0	$T_{12}^* = 1.1362$	2.0229	839.7144	
	2.0229	1.1045	≤ 0	≤ 0	> 0	$T_{12}^* = 1.1530$	2.0779	839.6451	
	2.0779	1.1193	≤ 0	≤ 0	> 0	$T_{12}^* = 1.1575$	2.0928	839.6402	

Table 9. *Cont.*

$\hat{c}_h$	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
6	2.0928	1.1232	≤ 0	≤ 0	> 0	T_{12}^* = 1.1587	2.0968	839.6399	9
	2.0968	1.1243	≤ 0	≤ 0	> 0	T_{12}^* = 1.1590	2.0977	839.6398	
	2.0977	1.1245	≤ 0	≤ 0	> 0	T_{12}^* = 1.1591	2.0981	839.6398	
	2.0981	1.1246	≤ 0	≤ 0	> 0	T_{12}^* = 1.1591	2.0981	839.6398	
9	0.5000	0.4818	≤ 0	> 0	> 0	T_{11}^* = 0.8813	1.3038	868.1933	9
	1.3038	0.8684	≤ 0	> 0	> 0	T_{11}^* = 1.0334	1.7104	859.1488	
	1.7104	1.0122	≤ 0	> 0	> 0	T_{11}^* = 1.0838	1.8613	858.3665	
	1.8613	1.0586	≤ 0	> 0	> 0	T_{11}^* = 1.0984	1.9066	858.3058	
	1.9066	1.0718	≤ 0	> 0	> 0	T_{11}^* = 1.1024	1.9192	858.3014	
	1.9192	1.0754	≤ 0	> 0	> 0	T_{11}^* = 1.1035	1.9227	858.3011	
	1.9277	1.0764	≤ 0	> 0	> 0	T_{11}^* = 1.1037	1.9233	858.3011	
	1.9233	1.0766	≤ 0	> 0	> 0	T_{11}^* = 1.1038	1.9236	858.3010	
	1.9236	1.0767	≤ 0	> 0	> 0	T_{11}^* = 1.1038	1.9236	858.3010	

Table 10. Effect of carbon emissions due to purchasing cost ($\hat{c}_p$) when $A = 200$, $\hat{A}$ =150, and $\hat{C}_h$ =3.

$\hat{c}_p$	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
15	0.5000	0.4848	≤ 0	> 0	> 0	T_{11}^* = 0.9112	1.3807	945.4060	4
	1.3807	0.8979	≤ 0	> 0	> 0	T_{11}^* = 1.1094	1.9453	931.6980	
	1.9453	1.0829	≤ 0	≤ 0	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2486	930.0708	
	2.2486	1.1627	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2486	930.0708	
25	0.5000	0.4818	≤ 0	> 0	> 0	T_{11}^* = 0.8895	1.3300	1058.0	5
	1.3300	0.8786	≤ 0	> 0	> 0	T_{11}^* = 1.0901	1.8937	1042.4	
	1.8937	1.0681	≤ 0	≤ 0	> 0	T_{12}^* = 1.1771	2.1826	1040.3	
	2.1826	1.1463	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2636	1040.1	
	2.2636	1.1663	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2636	1040.1	

Table 11. Effect of the advance payment period (t_o) when $\xi_o = 0.25$.

t_o	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
0.15	0.2500	0.3263	≤ 0	> 0	> 0	T_{11}^* = 0.8676	1.2679	840.9898	4
	1.2679	0.8543	≤ 0	> 0	> 0	T_{11}^* = 1.1019	1.9094	821.9345	
	1.9094	1.0726	≤ 0	≤ 0	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2299	820.0601	
	2.2299	1.1581	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2299	820.0601	
0.3	0.25	0.4763	≤ 0	> 0	> 0	T_{11}^* = 0.9092	1.0187	850.2749	5
	1.0187	0.8987	≤ 0	> 0	> 0	T_{11}^* = 1.1049	1.4844	836.3766	
	1.4844	1.0859	≤ 0	≤ 0	> 0	T_{12}^* = 1.1847	1.7038	834.6657	
	1.7038	1.1601	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	1.7481	834.5371	
	1.7481	1.1741	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	1.7481	834.5371	

Table 12. Effect of the advance payment period (t_o) when $\xi_o = 0.5$.

t_o	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
0.15	0.5000	0.4818	≤ 0	> 0	> 0	T_{11}^* = 0.9363	1.4394	832.6843	5
	1.4394	0.5789	≤ 0	> 0	> 0	T_{11}^* = 0.9794	1.5537	828.7671	
	1.5537	0.9602	≤ 0	≤ 0	> 0	T_{12}^* = 11.1460	2.0495	820.7521	
	2.0495	1.1117	≤ 0	≤ 0	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2299	820.0601	
	2.2299	1.1581	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	2.2299	820.0601	
0.30	0.5000	0.6318	≤ 0	> 0	> 0	T_{11}^* = 0.9814	1.1797	843.2152	4
	1.1797	0.9684	≤ 0	≤ 0	> 0	T_{12}^* = 1.1357	1.5668	835.4980	
	1.5668	1.1147	≤ 0	≤ 0	≤ 0	$T_{12}^* = \lambda$ = 1.2	1.7481	834.5371	
	1.7481	1.1741	≤ 0	x	≤ 0	$T_{12}^* = \lambda$ = 1.2	1.7481	834.5371	

Table 13. Effect of the advance payment period (t_o) when $\xi_o = 0.75$.

t_o	ξ_0	$t_d(\xi)$	Ω_0	Ω_1	Ω_2	T^*	$\xi^*\|T^*$	$\overline{TC}_1^*(\xi^*\|T^*)$	Iterations
0.15	0.7500	0.6191	≤ 0	> 0	> 0	T_{11}^* = 0.9977	1.6038	827.3666	4
	1.6038	0.9773	≤ 0	≤ 0	> 0	T_{12}^* = 1.1527	2.0714	820.6230	
	2.0714	1.1175	≤ 0	≤ 0	≤ 0	T_{12}^* = 1.2000	2.2299	820.0601	
	2.2299	1.1581	≤ 0	x	≤ 0	T_{12}^* = 1.2000	2.2299	820.0601	
0.30	0.7500	0.7691	≤ 0	> 0	> 0	T_{11}^* = 1.0455	1.3329	838.9471	4
	1.3329	1.0297	≤ 0	≤ 0	> 0	T_{12}^* = 1.1618	1.6389	834.9723	
	1.6389	1.1390	≤ 0	x	≤ 0	T_{12}^* = 1.2000	1.7481	834.5371	
	1.7481	1.1741	≤ 0	x	≤ 0	T_{12}^* = 1.2000	1.7481	834.5371	

Example 8. *The data are similar to Example 3: let λ= 3 years; then, based on the algorithm, the optimal solutions are shown in* Table 14.

Table 14. Effect of the number of installments (n).

n	Ψ_2	ξ_0	$t_d(\xi)$	Ω_3	Ω_4	Ω_5	T^*	$\xi^*\|T^*$	$\overline{TC}_2^*(\xi^*\|T^*)$	Iterations
3	2.4107	0.5000 0.6740	0.8318 0.9291	≤ 0 ≤ 0	> 0 > 0	> 0 > 0	T_{21}^* = 0.9419 T_{21}^* = 0.9755	0.6740 0.7350	840.2859 839.9702	4
		0.7350	0.9613	≤ 0	> 0	≤ 0	T_{21}^* = 0.9863	0.7550	839.9386	
		0.7550	0.9716	≤ 0	> 0	> 0	T_{21}^* = 0.9898	0.7620	839.9353	
		0.7620	0.9752	≤ 0	> 0	> 0	T_{21}^* = 0.9910	0.7640	839.9349	
		0.7640	0.9763	≤ 0	> 0	> 0	T_{21}^* = 0.9914	0.7650	839.9349	
		0.7650	0.9768	≤ 0	> 0	> 0	T_{21}^* = 0.9915	0.7650	839.9349	
13	2.2852	0.5000	0.8318	≤ 0	> 0	> 0	T_{21}^* = 0.9370	0.6660	867.1447	7
		0.6660	0.9248	≤ 0	> 0	> 0	T_{21}^* = 0.9694	0.7250	866.8418	
		0.7250	0.9561	≤ 0	> 0	> 0	T_{21}^* = 0.9802	0.7450	866.8100	
		0.7450	0.9665	≤ 0	> 0	> 0	T_{21}^* = 0.9837	0.7510	866.8067	
		0.7510	0.9696	≤ 0	> 0	> 0	T_{21}^* = 0.9847	0.7530	866.8065	
		0.7530	0.9706	≤ 0	> 0	> 0	T_{21}^* = 0.9851	0.7540	866.8064	
		0.7540	0.9711	≤ 0	> 0	> 0	T_{21}^* = 0.9852	0.7540	866.8064	

From the above tables, we can notice the following things:

(1) Table 7 indicates that an increase in ordering cost (A) greatly impacts the replenishment cycle time and preservation technology, as well as leads to a marked increase in the retailer's costs. It implies that with increasing ordering costs, the retailer's expenses also grow. As a result, if the retailer has invested in preservation technology, they will tend to place fewer and larger orders.

(2) Table 8 indicates that while rising carbon emissions from ordering costs $(\hat{A})$ have only a minor effect on replenishment cycle time and preservation technology, they lead to a significant increase in the retailer's cost. This implies that lowering carbon emissions could help reduce the retailer's cost.

(3) Table 9 indicates that while higher carbon emissions due to holding cost $(\hat{c}_h)$ have a negligible impact on replenishment cycle time and preservation technology, they substantially increase the retailer's costs. This implies that reducing carbon emissions could lead to lower cost for the retailer.

(4) Table 10 indicates that while an increase in carbon emissions due to purchasing cost $(\hat{c}_p)$ has little impact on replenishment cycle time and preservation technology, it significantly raises the retailer's costs. Therefore, reducing carbon emissions may be effective in lowering the retailer's costs.

(5) Tables 11–13 indicate that when the advance payment period (t_o) is lengthened, the retailer's cost rises, assuming the replenishment cycle time remains unchanged. This implies that extending the advance payment period leads to higher interest expenses due to the longer payment duration.

(6) Table 14 indicates that as the number of installments increases (n), both the replenishment cycle time and the efficiency of preservation technology decrease, while the retailer's cost rises. This indicates that a higher number of installments leads to a shorter non-deterioration period because of diminished preservation technology, so the retailer should take the number of installments into account when adopting Policy 2.

8. Managerial Insights

The insights discussed above can reveal valuable managerial insights, offering a deeper understanding of the intricate interplay between deterioration rates, preservation technology, green technology, carbon regulations, and key performance metrics within the supply chain. By integrating these insights into their decision-making processes, supply chain managers can minimize costs, lower carbon emissions, and create a sustainable and eco-friendly supply chain. This approach can enable the manager to balance costs more effectively, resulting in improved financial success.

- When managing non-instantaneous deteriorating items with a maximum lifetime, retailers must carefully consider the pricing discount strategy concerning the number of installments. A key consideration is that as the number of installments decreases, total costs also decrease. This delicate balance in the pricing discount strategy is essential for efficient inventory management and cost-effectiveness in the retailing of such items.
- The sensitivity analysis on various amounts of carbon emissions yields valuable insights into the connection between carbon regulations, green technology, and total cost. A notable finding is that higher amounts of carbon emissions lead to an increase in both the adoption of green technology and associated costs. It reveals how varying carbon emission levels impact the delicate balance between environmental sustainability and the economic considerations of total costs.
- The observed significant effect of the advance payment period becomes critical, as it contributes to an increment in the interest charge. In this scenario, the retailer's attention should be directed towards managing the advance payment period, because its extension results in an increase in the interest charge and subsequently the overall retailer's costs. Therefore, the period of advance payment may become crucial for

minimizing costs in management, and retailers must carefully consider the trade-offs between the pricing discount strategies and the associated increase in costs, emphasizing the need for a balanced approach to achieve optimal outcomes.

9. Conclusions

The main purpose of this study is to investigate a replenishment problem, emphasizing the integration of joint preservation technology, investment in green operations, and replenishment cycle time for non-instantaneous deteriorating items with expiration dates within the framework of joint emission regulations. The investigation explores the complex impact of preservation technology, which affects the deterioration rate and influences the duration of the non-deterioration period. Herein, we take into account the concept that the retailer's prepayment strategy can significantly impact the overall purchasing cost. Specifically, the retailer has the option to make a single prepayment, leading to the accrual of the maximum possible percentage of price discount. Conversely, the retailer may choose to prepay a portion of the purchasing cost over multiple installments, with the percentage of the price discount contingent upon the number of equal installments.

Additionally, the model has been developed within two distinct environments, each characterized by the combination of joint carbon regulation and a specific policy framework and denoted as Policy 1 and Policy 2. The objective of this formulation is the minimization of retailer costs, a task that proves challenging to achieve through analytical means due to the inherent complexity of the problem. Not surprisingly, it is very difficult to prove that the retailer's cost is jointly convex in (ξ, T) analytically. For this, a relevant algorithm has been presented to identify optimal solutions for minimizing retailer cost effectively within the specified environments. The incorporation of a sensitivity analysis enhances the research by providing insights into the model's robustness and adaptability under varying conditions, offering a nuanced understanding of the impact of different parameters on the optimal solution.

In summary, this study is expected to contribute valuable knowledge to the field by addressing the intricate relationships between joint preservation technology, green investment, replenishment cycle time, and prepayment strategies for cost minimization in retail operations. This study mainly focuses on the impacts of preservation technology investment, green technology, and advanced purchase discounts on operational decisions. The model is limited, since the price was set as a constant for items. Dynamic pricing is an effective strategy for managing demand. Therefore, exploring dynamic pricing based on the demand rate for non-instantaneous deteriorating items would be worthwhile. Additionally, future research could consider non-instantaneous deteriorating items with expiration dates in practical situations, such as non-linear holding costs and environmental factors like the enterprise's social responsibility.

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Appendix A. Notation and Assumptions

Appendix A.1. Notation

This study organizes the notation into three categories: constant parameters, functions, and decision variables.

Constant Parameters	
A	The ordering cost (\$/order)
$\hat{A}$	The amount of carbon emissions per order (units)
c_h	The holding cost (excluding interest charged) per unit per unit time in dollars. (\$/unit)
$\hat{c}_h$	The amount of carbon emissions per unit per unit time in inventory (units)
c_p	The purchasing cost per unit in dollars (\$/unit)
$\hat{c}_p$	The amount of carbon emissions associated per unit purchased (units)
D	The demand rate (units)
t_o	The advance payment period in units of time (unit time)
t_1	The permissible delay in payment period (unit time)
λ	The time to expiration date in units of time, where $\lambda > 0$ (unit time)
Q	The ordering quantity (units)
s	The selling price per unit in dollars, where $s > c_p > 0$ (\$/unit)
α	The percent of purchasing cost that must be paid as multiple prepayments, where $0 \leq \alpha \leq 1$ (%)
β	The price discount for purchasing cost due to all the payment is paid in advance (%)
n	The number of instalments defined for prepayments during the lead time before receiving the order
ϕ	The interest rate due to instalment-based payment in advance (%)
ϕ_L	The interest rate on loan amount (%)
I_d	The interest that can be earned per \$ in a unit time ($\phi_L > I_d$) (%)
δ_1	The carbon tax under the carbon tax policy (\$/unit)
δ_2	The carbon price under the carbon-and-trade policy (carbon trading price) (\$/unit)
Z	The carbon emissions cap in the carbon cap-and-trade policy (units)
Decision variables	
ξ	The preservation technology cost per unit time for reducing the deterioration rate in order to preserve the products (\$/unit)
G	The green technology cost per unit time for emissions reduction (\$/unit)
T	The length of the inventory cycle, where $t_d(\xi) \leq T$ (unit time)
Function variables	
$m(\xi)$	The proportion of reduced deterioration rate, where $0 \leq m(\xi) \leq 1$(%)
$t_d(\xi)$	The non-deterioration period with preservation technology investment (unit time)
$\theta(t)$	The time-varying rate of deterioration, where $0 \leq \theta(t) \leq 1$
$I_1(t)$	The inventory level at time $t \in [0, t_d(\xi)]$ in which the product has no deterioration
$I_2(t)$	The inventory level at time $t \in [t_d(\xi), T]$ in which the product has deterioration
$R(G)$	The carbon reduction function
$TC(T, \xi, G)$	The total cost per unit time of inventory system (\$/unit time)

Appendix A.2. Assumptions

(1) Demand for the product is constant with time.
(2) Shortages are not permitted.
(3) The time horizon of the inventory system is infinite.
(4) The replenishment rate is infinite.
(5) The preservation technology investment affects not only the deterioration rate but also the length of the non-deterioration period.
(6) The proportion of reduced deterioration rate, $m(\xi)$, is a continuous, concave, increasing function of the retailer's capital investment. Note that $m'(\xi) > 0$ and $m''(\xi) < 0$ imply the diminishing marginal productivity of capital.
(7) The deterioration rate depends on the time t and the product's expired date λ. We assume that $\theta(t - t_d(\xi)) = \dfrac{1}{1 + \lambda - (t - t_d(\xi))}$, where $t_d(\xi) \leq t \leq \lambda$
(8) There is no repair or replacement of deteriorated units during the inventory cycle.

(9) The carbon is emitted due to the ordering, purchasing process, and storage process.
(10) If the carbon emissions do not exceed the carbon cap Z, the retailer will accumulate revenue by selling the extra carbon allowance.
(11) Green technology and its appropriate use can make a good result for carbon reduction: the reduction is presented by a function $R(\mathrm{G})$ as follows: $R(\mathrm{G}) = aG - bG^2$, where $G < \frac{a}{b}$, as in Topal et al. [12] and Xiang and Lawley [75].
(12) According to Benjaafar et al. [76], the total amount of carbon emissions per replenishment cycle includes the fixed carbon emissions associated with placing an order $\hat{A}$, which is the variable amount of carbon emissions associated with each unit $\hat{c}_p$ multiplied by the order quantity Q, as well as the integration of the amount of carbon emissions associated with the storage of each unit held per unit of time $\hat{c}_h$ multiplied by the inventory level $I(t)$ throughout the replenishment cycle. Therefore, the total amount of carbon emissions per replenishment cycle without carbon reduction investment is given by

$$CE = \hat{A} + \hat{c}_p Q + \hat{c}_h [\int_0^{t_d(\xi)} I_1(t)dt + \int_{t_d(\xi)}^{T} I_2(t)dt]$$

(13) The supplier offers a discount on the purchasing cost of the products according to the number of instalment decisions. Herein, the supplier gives the retailer a maximum percentage of discount (β%) when the retailer settles the entire purchasing cost with one installment. Otherwise, the supplier offers a lower discount rate based on the number of installments $(\frac{\beta}{n}\%)(n \geq 2)$ when the retailer prepays a certain fraction of the purchasing cost.
(14) The retailer receives a loan from a third party such as bank to pay the prepayments and starts to pay back its principal and interest whenever he/she receives payments from the customers. In addition, when the number of installments is more than one, the sales revenue can generate interest income if it is greater than the amount of loan during the delay payment time.

Appendix B. Proof of Lemma 1

(1-1) The first-order derivative of $H_{11}(T,\xi)$ for a fixed ξ with respect to T is given by

$$\frac{\partial H_{11}(T,\xi)}{\partial T} = (1+\lambda)^{(1-m(\xi))} m(\xi) T (1-m(\xi))(1+\lambda-T+t_d(\xi))^{m(\xi)-2} > 0 \text{ if } T > 0$$

So, $H_{11}(T,\xi)$ is increasing when $T > 0$. After that, $H_{11}(T,\xi) > H_{11}(t_d(\xi),\xi) = 0$ if $T > t_d(\xi)$.

Furthermore, for any given ξ, $H_{11}(T,\xi) > 0$ if $T \geq t_d(\xi)$.

(1-2) The first-order derivative of $H_{21}(T,\xi)$ for a fixed ξ with respect to T is given by

$$\begin{aligned}\frac{\partial H_{21}(T,\xi)}{\partial T} &= (1+\lambda)^{(1-m(\xi))} t_d(\xi) m(\xi) T (1-m(\xi))(1+\lambda-T+t_d(\xi))^{m(\xi)-2} + \frac{m(\xi)}{2-m(\xi)} T \\ &+ \frac{m(\xi)}{2-m(\xi)} (1+\lambda)^{(2-m(\xi))} T (1-m(\xi))(1+\lambda-T+t_d(\xi))^{m(\xi)-2} \quad \text{if } T > 0 \\ &> 0\end{aligned}$$

So, $H_{21}(T,\xi)$ is increasing when $T > 0$. After that,

$H_{21}(T,\xi) > H_{21}(t_d(\xi),\xi) = \frac{m(\xi)}{2} t_d^2(\xi) > 0$ if $T > t_d(\xi)$. Furthermore, for any given ξ, $H_{21}(T,\xi) > 0$ if $T \geq t_d(\xi)$.

(2-1) The first-order derivative of $H_{12}(T,\xi)$ for a fixed ξ with respect to T is given by

$$\frac{\partial H_{12}(T,\xi)}{\partial T} = (1+\lambda)^{1-m(\xi)} m(\xi) T^2 (1-m(\xi))(2-m(\xi))(1+\lambda-T+t_d(\xi))^{m(\xi)-3} > 0 \text{ if } T > 0.$$

So, $H_{21}(T,\xi)$ is increasing when $T > 0$. After that,

$$H_{12}(T,\xi) > H_{12}(t_d(\xi),\xi) = t_d^2(\xi)m(\xi)(1-m(\xi))(1+\lambda)^{-1} > 0 \text{ if } T{>}t_d(\xi).$$

Furthermore, for any given ξ, $G_{11}(\xi T) > 0$ if $T \geq t_d(\xi)$

(2-2) The first-order derivative of $H_{22}(T,\xi)$ for a fixed ξ with respect to T is given by

$$\begin{aligned}\frac{\partial H_{22}(T,\xi)}{\partial T} &= (1+\lambda)^{1-m(\xi)}t_d(\xi)m(\xi)(1-m(\xi))(2-m(\xi))T^2(1+\lambda-T+t_d(\xi))^{m(\xi)-3} \\ &+(1+\lambda)^{2-m(\xi)}m(\xi)T^2(1+\lambda-T+t_d(\xi))^{m(\xi)-3} \qquad \text{if } T > 0 \\ &> 0\end{aligned}$$

So, we have that $H_{22}(T,\xi)$ is increasing when $T > 0$. After that,

$$H_{22}(T,\xi) > H_{22}(t_d(\xi),\xi) = t_d^3(\xi)m(\xi)(1+\lambda)^{-1} > 0 \text{ if } T \geq t_d(\xi).$$

Furthermore, for any given ξ, $H_{22}(T,\xi) > 0$ if $T \geq t_d(\xi)$.

Incorporating the above argument, we have completed the proof of Lemma 1.

Appendix C. Proof of Theorem 2

(1-1) By the convexity of $\overline{TC}_{1j}(T,\xi)$ $(j=1,2)$ for any given ξ, if $\Omega_2 \leq 0$, which implies that $\Omega_0 \leq 0$ and $\Omega_1 \leq 0$, so we have $\frac{\partial \overline{TC}_{1j}(T|\xi)}{\partial T} \leq 0$ $(j=1,2)$ on $T \in [t_d(\xi),\lambda]$. Additionally, $\overline{TC}_{11}(\Psi_1|\xi) = \overline{TC}_{12}(\Psi_1|\xi)$. Furthermore, $\overline{TC}_1(T|\xi)$ is decreasing on $[t_d(\xi),\lambda]$. Furthermore, $T_1^* = T_{12}^* = \lambda$.

(1-2) By the convexity of $\overline{TC}_{1j}(T,\xi)$ $(j=1,2)$ for any given ξ, if $\Omega_0 \leq 0, \Omega_1 \leq 0$, and $\Omega_2 > 0$, which implies that there exists a unique value of T such that $\frac{\partial \overline{TC}_{12}(T|\xi)}{\partial T}|_{T=T_{12}^*} = 0$ on $T \in [\Psi_1,\lambda]$ and $\overline{TC}_{11}(T|\xi)$ is decreasing on $[t_d(\xi),\Psi_1]$. Additionally, $\overline{TC}_{11}(\Psi_1|\xi) = \overline{TC}_{12}(\Psi_1|\xi)$. So, we have that $\overline{TC}_1(T|\xi)$ is decreasing on $[t_d(\xi),T_{12}^*]$ and increasing on $[T_{12}^*,\lambda]$. Furthermore, $T_1^* = T_{12}^*$.

(1-3) By the convexity of $\overline{TC}_{1j}(T,\xi)$ $(j=1,2)$ for any given ξ, if $\Omega_0 \leq 0, \Omega_1 > 0$, and $\Omega_2 > 0$, which implies that there exists a unique value of T such that $\frac{\partial \overline{TC}_{11}(T|\xi)}{\partial T}|_{T=T_{11}^*} = 0$ on $T \in [t_d(\xi),\Psi_1]$ and $\overline{TC}_{12}(T|\xi)$ is increasing on $[\Psi_1,\lambda]$. Additionally, $\overline{TC}_{11}(\Psi_1|\xi) = \overline{TC}_{12}(\Psi_1|\xi)$. So, we have that $\overline{TC}_1(T|\xi)$ is decreasing on $[t_d(\xi),T_{11}^*]$ and increasing on $[T_{11}^*,\lambda]$. Furthermore, $T_1^* = T_{11}^*$.

(1-4) By the convexity of $\overline{TC}_{1j}(T,\xi)$ $(j=1,2)$ for any given ξ, if $\Omega_0 > 0$, which implies that $\Omega_1 > 0$ and $\Omega_2 > 0$, so we have that $\frac{\partial \overline{TC}_{1j}(T|\xi)}{\partial T} > 0$ $(j=1,2)$ on $T \in [t_d(\xi),\lambda]$. Additionally, $\overline{TC}_{11}(\Psi_1|\xi) = \overline{TC}_{12}(\Psi_1|\xi)$. Furthermore, $\overline{TC}_1(T|\xi)$ is increasing on $[t_d(\xi),\lambda]$. Furthermore, $T_1^* = T_{11}^* = t_d(\xi)$.

(2-1) If $\varepsilon_1 < \varepsilon_2$, by Theorem 1, we have that $\overline{TC}_{ij}(T,\xi)$ $(i=1,2$ and $j=1,2)$, is increasing on $[t_d(\xi),\lambda]$ for any given ξ. Additionally, $\overline{TC}_{11}(\Psi_1|\xi) = \overline{TC}_{12}(\Psi_1|\xi)$. So, $\overline{TC}_1(T|\xi)$ is increasing on $[t_d(\xi),\lambda]$, and $T_1^* = T_{11}^* = t_d(\xi)$.

Incorporating the above argument, we have completed the proof of Theorem 2.

Appendix D. Proof of Lemma 2

(A) For any given T, $\frac{\partial TC_{1i}^2(\xi|T)}{\partial \xi^2} \leq 0$ $(i=1,2)$ and $\Omega_{11} \leq 0$, which imply that $\Omega_{12} \leq 0$, so we obtain $\frac{\partial TC_{1i}(\xi|T)}{\partial \xi} \leq 0$ $(i=1,2)$. Furthermore, $TC_{1i}(\xi|T)$ $(i=1,2)$ is decreasing on $[0,t_d^{-1}(T)]$, and $\xi_1^* = t_d^{-1}(T)$.

(B) For any given T, $\frac{\partial TC_{1i}^2(\xi|T)}{\partial \xi^2} \le 0$ $(i=1,2)$, $\Omega_{11} > 0$, and $\Omega_{12} \le 0$, which imply that there exists a unique value of ξ such that $\frac{\partial TC_{1i}(\xi|T)}{\partial \xi}|_{\xi=\xi_1^*} = 0$ $(i=1,2)$. For this, we obtain that $TC_{1i}(\xi|T)$ $(i=1,2)$ is increasing on $[0,\xi_1^*]$ and decreasing on $[\xi_1^*, t_d^{-1}(T)]$; then, $\xi^*= 0$ or $t_d^{-1}(T)$ associated with the least cost.

(C) For any given T, $\frac{\partial TC_{1i}^2(\xi|T)}{\partial \xi^2} \le 0$ $(i=1,2)$ and $\Omega_{12} > 0$, which imply that $\Omega_{11} > 0$, so we obtain $\frac{\partial TC_{1i}(\xi|T)}{\partial \xi} > 0$ $(i=1,2)$. Furthermore, $TC_{1i}(\xi|T)$ $(i=1,2)$ is increasing on $[0,t_d^{-1}(T)]$, and $\xi_1^*= 0$.

Incorporating the above argument, we have completed the proof of Lemma 2.

Appendix E. Proof of Lemma 3

(A) For any given T, $\frac{\partial TC_{1i}^2(\xi|T)}{\partial \xi^2} > 0$ $(i=1,2)$ and $\Omega_{11} > 0$, which imply that $\Omega_{12} > 0$, so we obtain $\frac{\partial TC_{1i}(\xi|T)}{\partial \xi} > 0$ $(i=1,2)$. Furthermore, $TC_{1i}(\xi|T)$ $(i=1,2)$ is increasing on $[0,t_d^{-1}(T)]$, and $\xi_1^*= 0$.

(B) For any given T, $\frac{\partial TC_{1i}^2(\xi|T)}{\partial \xi^2} > 0$ $(i=1,2)$, $\Omega_{11} \le 0$ and $\Omega_{12} > 0$ which imply that there exists a unique value of ξ such that $\frac{\partial TC_{1i}(\xi|T)}{\partial \xi}|_{\xi=\xi_1^*} = 0$ $(i=1,2)$. For this, we obtain that $TC_{1i}(\xi|T)$ $(i=1,2)$ is decreasing on $[0,\xi_1^*]$ and increasing on $[\xi_1^*, t_d^{-1}(T)]$; then, $\xi^* = \xi_1^*$.

(C) For any given T, $\frac{\partial TC_{1i}^2(\xi|T)}{\partial \xi^2} > 0$ $(i=1,2)$ and $\Omega_{12} \le 0$, which imply that $\Omega_{11} \le 0$, so we obtain $\frac{\partial TC_{1i}^2(\xi|T)}{\partial \xi} \le 0$ $(i=1,2)$. Furthermore, $TC_{1i}(\xi|T)$ $(i=1,2)$ is decreasing on $[0, t_d^{-1}(T)]$; and $\xi^* = t_d^{-1}(T)$.

Incorporating the above argument, we have completed the proof of Lemma 3.

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