A UNIFIED PRESENTATION OF CERTAIN BILATERAL GENERATING FUNCTIONS FOR A CLASS OF POLYNOMIALS

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ABSTRACT

The object of this paper is to present rather systematically several general classes of bilateral generating functions which are applicable to such familiar orthogonal polynomials as the Jacobi, Laguerre, Hermite, and Bessel polynomials, and indeed also to their numerous interesting unifications and generalizations introduced and studied in the literature. Several remarks, comments, and observations, relevant to the present discussion, are also included.

1. INTRODUCTION

Let $f_k : \mathbb{C} \to \mathbb{C}$ be a complex-valued polynomial function of a complex variable z $(k = 1, \cdots, n)$. Also let

$$M_f = \max_{|z|=1} |f(z)|. \tag{1.1}$$

In terms of the degree d_k of the polynomial $f_k(z)$ $(k = 1, \cdots, n)$, Rassias [6] gave an interesting two-sided inequality for $M_{f_1 \cdots f_n}$. Indeed, just as the upper and lower bounds derived earlier by Srivastava and Brenner [7] for a general system of orthogonal polynomials, Rassias's result would apply easily to various members of the family of classical orthogonal polynomials (see, *e.g.*, [1], [5], and [13]) led by the Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \binom{\beta+n}{k} \left(\frac{z-1}{2}\right)^k \left(\frac{z+1}{2}\right)^{n-k} \tag{1.2}$$

$$= \binom{\alpha+n}{n} \, {}_2F_1\left[\begin{matrix} -n, \ \alpha+\beta+n+1; \\ \alpha+1; \end{matrix} \ \frac{1-z}{2}\right],$$

where ${}_2F_1$ denotes the Gaussian hypergeometric function defined, in the special case $p = 2$ and $q = 1$, by the power series:

$${}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \end{matrix} \ z\right] = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m} \frac{z^m}{m!} \tag{1.3}$$

$$(p \leqq q, \ |z| < \infty; \ p = q+1, \ |z| < 1; \ \beta_j \neq 0, -1, -2, \cdots \ (j = 1, \cdots, q))$$

with $(\lambda)_m = \Gamma(\lambda+m)/\Gamma(\lambda)$.

Some other members of the family of classical orthogonal polynomials include such special cases of the Jacobi polynomials as the Gegenbauer (or ultraspherical) polynomials, the Legendre (or spherical) polynomials, and the Tchebycheff polynomials of the first and second kinds. Indeed we also have, in the same family, the classical Laguerre polynomials:

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{(-z)^k}{k!} \tag{1.4}$$

$$= \binom{\alpha+n}{n} \, {}_1F_1\left[\begin{matrix} -n; \\ \alpha+1; \end{matrix} \ z\right],$$

the classical Hermite polynomials:

$$(1.5) \qquad H_n(z) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} (2z)^{n-2k}$$

$$= (2z)^n \, {}_2F_0\left[\begin{matrix} -\tfrac{1}{2}n, \; -\tfrac{1}{2}n+\tfrac{1}{2}; \\ \_\_\_\_\_\_\_\_; \end{matrix} \; -\frac{1}{z^2}\right],$$

and the Bessel polynomials (*cf.* [2] and [3]):

$$(1.6) \qquad y_n(z,\alpha,\beta) = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n+k-2}{k} k! \left(\frac{z}{\beta}\right)^k$$

$$= {}_2F_0\left[\begin{matrix} -n, \; \alpha+n-1; \\ \_\_\_\_\_\_\_\_; \end{matrix} \; -\frac{z}{\beta}\right],$$

which are all limiting cases of the Jacobi polynomials (see, for details, [13, p. 103, Equation (5.3.4)] and [10, p. 99, Problems 35 and 36]).

The importance of a systematic study of various properties and characteristics of the classical Jacobi polynomials (including, for example, linear, bilinear, and bilateral generating functions (*cf.* [4] and [10]) associated with them) led Srivastava and Popov [12] recently to a class of mixed multilateral generating functions for the Jacobi polynomials, which can indeed be suitably applied to derive numerous further results involving Jacobi polynomials and some of their aforementioned relatives.

A familiar unification of each of the classes of orthogonal polynomials named above, and of their various known generalizations studied in the literature (*cf.*, *e.g.*, [10]), is provided by the sequence [11, p. 307, Equation (5)]:

$$(1.7) \quad S_n^{(\alpha,\beta)}[x,a,b,c,d;\gamma,\delta;w(x)] = \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!\; w(x)} D_x^n\left\{(ax+b)^{\gamma n+\alpha}(cx+d)^{\delta n+\beta}\, w(x)\right\}$$

$$\left[D_x = \frac{d}{dx}; \qquad n = 0,1,2,\cdots\right],$$

where the parameters a, b, c, d, α, β, γ, δ are arbitrary constants, real or complex, and

$w(x)$ is independent of n and differentiable any number of times. For example, it is easily observed from (1.7) and the known Rodrigues formulas for the Jacobi and Bessel polynomials that

$$S_n^{(\alpha,\beta)}[x,a,-a,c,c;1,1;C] = (2ca)^n P_n^{(\alpha,\beta)}(x) \tag{1.8}$$

for any nonzero constant C, and

$$S_n^{(\alpha-2,0)}[x,a,0,c,d;2,0;\exp(-\beta/x)] = \frac{a^{2n}\beta^n}{n!} y_n(x,\alpha,\beta). \tag{1.9}$$

Indeed, in each of these and other cases of reducibility, we can have numerous alternative sets of choices for the various parameters involved.

For the sequence of functions defined by (1.7), various classes of linear, bilinear, bilateral, and mixed multilateral generating functions are given in the literature. The most general results of this type for the sequence defined by (1.7) were announced recently by Srivastava and Handa [9]. The object of this paper is to present a detailed demonstration of each of the earlier results and to recall how these general results can indeed be applied to derive the corresponding generating functions for various systems of orthogonal polynomials and for numerous other polynomials associated with them.

2. General Bilateral Generating Functions

The main results on bilateral generating functions for the sequence defined by (1.7) are given by Theorems 1, 2, and 3 below (*cf.* [9]).

Theorem 1. *Corresponding to a non-vanishing function* $\Omega_\mu(y_1,\cdots,y_s)$ *of* s *variables* $y_1, \cdots, y_s$ $(s \geqq 1)$ *and of (complex) order* μ, *let*

$$F_{m,p,q}^{(\rho,\sigma)}[x;y_1,\cdots,y_s;t] = \sum_{n=0}^{\infty} a_n S_{m+qn}^{(\alpha-\rho qn,\beta-\sigma qn)}[x,a,b,c,d;\gamma,\delta;w(x)] \cdot \Omega_{\mu+pn}(y_1,\cdots,y_s)t^n \qquad (a_n \neq 0), \tag{2.1}$$

where p and q are positive integers, and ρ and σ are complex parameters. Also, for an arbitrary integer $m \geqq 0$, *let*

$$(2.2)\qquad \Theta_{n,m,p,q}^{\alpha,\beta,\rho,\sigma}(x;y_1,\ldots,y_s;z) = \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k$$

$$\cdot\, S_{m+n}^{(\alpha-\gamma n+\rho qk,\beta-\delta n+\sigma qk)}[x,a,b,c,d;\gamma,\delta;w(x)]\,\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k.$$

Then

$$(2.3)\qquad \sum_{n=0}^{\infty} \Theta_{n,m,p,q}^{\alpha,\beta,\rho,\sigma}(x;y_1,\ldots,y_s;z)\,t^n = \left[1+at(ax+b)^{\gamma-1}(cx+d)^{\delta}\right]^{\alpha}$$

$$\cdot \left[1+ct(ax+b)^{\gamma}(cx+d)^{\delta-1}\right]^{\beta} \frac{w(\xi)}{w(x)}\,F_{m,p,q}^{(\gamma-\rho,\delta-\sigma)}\Bigg[\xi;y_1,\ldots,y_s;$$

$$zt^q\left\{1+at(ax+b)^{\gamma-1}(cx+d)^{\delta}\right\}^{(\rho-\gamma)q}\left\{1+ct(ax+b)^{\gamma}(cx+d)^{\delta-1}\right\}^{(\sigma-\delta)q}\Bigg],$$

provided that each side exists; here, for convenience,

$$(2.4)\qquad \xi \equiv \xi(x,t) = x + t(ax+b)^{\gamma}(cx+d)^{\delta}.$$

Theorem 2. *Under the hypotheses of Theorem 1, let*

$$(2.5)\qquad G_{m,p,q}^{(\rho)}[x;y_1,\ldots,y_s;t] = \sum_{n=0}^{\infty} a_n\,S_{m+qn}^{(\alpha-\rho qn,\beta)}[x,a,b,c,d;\gamma,1;w(x)]$$

$$\cdot\,\Omega_{\mu+pn}(y_1,\ldots,y_s)\,t^n \qquad (a_n \neq 0),$$

and define

$$(2.6)\qquad \Phi^{\alpha,\beta,\rho}_{n,m,p,q}(x;y_1,\ldots,y_s;z) = \sum_{k=0}^{[n/q]} \begin{bmatrix} m+n \\ n-qk \end{bmatrix} a_k$$

$$\cdot\, S^{(\alpha-\gamma n+\rho qk,\beta)}_{m+n}[x,a,b,c,d;\gamma,1;w(x)]\,\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k$$

for an arbitrary integer $m \geqq 0$.

Then

$$(2.7)\qquad \sum_{n=0}^{\infty} \Phi^{\alpha,\beta,\rho}_{n,m,p,q}(x;y_1,\ldots,y_s;z)\,t^n = \left[1+t(ad-bc)(ax+b)^{\gamma-1}\right]^{\alpha}$$

$$\cdot \left[1-ct(ax+b)^{\gamma}\right]^{-\alpha-\beta-m-1} \frac{w(\eta)}{w(x)}\, G^{(\gamma-\rho)}_{m,p,q}\Big[\eta;y_1,\ldots,y_s;$$

$$zt^q\left\{1+t(ad-bc)(ax+b)^{\gamma-1}\right\}^{(\rho-\gamma)q}\left\{1-ct(ax+b)^{\gamma}\right\}^{(\gamma-\rho-1)q}\Big],$$

where

$$(2.8)\qquad \eta \equiv \eta(x,t) = [x+dt(ax+b)^{\gamma}]/[1-ct(ax+b)^{\gamma}].$$

Theorem 3. *Under the hypotheses of Theorem 1, let*

$$(2.9)\qquad H^{(\sigma)}_{m,p,q}[x;y_1,\ldots,y_s;t] = \sum_{n=0}^{\infty} a_n\, S^{(\alpha,\beta-\sigma qn)}_{m+qn}[x,a,b,c,d;1,\delta;w(x)]$$

$$\cdot\, \Omega_{\mu+pn}(y_1,\ldots,y_s)\,t^n \qquad (a_n \neq 0),$$

and define

$$(2.10)\qquad \Psi^{\alpha,\beta,\sigma}_{n,m,p,q}(x;y_1,\ldots,y_s;z) = \sum_{k=0}^{[n/q]} \begin{bmatrix} m+n \\ n-qk \end{bmatrix} a_k$$

$$\cdot\, S_{m+n}^{(\alpha,\beta-\delta n+\sigma qk)}[x,a,b,c,d;1,\delta;w(x)]\,\Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k$$

for an arbitrary integer $m \geqq 0$.

Then

$$(2.11)\qquad \sum_{n=0}^{\infty} \Psi_{n,m,p,q}^{\alpha,\beta,\sigma}(x;y_1,\ldots,y_s;z)\,t^n = \left[1+t(bc-ad)(cx+d)^{\delta-1}\right]^{\beta}$$

$$\cdot \left[1-at(cx+d)^{\delta}\right]^{-\alpha-\beta-m-1}\,\frac{w(\zeta)}{w(x)}\,H_{m,p,q}^{(\delta-\sigma)}\Bigg[\zeta;y_1,\ldots,y_s;$$

$$zt^q\left\{1+t(bc-ad)(cx+d)^{\delta-1}\right\}^{(\sigma-\delta)q}\left\{1-at(cx+d)^{\delta}\right\}^{(\delta-\sigma-1)q}\Bigg],$$

where

$$(2.12)\qquad \zeta \equiv \zeta(x,t) = [x+bt(cx+d)^{\delta}]/[1-at(cx+d)^{\delta}].$$

3. Demonstrations

The proofs of Theorems 1, 2, and 3 are based rather heavily upon certain generating relationships which we recall here as the following

Lemma (Srivastava and Panda [11]; see also Srivastava [7]). *Let* ξ, η, *and* ζ *be defined by Equations* (2.4), (2.8), *and* (2.12), *respectively.*

Then, for every non-negative integer m,

$$(3.1)\qquad \sum_{n=0}^{\infty}\begin{bmatrix} m+n \\ n\end{bmatrix} S_{m+n}^{(\alpha-\gamma n,\beta-\delta n)}[x,a,b,c,d;\gamma,\delta;w(x)]\,t^n = \left[1+at(ax+b)^{\gamma-1}(cx+d)^{\delta}\right]^{\alpha}$$

$$\cdot \left[1+ct(ax+b)^{\gamma}(cx+d)^{\delta-1}\right]^{\beta}\,\frac{w(\xi)}{w(x)}\,S_m^{(\alpha,\beta)}[\xi,a,b,c,d;\gamma,\delta;w(\xi)],$$

$$(3.2)\qquad \sum_{n=0}^{\infty} \begin{bmatrix} m+n \\ n \end{bmatrix} S_{m+n}^{(\alpha-\gamma n,\beta)}[x,a,b,c,d;\gamma,1;w(x)]\,t^n = \left[1+t(ad-bc)(ax+b)^{\gamma-1}\right]^{\alpha}$$

$$\cdot \left[1-ct(ax+b)^{\gamma}\right]^{-\alpha-\beta-m-1} \frac{w(\eta)}{w(x)} S_m^{(\alpha,\beta)}[\eta,a,b,c,d;\gamma,1;w(\eta)],$$

and

$$(3.3)\qquad \sum_{n=0}^{\infty} \begin{bmatrix} m+n \\ n \end{bmatrix} S_{m+n}^{(\alpha,\beta-\delta n)}[x,a,b,c,d;1,\delta;w(x)]\,t^n = \left[1+t(bc-ad)(cx+d)^{\delta-1}\right]^{\beta}$$

$$\cdot \left[1-at(cx+d)^{\delta}\right]^{-\alpha-\beta-m-1} \frac{w(\zeta)}{w(x)} S_m^{(\alpha,\beta)}[\zeta,a,b,c,d;1,\delta;w(\zeta)].$$

Proof of Theorem 1. Denoting the left side of the assertion (2.3) by Δ, and substituting for the Θ-polynomials from (2.2), we observe that

$$(3.4)\qquad \Delta = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/q]} \begin{bmatrix} m+n \\ n-qk \end{bmatrix} a_k\, S_{m+n}^{(\alpha-\gamma n+\rho qk,\beta-\delta n+\sigma qk)}[x,a,b,c,d;\gamma,\delta;w(x)]$$

$$\cdot\, \Omega_{\mu+pk}(y_1,\ldots,y_s)\,z^k$$

$$= \sum_{k=0}^{\infty} a_k\, \Omega_{\mu+pk}(y_1,\ldots,y_s)\,(zt^q)^k$$

$$\cdot \sum_{n=0}^{\infty} \begin{bmatrix} m+qk+n \\ n \end{bmatrix} S_{m+qk+n}^{(\alpha+(\rho-\gamma)qk-\gamma n,\beta+(\sigma-\delta)qk-\delta n)}[x,a,b,c,d;\gamma,\delta;w(x)]\,t^n,$$

where we have inverted the order of the double summation involved.

If we now sum the inner series in (3.4) by appealing to the generating relation (3.1), with m, α, and β replaced by $m+qk$, $\alpha+(\rho-\gamma)qk$, and $\beta+(\sigma-\delta)qk$, respectively $(k=0,1,2,\cdots)$, we shall find that

$$(3.5)\qquad \Delta = \left[1+at(ax+b)^{\gamma-1}(cx+d)^{\delta}\right]^{\alpha}\left[1+ct(ax+b)^{\gamma}(cx+d)^{\delta-1}\right]^{\beta}\frac{w(\xi)}{w(x)}$$

$$\cdot\sum_{k=0}^{\infty} a_k\, S_{m+qk}^{(\alpha+(\rho-\gamma)qk,\beta+(\sigma-\delta)qk)}[\xi,a,b,c,d;\gamma,\delta;w(\xi)]\ \Omega_{\mu+pk}(y_1,\ldots,y_s)$$

$$\cdot\left[zt^{q}\left[1+at(ax+b)^{\gamma-1}(cx+d)^{\delta}\right]^{(\rho-\gamma)q}\left[1+ct(ax+b)^{\gamma}(cx+d)^{\delta-1}\right]^{(\sigma-\delta)q}\right]^{k},$$

where ξ is given by (2.4). Interpretation of the infinite series occurring in (3.5) by means of the definition (2.1) yields the right side of the assertion (2.3).

This evidently completes the proof of Theorem 1 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 1 holds true for those values of the various parameters and variables involved for which each side of the assertion (2.3) exists.

Proofs of Theorems 2 and 3. The proofs of Theorems 2 and 3 run parallel to that of Theorem 1, which is already detailed above. Indeed, instead of (3.1), our proofs of Theorems 2 and 3 would make use of the generating relations (3.2) and (3.3), respectively.

Each of the cases of reducibility of the sequence defined by (1.7), which are illustrated by (1.8) and (1.9) and also described fairly adequately by Srivastava and Panda [11, p. 308], can indeed be shown to lead from Theorems 1, 2, and 3 to the corresponding generating functions for simpler (and, quite frequently, more familiar) sequences of special functions and polynomials (see also Remark 3 below). For instance, in one of such circumstances, Theorem 1 will readily yield the main result of Srivastava and Popov [12].

We conclude this section by recalling the following relevant remarks concerning Theorems 1, 2, and 3 (*cf.* [9]).

Remark 1. Since [11, p. 308, Equation (10)]

$$(3.6)\qquad S_n^{(\alpha,\beta)}[x,a,b,c,d;\gamma,\delta;w(x)] = \left[ab^{\gamma+\delta-1}\,c^{\gamma-1}\,d^{1-\delta}\right]^{-n}$$

$$\cdot\, S_n^{(\beta,\alpha)}\left[\frac{bcx}{ad}, a, b, a^2d, b^2c; \delta, \gamma; w\left[\frac{adx}{bc}\right]\right],$$

the generating relationships (3.2) and (3.3), and hence also Theorem 2 and Theorem 3, are essentially equivalent.

Remark 2. For $\rho = \sigma = 0$, the assertions (2.3), (2.7), and (2.11) would immediately yield certain bilateral generating functions due to Srivastava ([7, Part II, pp. 243–244, Corollaries 19, 20, and 21]; see also [10, pp. 435–436]).

Remark 3. By appealing to the known relationships between the sequence (1.7) and its numerous special cases (*cf.* [11, p. 308]), it is not difficult to derive, from Theorems 1, 2, and 3, analogous results on generating functions for such familiar orthogonal polynomials as Jacobi, Laguerre, Hermite, and Bessel polynomials, and also for the various interesting generalizations of these polynomials studied in the literature (*cf.* [2], [3], [10], and [13]). The details involved may be omitted.

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