The Representation Theory of the Symmetric Groups

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Abstract

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This paper forms an introductory account of the irreducible representations of the permutation group using Young Tableaux as the tool to achieve this. The basics of C*-Algebra theory and Young Tableaux are provided including a brief history of the two subjects. This paper provides a straightforward development of the subject up to the main result which says that restricting the irreducible representations of S_n corresponding to the Young diagrams of shape λ to S_{n-1} decomposes as the direct sum of the irreducible representations of S_{n-1} corresponding to the Young diagrams formed by removing one box from λ.

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Chapter 1

Introduction

In this paper, we will look to find the irreducible representations of the C*-algebra generated by the symmetric group S_n. The reason behind studying the representations of the symmetric group is largely to better understand the group itself. The idea of considering the representations of a group G as a group of linear operators to ultimately gain information on G itself is an important instance of the general process of 'linearization' that is employed systematically in many areas of mathematics.

As a very vague and general rule, the only class of problems that we are able to resolve fully are those which are linear, meaning that they involve vector spaces (ideally finite dimensional) and linear maps between them. Often, when we deal with objects that are intrinsically non-linear, a widespread strategy is to try to attach to them some sort of linear 'tool' that in some way preserve, in a different form, some or all of the structure of the original object. An example of this is the contruction of the tensor product which reduces multilinear algebra to linear algebra. Another example is the construction of the homotopy and (co)homology groups of a topological space. This allows us to better understand topology via groups and linear spaces.

The linear representations of a group G respond to the same principle; they are a 'linear' set of data attached to the group which can hopefully serve to characterize G. As such, this paper will begin with an introduction to the history and basic theories of C*-algebras. Throughout the second chapter, we will define a C*-algebra, highlight some key examples, and work through a variety of theorems leading up to a central result which states that for all finite groups, their group C*-algebra is isomorphic to a direct sum of matrices. We will also introduce Bratteli diagrams, a diagrammatical tool which will prove useful in illustrating some of the main ideas in the fourth chapter.

In the third chapter we will introduce Young tableaux. We will be using Young tableax in order to describe the irreducible representations of S_n and so, starting with a brief history of the subject, this chapter goes on to provide defintions, orderings, and a variety of results which will be used to prove the correspondence between the representations and the tableaux.

In the fourth chapter we work towards the main result of the paper. That is that restricting the irreducible representations of S_n corresponding to the Young diagrams of shape λ, a partition of n, to S_{n-1} decomposes as the direct sum of the irreducible representations of S_{n-1} corresponding to the Young diagrams formed by removing one box from λ. This chapter will tie in results from the first two chapters and will illustrate the main result with a Bratteli diagram.

Finally, in the concluding chapter we will briefly describe other results and applications of Young tableaux including a conjecture posed by Anatoly Vershik and Sergei Kerov which is now a proven result.

Chapter 2

C*Algebra Basics

History of C*-algebras

For a historical perspective on C*-algebras, it is interesting to remark that the ideology of noncommutative geometry is closely related to that of Quantum Mechanics. This analogy played an important role in shaping the field and was brought forth in the following way; from 1900's onward, physicists began to recognize that classical physics were not capable of describing all of nature. It would end up being the theory of Quantum mechanics that would replace classical mechanics and this was initially discovered in two seemingly different halves.

The first half was Quantum observables and was discovered in 1925 by Werner Karl Heisenberg using a principle of reinterpretation of the observables of classical mechanics. Heisenburg introduced a mathematical structure for Quantum observables which he recognized as noncommutative in nature, in fact his quantum observables were represented as infinite matrices.

The second half of quantum mechanics was called wave mechanics and was discovered in 1926 by Erwin Schrodinger. Finding the possible relationship between these two halves was the subject of much discussion at the time. It was John von Neumann in 1927 who connected Quantum observables and wave mechanics. In the process he defined the abstract concept of a Hilbert space, which at the time had only appeared in examples, and formulated Quantum mechanics in this language. What von Neumann saw was that the wave functions were unit vectors in a certain Hilbert space and quantum observables were linear operators on a different Hilbert space. Using a unitary transformation between these spaces, von Neumann was able to provide the mathematical equivalence between the two halves and thus provide the formulation for Quantum mechanics.

Much of von Neumann's formulation has found their way into the theory of C*-algebras. For example, the Quantum observables of a given physical system are the self-adjoint linear operators on a Hilbert space $\mathcal{H}$, the states of the system are positive trace-class operators on $\mathcal{H}$ and the expected value of an observable in a given state is given by the trace of the state multiplied by the observable. As this relates to C*-algebras, if we let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators on a Hilbert space $\mathcal{H}$ (where self-adjoit elements are bounded observables) with unit operator, then $\mathcal{B}(\mathcal{H})$ is a C*-algebra.Thus the study of C*-algebras came as a direct result of the study of Quantum mechanics.[8]

Basics of C*-algebras

We first begin with the definition of an algebra.

Definition 2.1. *An **algebra** over a field is a non-empty set together with operations of multiplication, addition, and scalar multiplication by elements in the underlying field. In other words, it is a vector space equipped with bilinear product. An algebra need not be associative but for the purposes of this paper we can assume the term algebra to mean a linear associative algebra with the scalars in the complex field. An algebra A is said to be a **normed algebra** when it has a norm which makes it into a normed linear space and the norm satisfies the condition*

$$\|a \cdot b\| \leq \|a\|\|b\|$$

Adding further to the definition, a normed algebra A which is also a Banach space is called a **Banach algebra**.

Definition 2.2. *Let A be an algebra over (C). An **involution** on A is a map $* : A \to A$ where for λ, μ in $\mathbb{C}$ and a, b in A; we have*

1. $(a^*)^* = a$,

2. $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$,

3. $(ab)^* = b^* a^*$.

If A is a Banach algebra with involution and also satisfies

4. $\|a^* a\| = \|a\|^2$, *for all a in A*

then A is called a ***C*-algebra****.*

Condition (4) gives a very strong link between the algebraic and topological structures of a C*-algebra. This condition is often referred to as the C*-conditon.

Next, we introduce some terminology for elements in a C*-algebra.

Definition 2.3. *Let A be a C*-algebra,*

1. *An element a is called* ***self-adjoint*** *if $a^* = a$.*

2. *An element a is called* ***normal*** *if $a^* a = aa^*$.*

3. *An element p is a* ***projection*** *if $p^2 = p = p^*$*

4. *We say that A is* ***unital*** *if that algebra has a unit element that is an identity for the multiplication. We denote such an element by 1.*

5. *Assuming that A is unital, an element u is a unitary if $u^* u = uu^* = 1$. That is, u is invertible and $u^{-1} = u^*$.*

6. *Assuming that A is unital, an element u is called an* ***isometry*** *if $u^* u = 1$.*

7. *An element u is a called a* ***partial isometry*** *if $u^* u$ is a projection.*

8. *An element u is called* ***positive*** *if it may be written as $a = b^* b$ for some b in A. In this case, we often write $a \geq 0$.*

We next define the centre of a C*-algebra. This will be useful for one of our main results which states that for a finite group, linear combinations of group elements over their conjugacy classes spans the centre of the group algebra (which we will define a little bit later).

Definition 2.4. *The **centre** of a C*-algebra A (denoted Z(A)) is the set of elements a in A which commute with every element in the C*-algebra. Notationally we have,*

$$Z(A) = \{a \in A \mid ab = ba \;\; \forall b \in A\}$$

Some Trivial Consequences

Let A be a C*-algebra. The following results hold.

1. In A, the involution is isometric.

   Proof. Let a be in A and assume $a \neq 0$. Then $\|a\|^2 = \|a\| \cdot \|a\| = \|a^*a\| \leq \|a^*\| \cdot \|a\|$. Dividing by $\|a\|$ implies $\|a\| \leq \|a^*\|$. Applying this to a^* gives $\|a^*\| \leq \|a^{**}\| = \|a\|$ and so $\|a\| = \|a^*\|$. □

   .

2. If A has a unit 1, then $1 = 1^*$

   Proof. First note that in any algebra the identity is unique. Then for any a in A, $a1^* = (1a^*)^* = (a^*)^* = a$ and $1^*a = (a^*1)^* = (a^*)^* = a$ so 1^* is the identity and $1 = 1^*$. □

3. If A has identity 1, then $\|1\| = 1$.

   Proof. $\|1\| = \|1^*1\| = \|1\|^2$ by the C*-condition, so $\|1\| = 1$ □

Now that we have a bit of a sense of some of the properties of C*algebras. Let's look at a few of the most foundational examples.

Some Examples of C*-algebras

Example 2.5. $\mathbb{C}$, *the complex numbers, form a C*algebra.*

Example 2.6. *Consider a compact Hausdorff space X and let $C(X)$ denote the space of all continuous, complex-valued functions on X.*

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous}\}$$

The norm will be the usual supremum norm $\|f\| = sup\{|f(x)| \mid x \in X\}$. Involution is defined by pointwise complex conjugation, addition and multiplication are defined pointwise. $C(X)$ is a commutative C-algebra with unit (namely the identity function $1(x) = 1$, $\forall x \in X$.*

Example 2.7. *Let $\mathcal{H}$ be a complex Hilbert space with inner product denoted by $< \cdot, \cdot >$. The collection of bounded linear operators on $\mathcal{H}$ is a C*-algebra and is denoted by $\mathcal{B}(\mathcal{H})$. The product is by composition, the adjoint is definted for any operator a on $\mathcal{H}$ by the equation $< a^*\xi, \eta >=< \xi, a\eta >$, for all ξ and η in $\mathcal{H}$. Finally, the norm is given by*

$$\|a\| = sup\{\|a\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\}$$

for any a in $\mathcal{B}(\mathcal{H})$.

In the special case where the Hilbert space is of finite dimension n, after choosing an orthonormal basis, every operator can be represented as an $n \times n$ matrix and so we have the following example.

Example 2.8. *Consider the set $M_n(\mathbb{C})$ of $n \times n$ complex matrices for some positive integer n. This is a noncommutative C*-algebra. The norm is given by $\|M\| = sup\{\|Mv\|_2 \mid v \in \mathbb{C}^n, \|v\|_2 \leq 1\}$ where $\| \cdot \|$ is the usual l_2 norm on $\mathbb{C}^n$. Involution is defined as $(m_{i,j})^* = \bar{m}_{j,i} \, \forall m_{i,j} \in M$ and multiplication is defined using the usual algebraic operations for matrices.*

Spectrum

Before moving on to the main results of C*-algebras, we introduce the notion of spectrum. This will prove useful in one of the proofs included in the following section and so we take the time here to introduce and work with it a bit.

Definition 2.9. *Let A be a Banach algebra with unit 1. Let $a \in A$ and $\lambda \in \mathbb{C}$. Then the **spectrum** of an element a is defined to be the set of all* *complex numbers λ such that $\lambda 1 - a$ is not invertible in A. The spectrum of a is denoted spec(a), thus*

$$spec(a) = \{\lambda \mid \lambda 1 - a \text{ is singular}\}$$

*The **spectral radius** of a, denoted $r(a)$ is $r(a) = sup\{|\lambda| \mid \lambda \in spec(a)\}$.*

Example 2.10. *Consider the C*-algebra $M_n(\mathbb{C})$ and recall from linear algebra that the following two conditions are equivalent for a matrix $M \in M_n(\mathbb{C})$.*

1. *M is invertible*

2. *$det(M) \neq 0$*

This allows us to compute the spectrum of M by simply finding the zeros of $det(\lambda - M)$. Now consider the C-algebra $\mathcal{B}(\mathcal{H})$ mentioned in 2.7, such a result fails when $\mathcal{H}$ is not finite-dimensional as the determinant function fails to exist. Thus it is important to study spectrum as it relates to general algebras.*

We next quote two fundamental results regarding the spectrum of an arbitrary element a of a Banach algebra A.

Theorem 2.11. *Let A be a Banach algebra with a in A. Then, spec(a) is a unital non-empty compact subset of $\mathbb{C}$.*

Proof. To show $spec(a)$ is bounded, consider $\lambda \in \mathbb{C}$ s.t. $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$ and so using that for a unital Banach algebra A if $\|x\| \leq 1$ then $(1-x)^{-1}$ exists, we have $\lambda(1 - \lambda^{-1}a) = \lambda 1 - a$ has inverse so $\lambda \notin spec(a)$. Hence $spec(a)$ is contained in the ball with center 0 and radius $\|a\|$.

To show that spec(a) is closed we use the following.

Theorem 2.12. *Let A be a Banach algebra and let G be the set of invertible unital elements of A. Then G is open.*

Proof. We need to show that for $a \in G, \exists x \in A$ s.t. $(a + x) \in G$. Write $a + x = a(1 + a^{-1}x)$ and if $\|a^{-1}x\| < 1$ then $(1 + a^{-1})$ is invertible and so is $a + x$. Thus if $\|x\| < \frac{1}{\|a^{-1}\|}$, $(a + x) \in G$. Therefore the open ball with center a and radius $\|a^{-1}\|^{-1}$ is a subset of G. □

Let $res(a)$ be defined as follows, $res(a) = \{\lambda \mid \lambda 1 - a \textit{ is invertible}\} = \{\lambda \mid \lambda \notin spec(a)\}$. Then $res(a)$ is open since it is the pre-image of the open set of G of invertible elements under the continuous map $\lambda \mapsto \lambda 1 - a$. Thus $spec(a)$ is closed and therefore compact.

To show the spectrum is non-empty, first let $\lambda, \mu \in res(a)$ and note

$$
\begin{aligned}
(\lambda 1 - a)^{-1} - (\mu 1 - a)^{-1} &= [(\mu 1 - a)(\mu 1 - a)^{-1}](\lambda 1 - a)^{-1} \\
&\quad -[(\lambda 1 - a)(\lambda 1 - a)^{-1}](\mu 1 - a)^{-1} \\
&= (\lambda 1 - a)^{-1}(\mu 1 - a)^{-1}[(\mu 1 - a) - (\lambda 1 - a)] \\
&= (\lambda 1 - a)^{-1}(\mu 1 - a)^{-1}(\mu - \lambda)
\end{aligned}
$$

Now take a non-zero linear functional ϕ and let $f(\lambda) = \phi((\lambda 1 - a)^{-1})$. Then f is analytic on $res(a)$. So if $spec(a)$ were empty, f would be entire. To see f is bounded consider

$$\lim_{|\lambda| \to \infty} |f(\lambda)| = \lim_{|\lambda| \to \infty} |\phi((\lambda 1 - a)^{-1}| = \lim_{|\lambda| \to \infty} \frac{1}{|\lambda|}(\phi((1 - \frac{a}{\lambda})^{-1})) = 0.$$

Hence, by Liouville's Theorem f is constant and its limit as $|\lambda| \to \infty$ shows $f = 0$. However, by the Hahn-Banach Theorem we can choose ϕ so that $\phi(-a^{-1}) = \phi((0 - a^{-1})) = f(0) \neq 0$ which is a contradiction. Thus f cannot be entire. Therefore $spec(a)$ is non-empty. $\square$

The second fundamental result is the following.

Theorem 2.13. *Let $a \in A$. The sequence $\|a^n\|^{\frac{1}{n}}$ is bounded by $\|a\|$ and has limit r(a). In particular r(a) is finite and r(a) $\leq \|a\|$.*

Proof. We will not give a complete proof but will demonstrate part of the argument. First, from the proof of 2.11 we have that for $\lambda \in spec(a)$ this implies $|\lambda| \leq \|a\|$ and so $r(a) \leq \|a\|$ follows immediately. To see $\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = r(a)$ we let $\lambda \in \mathbb{C}$ s.t. $|\lambda| > \limsup_n \|a^n\|^{\frac{1}{n}}$. Then the series $\sum_{n=0}^{\infty} \frac{a}{\lambda^{n+1}} a^n$ is convergent in A (geometric). We show $(\lambda 1 - a) \sum_{n=0}^{\infty} \frac{a}{\lambda^{n+1}} a^n = \lim_n 1 - \lambda^{-N} a^{-N} = 1 - 0 = 1$. Thus $(\lambda 1 - a)$ is invertible so $\lambda \notin spec(a)$. $\square$

Definition 2.14. *Let A be a unital algebra over $\mathbb{C}$. Then $\mathcal{M}(A)$ is the set of non-zero homomorphisms to $\mathbb{C}$.*

Theorem 2.15. *Let A be a commutative, unital C*-algebra. The function $\hat{}$ sending $a \in A$ to $\hat{a} \in C(\mathcal{M}(A))$ defined by $\hat{a}(\phi) = \phi(a), \phi \in \mathcal{M}(A)$ is an isometric *-isomorphism from A to $C(\mathcal{M}(A))$.*

Proof. The weak-* topology is defined so that a net ϕ_α converges to ϕ iff $\phi_\alpha(a)$ converges to $\phi(a)$ $\forall a \in A$. We now state a lemma without proof.

Lemma 2.16. *Let A be a unital commutative C*-algebra. The set $\mathcal{M}(A)$ is a weak-* compact subset of the unit ball of the dual space A^*.*

The fact that $\hat{a}$ is a continuous function follows from the definition of the weak-* topology. Next note that for $\phi \in \mathcal{M}(A)$ and $a, b \in A$

1. $\widehat{ab}(\phi) = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi)\hat{b}(\phi)$
2. $\widehat{(a+b)}(\phi) = \phi(a+b) = \phi(a) + \phi(b) = \hat{a}(\phi) + \hat{b}(\phi)$

thus $\hat{}$ is a homomorphism. To see that $\hat{}$ is isometric first suppose a is self-adjoint. Then $\|a\| = r(a)$ (which can be shown using the C*-condition which gives $\|a^2\| = \|a\|^2$) and $\|a\| = r(a) = sup\{|\phi(a)| \mid \phi \in \mathcal{M}(A)\} = sup\{|\hat{a}(\phi)| \mid \phi \in \mathcal{M}(A)\} = \|\hat{a}\|$. For arbitrary a, note that a^*a is self-adjoint, then $\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\widehat{a^*a}\|^{\frac{1}{2}} = \|\hat{a}^*\hat{a}\|^{\frac{1}{2}} = \|\hat{a}\|$.
Finally, we must show that $\hat{}$ is surjective. To do this we show that the range separates points. Let $\phi, \psi \in \mathcal{M}(A)$ with $\phi \neq \psi$. Then $\exists a \in A$ s.t. $\phi(a) \neq \psi(a) \implies \hat{a}(\phi) \neq \hat{a}(\psi)$ thus $\hat{a}$ separates points. Now since the range is a unital *-algebra and the range separates points, we can use the Stone-Weierstrass Theorem which states that a unital *-subalgebra of $C(\mathcal{M}(A))$ which separates points is dense. Thus since $\hat{}$ is isometric and A is by definition complete, we conclude the range is closed and hence $\hat{}$ is onto. □

Proposition 2.17. *Let B be a subalgebra of A which contains the unit 1 (i.e. $1 \in B \subseteq A$) and let a be in A. Then $spec_B(a) = spec_A(a)$.*

Definition 2.18. *Let B be a unital C*-algebra and let a be a normal element of B. Now let A be the C*-subalgebra of B generated by a and the unit. For each f in $C(spec(a))$ (that is the continuous functions on the spectrum of a), we let $f(a)$ be the unique element of A such that*

$$\phi(f(a)) = f(\phi(a)),$$

for all ϕ in $\mathcal{M}(A)$.

The following is a restatement of Theorem 2.14.

Corollary 2.19. *Let B be a unital C*-algebra and let a be a normal element of A. Then the map that sends f to $f(a)$ is an isometric *-isomorphism from $C(spec(a))$ to A, where A is the C*-subalgebra of B generated be a and the unit. Moreover, if $f(z) = \sum_{k,l}(a_{k,l})(z^k)(\bar{z}^l)$ is any polynomial in z and $\bar{z}$, then*

$$f(a) = \sum_{k,l}(a_{k,l})(a^k)(a^*)^l.$$

Corollary 2.20. *If A is a C*-algebra, then its norm is unique. That is to say, if a *-algebra possesses a norm in which it is a C*-algebra, then it possesses only one such norm.*

Proof. Let a be in A, if a is self-adjoint then $\|a\| = r(a)$.
Otherwise, using the C*-condition as well as the fact that a^*a is self-adjoint, we see that $\|a\| = \|a^*a\|^{1/2} = r(a^*a)^{1/2}$. The spectral radius depends on the algebraic structure of A and so we are done. □

Main Results of C*algebras

We now state several results of C*-algebras (representations of *-algebras, and group C*-algebras). We will be focusing on definitions and results pertaining to finite dimensional C*-algebras as well as group C*-algebras and their representations.

Theorem 2.21. *If N is a positive integer, then the centre of $M_N(\mathbb{C})$ is the scalar multiples of the identity.*

Proof. Let E_{ij} be the matrix in $M_N(\mathbb{C})$ with 1 in entry (i, j) and 0 everywhere else. Consider a matrix M in $M_N(\mathbb{C})$, so $E_{ii}ME_{jj} = m_{ij}E_{ij}$, where m_{ij} is the ij^{th} entry of M. Now assume M is in the centre of $M_N(\mathbb{C})$. Then $E_{ii}ME_{jj} = ME_{ii}E_{jj} = 0$ if $i \neq j$ which implies $m_{ij} = 0$ if $i \neq j$ so the off diagonals of a matrix in the centre are zero. Next, we see that $ME_{ij} = m_{ii}E_{ij}$ and also $E_{ij}M = m_{jj}E_{ij}$, but M is in the centre so $ME_{ij} = E_{ij}M$ and $m_{ii} = m_{jj}$. Thus the diagonal entries are all the same and we are done. □

Theorem 2.22. *Let A be a finite dimensional C*algebra. Then there exist positive integers K and $N_1, ..., N_K$ such that*

$$A \cong \oplus_{i=1}^{K} M_{N_i}(\mathbb{C}).$$

Moreover, K is the dimension of the centre of A and $N_1, ..., N_K$ are unique up to permutation.

Definition 2.23. *Let A be a *-algebra. A* ***representation*** *of A is a pair, (π,H), where H is a Hilbert space and $\pi : A \to \mathcal{B}(H)$ is a *-homomorphism. We say that π is a representation of A on H.*

Definition 2.24. *Consider the representation (π,H) of the *-algebra A. We say that a closed subspace $\mathcal{N} \subset H$ is* ***invariant*** *if $\pi(a)(\mathcal{N}) \subset \mathcal{N}$ for all a in A.*

Proposition 2.25. *Let A be a *-algebra and let (π, H) be a representation of A. Then a closed subspace $\mathcal{N}$ is invariant if and only if $\mathcal{N}^{\perp}$ is invariant.*

Proof. First let $\mathcal{N}$ be invariant and consider ξ in $\mathcal{N}^{\perp}$ and a in A. It suffices to show that $\pi(a)\xi$ is again in $\mathcal{N}^{\perp}$. Let $\eta \in \mathcal{N}$, then we have

$$0 =< \pi(a)\xi, \eta >=< \xi, \pi(a)^*\eta >=< \xi, \pi(a^*)\eta >,$$

Since $\pi(a^*)\mathcal{N} \subset \mathcal{N}$.
To show the other direction, if $\mathcal{N}^{\perp}$ is invariant so is $\mathcal{N}$ since $(\mathcal{N}^{\perp})^{\perp} = \mathcal{N}$. □

Definition 2.26. *A representation (π, H) of a *-algebra A is called* ***non-degenerate*** *if the only vector ξ in H such that $\pi(a)\xi = 0$ for all a in A is $\xi = 0$. Otherwise, we say that the representation is* ***degenerate****.*

Proposition 2.27. *A representation (π, H) of a unital *-algebra is non-degenerate if and only if $\pi(1) = 1$.*

Theorem 2.28. *Every representation of a *-algebra is a direct sum of a non-degenerate representation and the zero representation on some Hilbert space.[4]*

This means that we can restrict our attention to non-degenerate representations. A particularily nice class of representations are those which are cyclic. Below is the definition of a cyclic representation.

Definition 2.29. *Consider the representation (π, H) of a *-algebra A. We say that a vector ξ in H is* ***cyclic*** *if the linear space $\pi(A)\xi$ is dense in H. A representation is called cyclic if it has a cyclic vector.*

It turns out that the representations without non-trivial invariant subspaces are of particular interest. We now turn our attention here.

Definition 2.30. *A representation (π,H) of A is called **irreducible** if the only invariant subspaces $\mathcal{N} \subset H$ are 0 and H. Otherwise, it is called **reducible***

Proposition 2.31. *A non-degenerate representation of a *-algebra is irreducible if and only if every non-zero vector is cyclic.*

Proposition 2.32. *A non-degenerate representation of a *-algebra is irreducible if and only if the only positive operators which commute with its image are scalars.*

Proof. First we assume that the only positive operators which commute with the image of a non-degenerate representation are scalars. Now suppose that (π, H) is a reducible representation of the *-algebra A and let $\mathcal{N}$ be a non-trivial closed invariant subspace of H. Let p be the orthogonal projection onto $\mathcal{N}$ (i.e. $p\xi = \xi$ for all ξ in $\mathcal{N}$ and $p\xi = 0$ for all ξ in $\mathcal{N}^{\perp}$).Since p is a projection, $p = p^* = p^2$ which means that p is positive. Moreover, since both $\mathcal{N}$ and $\mathcal{N}^{\perp}$ are non-empty, the operator p is not a scalar. We now check to see that it commutes with $\pi(a)$ for any a in A. If ξ is in $\mathcal{N}$ then we know that $\pi(a)\xi$ is also in $\mathcal{N}$ and so:

$$(p\pi(a))\xi = p(\pi(a)\xi) = \pi(a)\xi = \pi(a)(p\xi) = (\pi(a)p)\xi.$$

Now if we consider ξ in $\mathcal{N}^{\perp}$, we know that $\pi(a)\xi$ is also in $\mathcal{N}^{\perp}$ and so:

$$(p\pi(a))\xi = p(\pi(a)\xi) = 0 = \pi(a)(0) = \pi(a)(p\xi) = (\pi(a)p)\xi.$$

Since every vector in H is the sum of two as above, we see that $p\pi(a) = \pi(a)p$. Thus p commutes with the image of (π, H) so we have our contradiction and (π, H) is irreducible.

Next, we assume that our non-degenerate representation of the *-algebra is irreducible. Let h be some positive, non-scalar operator on H which commutes with every element of $\pi(a)$. From Corollary 2.19, h is a scalar if its spectrum consists of a single point. Since this is not the case, the spectrum must consist of at least two points. So we can then find non-zero continuous functions f, g on $spec(h)$ whose product is zero. Since f is non-zero on the spectrum of h, the operator $f(h)$ is non-zero. Let the closure of its range be denoted by $\mathcal{N}$. This is a non-zero subspace of H. Similarily, $g(h)$ is also a non-zero operator but it will be zero on the range of $f(h)$ and hence $\mathcal{N}$. This then implies that $\mathcal{N}$ is a proper subspace of H (since $g(h)$ is non zero). Next, since h commutes with $\pi(a)$, so does $f(h)$. Let a be in A, then for any $\epsilon > 0$ we can find a polynomial $p(x)$ such that $\|p - f\|_\infty < \epsilon$ in $C(spec(h))$ which then means that $\|p(h) - f(h)\| < \epsilon$. Also, since h commutes with $\pi(a)$ so does $p(h)$. We will now try and show that $\mathcal{N}$ is invariant under $\pi(a)$ to give us the desired contradiction. It suffices to check that the range of $f(h)$ is invariant under $\pi(a)$. So let $\xi \in H$, then we have:

$$\pi(a)(f(h)\xi) = \pi(a)f(h)\xi = f(h)\pi(a)\xi \in f(h)H.$$

This shows that $\mathcal{N}$ is invariant under $\pi(a)$ and thus that (π, H) is reducible (a contradiction). Thus the only operators which commute with $\pi(a)$ are scalars. □

We now turn our attention to group C*algebras. Starting with group representations we move to understand the representations of finite group C*algebras.

Definition 2.33. *Let G be a group. A **unitary representation** of G is a pair (u,$\mathcal{H}$), where $\mathcal{H}$ is a complex Hilbert space and u is a group homomorphism from G to the group of unitary operators on $\mathcal{H}$ (denoted $\mathcal{U}(\mathcal{H})$), with the product as group operation. We say that u is a unitary representation of G on $\mathcal{H}$ with the image of an element g in G under the map u written as u_g*

As you might expect, there is an exact analogue of invariant subspaces and irreducible representations for unitary representations.

Definition 2.34. *Let (u,$\mathcal{H}$) be a unitary representation of the discrete group G. We say that a closed subspace $\mathcal{N} \subset \mathcal{H}$ is **invariant** for u if $u_g\mathcal{N} \subset \mathcal{N}$, for all $g \in G$. The representation is **irreducible** if its only closed invariant subspaces are 0 and $\mathcal{H}$ and called **reducible** otherwise.*

We now define a group algebra. For the moment, we do not need to assume the group is finite. Rather, our construction has a finiteness condition built in.

Definition 2.35. *Let G be a group. Its (complex) **group algebra** consists of all formal sums $\sum_{g\in G} a_g g$ where $a_g \in \mathbb{C}$ and $a_g = 0$ for all but finitely many $g \in G$. We denote the group algebra $\mathbb{C}G$. Defining the product $a_g g \cdot a_h h = a_g a_h gh$, for all $g, h \in G$ and $a_g, a_h \in \mathbb{C}$ and extending by linearity, $\mathbb{C}G$ becomes a complex algebra. If we define the involution by $g^* = g^{-1}$ and extend to be conjugate linear, $\mathbb{C}G$ becomes a complex *-algebra.*

Note that the group G is contained in its group algebra $\mathbb{C}G$.

Theorem 2.36. *Let G be a discrete group with $u : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G on the Hilbert space $\mathcal{H}$. Then u has a unique extension to a unital representation of $\mathbb{C}G$ on $\mathcal{H}$ defined by*

$$\pi_u \left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g u_g,$$

for $\sum_{g \in G} a_g g$ in $\mathbb{C}G$. Moreover, if $\pi : \mathbb{C}G \to \mathcal{B}(\mathcal{H})$ is a unital representation then its restriction to G is a unitary representation of G. Lastly, the representation u is irreducible iff π_u is.

Proposition 2.37. *Let G be a discrete group and let $(\lambda, l^2(G))$ be the left regular representation of G which is defined as*

$$(\lambda_g \xi)(h) = \xi(g^{-1}h), \qquad \forall\ g, h \in G.$$

Then the associated representation of $\mathcal{C}(G)$ is

$$\left[\pi_\lambda \left(\sum_{g \in G} a_g g \right) \xi \right](h) = \sum_{g \in G} a_g \xi(g^{-1}h),$$

for all $\sum_{g \in G} a_g g$ in $\mathbb{C}G$, h in G and ξ in $l^2(G)$.

Theorem 2.38. *Let G be a group. The left regular representation π_λ of $\mathbb{C}G$ is injective.*

Proof. Consider an element a of $\mathbb{C}G$, $a = \sum_{g \in G} a_g g$. Let a be in the kernel of π_λ. Next fix a g_0 in G and let ξ be the function that is one at the unit of G and zero everywhere else and let η be the function that is one at g_0 and zero everywhere else. We then see that:

$$0 =< \pi(a)\xi, \eta >=< \pi_\lambda \left(\sum_{g \in G} a_g g \right) \xi, \eta >= \sum_h \sum_g a_g \xi(g^{-1}h)\eta(h) = a_{g_0}.$$

Since g_0 was arbitrary, it follows that $a = 0$. □

Theorem 2.39. *Let G be a discrete group on a unital C*-algebra. Then the map from $\mathbb{C}G$ to $\mathbb{C}$ defined by $\tau\left(\sum_{g\in G} a_g g\right) = a_e$ (where e is the identity element of G), for any $\sum_{g\in G} a_g g$ in $\mathbb{C}G$, is a faithful trace.*

Note that a linear functional $\tau : A \to \mathcal{C}$ is a trace if $\tau(ab) = \tau(ba)$ and $\tau(a^*a) \geq 0$ for all $a, b \in A$. We say τ is faithful if $\tau(a^*a) = 0$ implies $a = 0$.

Proof. We can see that the trace τ is conjugate linear. We need to check that it is positive and faithful ($\tau(a^*a) = 0$ only for $a = 0$) as well as verify the trace property ($\tau(ab) = \tau(ba)$). For any $a = \sum_{g\in G} a_g g$ in $\mathbb{C}G$, we have

$$\begin{aligned} a^*a &= (\sum_{g\in G} a_g g)^*(\sum_{h\in G} a_h h) \\ &= (\sum_{g\in G} \bar{a_g} g^{-1})(\sum_{h\in G} a_h h) \\ &= \sum_{g,h} \bar{a_g} a_h g^{-1} h. \end{aligned}$$

So $\tau(a^*a)$ is the coefficient of e which is the sum over $g^{-1}h = e$ or $h = g$. Thus

$$\tau(a^*a) = \sum_{g\in G} \bar{a_g} a_g = \sum_{g\in G} |a_g|^2 \geq 0,$$

and so for $\tau(a^*a) = 0$ it must be the case that $a = 0$.

We have shown both positive and faithful, now we verify the trace property. Let g, h be in G (and so also in $\mathbb{C}G$). Consider $\tau(gh)$, this is one when $g^{-1} = h$ and zero otherwise. The same holds for $\tau(hg)$ and so we see that $\tau(gh) = \tau(hg)$, extending by linearity we see that the trace property holds. □

In order to make the group algebra into a C*-algebra we need to have a norm. It turns out that when G is finite only one such norm exists.

Theorem 2.40. *Let G be a finite group. Then there is a unique norm on $\mathbb{C}G$ such that $\mathbb{C}G$ is a finite dimensional C*-algebra.*

Proof. Using the left-regular representation, we define the norm on $\mathbb{C}G$ to be $\|a\| = \|\pi_\lambda(a)\|$ for $a \in \mathbb{C}G$. This is a norm because the left-regular representation is injective (Theorem 2.38) and it is complete because the image is finite dimensional. □

We are now able to completely describe the C*algebra of a finite group. We do this using the conjugacy classes of the group.

Definition 2.41. *Let G be a group. Recall that two elements $g, h \in G$ are* ***conjugate*** *if there exists an element u such that $ugu^{-1} = h$. Conjugacy is an equivalence relation and the equivalence class of an element g is called a* ***conjugacy class****.*

Looking ahead, we will be representing the conjugacy classes of S_n by partitions of n. So we state the following two Theorems to help us relate the partitions of n to the conjugacy classes of S_n.

Theorem 2.42. *For any cycle $(i_1\ i_2\ ...i_k)$ in S_n and any permutation σ in S_n,*

$$\sigma(i_1\ i_2\ ...i_k)\sigma^{-1} = (\sigma(i_1)\ \sigma(i_2)...\sigma(i_k)).$$

Proof. Let $\pi = \sigma(i_1\ i_2\ ...i_k)\sigma^{-1}$. We show that

1. π sends $\sigma(i_1)$ to $\sigma(i_2)$, $\sigma(i_2)$ to $\sigma(i_3)$, ..., and $\sigma(i_k)$ to $\sigma(i_1)$.

2. π does not move any number other than $\sigma(i_1), ..., \sigma(i_k)$.

To show the first point, we have

$$\pi(\sigma(i_1)) = \sigma(i_1\ i_2\ ...i_k)\sigma^{-1}(\sigma(i_1)) = \sigma(i_1\ i_2\ ...i_k)(i_1) = \sigma(i_2).$$

Note that the (i_1) at the end is not a 1-cycle, rather it denotes the point where a permutation is being evaluated. Similarly, we see that $\pi(\sigma(i_2)) = \sigma(i_3)$, ..., and $\pi(\sigma(i_k)) = \sigma(i_1)$.
For the second point we consider some number α where $\alpha \neq \sigma(i_1), ..., \alpha \neq \sigma(i_k)$. Since $\alpha \neq \sigma(i_j)$ for all $j = 1, ..., k$, we know that $\sigma^{-1}(\alpha)$ is not i_j for any $j = 1, ..., k$. Therefore $\pi(\alpha) = \alpha$ and we are done. □

Theorem 2.43. *All cycles of the same length in S_n are conjugate.*

Proof. Consider any two cycles of length k in S_n. Denote them by $(a_1\, a_2 ... a_k)$, and $(b_1\, b_2 ... b_k)$. Now choose a permutation σ in S_n such that $\sigma(a_1) = b_1, ..., \sigma(a_k) = b_k$ and extend σ to be an arbitrary bijection from the complement of the set $\{a_1, ..., a_k\}$ to the complement of $\{b_1, ..., b_k\}$. Now, using Theorem 2.42 we see that conjugation by σ takes the first k-cycle to the second. □

The cycle type of a permutation in S_n is just a set of positive integers which add up to n which is exactly a partition of n obtained by listing the lengths of the cycles in the permutation. Every permutation is written uniquely as a product of disjoint cycles. The cycle type determines the conjugacy class of the permutation uniquely. Thus we can conclude that the number of conjugacy classes in S_n are equal to the number of partitions of n. This will prove to be extremely useful in Chapter 4 since the irreducible representations of S_n can be determined by the conjugacy classes of S_n.

Theorem 2.44. *Let G be a finite group with conjugacy classes $C_1, C_2, ..., C_K$. For each $1 \leq i \leq K$, define*

$$c_i = \sum_{g \in C_i} g \quad \in \mathbb{C}G.$$

The set $\{c_1, ..., c_K\}$ is linearly independent and span$\{c_1, ..., c_K\}$ is the centre of $\mathbb{C}G$. In particular, $\mathbb{C}G$ is isomorphic to $\mathbb{C}G \cong \oplus_{i=1}^{K} M_{n_i}(\mathbb{C})$ and

$$\sum_{i=1}^{K} n_i^2 = |G|.$$

Proof. By Theorem 2.22, we get that $\mathbb{C}G$ is isomorphic to $\mathbb{C}G \cong \oplus_{i=1}^{K} M_{n_i}(\mathbb{C})$ for some positive integers $n_1, ..., n_K$.
Now suppose that $a = \sum\limits_{g \in G} a_g g$ is in the centre of $\mathbb{C}G$ and let h be another element of G and hence is also an element of $\mathbb{C}G$. Since h is invertible and a is in the centre of G, we have

$$\sum_{g \in G} a_g g = a = hah^{-1} = \sum_{g \in G} a_g hgh^{-1}.$$

And so for any g in G we have $a_g = a_{h^{-1}gh}$ which means that the function a is constant on conjugacy classes in G. Therefore it is a linear combination

of the c_i.
The same computation shows that each c_i commutes with every group element and since the group elements span the group algebra, each c_i is in the centre of $\mathbb{C}G$. Finally, since $\mathbb{C}G \equiv \oplus_{i=1}^{K} M_{n_i}(\mathbb{C})$ and $dim(\mathbb{C}G) = \#G$ (true since G is a linear basis for $\mathbb{C}G$) we have $dim(\oplus_{i=1}^{K} M_{n_i}(\mathbb{C})) = dim(\mathbb{C}G) = \#G$. Since $dim(M_{n_i}(\mathbb{C})) = n_i^2$, we therefore have

$$\sum_{i=1}^{K} n_i^2 = |G|.$$

□

Example 2.45. $\mathbb{C}S_3 \cong M_2 \oplus \mathbb{C} \oplus \mathbb{C}$
This comes directly from Theorem 2.38 since there are three conjugacy classes in S_3 and there is only one set of three numbers whose squares sum to $dim(\mathbb{C}S_3) = 6$. Namely $1^2 + 1^2 + 2^2$.

Example 2.46. $\mathbb{C}S_4 \cong M_3 \oplus M_3 \oplus M_2 \oplus \mathbb{C} \oplus \mathbb{C}$
Since there are five conjugacy classes in S_4 and $dim(\mathbb{C}S_4) = 24 = 3^2 + 3^2 + 2^2 + 1^1 + 1^2$.

While it may be easy enough to find $\mathbb{C}S_n$ for small n, it becomes much more complicated as early as when $n = 5$ to figure out the direct sum. We will soon determine a better way to break down our group algebras into irreducible representations. Before we move on to that, we will look at how a finite dimensional algebra can be written diagrammatically as a bunch of points, one for each matrix summand. We start by stating a Lemma which will help in our description of Bratteli diagrams.

Lemma 2.47. *If $\rho : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a *-homomorphism then there exists a unique $k \geq 0$ with $km \leq n$ and a unitary u in $M_n(\mathbb{C})$ such that*

$$\rho(a) = u \begin{bmatrix} a & 0 & ... & ... & ... & ... & 0 \\ 0 & a & ... & ... & ... & ... & 0 \\ 0 & 0 & . & ... & ... & ... & 0 \\ 0 & 0 & 0 & . & ... & ... & 0 \\ 0 & 0 & 0 & 0 & a & ... & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix} u^*,$$

$a \in M_m(\mathcal{C})$, where a appears k times.

As we saw in Theorem 2.44, a finite dimensional C*-algebra is isomorphic to a direct sum of square matrices. Bratteli's idea is that we represent by K dots the number of matrices. Using Lemma 2.47, we can represent the *-homomorphism ρ by k edges.

Definition 2.48. *A **Bratteli diagram**, consists of a sequence of finite, pairwise disjoint, non-empty sets which we call the vertices and are denoted by $\{V_n\}_{n=0}^{\infty}$. It consists of a sequence of finite non-empty sets $\{E_n\}_{n=1}^{\infty}$ called the edges, and it consists of maps $i : E_n \to V_{n-1}$ and $t : E_n \to V_n$ called the initial and terminal maps respectively. We let V and E denote the union of these sets and denote the diagram by (V, E). We will assume that V_0 has exactly one element v_0 and that $i^{-1}\{v\}$ is non-empty for every v in V, and that t^{-1} is non-empty for every $v \neq v_0$ in V.*

In the next section we will introduce Young tableaux, a tool which we will use to describe the conjugacy classes of S_n for all n. Once we have determined the conjugacy classes we will have determined the irreducible representations and we will describe what happens to the representations when restricted to S_{n-1}. After this is done we will use a Bratteli diagram to visually describe how $\mathbb{C}S_1 \subset \mathbb{C}S_2 \subset ... \subset \mathbb{C}S_n$ and similarly how their irreducible representations are related.

Chapter 3

Young Tableaux

We begin with a very brief description of the history of Young tableaux. In mathematics, a Young tableau is a combinatorial object which has uses in representation theory as well as algebraic geometry. They were introduced in 1900 by a mathematician at Cambridge University named Alfred Young and in 1903, Georg Frobenius applied them to the study of the symmetric group.

Young made his debut in mathematics with the computation of the concomitants of binary quartics. Young realized as he proceeded to derive a systematic method for computing the syzygies among the invariants of such quartics that the methods developed by Clebsch and Gordan could not be pushed much farther. So he went into a period of self-searching after which he published the first two papers in the series "Quantitative Substitutional Analysis". In these papers, Young outlined the theory of representations of the symmetric group and proved that the number of irreducible representations of the symmetric group of order n equals the number of partitions of n.

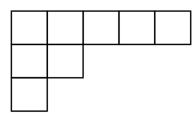
Young's combinatorial construction of the irreducible representations of the symmetric group made no appeal to the theory of group representations developed by Frobenius and as such, he was irritated by Young's results. So Frobenius carefully studied Young's papers and rederived the results following the precepts of his own theory of group characters. He even discovered the character formula which now bears his name. Matters would seem to become even worse for Young since his two papers on substitutional analysis were published around the exact same time as the thesis of Frobenius' best student Issai Schur. The thesis showed results in which all irredicible representations of the general linear group were explicitely determined on the basis of their traces, now called Schur functions. Although the two papers and the thesis did not overlap, they contained rather close results. It would be a while before Young published again.

Some twenty years later, Young published his third paper in the series. In this paper the notion of standard tableaux was introduced, their number was computed, and their relation to representation theory was described. A new proof of Frobenius' character formula was given using purely combinatorial techniques.

"Alfred Young believed his greatest contribution to mathematics was the application of representation theory to the computation of invariants of binary forms. If he had been told that one day we would mention his name with reverence in connection with the notion of standard tableaux, he probably would have winced." (Gian-Carlo Rota)[6]

Definition 3.1. *A **Young diagram** is a collection of n boxes arranged in left-justified rows which are weakly decreasing in number of boxes in each row. We usually denote the Young diagram (whose shape is given by the number of boxes in each row) as λ and sometimes write it as $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ where λ_i denotes the number of boxes in row i (so that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$). In other words,if n is the number of boxes then λ gives a partition of n (sometimes written as $\lambda \vdash n$) i.e. $\sum_{i=1}^{k} \lambda_i = n$ and conversely, every partition of n corresponds to some Young diagram λ.*

Example 3.2. $\lambda = (5, 2, 1)$, $n = 8$



There are two important partial orderings on Young diagrams. The first is probably the most obvious and produces a total ordering of Young diagrams of size n. The second is perhaps not so obvious and does not induce a total ordering.

Definition 3.3. *For Young diagrams $\lambda = (\lambda_1, ..., \lambda_k)$ and $\lambda' = (\lambda'_1, ..., \lambda'_r)$ both of size n, the **lexicographic** ordering denoted $\lambda' < \lambda$ means that the first i for which $\lambda_i \neq \lambda'_i$ has $\lambda'_i < \lambda_i$. If neither one is greater than the other then we say $\lambda = \lambda'$.*

Example 3.4. *Let $\lambda = (3, 1, 1, 1)$ and $\lambda' = (2, 2, 2)$. Then both λ and λ' partition $n = 6$, and $\lambda > \lambda'$ since $3 > 2$.*

Definition 3.5. *For Young diagrams $\lambda = (\lambda_1, ..., \lambda_k)$ and $\lambda' = (\lambda'_1, ..., \lambda'_r)$ both of size n, the **dominance** ordering, denoted $\lambda' \trianglelefteq \lambda$, means that*

$$\lambda'_1 + ... + \lambda'_i \leq \lambda_1 + ... + \lambda_i \quad \forall i.$$

We say that λ dominates λ'.

Example 3.6. *Let $\lambda = (3, 1, 1)$ and $\lambda' = (2, 2, 1)$. Then both λ and λ' partition $n = 5$, and $\lambda' \trianglelefteq \lambda$ since*

$$3 \geq 2$$
$$3 + 1 = 4 \geq 2 + 2 = 4$$
$$3 + 1 + 1 = 5 \geq 2 + 2 + 1 = 5$$

Example 3.7. *Consider $\lambda = (3,1,1,1)$ and $\lambda' = (2,2,2)$ from Example 3.4. Notice that neither $\lambda \trianglelefteq \lambda'$ nor $\lambda' \trianglelefteq \lambda$, thus $(3,1,1,1)$ and $(2,2,2)$ are not comparible in the dominance ordering.*

It seems reasonable then, that the purpose of writing a Young diagram, instead of just the partition, is to put something in the boxes. The following defintion is slighty altered from the common definition of Young tableau as we are building towards finding the representations of S_n and thus only have use for Young diagrams filled with the numbers $1, ..., n$.

Definition 3.8. *A* ***Young tableau*** *is a Young diagram with the numbers $1, ..., n$ put in the boxes using each number exactly once. A Young tableau (of shape λ, where $\lambda \vdash n$) which arranges numbers in such a way that they are increasing accross the rows and down the columns is called a* ***standard tableau****.*

Example 3.9. *Standard Tableau T:*
$$\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 7 & 8 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Definition 3.10. *The* ***column word*** *of a tableau T (denoted $w_{col}(T)$) consists of the elements of T reading the entries from bottom to top and from left to right. That is, you write down the elements starting from the bottom of the left column then write the elements of the second column starting at the bottom and continuing on until you reach the top of the final column.*

Example 3.11. *The tableau T from Example 3.9 has column word $w_{col}(T) =$* 62153478

This means that a standard tableau can be reconstructed from its column word since the first element of the first column is always a 1 we can easily see what elements are contained in the first column from the column word. The second column must be of the same size or smaller than the first column and will stop once we get to a number which is bigger than the previous number.

Example 3.12. *The column word $w_{col}(T) = 521843967$ corresponds to the tableau T below:*

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 7 \\ \hline 2 & 4 & 9 \\ \cline{1-3} 5 & 8 \\ \cline{1-2} \end{array}$$

We define a linear ordering on the set of all tableaux with n boxes. We say that for tableaux T and $T^{'}$ that $T' > T$ if either

1. the shape of $T^{'}$ is greater than the shape of T in the lexicographic sense of ordering, or

2. T and $T^{'}$ have the same shape and the largest entry that occurs in a different box in the two numberings occurs earlier in the column word of $T^{'}$.

Example 3.13.

$$\begin{array}{lll} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{array} > \begin{array}{lll} 1 & 2 & 5 \\ 4 & 3 & \\ 6 & & \end{array} > \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & \\ 6 & & \end{array} > \begin{array}{lll} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{array} > \begin{array}{lll} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & & \end{array}$$

The action of S_n on tableaux

Consider a Young tableau T (of size n) and the permutation $\sigma \in S_n$. Then the symmetric group acts on the set of such tableaux with $\sigma \cdot T$ being the tableau that puts $\sigma(i)$ in the box in which T puts i, $1 \leq i \leq n$.

Example 3.14. *Consider the Young tableau T of shape $\lambda = (3, 2, 1)$*

$$T = \begin{array}{lll} 1 & 4 & 5 \\ 2 & 6 & \\ 3 & & \end{array}$$

Now consider $\sigma = (1\ 3\ 5)$ in S_6. Then,

$$\sigma \cdot T = \begin{array}{lll} 3 & 4 & 1 \\ 2 & 6 & \\ 5 & & \end{array}$$

Notice that T is a standard tableau but that $\sigma \cdot T$ is no longer in standard form. From this action arises two subgroups of S_n which we define below.

Definition 3.15. *Let T be a Young tableau of size n. We define two subgroups of S_n. The first, $R(T)$ consists of the permutations which permute the entries of each row of T among themselves. We call this the* ***row group*** *of T. If $\lambda = (\lambda_1, ..., \lambda_k)$ is the shape of T then $R(T)$ is isomorphic to a product of symmetric groups $S_{\lambda_1} \times S_{\lambda_2} \times ... \times S_{\lambda_k}$. Similarily, we have $C(T)$ which consists of the permutations preserving the columns of T. We call this the* ***column group*** *of T.*

The following is a lemma regarding how a permutation in these subgroups effects a Young tableau.

Lemma 3.16. *Let T be a standard Young tableau of shape λ and size n. Then for any $\sigma \in R(T)$ and $\alpha \in C(T)$, the following hold true:*

1. *$\sigma \cdot T \geq T$ and*
2. *$\alpha \cdot T \leq T$.*

Proof. To compare T and $\sigma \cdot T$, we must examine the largest integer which appears in different places. This is just the largest i with $\sigma(i) \neq i$. Since the elements of the row of T to the right of i are greater than i, they are fixed by σ. So σ must move i to the left. In the linear ordering, left columns appear earlier in the column word. Thus $\sigma \cdot T \geq T$. A similar argument shows that *alpha* moves the largest element i up in the column word making $\alpha \cdot T \leq T$. □

We now show that the group of permutations preserving the elements in the columns of $\sigma \cdot T$ equals $\sigma \cdot C(T) \cdot \sigma^{-1}$.

Lemma 3.17. *Let T be a Young tableau of shape λ and size n and let σ be a permutation in S_n. Then*

$$C(\sigma \cdot T) = \sigma \cdot C(T) \cdot \sigma^{-1}$$

and

$$R(\sigma \cdot T) = \sigma \cdot R(T) \cdot \sigma^{-1}.$$

Proof. Given $\sigma \in S_n$ and $\alpha \in C(T)$, let $\{a_1, a_2, ..., a_k\}$ be the columns of T so that $\sigma(a_1), ..., \sigma(a_k)$ are the columns of $\sigma \cdot T$. It follows that

$$\sigma\alpha\sigma^{-1}\sigma(a_i) = \sigma\alpha(a_i) = \sigma(a_i).$$

So we have containment one way, i.e. $\sigma \cdot C(T) \cdot \sigma^{-1} \subseteq C(\sigma \cdot T)$. Rearranging, we then get opposite containment

$$\sigma^{-1}\sigma C(T)\sigma^{-1}\sigma \subseteq \sigma^{-1}C(\sigma T)\sigma$$
$$\Leftrightarrow C(T) \subseteq \sigma^{-1}C(\sigma \cdot T)\sigma \subseteq C(\sigma\sigma^{-1} \cdot T) = C(T).$$

Therefore $C(\sigma T) = \sigma C(T)\sigma^{-1}$. A similar argument shows that $R(\sigma \cdot T) = \sigma \cdot R(T) \cdot \sigma^{-1}$.

□

Lemma 3.18. *Let T and T' be tableaux of shapes λ and λ' respectively. Assume λ does not strictly dominate λ'. Then exactly one of the following occurs:*

1. *There is a column of T and a row of T' with two integers in common.*
2. *$\lambda = \lambda'$ and there is some $\alpha' \in R(T')$ and some $\sigma \in C(T)$ such that $\alpha' \cdot T' = \sigma \cdot T$.*

Proof. Assume (1) is false. Then the entries of the first row of T' must appear in different columns of T, and so there is an element in $C(T)$, let's call it σ_1 such that $\sigma_1 \cdot T$'s first row contains the entries of the first row of T', in particular, $\lambda'_1 \leq \lambda_1$. Similarly, the entries of the second row of T' must occur in different columns of T and thus also of $\sigma \cdot T$. So there is an element in $C(T) = C(\sigma_1 \cdot T)$, called σ_2 so that the entries in the first two rows of T' contain the first two rows of $\sigma_2\sigma_1 \cdot T$, in particular, $\lambda'_1 + \lambda'_2 \leq \lambda_1 + \lambda_2$. Continuing this way we have $\sigma_1, \sigma_2, ..., \sigma_k \in C(T)$ such that the entries of the first k rows of T' appear in the first k rows of T. The shape λ equals the shape $\sigma_k...\sigma_2\sigma_1 T$ so

$$\lambda'_1 + ... + \lambda'_k \leq \lambda_1 + ... + \lambda_k.$$

This will be true for all k so λ dominates λ' which is a contradiction unless $\lambda = \lambda'$.
(2) If $\lambda = \lambda'$ and if k is the number of rows in λ, then $\sigma_k...\sigma_2\sigma_1 T$ and T' have the same entries in each row. Let $\sigma_k...\sigma_1 = \sigma \in C(T)$ then there is a row permutation $\alpha' \in R(T')$ such that $\alpha' \cdot T' = \sigma \cdot T$. □

This leads to the following corollary.

Corollary 3.19. *If T and T' are standard tableaux of shape λ and λ' respectively with $T' > T$, then there is a pair of integers in the same row of T' and the same column of T.*

Proof. Since $T' > T$, λ cannot dominate λ'. Assume there is no such pair of integers, then by part 2 of Lemma 3.18, $\alpha' \cdot T' = \sigma \cdot T$ for some α' in $R(T)$ and some σ in $C(T)$. But since T and T' are standard tableaux

$$\sigma \cdot T \leq T \text{ and } \alpha' \cdot T' \geq T', \quad \text{by Lemma 3.16.}$$

Which means $T' \leq \alpha' \cdot T' = \sigma \cdot T \leq T$ a contradiction. Thus such a pair must exist. □

Next, we look to define an equivalence relation on Young tableaux.

Definition 3.20. *A **tabloid** is an equivalence class of Young tableaux where T is equivalent to T' if their corresponding rows contain the same entries. We denote the tabloid determined by the tableau T by $[T]$.*

So $[T] = [T']$ when $T' = \sigma \cdot T$ for some $\sigma \in R(T)$.

Lemma 3.21. *For each tabloid $[T]$ there is a unique representative tableau, which we denote by $T^\sharp$, which has the numbering of the rows in increasing order.*

Example 3.22. *Consider S_3 and the Young diagram $\lambda = (2, 1)$. Then we have the following three tabloids:*

1. $[T_1^\sharp] = \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right] = \left\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right\}$

2. $[T_2^\sharp] = \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right] = \left\{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right\}$

3. $[T_3^\sharp] = \left[\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right] = \left\{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\right\}$

Lemma 3.23. *Let T be a tableau of size n and let σ be in S_n. Then*

$$[\sigma \cdot T] = \sigma \cdot [T].$$

Proof. Consider $T' \in [\sigma \cdot T]$, then $T' = \alpha\sigma \cdot T$ for some $\alpha \in R(\sigma \cdot T)$. Then from Lemma 3.17 we know that $\alpha = \sigma\beta\sigma^{-1}$ for some $\beta \in R(T)$. Thus

$$T' = \sigma\beta\sigma^{-1}\sigma \cdot T = \sigma\beta \cdot T,$$

and so T' is in $\sigma \cdot [T]$. □

We can now state that the symmetric group S_n acts on the set of tabloids by the formula

$$\sigma \cdot [T] = [\sigma \cdot T^\sharp].$$

This is well- defined by Lemma 3.23.

Example 3.24. *Let T be the tableau $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. Then consider the permuations $(1\,2)\,,(2\,3)\,,$ and $(1\,2\,3)$ in S_3. We observe the action of each of these permuations on the tabloid $[T]$:*

$$(i) \quad (1\,2) \cdot [T] = (1\,2) \cdot \{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\}$$

$$= \{(1\,2) \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, (1\,2) \cdot \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\}$$

$$= \{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\}$$

$$= \left[\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right]$$

$$= \left[(1\,2) \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right] = [(1\,2) \cdot T]$$

$$(ii) \quad (2\,3) \cdot [T] = (2\,3) \cdot \{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\}$$

$$= \{(2\,3) \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, (2\,3) \cdot \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\}$$

$$= \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \right\}$$

$$= \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right]$$

$$= \left[(2\,3) \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right] = [(2\,3) \cdot T]$$

$$(iii) \quad (1\,2\,3) \cdot [T] = (1\,2\,3) \cdot \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}$$

$$= \left\{ (1\,2\,3) \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, (1\,2\,3) \cdot \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}$$

$$= \left\{ \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right\}$$

$$= \left[\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \right]$$

$$= \left[(1\,2\,3) \cdot \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right] = [(1\,2\,3) \cdot T]$$

We now have enough tools to work with in order to understand the irreducible representations of S_n. In the next chapter, we will look at the vector space with basis the tabloids and build the irreducible representations of the symmetric group from subspaces of this vector space. We will also see how to restrict our representations of S_n to S_{n-1}.

Chapter 4

Representations of Permutation Groups

We begin this chapter by defining a vector space with basis the tabloids. From here, we restrict our attention to specific elements of this vector space and the subspace that their span forms. We can then state what are the irreducible representations of S_n and work to prove that these elements do in fact yield all of them.

For a fixed shape λ, where λ is a partition of n, consider the complex vector space with basis the tabloids $[T]$ of shape λ. We denote this space by U_λ. This is an inner product space with tabloids as an orthonormal basis. Since S_n acts on the set of tabloids, it also acts on U_λ.

The following definition will be used to build a subspace of U_λ.

Definition 4.1. *Let T be a tableau of shape λ and size n. The* ***Young symmetrizers****, denoted a_T, b_T, and c_T are elements of $\mathbb{C}S_n$ and are defined by:*

$$\begin{aligned} a_T &= \sum_{\sigma \in R(T)} \sigma, \\ b_T &= \sum_{\alpha \in C(T)} sgn(\alpha)(\alpha), \\ c_T &= a_T \cdot b_T \end{aligned}$$

Note that in this paper we will only be using b_T.

Lemma 4.2. *Let T be a tableau of shape λ and size n. Then $b_T^2 = |C(T)|b_T$.*

Proof.

$$\begin{aligned} b_T^2 &= \sum_{\alpha,\beta \in C(T)} sgn(\alpha)sgn(\beta)(\alpha)(\beta) \\ &= \sum_{\alpha \in C(T)} \sum_{\beta \in C(T)} sgn(\alpha\beta)(\alpha\beta) \\ &= \sum_{\alpha \in C(T)} b_T \\ &= |C(T)|b_T. \end{aligned}$$

□

Definition 4.3. *For each tableau T of shape λ we use b_T, one of the Young symmetrizers in Definition 4.1, to define an element $v_T \in U_\lambda$ by:*

$$v_T = b_T \cdot [T] = \sum_{\alpha \in C(T)} sgn(\alpha)[\alpha \cdot T].$$

Example 4.4. *Consider the tabloids* $[T_1^\sharp] = \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right]$, $[T_2^\sharp] = \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right]$, *and* $[T_3^\sharp] = \left[\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right]$. *Then:*

1. $v_{T_1^\sharp} = \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}\right]$

2. $v_{T_2^\sharp} = \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right]$

3. $v_{T_3^\sharp} = \left[\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right]$

Note that $v_{T_3^\sharp} = -v_{T_2^\sharp}$. In fact, it will be proven later in Proposition 4.16, that it is always the case that $v_{T^\sharp}$ is a linear combination of the others when $T^\sharp$ is not in standard form. That is, the columns are not increasing.

Definition 4.5. *We define the* ***Specht module*** *V_λ to be the subspace of U_λ spanned by the elements v_T, as T varies over all Young tableaux of shape λ.*

Example 4.6. *Thus for $\lambda = (2,1)$, $V_\lambda = span\{v_{T_1^\sharp}, v_{T_2^\sharp}\}$ with $v_{T_1^\sharp}, v_{T_2^\sharp}$ as defined in the previous example although this set is not orthonormal.*

We want to show that V_λ is an irreducible representation of S_n for every $\lambda \vdash n$. To do this, we begin by showing that V_λ is preserved by S_n.

Lemma 4.7. *Let T be a tableau of shape λ of size n and let σ be a permutation in S_n. Then*

$$\sigma \cdot v_T = v_{\sigma \cdot T}.$$

Proof.

$$\sigma \cdot v_T = \sum_{\alpha \in C(T)} sgn(\alpha)[\sigma\alpha T] = \sum_{\alpha \in C(T)} sgn(\alpha)[\sigma\alpha\sigma^{-1}\sigma T].$$

Let $\alpha' = \sigma\alpha\sigma^{-1}$, then from Lemma 3.17, $\alpha' \in C(\sigma T)$. Also,

$$sgn(\alpha) = sgn(\sigma\alpha\sigma^{-1}) = sgn(\alpha')$$

so it then follows that

$$\sigma \cdot v_T = \sum_{\alpha' \in C(\sigma T)} sgn(\alpha')[\alpha'\sigma T] = v_{\sigma \cdot T}.$$

□

Let $A = \mathbb{C}[S_n]$ denote the group ring of S_n which consists of all complex linear combinations $\sum x_\sigma \sigma$ with multiplication determined by composition in S_n. Then

Lemma 4.8. *$V_\lambda = A \cdot v_T$ for any numbering T of λ.*

Proof. Let $\sum x_\sigma \sigma$ be in A. Then

$$\sum_{\sigma \in S_n} x_\sigma \sigma \cdot v_T = \sum_{\sigma \in S_n} x_\sigma v_{\sigma \cdot T} \in V_\lambda.$$

Hence, $A \cdot v_T \subseteq V_\lambda$. If T' is any tableau, $T' = \sigma \cdot T$ for some $\sigma \in S_n$. So $v_{T'} = v_{\sigma \cdot T} = \sigma \cdot v_T$. Thus $V_\lambda \subseteq A \cdot v_T$ and we are done.

□

Corollary 4.9. *For any Young diagram λ of size n, $V_\lambda \subseteq U_\lambda$ is invariant under S_n.*

Lemma 4.10. *Let T and T' be Young tableaux of shapes λ and λ' respectively where λ does not strictly dominate λ'. If there is a pair of integers in a row of T' and a column of T, then $b_T \cdot [T'] = 0$. If there is no such pair, then $b_T \cdot [T'] = \pm v_T$.*

Proof. If there is such a pair of integers, let t be the transposition that permutes them. Since $t \in C(T)$, we have $b_T \cdot t = \sum\limits_{\sigma \in C(T)} sgn(\sigma)\sigma \cdot t = -b_T$ and since $t \in R(T')$, we also have $t \cdot [T'] = [T']$. Thus, we conclude

$$b_T \cdot [T'] = b_T(t \cdot [T']) = (b_T \cdot t)[T'] = -b_T \cdot [T']$$

which immediately implies $b_T \cdot [T'] = 0$.
Otherwise, we are in (2) of Lemma 3.18 and so $\alpha' \cdot T' = \sigma \cdot T$ for some α' in $R(T')$ and some σ in $C(T)$. Then

$$\begin{aligned} b_T \cdot [T'] &= b_T \cdot [\alpha' T'] \\ &= b_T \cdot [\sigma \cdot T] \\ &= b_T \cdot \sigma[T] \\ &= sgn(\sigma) \cdot b_T[T] \\ &= sgn(\sigma) \cdot v_T \\ &= \pm v_T. \end{aligned}$$

□

Corollary 4.11. *If T and T' are standard tableaux with $T' > T$, then $b_T \cdot [T'] = 0$.*

Proof. This follows immediately from Corollary 3.19. □

Lemma 4.12. *For any tableau T of λ, we have*

1. $b_T \cdot U_\lambda = b_T \cdot V_\lambda = \mathbb{C}v_T \neq 0$,

2. $b_T \cdot U_{\lambda'} = b_T \cdot V_{\lambda'} = 0, \quad if\ \lambda' > \lambda$.

Proof. 1. $b_T \cdot U_\lambda = b_T \cdot span\{[T'] \mid T'\ tableau\ of\ shape\ \lambda\} \subseteq span\{b_T \cdot [T']\} \subseteq span\{v_T\}$ by Lemma 4.10. Recall by Lemma 4.2 that $b_T^2 = \#C(T)b_T$, so $b_T \cdot V_\lambda \supseteq b_T \cdot v_T = b_T \cdot b_T[T] = \#C(T) \cdot b_T[T] = \#C(T)v_T \neq 0$.

2. If $\lambda' > \lambda$, then $T' > T$, $\forall T$, T' standard tableaux of shape λ, λ' and so $b_T \cdot [T'] = 0$ by Corollary 4.11.

□

It is these same equations that imply that each V_λ is irreducible.

Proposition 4.13. *For each partition λ of n, V_λ is an irreducible representation of S_n.*

Proof. Assume V_λ is not irreducible, and $V_\lambda = W_1 \bigoplus W_2$, where W_1 and W_2 are subspaces of V_λ which are invariant under S_n. Let T be a tableau, then from (1) of Lemma 4.12, $b_T \cdot W_i \subseteq b_T \cdot V_\lambda = \mathbb{C} \cdot v_T$. So $b_T \cdot W_1$ and $b_T \cdot W_2$ are each either zero or one dimensional. Observe that $\mathbb{C} \cdot v_T = b_T \cdot V_\lambda = b_T \cdot W_1 \oplus b_T \cdot W_2$, so one must equal $\mathbb{C} \cdot v_T$. As $b_T \cdot W_i \subseteq W_i$, one of W_1 and W_2 contains v_T. If W_1 contains v_T then $V_\lambda = A \cdot v_T = W_1$. Therefore each V_λ is irreducible.

□

Proposition 4.14. *For all partitions λ and λ' of n, if the representation of S_n restricted to V_λ is unitarily equivalent to its restriction to $V_{\lambda'}$ then the partitions are the same, that is $\lambda = \lambda'$.*

Note: Going forward, we will denote the representation of S_n restricted to V_λ as π_λ.

Proof. If V_λ is unitarily equivalent to $V_{\lambda'}$ then there exists a unitary operator $U : V_\lambda \to V_{\lambda'}$ such that

$$U\pi_\lambda(\sigma) = \pi_{\lambda'}(\sigma)U.$$

If $\lambda \neq \lambda$, then without loss of generality we can assume that $\lambda' > \lambda$ and thus by Corollary 4.11 we have that

$$\pi_\lambda(b_T) \cdot V_{\lambda'} = 0,$$

for all T of shape λ. Now we start dealing with λ and λ', we'll let $\pi_\lambda : \mathbb{C}S_n \to \mathcal{B}(V_\lambda)$, where $\mathcal{B}(V_\lambda)$ are the bounded operators on V_λ. Then,

$$U(|C(T)|v_T) = U\pi_\lambda(b_T)v_T = \pi_{\lambda'}(b_T)Uv_T \in \pi_{\lambda'}(b_T)V_{\lambda'} = 0,$$

a contradiction. So $\lambda = \lambda'$. □

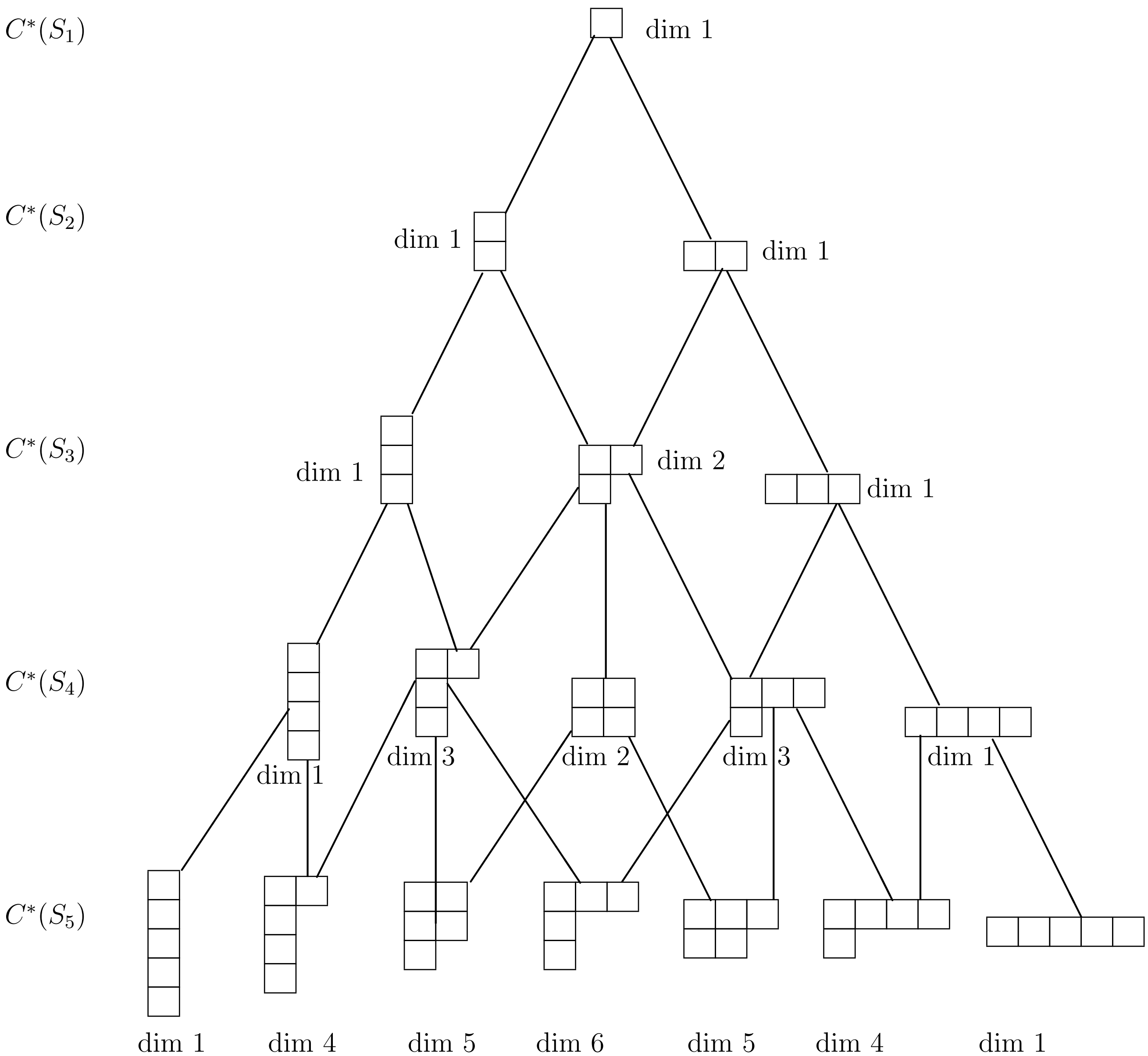
Proposition 4.15. *Every irreducible representation π_λ of S_n is isomorphic to exactly one V_λ for λ a partition of n.*

Proof. There is an irreducible representation π_λ for each partition λ of n. Since we have V_λ for each partition of n and since the number of partitions of n equals the number of conjugacy classes of S_n which equals the number of irreducible representations of S_n, these are all of them. □

We next want to explore what happens when we restrict the irreducible representations of S_n to S_{n-1}. First we must understand how S_n is related to S_{n-1}. We can see that $S_{n-1} \subseteq S_n$ by regarding it as the subgroup with $\sigma \in S_n$ such that $\sigma(n) = n$. Now consider $\lambda \vdash n$ and let S be the set of Young diagrams (of size (n-1)) formed by removing one box, x, from λ (x must be at the end of both its row and column). We will show that restricting π_λ to S_{n-1} decomposes as the direct sum of the irreducible representations of S_{n-1} corresponding to the Young diagrams in S (each occurring exactly once in the sum).

$$i.e. \qquad \pi_\lambda|_{S_{n-1}} = \oplus_{\lambda' \in S}\ \pi_{\lambda'}.$$

Here is a Bratteli diagram to better illustrate the process.



Recall from Examples 2.45 and 2.46 that $\mathbb{C}S_3 \cong \mathbb{C} \oplus M_2 \oplus \mathbb{C}$ and $\mathbb{C}S_4 \cong \mathbb{C} \oplus M_3 \oplus M_2 \oplus M_3 \oplus \mathbb{C}$. This is illustrated here.

To show what happens when we restrict our representations to S_{n-1}, we first need to think of a tableau T as a bijective map from the boxes x in λ to the set $\{1, ..., n\}$. For example if the box x contains the number 4, we say $T(x) = 4$. We will denote $T|_{\{1,...,n-1\}}$ as $T \setminus \{x\}$. To begin with, we are going to prove the following,

Theorem 4.16. *Let λ be a Young diagram of size n. Then the set $\{v_T \mid T\, is\, a\, standard\, tableau\, of\, shape\, \lambda\}$ spans V_λ.*

Most of the work will be done in proving the following:

Lemma 4.17. *If T is a tableau of shape λ, with increasing columns, then either T is in standard form or v_T is a linear combination of $v_{T_1}, v_{T_2}, ..., v_{T_k}$ where each T_k is of shape λ and has increasing columns with $T > T_i$ for $1 \leq i \leq k$.*

Before proving Lemma 4.17, let us see how the theorem follows from the lemma.

Proof of Theorem 4.16. Let T be any tableau. From Lemma 4.7, $v_T = sgn(\sigma)v_{\sigma \cdot T}$ for any σ in $C(T)$. So we may replace T by $\sigma \cdot T$ which has increasing columns. Thus we assume T has increasing columns. If T is in standard form, we are done. If not, by Lemma 4.17, v_T is a linear combination of v_{T_1},v_{T_2},...,v_{T_k} with each T_i having increasing columns. If any of the T_i are standard, leave them alone. If not, apply the Lemma again and repeat. This process must terminate since if it did not we would construct an infinite sequence of tableaux with $T > T_1' > T_2' > T_3' >$ ... which is clearly impossible as there are only finitely many different T. So we have that $V_\lambda = span\{v_T \mid T\, is\, a\, tableau\, of\, shape\, \lambda\} = span\{v_T \mid T\, is\, a\, standard\, tableau\, of\, shape\, \lambda\}$.

□

Let us turn to the proof of the Lemma. We will first introduce some new notation and then two subsequent lemmas and their proofs which will lead us to a proof of Lemma 4.17.

Assume that T is not in standard form but has increasing columns. Then there is some place where a row is not increasing. Find a spot in a row of T where b is immediately to the left of a and $b > a$ (where $b, a \in \{1, ..., n\}$). We will let A be the set consisting of the numbers of T in the column of a which lie above a (and include a) and let B be the set consisting of the numbers of T in the column of b which lie below b (and include b). We call these boxes a skew column.

Example: 4.18

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 4 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \qquad \begin{array}{l} a = 3 \text{ and } b = 4 \\ A = \{2, 3\} \\ B = \{4, 6\} \end{array}$$

Observe that, starting at the top of the column of a, going down the skew column, with a jag left at a, the numbers increase since $b > a$ and T has increasing columns. Let S_A,S_B and $S_{A\cup B}$ be the permutations of A,B, and $A \cup B$ respectively, or more accurately, the elements of S_n which fix their respective complements. Notice that $S_A \times S_B$ is a subgroup of both $S_{A\cup B}$ and $C(T)$. In fact, $S_{A\cup B} \cap C(T) = S_A \times S_B$. Define D to be the set of all permutations δ in $S_{A\cup B}$ such that $\delta \cdot T$ has increasing columns in the region covered by $A \cup B$.

Example: 4.19

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 4 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array} \qquad D = \{e, (234), (34), (2364), (364), (24)(36)\}$$

We will now describe a method to produce all elements of D. Note that $\sum\limits_{\delta\in D} sgn(\delta)\delta \in \mathbb{C}S_n$ is called the Garnir element and is usually written $g_{A,B}$.
Pick subsets $A_0 \subseteq A$, $B_0 \subseteq B$, with $\#A_0 = \#B_0$. Going down the skew-column, we first list the elements of $A \setminus A_0$ in order, then the elements of B_0 in order. This should fill up the right part of the skew-column since $\#A_0 = \#B_0$. Next, list the elements of A_0 in order and finally the elements of $B \setminus B_0$ in order.

Example: 4.20

$$\underset{T}{\begin{array}{|c|c|c|}\hline 1 & 2 & 7 \\ \hline 4 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}} \quad \longmapsto \quad \underset{\delta \cdot T}{\begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}} \qquad \begin{array}{l} A_0 = \{2\} \text{ and } B_0 = \{6\} \\ \delta = (2364) \end{array}$$

Notice the second column is only increasing in $A \cup B$. First notice that this construction of δ ensures that δ is in D. Next, observe that $B_0 = \delta(A) \cap B$ and $A_0 = \delta(B) \cap A$. This means that A_0,B_0 can be recovered from δ in D. The set D is not a subgroup of $S_{A\cup B}$; it is however a convenient list of the right cosets of $S_A \times S_B$ in $S_{A\cup B}$.

Lemma 4.18. *For σ in $S_{A\cup B}$ with A,B as described previously, there are unique elements δ of D, and τ of $S_A \times S_B$ such that $\sigma = \delta\tau$ and $S_{A\cup B} = D(S_A \times S_B)$.*

Proof. Let $A_0 = \sigma(B) \cap A$ and $B_0 = \sigma(A) \cap B$. We have $\#A_0 = \#B_0$ since the number of elements of A moved to B by σ must equal the number of elements of B moved to A. Now let δ be the element of D associated to A_0,B_0. Then we see

$$\delta(B) \cap A = A_0 = \sigma(B) \cap A$$

and

$$\delta(A) \cap B = B_0 = \sigma(A) \cap B.$$

Next, look at

$$\begin{aligned} \sigma(A) \cap A &= A \setminus (\sigma(B) \cap A) \\ &= A \setminus (\delta(B) \cap A) \\ &= \delta(A) \cap A. \end{aligned}$$

Then we consider,

$$\begin{aligned} \delta^{-1}\sigma(A) &= \delta^{-1}\sigma((A \cap \sigma^{-1}(A)) \cup (A \cap \sigma^{-1}(B))) \\ &= \delta^{-1}((\sigma(A) \cap A) \cup (\sigma(A) \cap B)) \\ &= \delta^{-1}((\delta(A) \cap A) \cup (\delta(A) \cap B)) \\ &= (A \cap \delta^{-1}(A)) \cup (A \cap \delta^{-1}(B)) \\ &= A. \end{aligned}$$

So let $\tau := \delta^{-1}\sigma$ then $\tau(B) = \delta^{-1}\sigma(B)$ which gives that τ is in $S_A \times S_B$. $\square$

Lemma 4.19. *Let T be a tableau of shape λ and size n. If σ is any element of $C(T)$, then with A,B as described previously*

$$\sum_{\alpha \in S_{A\cup B}} sgn(\alpha)\alpha \cdot [\sigma \cdot T] = 0.$$

Proof. Let l be the length of the entire column of λ which contains B. Notice that $\#B+\#A=l+1$. In $\sigma\cdot T$, the elements of B occupy $\#B$ boxes and the elements of A occupy $\#A$ boxes. This means that there is some row which contains $b'\in B$ and $a'\in A$ adjacent (i.e. in the same row of $\sigma\cdot T$). Thus $(a'b')$ is in $R(\sigma\cdot T)$ and also in $S_{A\cup B}$. Thus we have,

$$\begin{aligned}\sum_{\alpha\in S_{A\cup B}} sgn(\alpha)\alpha\cdot[\sigma\cdot T] &= \sum_{\alpha\in S_{A\cup B}} sgn(\alpha)\alpha\cdot[(a'b')\sigma\cdot T]\\ &= \sum_{\alpha\in S_{A\cup B}} sgn(\alpha)\alpha(a'b')\cdot[\sigma\cdot T]\\ &= -\sum_{\alpha\in S_{A\cup B}} sgn(\alpha(a'b'))\alpha(a'b')[\sigma\cdot T]\\ &= -\sum_{\alpha\in S_{A\cup B}} sgn(\alpha)\alpha\cdot[\sigma\cdot T].\end{aligned}$$

And so we are done. □

Lemma 4.20. *Let T be a tableau of shape λ and size n. Then*

$$\sum_{\delta\in D} sgn(\delta)v_{\delta\cdot T}=0.$$

Proof. Consider

$$\begin{aligned}\sum_{\delta\in D} sgn(\delta)v_{\delta\cdot T} &= \sum_{\delta\in D}\sum_{\sigma\in C(\delta\cdot T)} sgn(\delta\sigma)\sigma\cdot[\delta\cdot T]\\ &= \sum_{\delta\in D}\sum_{\sigma\in C(T)} sgn(\delta\sigma)\delta\sigma\delta^{-1}\cdot[\delta\cdot T]\\ &= \sum_{\delta\in D}\sum_{\sigma\in C(T)} sgn(\delta\sigma)\delta\sigma\cdot[T]\\ &= \sum_{\sigma\in C(T)} sgn(\sigma)\sum_{\delta\in D} sgn(\delta)\delta\sigma\cdot[T].\end{aligned}$$

In order to use Lemma 4.19 we recall that $S_A\times S_B$ is a subgroup of $C(T)$ and choose a list of representatives $\sigma_1,...,\sigma_m$ of the right cosets of $S_A\times S_B$. Then we can write $C(T)$ as the disjoint union over i of $(S_A\times S_B)\sigma_i$. Thus,

$$\begin{aligned}\sum_{\sigma\in C(T)} sgn(\sigma)\sum_{\delta\in D} sgn(\delta)\delta\sigma\cdot[T] &= \sum_{i=1}^{m}\sum_{\tau\in S_A\times S_B}\sum_{\delta\in D} sgn(\delta\tau\sigma_i)\delta\tau\sigma_i\cdot[T]\\ &= \sum_{i=1}^{m} sgn(\sigma_i)\sum_{\tau\in S_A\times S_B}\sum_{\delta\in D} sgn(\delta\tau)\delta\tau\cdot[\sigma_i\cdot T]\end{aligned}$$

We know from Lemma 4.18 that for all σ in $S_{A\cup B}$ there exist unique δ in D and τ in $S_A \times S_B$ such that $\sigma = \delta\tau$. So the two inner sums become the sum over $S_{A\cup B}$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{m} sgn(\sigma_i) \sum_{\tau\in S_A\times S_B} \sum_{\delta\in D} sgn(\delta\tau)\delta\tau \cdot [\sigma_i \cdot T] &= \sum_{i=1}^{m} sgn(\sigma_i) \sum_{\sigma\in S_{A\cup B}} sgn(\sigma)\sigma \cdot [\sigma_i \cdot T] \\
&= \left(\sum_{i=1}^{m} sgn(\sigma_i) \right) (0) \\
&= 0
\end{aligned}$$

□

The result of the Lemma 4.20 can be re-written as

$$\begin{aligned}
0 = \sum_{\delta\in D} sgn(\delta) v_{\delta\cdot T} &= sgn(1) v_{1\cdot T} + \sum_{1\neq\delta\in D} sgn(\delta) v_{\delta\cdot T} \\
v_T &= - \sum_{1\neq\delta\in D} sgn(\delta) v_{\delta\cdot T}.
\end{aligned}$$

In fact, we can also conclude that $\delta \cdot T < T$ for each δ in D however, $\delta \cdot T$ need not have increasing columns (outside of $A \cup B$). See example 4.20 for an instance of this.

Let's return now to the proof of Lemma 4.17. Recall, Lemma 4.17 stated if T is a tabloid of shape λ, with increasing columns, then either T is in standard form or v_T is a linear combination of $v_{T_1}, ..., v_{T_k}$, where each T_k is of shape λ and has increasing columns with $T > T_i$ for $1 \leq i \leq k$.

Proof of Lemma 4.17: Using the same notation for A,B, and δ as used earlier, let $B_0 = \delta(A) \cap B$, $A_0 = \delta(B) \cap A$ and let b_0 be the largest element of B_0. Let α be the element of $C(\delta \cdot T)$ such that $\alpha\delta \cdot T$ has increasing columns. Notice α will move some elements of the left column up, and some other elements down but it won't move any element greater than b_0. In the right column, α may move b_0 down and some other elements i up, but only if $i < b_0$. So we can conclude that b_0 is the largest element which can be moved by α since δ moved it to the next column. This also means that b_0 in T occurs earlier in the column word than in $\alpha\delta \cdot T$; that is $T > \alpha\delta \cdot T$.

Finally, we note $v_{\alpha\delta\cdot T} = sgn(\alpha) v_{\delta\cdot T}$ since α is in $C(\delta \cdot T)$. Thus using that

$v_T = -\sum_{1 \neq \delta \in D} sgn(\delta) v_{\delta \cdot T}$ from Lemma 4.20, substituting in $v_{\alpha\delta \cdot T}$ for $v_{\delta \cdot T}$ we have,

$$v_T = -\sum_{1 \neq \delta \in D} sgn(\delta\alpha) v_{\alpha\delta \cdot T}$$

where $T > \alpha\delta \cdot T$ and $\alpha\delta \cdot T$ has increasing column and so we are done. □

Now that we have shown that the set $\{v_T \mid T\ is\ a\ standard\ tableau\ of\ shape\ \lambda\}$ spans V_λ, all that is left to see that this set is in fact a basis for V_λ is to show that the set is linearly independent.

Theorem 4.21. *The set* $\{v_T \mid T\ is\ a\ standard\ tableau\ of\ shape\ \lambda\}$ *is linearly independent.*

Proof. Suppose $T_1 > T_2 > ... > T_k$ are standard tableaux with $x_1 v_{T_1} + ... + x_k v_{T_k} = 0$. The element $x_1[T_1]$ appears in the term $x_1 v_{T_1}$. We know from Lemma 3.16 that if $1 \neq \alpha \in C(T_1)$ then $\alpha T_1 < T_1$. We also know that for any $i > 1$ and $\alpha \in C(T_i)$, $\alpha T_i \leq T_i \leq T_1$. If we have either $[\alpha \cdot T_1] = [T_1]$ or $[\alpha \cdot T_i] = [T_1]$ then there is a σ in $R(T_1)$ with $\alpha \cdot T_1 = \sigma \cdot T_1$ in the former case or $\alpha \cdot T_i = \sigma \cdot T_1$ in the latter. This contradicts Lemma 3.16:

$$\sigma \cdot T_1 > T_1 > \alpha \cdot T_1 \ or\ \alpha \cdot T_i.$$

We conclude that $[T_1]$ can only appear once in the expression

$$x_1[v_{T_1}] + ... + x_k[v_{T_k}]$$

with coefficient x_1 and hence $x_1 = 0$. Applying the same argument to $T_2 > ... > T_k$ implies $x_2 = 0$ and so on. Therefore, $x_1 = x_2 = ... = x_k = 0$ and the set is linearly independent. □

Restricting the Representations

Now we have proven Lemma 4.17 which we use to prove that the set $\{v_T \mid T\ is\ a\ standard\ tableau\ of\ shape\ \lambda\}$ spans V_λ (Thm. 4.16). We now show what happens when we restrict a representation of S_n to S_{n-1}. We begin by defining corners of Young diagrams.

Let λ be a Young diagram. We say a box x is a corner of λ if it is in the bottom of its column and the right of its row. This implies that $\lambda \setminus \{x\}$ is also a Young diagram. Let $x_1, ..., x_l$ be the corners of λ ordered from right to left (or top to bottom). Let $\lambda_i = \lambda \setminus \{x_i\}$.

Theorem 4.22. *For any λ, the restriction of π_λ to S_{n-1} is unitarily equivalent to $\oplus_{i=1}^{l}\pi_{\lambda_i}$.*

For $1 \leq i \leq l$, let B_i be the set of v_T over all T, where T is a standard tableau with n in box x_i. This means $\cup_{i=1}^{l} B_i$ is a basis for V_λ. Also, define $V_i = span(B_1 \cup ... \cup B_i)$.

Lemma 4.23. *Consider B_i and V_i as defined above. Then,*

1. $V_1 \subseteq V_2 \subseteq ... \subseteq V_l = V_\lambda$
2. *For all i, V_i is invariant under $\pi_\lambda(S_{n-1})$.*

Proof. The first part is clear immediately. For the second, let v_T be in $B_i \in V_i$, recall that T is standard with n in box x_i. Then for any σ in S_{n-1}, $\sigma \cdot T$ also has n in box x_i and so by Lemma 4.17 $\sigma \cdot v_T = v_{\sigma \cdot T}$ is a linear combination of $v_{T'}$ with T' in standard form and $T' \leq \sigma \cdot T$. If T' is standard with n in box x_j $(j \neq i)$, then clearly n is the largest integer appearing in different boxes in $\sigma \cdot T$ and T' and $T' \leq \sigma \cdot T$ means n appears later in the column word of T' than of $\sigma \cdot T$. This implies $j < i$, so $v_{T'} \in V_j \subseteq V_i$ and $\pi_\lambda(\sigma) v_T \in V_i$. □

Now, we define a map for each $1 \leq i \leq l$, $P_i : V_i \to V_{\lambda_i}$ by

$$P_i([T]) = \begin{cases} [T^\sharp|_{\lambda_i}], & if\ T^\sharp\ has\ n\ in\ box\ x_i \\ 0, & else \end{cases}$$

This map is well defined since $[T] = [T']$ if and only if $T^\sharp = (T')^\sharp$ and also since if T has n in box x_i, then $T|_{\lambda_i}$ is a tableau of shape λ_i. The following Lemma gives a variety of useful properties about the map P_i.

Lemma 4.24. *Let $1 \leq i \leq l$. Then,*

1. $P_i(V_{i-1}) = 0$.
2. *If T is a standard tableau of shape λ with n in box x_i, then $P_i(v_T) = v_{T|_{\lambda_i}}$.*
3. *If T' is a standard tableau of shape λ_i, then $P_i(v_T) = v_{T'}$, where T is the tableau of shape λ with $T|_{\lambda_i} = T'$, and n in box x_i.*
4. $P_i(V_i) = V_{\lambda_i}$.

5. *dim*(V_{λ_i}) = *dim*(V_i) - *dim*(V_{i-1}).

6. $P_i(\pi_\lambda(\sigma)) = \pi_{\lambda_i}(\sigma)P_i$, *for all* σ *in* S_{n-1}.

Proof. 1. Let T be a standard tableau with n in box x_j, where $j < i$ (i.e. box x_j appears higher up than box x_i and $v_T \in V_{i-1}$). Then for any σ in $C(T)$, $\sigma \cdot T$ has n in the same row as x_j or higher (since n was originally at the bottom of its column in box x_j). This implies $(\sigma \cdot T)^\sharp$ cannot have n in box x_i, so $P_i([\sigma \cdot T]) = 0$. As v_T is a linear combination of such elements we have that $P_i(v_T) = 0$ and thus $P_i(V_{i-1}) = 0$.

2. First recall that $v_T = \sum_{\sigma \in C(T)} sgn(\sigma)[\sigma \cdot T]$. As n is in box x_i in T, it remains in the same row in $\sigma \cdot T$ if and only if $\sigma(n) = n$ or, in other words, $\sigma \in S_{n-1}$. Otherwise, $\sigma \cdot T$ has n in a different row than x_i, so $P_i([\sigma \cdot T]) = 0$. So we have shown,

$$\begin{aligned} P_i(v_T) &= P_i\left(\sum_{\sigma \in C(T)} sgn(\sigma)[\sigma \cdot T]\right) \\ &= \sum_{\sigma \in C(T) \cap S_{n-1}} P_i(sgn(\sigma)[\sigma \cdot T]) \\ &= \sum_{\sigma \in C(T|_{\lambda_i})} sgn(\sigma)[\sigma \cdot T|_{\lambda_i}] \\ &= v_{T|_{\lambda_i}}. \end{aligned}$$

3. It is clear that T is a standard tableau and so the rest follows from 2.

4. We know that $B_1 \cup ... \cup B_i$ is a basis for V_i, and from (1) we know P_i maps the first $i-1$ sets to zero and the last bijectively onto a basis for V_{λ_i}. This also proves part 5.

5. See proof of 4.

6. Let T be any tableau of shape λ. As we want to look at linear maps on $[T]$, we may assume $T = T^\sharp$. If T has n in box x_i, we have

$$\begin{aligned} \pi_{\lambda_i}(\sigma)P_i([T]) &= \pi_{\lambda_i}(\sigma)[T|_{\lambda_i}] \\ &= [\sigma \cdot T|_{\lambda_i}]. \end{aligned}$$

and also

$$\begin{aligned} P_i(\pi_\lambda(\sigma)[T]) &= P_i([\sigma \cdot T]) \\ &= [(\sigma \cdot T)^\sharp|_{\lambda_i}] \\ &= [(\sigma \cdot T|_{\lambda_i})^\sharp] \\ &= [\sigma \cdot T|_{\lambda_i}], \end{aligned}$$

as $\sigma \cdot T$ also has n in box x_i.
If n is not in x_i in T, then $\pi_\lambda(\sigma)P_i([T]) = 0$. As the rows of T are increasing, n does not lie in the row of x_i in T. As $\sigma(n) = n$, n is also not in the row of x_i in $\sigma \cdot T$ and so $(\sigma \cdot T)^\sharp$ does not have n in box x_i. Thus we have $P_i(\pi_\lambda(\sigma))[T] = P_i([\sigma \cdot T]) = 0$.

□

We know each V_i is invariant under $\pi_\lambda(S_{n-1})$ and hence, so is $V_i^\perp$ and then so is $V_i \cap V_{i-1}^\perp$. It is easy to see then that

$$V_\lambda = V_1 \oplus (V_2 \cap V_1^\perp) \oplus ... \oplus (V_l \cap V_{l-1}^\perp),$$

that is, each subspace is $\pi(S_{n-1})$-invariant.

Lemma 4.25. *Let $Q_i = P_i|_{V_i \cap V_{i-1}^\perp}$. Then Q_i is a linear isomorphism from $V_i \cap V_{i-1}^\perp$ to V_{λ_i} and*

$$Q_i(\pi_\lambda(\sigma)) = \pi_{\lambda_i}(\sigma)Q_i,$$

for all σ in S_{n-1}.

Proof. By definition of Q_i we have,

$$Q_i(V_i \cap V_{i-1}^\perp) = P_i(V_i \cap V_{i-1}^\perp) \subseteq P_i(V_i) = V_{\lambda_i}$$

. On the other hand, from 5 of Lemma 4.24 we know that, $\dim(V_i \cap V_{i-1}^\perp) = \dim(V_i) - \dim(V_{i-1}) = \dim(V_{\lambda_i})$. Now let $v_T = B_i$ and write $v_T = v + w$, for $w \in V_{i-1}$ and $v \in V_i \cap V_{i-1}^\perp$. Then

$$P_i(v_T) = P_i(v) + P_i(w) = P_i(v) + 0,$$

since $P_i|_{V_{i-1}} = 0$. Then $P_i(B_i)$ is a basis for V_{λ_i} which implies that $P_i(V_i \cap V_{i-1}^\perp)$ contains a basis for V_{λ_i}. Thus Q_i is onto. □

Theorem 4.26. *Let $(\sigma, \mathcal{H})$, $(\tau, \mathcal{K})$ be unitary representations of the group G. Suppose $T : \mathcal{H} \to \mathcal{K}$ is an invertible linear map such that*

$$T\sigma(g) = \tau(g)T,$$

for all g in G. Then T^*T is a positive invertible operator and $U = T(T^*T)^{-1/2}$ is a unitary operator such that

$$U\sigma(g) = \tau(g)U,$$

for all g in G. That is, σ and τ are unitarily equivalent.

Proof. As T is invertible, so is T^* with $((T^*)^{-1} = (T^{-1})^*)$. So T^*T is also invertible and positive. Its spectrum is contained in $[a, b]$ for some $0 < a < b$. Notice that, for any g in G, we have

$$\begin{aligned} T^*T\sigma(g) &= T^*\tau(g)T \\ &= (\tau(g)^*T)^*T \\ &= (\tau(g^{-1})T)^*T \\ &= (T\sigma(g^{-1}))^*T \\ &= \sigma(g^{-1})^*T^*T \\ &= \sigma(g)T^*T. \end{aligned}$$

It follows that for any polynomial $p(t)$, $p(T^*T)$ commutes with $\sigma(G)$. Let $p_n(t)$ be a sequence of polynomials converging uniformly to $f(t) = t^{-1/2}$ on $[a, b]$. We have, for g in G,

$$\begin{aligned} (T^*T)^{-1/2}\sigma(g) &= \lim_n p_n(T^*T)\sigma(g) \\ &= \lim_n \sigma(g)p_n(T^*T) \\ &= \sigma(g)(T^*T)^{-1/2}. \end{aligned}$$

The final equation follows. Finally, we have

$$\begin{aligned} U^*U &= (T(T^*T)^{-1/2})^*(T(T^*T)^{-1/2}) \\ &= (T^*T)^{-1/2}T^*T(T^*T)^{-1/2} \\ &= I. \end{aligned}$$

Similarly, $UU^* = I$, therefore U is a unitary. □

Chapter 5

Further Applications

We've now seen how to describe the irreducible representations of S_n using Young tableaux and how to restrict them to S_{n-1} in order to get the irreducible representations of S_{n-1} and so on. But where do we go from here? There are many ways one could explore this topic further. For instance, the alternative uses for Young tableaux or perhaps looking at the representations of the alternating group A_n. What I will briefly describe here is a now proven conjecture made by Kerov and Vershik in regards to the dimensions of the irreducible representations of finite symmetric groups.

In 1985, Anatoly Vershik and Sergei Kerov conjectured that the dimensions of the irreducible representations of finite symmetric groups, after appropriate normalization, will converge to a constant with repsect to the Plancherel family of measures on the space of Young diagrams.

A brief description of the Plancherel measure is defined on the set of representations of a locally compact group (in our case S_n) and describes how the regular representation breaks up into irreducible unitary representations.As we have seen for the symmetric group S_n, the set of irreducible representations is in natural bijection with the set of integer partitions of n and the dimension of a given irreducible representation V_λ (corresponding to some partititon λ of n) equals the number of standard Young tableax of shape λ. In this case Plancherel measure is a measure on the set of integer partitions of given order n and is given by

$$\mu(\lambda) = \frac{(dim(V_\lambda))^2}{n!}.$$

The Vershik-Kerov Conjecture was stated as follows:

Conjecture 1. *Let n be a natural number and consider the set of Young diagrams (denoted $\mathbb{Y}$) of size n and, in particular, let λ be one of these Young diagrams. We will denote the Plancherel probability measure by*

$$\mathbb{P}l^{(n)}(\lambda) = \frac{dim^2(V_\lambda)}{n!}.$$

. In 1985 Vershik and Kerov showed that there exists two positive constants c_1, c_2 such that

$$\lim_{n\to\infty} \mathbb{P}l^{(n)}(\{\lambda \in \mathbb{Y} \mid c_1\sqrt{n} \leq -\log \mathbb{P}l^{(n)}(\lambda) \leq c_2\sqrt{n}\}) = 1$$

The conjecture is that the sequence of random variables $\frac{-\log \mathbb{P}l^{(n)}(\lambda)}{\sqrt{n}}$ converges to a constant according to the Plancherel measure.

Alexander Bufetov has now published a proof of the conjecture and thus the theorem is restated as follows:

Theorem 5.1. *There exists a constant $H > 0$ such that for any $\epsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}l^{(n)}\{\lambda \in \mathbb{Y} \mid \big|H + \frac{\log \mathbb{P}l^{(n)}(\lambda)}{\sqrt{n}}\big| \leq \epsilon\} = 1.$$

Going in another direction, we could study the hook-length formula which is a formula for the number of standard Young tableaux (denoted f^λ) of size n with shape λ. There are other formulas for f^λ but the hook-length formula is particularily simple and elegant. It was discovered in 1954 by J.S. Frame, G. de B. Robinson and R.M. Thrall by improving a much less convenient formula which expressed f^λ in terms of a determinant.

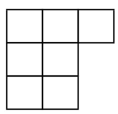
To state the formula, we must first introduce a bit of notation. Let $\lambda = (\lambda_1, ..., \lambda_m)$ be a partition of n. Consider a cell (i, j) of the Young diagram, that is the cell in the i^{th} row and j^{th} column. The **hook** $H_\lambda(i, j)$ is the set of cells (a, b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The **hook-length** $h_\lambda(i, j)$ is the number of cells in the hook $H_\lambda(i, j)$. In other words, the hook length of a cell (i,j) is the number of boxes to the right of (i,j) in row i plus the number of boxes below (i,j) in column $j+1$ (for the box (i,j) itself).

Then the hook-length formula expresses the number of standard Young tableaux of shape λ as,

$$f^{\lambda} = \frac{n!}{\prod h_{\lambda}(i,j)},$$

where the product is over all cells (i,j) of λ.

Example:



$h_{\lambda}(1,1) = 5$ $\qquad f^{\lambda} = \frac{7!}{5 \cdot 4 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 21$

$h_{\lambda}(1,2) = 4$

$h_{\lambda}(2,1) = 3$

This means that for $\lambda = (3,2,2)$ the corresponding irreducible representation V_{λ} has dimension 21. This can be proved using Theorem 4.22 and induction on n.

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