Certain Subclasses of Meromorphically Univalent Functions with Positive or Negative Coefficients

M-P. Chen, H. Irmak,
H.M. Srivastava, & C. Yu

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Ming-Po Chen
Institute of Mathematics
Academia Sinica
Taipei 11529, Taiwan, Republic of China
E-Mail: MAAPO@CCVAX.SINICA.EDU.TW

Hüseyin Irmak
Department of Mathematics Education
Hacettepe University
06532 Beytepe-Ankara, Turkey
E-Mail: IRMAK@ETI.CC.HUN.EDU.TR

H.M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
E-Mail: HMSRI@UVVM.UVIC.CA

Ching-Shu Yu
Department of Mathematics
Chung Yuan Christian University
Chung-Li 32033, Taiwan, Republic of China

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Abstract

The main object of the present paper is to introduce and study rather systematically two new subclasses $\mathcal{R}(p,\lambda,\alpha)$ and $\mathcal{C}(p,\lambda,\alpha)$ of meromorphically univalent functions with positive and negative coefficients, respectively. We first obtain a necessary and sufficient condition for a function to be in each of these classes. We then investigate the meromorphically starlikeness and meromorphically convexity of functions belonging to the classes $\mathcal{R}(p,\lambda,\alpha)$ and $\mathcal{C}(p,\lambda,\alpha)$. Several other properties and characteristics of functions in these classes are also derived.

1. Introduction and Definitions

Let $\mathcal{M}_p^{(A,B)}$ denote the class of functions $f(z)$ of the form:

$$f(z) := \frac{A}{z} + B\sum_{n=p}^{\infty} a_n\, z^n \quad (a_n \geqq 0;\ AB \neq 0;\ p \in \mathbb{N} := \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic *and* univalent in the *punctured* unit disk

$$\mathcal{D} := \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\},$$

and which have a simple pole at the origin ($z = 0$) with residue A there. A function $f(z) \in \mathcal{M}_p^{(A,B)}$ is said to be in the class $\mathcal{M}_p^{(A,B)}(\alpha;\lambda,\mu)$ if it also satisfies the inequality:

$$\Re\left\{\frac{(1-2\lambda)\,zf'(z) - \lambda z^2 f''(z)}{(\mu-1)\,f(z) + \mu z\, f'(z)}\right\} > \alpha \qquad (z \in \mathcal{D};\ 0 \leqq \alpha < 1) \tag{1.2}$$

for some suitably restricted real parameters λ and μ.

Two important subclasses of the class $\mathcal{M}_p^{(A,B)}(\alpha;\lambda,\mu)$ are worthy of mention. First of all, for $\lambda = \mu = 0$, the class $\mathcal{M}_p^{(A,B)}(\alpha;0,0)$ consists of *meromorphically starlike functions of order* α $(0 \leqq \alpha < 1)$ (with positive or negative coefficients depending upon the value of the nonzero constant B). On the other hand, when $\lambda = \mu = 1$, the class $\mathcal{M}_p^{(A,B)}(\alpha;1,1)$ would consist of *meromorphically convex functions of order* α $(0 \leqq \alpha < 1)$ (with positive or negative coefficients depending upon the value of the nonzero constant B) (*cf.*, *e.g.*, Duren [4] and Goodman [5]; see also Srivastava and Owa [6]). Some other subclasses of the class $\mathcal{M}_p^{(A,B)}$ were studied recently by (for example) Cho *et al.* ([2] and [3]) and Altintaş *et al.* [1].

In the present paper we propose to investigate various interesting properties and characteristics of functions belonging to the following subclasses $\mathcal{R}(p,\lambda,\alpha)$ and $\mathcal{C}(p,\lambda,\alpha)$ of the general class $\mathcal{M}_p^{(A,B)}(\alpha;\lambda,\mu)$ which we introduced above:

$$\mathcal{R}(p,\lambda,\alpha) := \mathcal{M}_p^{(-1,1)}(\alpha;\lambda,\lambda) \tag{1.3}$$

$$(p \in \mathbb{N};\ \lambda \geq 1/(p+1);\ 0 \leq \alpha < 1)$$

and

$$\mathcal{C}(p,\lambda,\alpha) := \mathcal{M}_p^{(1,-1)}(\alpha;1,1-\lambda) \tag{1.4}$$

$$(p \in \mathbb{N};\ 0 \leq \lambda \leq p;\ 0 \leq \alpha < 1).$$

Clearly, the class $\mathcal{R}(p,1,\alpha)$ consists of meromorphically convex functions $f(z)$ of order α $(0 \leq \alpha < 1)$ with *positive* coefficients, given by

$$f(z) = -\frac{1}{z} + \sum_{n=p}^{\infty} a_n\, z^n \qquad (a_n \geq 0;\ p \in \mathbb{N}). \tag{1.5}$$

Furthermore, since the condition $\lambda \geq 1/(p+1)$ $(p \in \mathbb{N})$ is not actually a requirement for the *definition* (1.3), we may set $\lambda = 0$ in (1.3) and observe that the class $\mathcal{R}(p,0,\alpha)$ consists of meromorphically starlike functions $f(z)$ of order α $(0 \leq \alpha < 1)$ with *positive* coefficients, given by (1.5). The class $\mathcal{C}(p,0,\alpha)$, on the other hand, consists of meromorphically convex functions $f(z)$ of order α $(0 \leq \alpha < 1)$ with *negative* coefficients, given by

$$f(z) = \frac{1}{z} - \sum_{n=p}^{\infty} a_n\, z^n \qquad (a_n \geq 0;\ p \in \mathbb{N}). \tag{1.6}$$

2. Coefficient Inequalities and Inclusion Properties

A necessary and sufficient condition for a function $f(z)$, given by (1.5), to be in the class $\mathcal{R}(p,\lambda,\alpha)$ is provided by

Theorem 1. *Let a function $f(z)$ be in the class $\mathcal{M}_p^{(-1,1)}$. Then the function $f(z)$ belongs to the class $\mathcal{R}(p,\lambda,\alpha)$ if and only if*

$$\sum_{n=p}^{\infty}(n+\alpha)(n\lambda+\lambda-1)\,a_n \leq 1-\alpha \qquad (0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N}). \tag{2.1}$$

The result is sharp.

Proof. First of all, suppose that the function $f(z)$, given by (1.5), is in the class $\mathcal{R}(p,\lambda,\alpha)$. Then it is easily seen from (1.5), (1.3), and (1.2) that

$$\Re\left\{\frac{1-\sum\limits_{n=p}^{\infty} n(n\lambda+\lambda-1)\,a_n\,z^{n+1}}{1+\sum\limits_{n=p}^{\infty}(n\lambda+\lambda-1)\,a_n\,z^{n+1}}\right\} > \alpha \tag{2.2}$$

$$(z \in \mathcal{D};\ 0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N}).$$

If we choose values of z on the real axis and let $z \to 1-$ through real values, we obtain the inequality:

$$\frac{1-\sum\limits_{n=p}^{\infty} n(n\lambda+\lambda-1)\,a_n}{1+\sum\limits_{n=p}^{\infty}(n\lambda+\lambda-1)\,a_n} \geq \alpha \tag{2.3}$$

$$(0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N}),$$

which readily yields the assertion (2.1) of Theorem 1.

Conversely, let us suppose that the inequality (2.1) holds true and let

$$z \in \partial\mathcal{D} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| = 1\}. \tag{2.4}$$

Then we find from the definition (1.5) that

$$\begin{aligned}
&\left|\frac{(1-2\lambda)\,zf'(z)-\lambda\,z^2f''(z)}{(\lambda-1)\,f(z)+\lambda z\,f'(z)}-1\right| \\
&\qquad\leq \frac{\sum\limits_{n=p}^{\infty}(n+1)(n\lambda+\lambda-1)\,a_n\,|z|^{n+1}}{1+\sum\limits_{n=p}^{\infty}(n\lambda+\lambda-1)\,a_n\,|z|^{n+1}} \\
&\qquad= \frac{\sum\limits_{n=p}^{\infty}(n+1)(n\lambda+\lambda-1)\,a_n}{1+\sum\limits_{n=p}^{\infty}(n\lambda+\lambda-1)\,a_n} \qquad (\text{since } z\in\partial\mathcal{D}) \\
&\qquad\leq 1-\alpha \qquad (0\leq\alpha<1;\ \lambda\geq 1/(p+1);\ p\in\mathbb{N}),
\end{aligned} \tag{2.5}$$

where we have made use of the inequality (2.1). Thus, by the maximum modulus theorem, we conclude from (2.5) that

$$f(z)\in\mathcal{R}(p,\lambda,\alpha).$$

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being given by

$$f(z)=-\frac{1}{z}+\frac{1-\alpha}{(p+\alpha)(p\lambda+\lambda-1)}\,z^p \qquad (p\in\mathbb{N}). \tag{2.6}$$

In precisely the same manner, we can prove

Theorem 2. *Let a function $f(z)$ be in the class $\mathcal{M}_p^{(1,-1)}$. Then the function $f(z)$ belongs to the class $\mathcal{C}(p,\lambda,\alpha)$ if and only if*

$$\sum_{n=p}^{\infty}\{n(n-\lambda\alpha)+\alpha(n-\lambda)\}\,a_n\leq 1-\alpha \qquad (0\leq\alpha<1;\ 0\leq\lambda\leq p;\ p\in\mathbb{N}). \tag{2.7}$$

The result is sharp, the extremal function being given by

$$f(z)=\frac{1}{z}-\frac{1-\alpha}{p(p-\lambda\alpha)+\alpha(p-\lambda)}\,z^p \qquad (p\in\mathbb{N}). \tag{2.8}$$

Corollary 1 and Corollary 2 below are rather immediate consequences of Theorem 1 and Theorem 2, respectively:

Corollary 1. *If* $f(z) \in \mathcal{R}(p, \lambda, \alpha)$, *then*

$$a_n \leq \frac{1-\alpha}{(n+\alpha)(n\lambda+\lambda-1)} \qquad (n \geq p;\ p \in \mathbb{N}). \tag{2.9}$$

Corollary 2. *If* $f(z) \in \mathcal{C}(p, \lambda, \alpha)$, *then*

$$a_n \leq \frac{1-\alpha}{n(n-\lambda\alpha)+\alpha(n-\lambda)} \qquad (n \geq p;\ p \in \mathbb{N}). \tag{2.10}$$

Next we prove

Theorem 3. *Let the function* $f(z)$ *defined by* (1.5) *and the function* $g(z)$ *defined by*

$$g(z) = -\frac{1}{z} + \sum_{n=p}^{\infty} b_n\, z^n \qquad (b_n \geq 0;\ p \in \mathbb{N}) \tag{2.11}$$

be in the same class $\mathcal{R}(p, \lambda, \alpha)$. *Then the function* $h(z)$ *defined by*

$$h(z) := (1-\xi)\, f(z) + \xi\, g(z) = -\frac{1}{z} + \sum_{n=p}^{\infty} c_n\, z^n \tag{2.12}$$

$$(c_n := (1-\xi)\, a_n + \xi\, b_n \geq 0;\ 0 \leq \xi \leq 1)$$

is also in the class $\mathcal{R}(p, \lambda, \alpha)$.

Proof. Suppose that each of the functions $f(z)$ and $g(z)$, involved in Theorem 3, is in the class $\mathcal{R}(p, \lambda, \alpha)$. Then, making use of (2.1) and (2.12), we observe that

$$\begin{aligned}
&\sum_{n=p}^{\infty} (n+\alpha)(n\lambda+\lambda-1)\, c_k \\
&\quad = (1-\xi) \sum_{n=p}^{\infty} (n+\alpha)(n\lambda+\lambda-1)\, a_n + \xi \sum_{n=p}^{\infty} (n+\alpha)(n\lambda+\lambda-1)\, b_n \\
&\quad \leq (1-\xi)(1-\alpha) + \xi(1-\alpha) \\
&\quad = 1-\alpha \qquad (0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N};\ 0 \leq \xi \leq 1),
\end{aligned} \tag{2.13}$$

which evidently completes the proof of Theorem 3.

Similarly, by employing Theorem 2 in place of Theorem 1, we obtain

Theorem 4. *Let the function* $f(z)$ *defined by* (1.6) *and the function* $g(z)$ *defined by*

$$g(z) = \frac{1}{z} - \sum_{n=p}^{\infty} b_n\, z^n \qquad (b_n \geq 0;\ p \in \mathbb{N}) \tag{2.14}$$

be in the same class $\mathcal{C}(p, \lambda, \alpha)$. *Then the function* $h(z)$ *defined by*

$$h(z) := (1-\xi)\, f(z) + \xi\, g(z) = \frac{1}{z} - \sum_{n=p}^{\infty} c_n\, z^n \tag{2.15}$$

$$(c_n := (1-\xi)\, a_n + \xi\, b_n \geq 0;\ 0 \leq \xi \leq 1)$$

is also in the class $\mathcal{C}(p, \lambda, \alpha)$.

The following results (Theorem 5 and Theorem 6) involve the *quasi-Hadamard product* (or *convolution*) of functions belonging to the classes $\mathcal{R}(p, \lambda, \alpha)$ and $\mathcal{C}(p, \lambda, \alpha)$, respectively. We first state

Theorem 5. *Let the function* $f(z)$ *defined by* (1.5) *and the function* $g(z)$ *defined by* (2.11) *be in the same class* $\mathcal{R}(p, \lambda, \alpha)$. *Then*

$$(f * g)(z) \in \mathcal{R}(p, \lambda, \beta),$$

where $(f * g)(z)$ *denotes the quasi-Hadamard product (or convolution) of* $f(z)$ *and* $g(z)$, *defined by*

$$(f * g)(z) := -\frac{1}{z} + \sum_{n=p}^{\infty} a_n\, b_n\, z^n \qquad (a_n \geq 0;\ b_n \geq 0;\ p \in \mathbb{N}) \tag{2.16}$$

and

$$\beta \leq 1 - \frac{(p+1)(1-\alpha)^2}{(p\lambda + \lambda - 1)(p+\alpha)^2 + (1-\alpha)^2} \qquad (0 \leq \alpha < 1;\ p \in \mathbb{N}). \tag{2.17}$$

The result is sharp for the functions $f(z)$ *and* $g(z)$ *given by*

$$f(z) = g(z) = -\frac{1}{z} + \frac{1-\alpha}{(p+\alpha)(p\lambda + \lambda - 1)}\, z^p \qquad (p \in \mathbb{N}). \tag{2.18}$$

Proof. Applying Theorem 1 to the functions $f(z)$ and $g(z)$, we obtain

$$\sum_{n=p}^{\infty} \frac{(n+\alpha)(n\lambda + \lambda - 1)}{1-\alpha}\, a_n \leq 1 \qquad (0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N}) \tag{2.19}$$

and

$$\sum_{n=p}^{\infty} \frac{(n+\alpha)(n\lambda+\lambda-1)}{1-\alpha}\, b_n \leq 1 \qquad (0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N}), \tag{2.20}$$

respectively.

In order to prove Theorem 5, it is sufficient to find the largest β such that

$$\sum_{n=p}^{\infty} \frac{(n+\beta)(n\lambda+\lambda-1)}{1-\beta}\, a_n\, b_n \leq 1 \qquad (0 \leq \beta < 1;\ \lambda \geq 1/(p+1);\ n \in \mathbb{N}). \tag{2.21}$$

Indeed, in view of the Cauchy-Schwarz inequality, we find from (2.19) and (2.20) that

$$\sum_{n=p}^{\infty} \frac{(n+\alpha)(n\lambda+\lambda-1)}{1-\alpha} \sqrt{a_n\, b_n} \leq 1 \qquad (0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N}). \tag{2.22}$$

Therefore, the inequality (2.21) holds true if

$$\sqrt{a_n\, b_n} \leq \frac{(1-\beta)(n+\alpha)}{(1-\alpha)(n+\beta)} \qquad (n \geq p;\ p \in \mathbb{N}), \tag{2.23}$$

that is, if

$$\frac{1-\alpha}{(n+\alpha)(n\lambda+\lambda-1)} \leq \frac{(1-\beta)(n+\alpha)}{(1-\alpha)(n+\beta)} \qquad (n \geq p;\ p \in \mathbb{N}), \tag{2.24}$$

which readily yields

$$\beta \leq 1 - \frac{(n+1)(1-\alpha)^2}{(n\lambda+\lambda-1)(n+\alpha)^2+(1-\alpha)^2} \qquad (n \geq p;\ p \in \mathbb{N}). \tag{2.25}$$

Finally, letting

$$\Phi(n) := 1 - \frac{(n+1)(1-\alpha)^2}{(n\lambda+\lambda-1)(n+\alpha)^2+(1-\alpha)^2} \qquad (n \geq p;\ p \in \mathbb{N}), \tag{2.26}$$

we see that $\Phi(n)$ is an increasing function of n. This shows, in conjunction with the inequality (2.25), that

$$\beta \leq \Phi(p) = 1 - \frac{(p+1)(1-\alpha)^2}{(p\lambda+\lambda-1)(p+\alpha)^2+(1-\alpha)^2} \tag{2.27}$$

$$(0 \leq \alpha < 1;\ \lambda \geq 1/(p+1);\ p \in \mathbb{N}),$$

which completes the proof of Theorem 5.

A similar consequence of Theorem 2 may be stated as

Theorem 6. *Let the function $f(z)$ defined by* (1.6) *and the function $g(z)$ defined by* (2.14) *be in the same class $\mathcal{C}(p,\lambda,\alpha)$. Then*

$$(f*g)(z) \in \mathcal{C}(p,\lambda,\beta),$$

where $(f*g)(z)$ denotes the quasi-Hadamard product (or convolution) of $f(z)$ and $g(z)$, defined by

$$(f*g)(z) := \frac{1}{z} - \sum_{n=p}^{\infty} a_n\, b_n\, z^n \qquad (a_n \geqq 0;\ b_n \geqq 0;\ p \in \mathbb{N}) \tag{2.28}$$

and

$$\beta \leqq 1 - \frac{(p-\lambda-p\lambda+p^2)(1-\alpha)^2}{[p(p-\lambda\alpha)+\alpha(p-\lambda)]^2+(p-\lambda-p\lambda)(1-\alpha)^2} \tag{2.29}$$

$$(0 \leqq \alpha < 1;\ p \in \mathbb{N}).$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = \frac{1}{z} - \frac{1-\alpha}{p(p-\lambda\alpha)+\alpha(p-\lambda)}\, z^p \qquad (p \in \mathbb{N}). \tag{2.30}$$

3. Growth and Distortion Theorems

The assertion (2.1) of Theorem 1 readily yields the following coefficient inequalities for a function $f(z)$ belonging to the class $\mathcal{R}(p,\lambda,\alpha)$:

$$\sum_{n=p}^{\infty} a_n \leqq \frac{1-\alpha}{(p+\alpha)(p\lambda+\lambda-1)} \qquad (0 \leqq \alpha < 1;\ \lambda > 1/(p+1);\ p \in \mathbb{N}) \tag{3.1}$$

and

$$\sum_{n=p}^{\infty} n\, a_n \leqq \frac{p(1-\alpha)}{(p+\alpha)(p\lambda+\lambda-1)} \qquad (0 \leqq \alpha < 1;\ \lambda > 1/(p+1);\ p \in \mathbb{N}). \tag{3.2}$$

By applying the coefficient inequalities (3.1) and (3.2), it is not difficult to prove

Theorem 7. *If $f(z) \in \mathcal{R}(p,\lambda,\alpha)$, then*

$$\frac{1}{|z|} - \frac{1-\alpha}{(p+\alpha)(p\lambda+\lambda-1)}|z|^p \leq |f(z)| \leq \frac{1}{|z|} + \frac{1-\alpha}{(p+\alpha)(p\lambda+\lambda-1)}|z|^p \tag{3.3}$$

$$(z \in \mathcal{D};\ 0 \leq \alpha < 1;\ \lambda > 1/(p+1);\ p \in \mathbb{N})$$

and

$$\frac{1}{|z|^2} - \frac{p(1-\alpha)}{(p+\alpha)(p\lambda+\lambda-1)}|z|^{p-1} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{p(1-\alpha)}{(p+\alpha)(p\lambda+\lambda-1)}|z|^{p-1} \tag{3.4}$$

$$(z \in \mathcal{D};\ 0 \leq \alpha < 1;\ \lambda > 1/(p+1);\ p \in \mathbb{N}).$$

Each of these results is sharp for the function $f(z)$ given by (2.6).

Similarly, by applying *analogous* coefficient inequalities derivable from the assertion (2.7) of Theorem 2, we obtain

Theorem 8. *If $f(z) \in \mathcal{C}(p,\lambda,\alpha)$, then*

$$\frac{1}{|z|} - \frac{1-\alpha}{p(p-\lambda\alpha)+\alpha(p-\lambda)}|z|^p \leq |f(z)| \leq \frac{1}{|z|} + \frac{1-\alpha}{p(p-\lambda\alpha)+\alpha(p-\lambda)}|z|^p \tag{3.5}$$

$$(z \in \mathcal{D};\ 0 \leq \alpha < 1;\ 0 \leq \lambda \leq p;\ p \in \mathbb{N})$$

and

$$\frac{1}{|z|^2} - \frac{p(1-\alpha)}{p(p-\lambda\alpha)+\alpha(p-\lambda)}|z|^{p-1} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{p(1-\alpha)}{p(p-\lambda\alpha)+\alpha(p-\lambda)}|z|^{p-1} \tag{3.6}$$

$$(z \in \mathcal{D};\ 0 \leq \alpha < 1;\ 0 \leq \lambda \leq p;\ p \in \mathbb{N}).$$

Each of these results is sharp for the function $f(z)$ given by (2.8).

A number of simpler distortion properties can easily be deduced from Theorem 7 and Theorem 8 by suitably specializing the parameter λ occurring in the assertions (3.3) to (3.6).

4. Meromorphically Starlikeness and Meromorphically Convexity of Functions in the Classes $\mathcal{R}(p,\lambda,\alpha)$ and $\mathcal{C}(p,\lambda,\alpha)$

We begin by proving

Theorem 9. *If* $f(z) \in \mathcal{R}(p,\lambda,\alpha)$, *then* $f(z)$ *is meromorphically starlike of order* γ $(0 \leq \gamma < 1)$ *in*

$$0 < |z| < r_1(p,\lambda,\alpha,\gamma) := \inf_n \left\{ \frac{(1-\gamma)(n+\alpha)(n\lambda+\lambda-1)}{(1-\alpha)(n-\gamma+2)} \right\}^{1/(n+1)} \tag{4.1}$$

$$(n \geq p;\ p \in \mathbb{N}).$$

The result is sharp for the functions $f(z)$ *given by*

$$f(z) = -\frac{1}{z} + \frac{1-\alpha}{(n+\alpha)(n\lambda+\lambda-1)}\, z^n \qquad (n \geq p;\ p \in \mathbb{N}). \tag{4.2}$$

Proof. We must show that

$$\left| -\frac{z\,f'(z)}{f(z)} - 1 \right| \leq 1-\gamma \tag{4.3}$$

$$\left(0 < |z| < r_1(p,\lambda,\alpha,\gamma);\ f(z) \in \mathcal{R}(p,\lambda,\alpha);\ 0 \leq \gamma < 1\right).$$

Indeed we find from the definition (1.5) that

$$\left| -\frac{z\,f'(z)}{f(z)} - 1 \right| \leq \frac{\sum\limits_{n=p}^{\infty} (n+1)\, a_n\, |z|^{n+1}}{1 - \sum\limits_{n=p}^{\infty} a_n\, |z|^{n+1}} \tag{4.4}$$

$$\leq 1-\gamma,$$

provided that

$$\frac{(n-\gamma+2)\,|z|^{n+1}}{1-\gamma} \leq \frac{(n+\alpha)(n\lambda+\lambda-1)}{1-\alpha} \tag{4.5}$$

$$(0 \leq \alpha < 1;\ 0 \leq \gamma < 1;\ \lambda > 1/(p+1);\ p \in \mathbb{N}).$$

Solving this last equation (4.5) for $|z|$, we obtain the inequality (4.1), and the proof of Theorem 9 is thus completed.

Since a function $f(z) \in \mathcal{M}_p^{(-1,1)}$ is meromorphically convex of order γ $(0 \leq \gamma < 1)$ if and only if $z f'(z)$ is meromorphically starlike of order γ $(0 \leq \gamma < 1)$, an immediate consequence of Theorem 9 may be stated as

Theorem 10. *If $f(z) \in \mathcal{R}(p, \lambda, \alpha)$, then $f(z)$ is meromorphically convex of order γ $(0 \leq \gamma < 1)$ in*

$$0 < |z| < r_2(p, \lambda, \alpha, \gamma) := \inf_n \left\{ \frac{(1-\gamma)(n+\alpha)(n\lambda + \lambda - 1)}{n(1-\alpha)(n-\gamma+2)} \right\}^{1/(n+1)} \tag{4.6}$$

$$(n \geq p;\ p \in \mathbb{N}).$$

The result is sharp for the functions $f(z)$ given by (4.2).

The derivations of Theorem 9 and Theorem 10 can be applied *mutatis mutandis* in order to obtain the following results for the class $\mathcal{C}(p, \lambda, \alpha)$.

Theorem 11. *If $f(z) \in \mathcal{C}(p, \lambda, \alpha)$, then $f(z)$ is meromorphically starlike of order δ $(0 \leq \delta < 1)$ in*

$$0 < |z| < r_3(p, \lambda, \alpha, \delta) := \inf_n \left\{ \frac{(1-\delta)[n(n-\lambda\alpha) + \alpha(n-\lambda)]}{(1-\alpha)(n-\delta+2)} \right\}^{1/(n+1)} \tag{4.7}$$

$$(n \geq p;\ p \in \mathbb{N}).$$

The result is sharp for the functions $f(z)$ given by

$$f(z) = \frac{1}{z} - \frac{1-\alpha}{n(n-\lambda\alpha) + \alpha(n-\lambda)} z^n \qquad (n \geq p;\ p \in \mathbb{N}). \tag{4.8}$$

Theorem 12. *If $f(z) \in \mathcal{C}(p, \lambda, \alpha)$, then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in*

$$0 < |z| < r_4(p, \lambda, \alpha, \delta) := \inf_n \left\{ \frac{(1-\delta)[n(n-\lambda\alpha) + \alpha(n-\lambda)]}{n(1-\alpha)(n-\delta+2)} \right\}^{1/(n+1)} \tag{4.9}$$

$$(n \geq p;\ p \in \mathbb{N}).$$

The result is sharp for the functions $f(z)$ given by (4.8).

Meromorphically starlikeness and meromorphically convexity of functions belonging to several simpler classes can indeed be deduced by suitably specializing the parameter λ occurring in Theorems 9 to 12 above.

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