RESULTS CONCERNING UNIQUENESS OF SOLUTIONS TO
THE STEADY STATE BOLTZMANN EQUATION

by

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ABSTRACT

Using a contraction mapping, solutions to the truncated steady-state one dimensional Boltzmann Equation in a bounded interval with initial boundary conditions are shown to be unique for small enough interval width. Associated results are the existence and uniqueness of a solution without the restriction of non-negativity. Also, a companion uniqueness result to the existence of a measure solution for the associated truncated measure boundary value problem is obtained for small enough interval width.

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1 Introduction and Preliminaries

1.1 Introduction

In 1872, Ludwig Boltzmann introduced an integro-differential equation to describe the behaviour of a rarefied gas [3]. This equation is fundamental in the kinetic theory of gases, and has been applied in different forms to such areas of study as the transport of electrons, neutrons, and phonons, as well as radiative transfer in the atmosphere [3]. These areas of interest are the target of a considerable amount of research [3].

A derivation of the Boltzmann equation can be found in [3] or [8]. To state the equation, consider a system (generally a gas) of N identical rigid spheres in a given volume. Each sphere has position $\vec{r} \in \mathbf{R}^3$ and velocity $\vec{v} \in \mathbf{R}^3$. The evolution of this gas in time $t \in \mathbf{R}$ is approximately described by the particle density function $f(\vec{r},\vec{v},t)$. The Boltzmann equation for this case is a non-linear integro-differential equation given (with the arguments suppressed) by

$$\frac{\partial}{\partial t} f + \vec{v} \cdot \vec{\nabla} f = \frac{1}{\lambda} \iint_{\mathbf{R}^3 S^2} (f'f'_* - ff_*) B(\vec{n},\vec{v}-\vec{w}) d\vec{n} d\vec{w}. \qquad (1.1.1)$$

Here, the notations $\vec{v}' = \vec{v} - \vec{n}\langle \vec{n},\vec{v}-\vec{w}\rangle$ and $\vec{w}' = \vec{w} + \vec{n}\langle \vec{n},\vec{v}-\vec{w}\rangle$ are used to define $f' = f(\vec{r},\vec{v}',t)$, $f'_* = f(\vec{r},\vec{w}',t)$, and $f_* = f(\vec{r},\vec{w},t)$. λ is a dimensionless parameter dependent on the number and size of the particles. $B: S^2 \times \mathbf{R}^3 \to \mathbf{R}$ is called the collision kernel.

To state the steady state problem, the change of variable seen above is completed via the following transformation $J:(\mathbf{R}^3 \times S^2 \times \mathbf{R}^3) \to (\mathbf{R}^3 \times S^2 \times \mathbf{R}^3)$ defined by $J(\vec{v},\vec{n},\vec{w}) = (\vec{v}',-\vec{n},\vec{w}')$ and referred to as the collision transformation. The notation $\langle \vec{n}, \vec{v}-\vec{w} \rangle$ denotes the Euclidean inner product in $\mathbf{R}^3$. Note that for fixed $\vec{n} \in S^2$, J is a linear transformation on $\mathbf{R}^6$. J is linear by definition and invertible by observing that $J^{-1} = J$. Hence, J is in the group of invertible linear transformations on $\mathbf{R}^6$. For any Lebesgue measurable $f \in L^1(\mathbf{R}^6)$ it holds [4] that $\int f(x)dx = |\det J| \int f \circ J(x)dx$. Since $J^{-1} = J$, it is true that $1 = \det I = \det J^2 = \det J \cdot \det J^{-1} = (\det J)^2$ implying $|\det J| = 1$.

For the case of N rigid spheres, B can be written explicitly as $|(\vec{v}-\vec{w})\cdot\vec{n}|$. In general, B is only dependent on $\|\vec{v}-\vec{w}\|$ and $|\langle \vec{n}, \vec{v}-\vec{w} \rangle|$ and thus is invariant under J. In the following, the full collision kernel is not dealt with and some assumptions must be made to B before any results can be obtained. Specifically, the truncation introduced in Arkeryd, Cercignani and Illner [1] must be assumed and incorporated into B. Letting $\vec{v} = (\xi, \eta, \zeta)$ and $\vec{w} = (\xi_*, \eta_*, \zeta_*)$ represent vectors in $\mathbf{R}^3$ with $x, y,$ and z components given by $\xi, \eta,$ and ζ or $\xi_*, \eta_*,$ and ζ_* respectively, this truncation can be written by including the factor

$$\chi_\delta(\vec{v},\vec{w},\vec{v}',\vec{w}') = \begin{Bmatrix} 1 \text{ if } \min\{|\xi|,|\xi_*|,|\xi'|,|\xi_*'|\} \geq \delta, \text{ and} \\ 0 \text{ otherwise} \end{Bmatrix}$$

for $\delta > 0$ arbitrary but fixed. In the above, ξ' and ξ_*' are the x-components of $\vec{v}'$ and $\vec{w}'$. A truncated collision kernel B_δ can be defined by

$$B_\delta(\vec{n},\vec{v},\vec{w}) = B(\vec{n},\vec{v}-\vec{w}) \cdot \chi_\delta(\vec{v},\vec{w},\vec{v}',\vec{w}').$$

By construction, χ_δ is invariant under the collision transformation using the property $(\vec{v}')' = \vec{v}$ and $(\vec{w}')' = \vec{w}$. The invariance of B under J allows B_δ to maintain this property. In the remaining chapters of this paper, B_δ is used instead of B; however to simplify notation the collision kernel will be denoted simply by B.

The problem dealt with herein is a special case of the full non-linear Boltzmann equation. It is obtained by restricting $\vec{r}$ to one spatial dimension, and considering a steady-state solution $f(x,\vec{v})$ independent of time. With $\frac{\partial f}{\partial t} = 0$, the equation (1.1.1) simplifies to

$$\xi \cdot \frac{d}{dx} f = \iint_{\mathbf{R}^3 S^2} (f'f'_* - ff_*) B(\vec{n},\vec{v},\vec{w}) d\vec{n} d\vec{w}, \qquad (1.1.2)$$

where $x \in [0,a]$ for some $a \in \mathbf{R}$ as in [1]. A further notational simplification can be reached by setting

$$C(f,f)(x,\vec{v}) = \iint_{\mathbf{R}^3 S^2} (f'f'_* - ff_*) B(\vec{n},\vec{v},\vec{w}) d\vec{n} d\vec{w}.$$

Equation (1.1.2) can then be written more succinctly as

$$\xi \cdot \frac{d}{dx} f = C(f,f). \qquad (1.1.3)$$

This ordinary integro-differential equation is known as the steady state Boltzmann equation in one dimension [1].

The main results of this paper can now be stated; uniqueness of a solution to an associated boundary value problem to (1.1.3) is proven in section 2.2, while uniqueness of a measure solution (to be defined in section 1.4) to the associated measure boundary value problem of (1.1.3) will be obtained in chapter 3.3.

1.2 The Truncated Problem

Uniqueness of solutions to the steady Boltzmann equation is unknown in general [1]. However, under some simple assumptions to the collision kernel, uniqueness of a solution can be obtained for the truncated problem with boundary conditions for small enough length of the interval $[0,a]$. In leading up to the statement of this simplified equation, some notation and definitions are needed.

Let $X = C([0,a], L^1(\mathbf{R}^3))$, and $X_+ = C([0,a], L^1_+(\mathbf{R}^3))$ be the function spaces of continuous mappings from $[0,a]$ with the usual topology into $L^1(\mathbf{R}^3)$, the space of integrable functions with respect to Lebesgue measure on $\mathbf{R}^3$. $L^1_+ \subseteq L^1$ is the subset of non-negative functions in L^1. Both of these spaces can be equipped with a norm given by

$$\|f(x,\cdot)\| = \sup_{x\in[0,a]} \int_{\mathbf{R}^3} |f(x,\vec{v})| d\vec{v} \text{ for } f \in X \text{ or } X_+.$$

The above is shown to be a norm on X in the Appendix, theorem A.1.

The first assumption on the collision kernel is a straightforward bound, given by $0 \le B(\vec{n},\vec{v},\vec{w}) \le b$ for some $b \in \mathbf{R}$, and all $(\vec{v},\vec{n},\vec{w}) \in \mathbf{R}^3 \times \mathrm{S}^2 \times \mathbf{R}^3$. The second assumption, made in 1.1, takes the form of a truncation to the collision kernel B.

Using this notation, the truncated boundary value problem for the steady Boltzmann equation in a slab can be stated in its entirety as

$$\xi \cdot \frac{d}{dx} f = C(f,f) \tag{1.2.1}$$

with the boundary conditions $f_0(\vec{v}), f_a(\vec{v}) \in L^1_+(\mathbf{R}^3)$, where

$$\begin{aligned} f(0,\vec{v}) &= f_0(\vec{v}) \text{ if } \xi > 0, \text{ and} \\ f(a,\vec{v}) &= f_a(\vec{v}) \text{ if } \xi < 0. \end{aligned} \tag{1.2.2}$$

Note that the functions $f_0(\vec{v}), f_a(\vec{v}) \in L^1_+(\mathbf{R}^3)$ are fixed and assumed to be non-negative. This is a physical constraint since only positive densities are considered to exist in reality [3].

In chapter 2, problem (1.2.1-2) is shown to have a unique solution in X_+, assuming that one exists. This is achieved via a contraction inequality depending linearly on the slab width a, which gives both existence and uniqueness of a solution in X via the Banach fixed point theorem. It is also important to observe that for $|\xi| < \delta$, truncation of the collision kernel implies $C(f,f) = 0$ for an arbitrary but fixed constant $\delta > 0$.

1.3 Review of Definitions and Results from Standard Analysis

In leading up to a statement of the measure problem in section 1.4, some definitions and results are necessary. This section is a review of the following definitions which are taken from Folland [4].

Let $(Y, \Im)$ denote a topological space. Y is called *locally compact* if every point $y \in Y$ has a compact neighborhood. Y is said to be *Hausdorff* if for every $x, y \in Y$ with $x \neq y$ there are disjoint open sets $U, V \in \Im$ such that $x \in U$ and $y \in V$.

For a topological space $(Y_*, \Im_*)$, a map $f: Y \to Y_*$ is called *continuous* if $f^{-1}(V) \in \Im$ for every open set $V \in \Im_*$. Denote the set of all such continuous functions $f: Y \to Y_*$ by $C(Y, Y_*)$ or simply $C(Y)$ if $Y_* = \mathbf{C}$ or $\mathbf{R}$. There are two other common subsets of $C(Y)$ which are defined as

$$C_c(Y) = \{f \in C(Y) \text{: supp}(f) \text{ is compact}\}, \text{ and}$$
$$C_0(Y) = \{f \in C(Y) \text{: } f \text{ vanishes at } \infty\}.$$

The *support* of f, denoted by $\operatorname{supp}(f)$, is the smallest closed set outside of which f vanishes. Hence, $\operatorname{supp}(f)=\operatorname{cl}\left(f^{-1}\left(\{0\}^c\right)\right)$. Also, f *vanishes at* ∞ if for every $\varepsilon>0$ it is true that $\{y\in Y:|f(y)|\geq\varepsilon\}$ is compact.

For any set Z and family of maps $\{f_\alpha:Z\to Y_\alpha\}_{\alpha\in A}$, where Y_α are some topological spaces indexed by an indexing set A, there is a unique topology on Z which has the property that any other topology in which all the f_α's are continuous contains Z. This topology is called the *weak topology generated by* $\{f_\alpha\}_{\alpha\in A}$.

Now let Y be a normed vector space over K where $K=\mathbf{R}$ or $\mathbf{C}$. A *Banach space* is a normed vector space which is complete with respect to the norm metric. A linear map from Y to K is called a *linear functional* on Y. A linear functional f on Y is said to be *bounded* if there exists a real number $C>0$ such that $|f(y)|\leq C\|y\|$ for all $y\in Y$. The space of bounded linear functionals on Y, denoted by Y^*, is called the *dual space* of Y and is a Banach space with the operator norm defined by

$$\|f\|=\sup_{\|y\|=1}\{|f(y)|\}.$$

The weak topology generated by Y^* is called simply the *weak topology on* Y.

For a normed vector space Y with its dual Y^* the topology generated by Y (considered as a subspace of Y^{**}) is called the *weak * topology* on Y^*. Convergence in the weak topology is called *weak convergence* while convergence in the weak * topology is simply the topology of pointwise convergence: for a net $\langle y_\alpha\rangle\subseteq Y$, $f_\alpha\to f$ if and only if $f_\alpha(y)\to f(y)$ for all $y\in Y$.

For notation and definitions involving measures, let (Y,B,μ) denote a measure space. If B contains the σ-algebra of Borel sets of Y then μ is said to be a *Borel* *measure* on Y. Further, for μ a Borel measure on Y and E a Borel subset of Y, μ is called *outer regular* on E if $\mu(E) = \inf\{\mu(U)\colon U \supseteq E,\ U \text{ open}\}$ and *inner regular* on E if $\mu(E) = \sup\{\mu(K)\colon K \subseteq E,\ K \text{ compact}\}$. A Borel measure which is finite on compact sets, outer regular on all Borel sets, and inner regular on all open sets is called a *Radon measure* on Y. The space of all bounded Radon measures on Y is denoted by $M(Y)$, which is a normed vector space with the norm $\mu \to |\mu|(Y)$ where $|\mu|$ is the total variation of μ. With this notation, a form of the Riesz Representation theorem can be stated from [4] as follows:

Theorem 1.3.1 (Riesz Representation) Let Y be a locally compact Hausdorff space, and for $\mu \in M(Y)$ and $f \in C_0(Y)$ let $I_\mu(f) = \int f d\mu$. Then the map $\mu \to I_\mu$ is an isometric isomorphism from $M(Y)$ to $C_0(Y)^*$.

The term *isometric* means that the map preserves the norms involved. In the theorem, $M(Y)$ is equipped with the total variation norm, and $C_0(Y)^*$ uses the usual operator norm.

To apply the Riesz Representation theorem, consider $M(\mathbf{R}^3) \cong C_0(\mathbf{R}^3)^*$ with the weak-* topology, and convergence of measures in $M(\mathbf{R}^3)$ written as $\mu_n \xrightarrow[wk*]{} \mu$ if $\int f d\mu_n \to \int f d\mu$ for all $f \in C_0(\mathbf{R}^3)$. It is in this notational setting that a measure formulation of the steady Boltzmann equation can be formulated.

1.4 Measure Formulation

In this section, problem (1.2.1-2) is reformulated to admit a measure solution to the associated measure boundary value problem [1]. In chapter 3, a solution to this problem is shown to be unique for small enough slab width a.

Let Z denote the subspace of functions in X,

$$Z = \left\{ \varphi(x,\vec{v}) \in X \;\middle|\; \begin{array}{l} \varphi \text{ is bounded and continuous, } \partial_x \varphi(x,\vec{v})/\xi \text{ is continuous,} \\ \varphi \text{ is Lipschitz continuous with respect to } \vec{v} \text{ independent of } x, \\ \text{supp}(\varphi) \text{ is compact and } \varphi(0,\vec{v}) = 0 \text{ if } \xi < 0,\ \varphi(a,\vec{v}) = 0 \text{ if } \xi > 0 \end{array} \right\}.$$

Elements of Z are called *admissible test functions.*

To obtain a measure formulation of the steady equation, it is necessary to start with equation (1.2.1). Multiplication of the equation by any $\varphi \in Z$ and integration by $\int_0^a \int_{\mathbf{R}^3} d\vec{v}dx$ yields

$$\int_0^a \int_{\mathbf{R}^3} \varphi(x,\vec{v})\xi \cdot \frac{d}{dx} f d\vec{v}dx = \int_0^a \int_{\mathbf{R}^3} \varphi(x,\vec{v}) C(f,f) d\vec{v}dx. \tag{1.4.1}$$

Integration of the left hand side of equation (1.4.1) by parts with respect to x yields

$$\int_{\mathbf{R}^3} \xi \cdot f(a,\vec{v})\varphi(a,\vec{v})d\vec{v} - \int_{\mathbf{R}^3} \xi \cdot f(0,\vec{v})\varphi(a,\vec{v})d\vec{v} - \int_0^a \int_{\mathbf{R}^3} \xi \cdot f(x,\vec{v}) \frac{\partial}{\partial x} \varphi(x,\vec{v}) d\vec{v}dx$$

$$= \int_0^a \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \int_{S^2} \varphi(x,\vec{v}) B(\vec{n},\vec{v},\vec{w}) [f'f'_* - ff_*] d\vec{n} d\vec{w} d\vec{v} dx. \tag{1.4.2}$$

Here $C(f,f)$ has been written in its full form and the above integration by parts must be done by changing the order of integration, integrating by parts, and changing back. This can be done via the Fubini-Tonelli Theorem [4].

The boundary conditions on $\varphi \in Z$ and condition 1.2.2 can be used to simplify the first two integrals in (1.4.2), and the collision transformation J can be applied to the right hand side to obtain

$$\int_{\xi<0} \xi \cdot f_a(\vec{v})\varphi(a,\vec{v})d\vec{v} - \int_{\xi>0} \xi \cdot f_0(\vec{v})\varphi(0,\vec{v})d\vec{v} - \int_0^a\int_{\mathbf{R}^3} \xi \cdot f(x,\vec{v})\frac{\partial}{\partial x}\varphi(x,\vec{v})d\vec{v}dx$$
$$= \int_0^a\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{S^2} (\varphi(x,\vec{v}') - \varphi(x,\vec{v}))B(\vec{n},\vec{v},\vec{w})[f(x,\vec{v})f(x,\vec{w})]d\vec{n}d\vec{w}d\vec{v}dx. \tag{1.4.3}$$

The right hand side can be written with $B \circ J = B$ by the assumptions on B in section 1.1. Considering the signed measures associated with the function $f \in X$ defined by $\mu(E) = \int_E f(\vec{v})d\vec{v},\ E \subseteq \mathbf{R}^3$ measurable, equation (1.4.3) becomes

$$\int_{\xi<0} \xi \cdot \varphi(a,\vec{v})d\mu_a^-(\vec{v}) - \int_{\xi>0} \xi \cdot \varphi(0,\vec{v})d\mu_0^+(\vec{v}) - \int_0^a\int_{\mathbf{R}^3} \xi \cdot \frac{\partial}{\partial x}\varphi(x,\vec{v})d\mu_x(\vec{v})dx$$
$$= \int_0^a\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{S^2} (\varphi(x,\vec{v}') - \varphi(x,\vec{v}))B(\vec{n},\vec{v},\vec{w})d\vec{n}d\mu_x(\vec{w})d\mu_x(\vec{v})dx. \tag{1.4.4}$$

With the notation μ_0^+ and μ_a^- representing the initial data as measures on $\xi > 0$ or $\xi < 0$, the definition of a measure solution given in [1] can now be stated.

Definition 1.4.1 Consider $M = M(\mathbf{R}^3)$ with the weak-* topology. A measure-valued function

$$[0,a] \to M(\mathbf{R}^3),$$
$$x \to \mu_x$$

is called a measure solution of (1.2.1) if $\mu_0\big|_{\{\vec{v}:\xi>0\}} = \mu_0^+$, $\mu_a\big|_{\{\vec{v}:\xi<0\}} = \mu_a^-$, $x \to \mu_x$ is continuous with respect to the weak-* topology on $M(\mathbf{R}^3)$ and (1.4.4) holds for all admissible test functions.

Using this definition of a measure solution, a truncation to the collision kernel can be introduced to state the steady measure problem. Let $\delta > 0$ be arbitrary but fixed and define a function $g^{\delta}:(S^2 \times \mathbf{R}^3 \times \mathbf{R}^3) \to [0,1]$ by

$$g^{\delta}(\vec{n},\vec{v},\vec{w}) = \begin{Bmatrix} 0 & \text{if } z \le \delta, \\ \delta^{-1}(z-\delta) & \text{if } \delta < z \le 2\delta, \text{ and} \\ 1 & \text{if } 2\delta \le z \end{Bmatrix}$$

where $z = \min\{|\xi|,|\xi_*|,|\xi'|,|\xi'_*|\}$. Define C^{δ} to be the collision operator C in (1.2.1) with the collision kernel B replaced by $B^{\delta}(\vec{n},\vec{v},\vec{w}) = B(\vec{n},\vec{v},\vec{w}) \cdot g^{\delta}(\vec{n},\vec{v},\vec{w})$.

The problem for which uniqueness is solved is then

$$\xi \frac{d}{dx}\mu_x = C^{\delta}(\mu_x,\mu_x), \tag{1.4.5}$$

with the boundary conditions

$$\mu_0\Big|_{\{\xi>0\}} = \mu_0^+,\ \mu_a\Big|_{\{\xi<0\}} = \mu_a^-. \tag{1.4.6}$$

For uniqueness to the problem (1.4.5-6), a norm or metric on this space of measures is necessary. To achieve this, define

$$D = \left\{ \varphi:\mathbf{R}^3 \to [0,1] \,\middle|\, \begin{matrix} \varphi \text{ is Lebesgue measurable, and} \\ \|\varphi(x)-\varphi(y)\| \le \|x-y\| \text{ for all } x,y \in \mathbf{R}^3 \end{matrix} \right\}.$$

Let the space of measure solutions be denoted by $X = C([0,a], M(\mathbf{R}^3))$, and although this notation is the same as that introduced in section 1.2, the meaning is clear in the following chapters from the context. The function $\rho: X \times X \to \mathbf{R}$ given for $\mu_{\bullet}, \gamma_{\bullet} \in X$ by

$$\rho(\mu_{\bullet},\gamma_{\bullet}) = \sup_{x\in[0,a]} \sup_{\varphi\in D} \left| \int_{\mathbf{R}^3} \varphi(\vec{v})d\mu_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v})d\gamma_x(\vec{v}) \right|$$

is shown in the Appendix theorem A.2 to be a metric on X if the equivalence of two measures is defined almost everywhere for each $x \in [0,1]$. This metric is called the *bounded Lipschitz distance* on X.

In Arkeryd, Cercignani and Illner [1] an existence result for the problem (1.4.5-6) is proved and part of this presentation is restated. This theorem is actually proved for a truncation which is weaker than that assumed here; however the similarities in the truncations allow the result to be applied to the problem (1.4.5-6).

Theorem 1.4.2 For any $\delta > 0$, the problem (1.4.5-6) has a measure solution. That is, there exists a measure $\mu_\bullet \in X$ such that

$$\xi \frac{d}{dx} \mu_x = C^\delta(\mu_x, \mu_x),$$

and

$$\mu_0\big|_{\{\xi>0\}} = \mu_0^+,\ \mu_a\big|_{\{\xi<0\}} = \mu_a^-.$$

In chapter 3, uniqueness for this problem (1.4.5-6) is proved via a contraction inequality using the bounded Lipschitz distance.

2 Uniqueness of the steady problem

2.1 The operator T

In this chapter, uniqueness of a nonnegative solution to the boundary value problem (1.2.1-2) is proved by a contractive mapping argument. Existence of such a solution is unknown at present, although both existence and uniqueness are obtained in section 2.2 for a solution in X . The first step in finding a contractive mapping is to construct an operator $T: X \to X$.

Fixing $g \in X$, the partial Cauchy problem (1.2.1-2) can be written as the first order non-homogeneous ordinary initial value problem

$$\xi \frac{df}{dx} = C(g,g) \tag{2.1.1}$$

with the boundary conditions

$$\begin{aligned} &f(0,\vec{v}) = g(0,\vec{v}) \text{ if } \xi > 0, \text{ and} \\ &f(a,\vec{v}) = g(a,\vec{v}) \text{ if } \xi < 0. \end{aligned} \tag{2.1.2}$$

Lemma 2.1.1 The problem (2.1.1-2) has a unique solution $f \in X$.

Proof: With $g \in X$ fixed, problem (2.1.1-2) is a first order non-homogeneous linear equation in x. Using the boundary conditions, a solution can be constructed explicitly [2] by integrating from 0 to x for $\xi > 0$ and from x to a for $\xi < 0$. For simplicity, $f(x,\vec{v})$ is taken to be 0 at $\xi = 0$ which will not affect the question of uniqueness since the plane $\xi = 0$ in $\mathbf{R}^3$ is a set of Lebesgue measure zero. Then,

$$f(x,\vec{v}) = \left\{ \begin{array}{l} \int_0^x \frac{1}{\xi} C(g,g)dt + g(0,\vec{v}) \text{ if } \xi > 0, \\ \int_x^a \frac{1}{|\xi|} C(g,g)dt + g(a,\vec{v}) \text{ if } \xi < 0, \text{ and} \\ 0 \text{ if } \xi = 0 \end{array} \right\}.$$

It needs to be shown that $f(x,\vec{v}) \in X$. Continuity with respect to x follows by integration, and is shown at the end of the proof. To show $f(x,\vec{v}) \in L^1(\mathbf{R}^3)$ for fixed x,

$$\begin{aligned} \|f(x,\vec{v})\|_{L^1} &= \int_{\mathbf{R}^3} |f(x,\vec{v}| d\vec{v} \\ &= \int_{\xi>0} \left| \int_0^x \frac{1}{\xi} C(g,g)dt + g(0,\vec{v}) \right| d\vec{v} + \int_{\xi<0} \left| \int_x^a \frac{1}{|\xi|} C(g,g)dt + g(a,\vec{v}) \right| d\vec{v} \\ &\le \int_{\xi>0} \left(\int_0^x \left| \frac{1}{\xi} C(g,g) \right| dt + |g(0,\vec{v})| \right) d\vec{v} + \int_{\xi<0} \left(\int_x^a \left| \frac{1}{|\xi|} C(g,g) \right| dt + |g(a,\vec{v})| \right) d\vec{v} \\ &\le \int_{\xi>0} \int_0^a \left| \frac{1}{\xi} C(g,g) \right| dt d\vec{v} + \int_{\xi>0} |g(0,\vec{v})| d\vec{v} + \int_{\xi<0} \int_0^a \left| \frac{1}{|\xi|} C(g,g) \right| dt d\vec{v} + \int_{\xi<0} |g(a,\vec{v})| d\vec{v} \\ &\le \int_{\mathbf{R}^3} \int_0^a \left| \frac{1}{|\xi|} C(g,g) \right| dt d\vec{v} + \int_{\mathbf{R}^3} |g(0,\vec{v})| d\vec{v} + \int_{\mathbf{R}^3} |g(a,\vec{v})| d\vec{v} \end{aligned}$$

interchanging the order of integration, and using the fact that $g \in X$ implies the integral exists for all $\xi \in \mathbf{R}$. Using the definition of the norm on X, the above is bounded by

$$\int_0^a \int_{\mathbf{R}^3} \left| \frac{1}{|\xi|} C(g,g) \right| d\vec{v} dt + 2\|g(x,\cdot)\| \tag{2.1.3}$$

$$\le \int_0^a \int_{\mathbf{R}^3} \int_{S^2} \int_{\mathbf{R}^3} \left| \frac{1}{|\xi|} B(\vec{n},\vec{v},\vec{w}) \left(g(t,\vec{v}')g(t,\vec{w}') - g(t,\vec{v})g(t,\vec{w}) \right) \right| d\vec{w} d\vec{n} d\vec{v} dt + 2\|g(x,\cdot)\|$$

$$\le \frac{b}{\delta} \int_0^a \int_{\mathbf{R}^3} \int_{S^2} \int_{\mathbf{R}^3} \left| \left(g(t,\vec{v}')g(t,\vec{w}') - g(t,\vec{v})g(t,\vec{w}) \right) \right| d\vec{w} d\vec{n} d\vec{v} dt + 2\|g(x,\cdot)\|$$

from the bounds on the collision kernel. By the triangle inequality, the above is

$$\le \frac{b}{\delta} \int_0^a \int_{\mathbf{R}^3} \int_{S^2} \int_{\mathbf{R}^3} \left(|g(t,\vec{v}')g(t,\vec{w}')| + |g(t,\vec{v})g(t,\vec{w})| \right) d\vec{w} d\vec{n} d\vec{v} dt + 2\|g(x,\cdot)\|$$

$$\leq \frac{2b}{\delta}\int_0^a \int_{\mathbf{R}^3}\int_{S^2}\int_{\mathbf{R}^3} |g(t,\vec{v})g(t,\vec{w})| d\vec{w} d\vec{n} d\vec{v} dt + 2\|g(x,\cdot)\|$$

by applying the collision transformation. Now, integrating the last equation yields a bound of

$$\frac{8\pi b}{\delta}\int_0^a \int_{\mathbf{R}^3} |g(t,\vec{v})| d\vec{v} \int_{\mathbf{R}^3} |g(t,\vec{w})| d\vec{w} dt + 2\|g(x,\cdot)\|$$

$$\leq \frac{8\pi b}{\delta}\int_0^a \|g(t,\cdot)\|^2 dt + 2\|g(x,\cdot)\|$$

$$= \frac{8\pi b}{\delta}\|g(x,\cdot)\|^2 \int_0^a dt + 2\|g(x,\cdot)\|$$

$$= \frac{8\pi b}{\delta}\|g(x,\cdot)\|^2 a + 2\|g(x,\cdot)\| < \infty$$

since $g \in X$ and $\|g(t,\cdot)\|$ is a real constant independent of t.

It remains to be shown that f maps $x \in [0,a]$ continuously into L^1 functions. This can be done in a similar manner of estimates to the series above. Without loss of generality choose $x, y \in \mathbf{R}$ such that $0 \leq y \leq x \leq a$. Then,

$$\|f(x,\vec{v}) - f(y,\vec{v})\|_{L^1} = \int_{\mathbf{R}^3} |f(x,\vec{v}) - f(y,\vec{v})| d\vec{v}$$

$$\leq \int_{\xi>0} \left| \int_y^x \frac{1}{\xi} C(g,g) dt \right| d\vec{v} + \int_{\xi<0} \left| -\int_y^x \frac{1}{|\xi|} C(g,g) dt \right| d\vec{v}$$

$$\leq \int_y^x \int_{\mathbf{R}^3} \left| \frac{1}{|\xi|} C(g,g) \right| d\vec{v} dt. \qquad (2.1.4)$$

Equation (2.1.4) has the same interior integral as equation (2.1.3). By using the same approximations following equation (2.1.3), it follows that (2.1.4) is bounded by

$$\frac{8\pi b}{\delta}\int_y^x \|g(t,\cdot)\|^2 dt = \frac{8\pi b}{\delta}\|g(t,\cdot)\|^2 |x-y|.$$

Since the norm of g is fixed, $|x-y|$ can be chosen as small as is necessary to force $\|f(x,\cdot) - f(y,\cdot)\|_{L^1}$ to small. In this sense f has the appropriate continuity properties, thus completing the proof of the lemma.

Define a mapping T from $X \to X$ by $Tg = f$ using Lemma 2.1.1 for the expression for f. Then T has the following property, stated as a theorem.

Theorem 2.1.2: Let $g_0(\vec{v}), g_a(\vec{v}) \in L^1_+(\mathbf{R}^3)$. Then for $g, \tilde{g} \in X$ satisfying

$$g(0,\vec{v}) = \tilde{g}(0,\vec{v}) = g_0(\vec{v}) \text{ if } \xi > 0, \text{ and}$$
$$g(a,\vec{v}) = \tilde{g}(a,\vec{v}) = g_a(\vec{v}) \text{ if } \xi < 0$$

the problem (2.1.1-2) satisfies the following inequality:

$$\|Tg(x,\cdot) - T\tilde{g}(x,\cdot)\| \le \frac{8\pi ba}{\delta}\left(\|g(x,\cdot)\| + \|\tilde{g}(x,\cdot)\|\right)\|g(x,\cdot) - \tilde{g}(x,\cdot)\|.$$

Proof: $Tg(x,\cdot) - T\tilde{g}(x,\cdot) \in X$ since X is closed under scalar multiplication and addition. By the linearity of the integral, this difference is given by

$$Tg(x,\cdot) - T\tilde{g}(x,\cdot) = \left\{ \begin{array}{l} \int_0^x \frac{1}{\xi}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] d\tau \text{ if } \xi > 0, \\ \int_x^a \frac{1}{|\xi|}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] d\tau \text{ if } \xi < 0, \text{ and} \\ 0 \text{ if } \xi = 0 \end{array} \right\}.$$

Then for any $x \in [0,a]$ it follows that

$$\int_{\mathbf{R}^3} \left|Tg(x,\vec{v}) - T\tilde{g}(x,\vec{v})\right| d\vec{v} =$$
$$= \int_{\xi>0} \left| \int_0^x \frac{1}{\xi}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] d\tau \right| d\vec{v} + \int_{\xi<0} \left| \int_x^a \frac{1}{|\xi|}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] d\tau \right| d\vec{v}$$
$$\le \int_{\xi>0} \int_0^x \left| \frac{1}{\xi}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] \right| d\tau d\vec{v} + \int_{\xi<0} \int_x^a \left| \frac{1}{|\xi|}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] \right| d\tau d\vec{v}$$
$$= \int_0^x \int_{\xi>0} \left| \frac{1}{\xi}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] \right| d\vec{v} d\tau + \int_x^a \int_{\xi<0} \left| \frac{1}{|\xi|}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] \right| d\vec{v} d\tau$$

via Fubini's theorem to change the order of integration. Since $0 \le x \le a$, the above equation is bounded by

$$\int_0^a \int_{\xi>0} \left| \frac{1}{\xi}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] \right| d\vec{v}\,d\tau + \int_0^a \int_{\xi<0} \left| \frac{1}{|\xi|}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] \right| d\vec{v}\,d\tau$$

$$= \int_0^a \int_{\mathbf{R}^3} \left| \frac{1}{|\xi|}\left[C(g,g) - C(\tilde{g},\tilde{g})\right] \right| d\vec{v}\,d\tau.$$

Using the property of the collision kernel that $C = 0$ if $|\xi| \le \delta$ the above is bounded by

$$\frac{1}{\delta}\int_0^a \int_{\mathbf{R}^3} \left|C(g,g) - C(\tilde{g},\tilde{g})\right| d\vec{v}\,d\tau.$$

Substituting in the full collision term, rewritten as a difference as

$$C(g,g) - C(\tilde{g},\tilde{g}) = \int_{\mathbf{R}^3}\int_{S^2} B(\vec{n},\vec{v},\vec{w})\left(g' g_*' - g g_* - \tilde{g}'\tilde{g}_*' + \tilde{g}\tilde{g}_*\right) d\vec{n}\,d\vec{w},$$

and rewriting $g'g_*' - \tilde{g}'\tilde{g}_*' = (g' - \tilde{g}')g_*' + \tilde{g}'(g_*' - \tilde{g}_*')$ and $\tilde{g}\tilde{g}_* - gg_* = (\tilde{g} - g)\tilde{g}_* + g(\tilde{g}_* - g_*)$,

it is obtained that

$$\frac{1}{\delta}\int_0^a \int_{\mathbf{R}^3} \left|C(g,g) - C(\tilde{g},\tilde{g})\right| d\vec{v}\,d\tau =$$

$$= \frac{1}{\delta}\int_0^a \int_{\mathbf{R}^3_{\vec{v}}}\int_{\mathbf{R}^3_{\vec{w}}}\int_{S^2} \left|B(\vec{n},\vec{v},\vec{w})\left((g' - \tilde{g}')g_*' + (g_*' - \tilde{g}_*')\tilde{g}' + (\tilde{g} - g)\tilde{g}_* + (\tilde{g}_* - g_*)g\right)\right| d\vec{n}\,d\vec{w}\,d\vec{v}\,d\tau$$

$$\le \frac{b}{\delta}\int_0^a \int_{\mathbf{R}^3_{\vec{v}}}\int_{\mathbf{R}^3_{\vec{w}}}\int_{S^2} \left(\left|(g' - \tilde{g}')g_*'\right| + \left|(g_*' - \tilde{g}_*')\tilde{g}'\right| + \left|(\tilde{g} - g)\tilde{g}_*\right| + \left|(\tilde{g}_* - g_*)g\right|\right) d\vec{n}\,d\vec{w}\,d\vec{v}\,d\tau$$

using the triangle inequality and the bounds on $B(\vec{n},\vec{v},\vec{w})$ from section 1.2. Applying the collision transformation J, the above becomes

$$= \frac{b}{\delta}\int_0^a |\det J| \int_{\mathbf{R}^3_{\vec{v}}}\int_{\mathbf{R}^3_{\vec{w}}}\int_{S^2} \left(\left|(g' - \tilde{g}')g_*'\right| + \left|(g_*' - \tilde{g}_*')\tilde{g}'\right|\right) \circ J(\vec{v},\vec{n},\vec{w})\, d\vec{n}\,d\vec{w}\,d\vec{v}\,d\tau$$

$$+ \frac{b}{\delta}\int_0^a \int_{\mathbf{R}^3_{\vec{v}}}\int_{\mathbf{R}^3_{\vec{w}}}\int_{S^2} \left(\left|(\tilde{g} - g)\tilde{g}_*\right| + \left|(\tilde{g}_* - g_*)g\right|\right) d\vec{n}\,d\vec{w}\,d\vec{v}\,d\tau$$

$$= \frac{b}{\delta}\int_0^a \int_{\mathbf{R}^3_{\vec{v}}}\int_{\mathbf{R}^3_{\vec{w}}}\int_{S^2} \left(\left|(g - \tilde{g})g_*\right| + \left|(g_* - \tilde{g}_*)\tilde{g}\right|\right) d\vec{n}\,d\vec{w}\,d\vec{v}\,d\tau$$

$$+ \frac{b}{\delta}\int_0^a \int_{\mathbf{R}^3_{\vec{v}}}\int_{\mathbf{R}^3_{\vec{w}}}\int_{S^2} \left(\left|(\tilde{g} - g)\tilde{g}_*\right| + \left|(\tilde{g}_* - g_*)g\right|\right) d\vec{n}\,d\vec{w}\,d\vec{v}\,d\tau$$

$$= \frac{b}{\delta}\int_0^a \int_{\mathbf{R}^3_{\vec{v}}}\int_{\mathbf{R}^3_{\vec{w}}}\int_{S^2} \left(\left|(g - \tilde{g})\right|\left(|g_*| + |\tilde{g}_*|\right) + \left|(g_* - \tilde{g}_*)\right|\left(|g| + |\tilde{g}|\right)\right) d\vec{n}\,d\vec{w}\,d\vec{v}\,d\tau.$$

Until now, the arguments have been suppressed to simplify the notation. To estimate the integral, they are written in yielding the above as

$$= \frac{b}{\delta} \int_0^a \int_{\mathbf{R}^3_{\vec{v}}} \int_{\mathbf{R}^3_{\vec{w}}} \int_{S^2} \Big(\left| g(\tau,\vec{v}) - \tilde{g}(\tau,\vec{v}) \right| \left(\left| g(\tau,\vec{w}) \right| + \left| \tilde{g}(\tau,\vec{w}) \right| \right) +$$

$$+ \left| g(\tau,\vec{w}) - \tilde{g}(\tau,\vec{w}) \right| \left(\left| g(\tau,\vec{v}) \right| + \left| \tilde{g}(\tau,\vec{v}) \right| \right) \Big) d\vec{n}\, d\vec{v}\, d\vec{w}\, d\tau$$

$$= \frac{4\pi b}{\delta} \int_0^a \int_{\mathbf{R}^3_{\vec{v}}} \left| g(\tau,\vec{v}) - \tilde{g}(\tau,\vec{v}) \right| \int_{\mathbf{R}^3_{\vec{w}}} \left(\left| g(\tau,\vec{w}) \right| + \left| \tilde{g}(\tau,\vec{w}) \right| \right) d\vec{w}\, d\vec{v}\, d\tau$$

$$+ \frac{4\pi b}{\delta} \int_0^a \int_{\mathbf{R}^3_{\vec{v}}} \left(\left| g(\tau,\vec{v}) \right| + \left| \tilde{g}(\tau,\vec{v}) \right| \right) \int_{\mathbf{R}^3_{\vec{w}}} \left| g(\tau,\vec{w}) - \tilde{g}(\tau,\vec{w}) \right| d\vec{w}\, d\vec{v}\, d\tau .$$

Using the norm on X, $\left\| f(x,\vec{v}) \right\| = \sup_{x\in[0,a]} \int_{\mathbf{R}^3} \left| f(x,\vec{v}) \right| d\vec{v}$, the above can be estimated by

$$\frac{4\pi b}{\delta} \int_0^a \left\| g(x,\cdot) - \tilde{g}(x,\cdot) \right\| \left(\left\| g(x,\cdot) \right\| + \left\| \tilde{g}(x,\cdot) \right\| \right) d\tau$$

$$+ \frac{4\pi b}{\delta} \int_0^a \left(\left\| g(x,\cdot) \right\| + \left\| \tilde{g}(x,\cdot) \right\| \right) \left\| g(x,\cdot) - \tilde{g}(x,\cdot) \right\| d\tau$$

$$= \frac{8\pi b}{\delta} \left\| g(x,\cdot) - \tilde{g}(x,\cdot) \right\| \left(\left\| g(x,\cdot) \right\| + \left\| \tilde{g}(x,\cdot) \right\| \right) \cdot a$$

Using $\left\| Tg(x,\cdot) - T\tilde{g}(x,\cdot) \right\| = \sup_{x\in[0,a]} \int_{\mathbf{R}^3} \left| Tg(x,\vec{v}) - T\tilde{g}(x,\vec{v}) \right| d\vec{v}$, and the above inequalities, the theorem is proved.

2.2 Uniqueness of a solution

Using the details proved in section 2.1, the main result of this chapter can now be stated.

Theorem 2.2.1: Assume that problem (1.2.1-2) has a solution in X_+. Then for small enough a, that solution is unique.

Proof: Suppose $\left(f,\tilde{f}\right)\in X_{+}\subset \mathrm{X}$, with $f,\tilde{f}$ satisfying the boundary conditions (1.2.2). Choose $0<R\in \mathbf{R}$ large enough such that $f_0(\vec{v}), f_a(\vec{v}), f$, and $\tilde{f}$ are all in $B_R(0)\subset X$. Then, letting $0<a<\dfrac{16\pi bR}{\delta}$, application of theorem 2.1.2 yields

$$\begin{aligned}\left\|Tf(x,\cdot)-T\tilde{f}(x,\cdot)\right\| &\le \frac{8\pi ba}{\delta}\left(\left\|f(x,\cdot)\right\|+\left\|\tilde{f}(x,\cdot)\right\|\right)\left\|f(x,\cdot)-\tilde{f}(x,\cdot)\right\| \\ &\le \frac{8\pi ba}{\delta}(R+R)\left\|f(x,\cdot)-\tilde{f}(x,\cdot)\right\| \\ &= \frac{16\pi bRa}{\delta}\left\|f(x,\cdot)-\tilde{f}(x,\cdot)\right\| \\ &= k\left\|f(x,\cdot)-\tilde{f}(x,\cdot)\right\|\end{aligned}$$

where $0\le k<1$ by substituting the value for a. Assuming $f,\tilde{f}$ satisfy problem (1.2.1-2), $Tf=f$, and $T\tilde{f}=\tilde{f}$. This implies $\left\|f-\tilde{f}\right\|\le k\left\|f-\tilde{f}\right\|$, hence it must be true that $\left\|f-\tilde{f}\right\|=0$, which is equivalent to $f(x,\vec{v})=\tilde{f}(x,\vec{v})$, almost everywhere in $\mathbf{R}^3$ for all $x\in[0,a]$. This gives uniqueness in $B_R(0)\subseteq X$. Therefore, the problem (1.2.1-2) has a unique solution in $B_R(0)\cap X_{+}$. This completes the proof.

In addition to uniqueness of a non-negative solution, existence can be obtained in the larger class of functions if negative solutions are allowed.

Theorem 2.2.2: For small enough a, there exists a unique solution in X to the problem (1.2.1-2).

Proof: The operator T maps the Banach space X continuously into itself, and is a contraction mapping for small enough a. Thus by the Banach fixed point theorem [7], T has a unique fixed point in X. Since any fixed point of T satisfies the problem (1.2.1-2), there is a unique solution.

The above theorem does not give a unique non-negative solution when coupled with theorem 2.2.1 since non-negative solutions were assumed in the hypothesis of the latter result. However, it at least allows the possibility that a non-negative solution may still exist, and if it does then it is known to be unique by theorem 2.2.1.

3 Uniqueness of the measure problem

In chapter 3, the problem (1.4.5-6) is shown to have a unique solution. This is done by a contraction using the bounded Lipschitz distance introduced in section 1.4. The main theorem found in section 3.3 of this chapter is straightforward; however, it involves several propositions and lemmas. These will be proved here before stating the final uniqueness result in 3.3.

3.1 Properties of the collision kernel and truncation

It is assumed throughout this chapter that the collision kernel $B(\vec{n},\vec{v},\vec{w})$ is bounded, non-negative and Lipschitz continuous with respect to $\vec{v}$ and $\vec{w}$. This is formulated by the existence of a positive constant $b>0$ such that:

(i) $0 \le \dfrac{B(\vec{n},\vec{v},\vec{w})}{b} \le 1$ for all $\vec{n} \in S^2$, $\vec{v},\vec{w} \in \mathbf{R}^3$,

(ii) $\left\| \dfrac{B(\vec{n},\vec{v},\vec{w})}{b} - \dfrac{B(\vec{n},\vec{y},\vec{w})}{b} \right\| \le \|\vec{v}-\vec{y}\|$ for all $\vec{v},\vec{y} \in \mathbf{R}^3$, and

(iii) $\left\| \dfrac{B(\vec{n},\vec{v},\vec{y})}{b} - \dfrac{B(\vec{n},\vec{v},\vec{w})}{b} \right\| \le \|\vec{y}-\vec{w}\|$ for all $\vec{y},\vec{w} \in \mathbf{R}^3$.

The first proposition concerns an important property of the truncation to the collision kernel introduced in section 1.4.

Proposition 3.1.1 Let $\vec{v},\vec{w},\vec{y}\in\mathbf{R}^3$. Then the following two inequalities hold for the function g^{δ}:

(1) $\left\|g^{\delta}(\vec{n},\vec{v},\vec{w})-g^{\delta}(\vec{n},\vec{y},\vec{w})\right\|\le\frac{2}{\delta}\|\vec{v}-\vec{y}\|$, and

(2) $\left\|g^{\delta}(\vec{n},\vec{v},\vec{w})-g^{\delta}(\vec{n},\vec{v},\vec{y})\right\|\le\frac{2}{\delta}\|\vec{w}-\vec{y}\|$.

Proof: For (1), let $z_1=\min\left\{|v_x|,|w_x|,\left|v_x-n_x\langle\vec{n},\vec{v}-\vec{w}\rangle\right|,\left|w_x+n_x\langle\vec{n},\vec{v}-\vec{w}\rangle\right|\right\}$ and $z_2=\min\left\{|y_x|,|w_x|,\left|y_x-n_x\langle\vec{n},\vec{y}-\vec{w}\rangle\right|,\left|w_x+n_x\langle\vec{n},\vec{y}-\vec{w}\rangle\right|\right\}$. Here, the subscript denotes the x-component of the appropriate vector. First, a general property of z_1 and z_2 is shown.

Claim: $0\le(z_2-z_1)\le2\|\vec{v}-\vec{y}\|$ for $0\le z_1\le z_2\le2\delta$. The inequality is true for the trivial case $z_2=z_1$. For $z_1<z_2$ there are only one of three possibilities.

(i) $z_1=|v_x|$. Then

$0<(z_2-z_1)=z_2-|v_x|\le|y_x|-|v_x|$ by the definition of z_2,

$\le|y_x-v_x|$ by the triangle inequality, and

$\le\|\vec{y}-\vec{v}\|\le2\|\vec{v}-\vec{y}\|$ by the definition of the norm.

(ii) $z_1=\left|v_x-n_x\langle\vec{n},\vec{v}-\vec{w}\rangle\right|$. Then

$$
\begin{aligned}
0<(z_2-z_1)&\le\left|y_x-n_x\langle\vec{n},\vec{y}-\vec{w}\rangle\right|-\left|v_x-n_x\langle\vec{n},\vec{v}-\vec{w}\rangle\right|\\
&\le\left|y_x-n_x\langle\vec{n},\vec{y}-\vec{w}\rangle-(v_x-n_x\langle\vec{n},\vec{v}-\vec{w}\rangle)\right|\\
&\le|y_x-v_x|+\left|n_x\langle\vec{n},\vec{v}-\vec{y}\rangle\right|\\
&\le\|\vec{y}-\vec{v}\|+|n_x|\cdot\|\vec{n}\|\cdot\|\vec{v}-\vec{y}\|\\
&\le2\|\vec{v}-\vec{y}\|\text{ since }\vec{n}\in S^2\text{ implies }\vec{n}\text{ has norm 1.}
\end{aligned}
$$

(iii) $z_1=\left|w_x+n_x\langle\vec{n},\vec{v}-\vec{w}\rangle\right|$. Then

$$
\begin{aligned}
0<(z_2-z_1)&\le\left|w_x+n_x\langle\vec{n},\vec{y}-\vec{w}\rangle\right|-\left|w_x+n_x\langle\vec{n},\vec{v}-\vec{w}\rangle\right|\\
&\le\left|w_x+n_x\langle\vec{n},\vec{y}-\vec{w}\rangle-(w_x+n_x\langle\vec{n},\vec{v}-\vec{w}\rangle)\right|\\
&\le\left|n_x\langle\vec{n},\vec{y}-\vec{v}\rangle\right|\le2\|\vec{v}-\vec{y}\|.
\end{aligned}
$$

The case $z_1=|w_x|$ has already been considered under the case $z_2=z_1$ because $|w_x|=z_1\le z_2\le|w_x|$. This proves the claim.

Now consider the following cases:

(i) $0 \le z_1 \le \delta$, and $0 \le z_2 \le \delta$. Then

$$\left\|g^{\delta}(\vec{n},\vec{v},\vec{w}) - g^{\delta}(\vec{n},\vec{y},\vec{w})\right\| = 0 \le \frac{2}{\delta}\|\vec{v} - \vec{y}\|.$$

(ii) $0 \le z_1 \le \delta$, and $\delta < z_2 \le 2\delta$. In this case,

$$\begin{aligned}\left\|g^{\delta}(\vec{n},\vec{v},\vec{w}) - g^{\delta}(\vec{n},\vec{y},\vec{w})\right\| &= \left|0 - \frac{1}{\delta}(z_2 - \delta)\right| = \frac{1}{\delta}(z_2 - \delta)\\ &\le \frac{1}{\delta}(z_2 - z_1) \le \frac{2}{\delta}\|\vec{v} - \vec{y}\| \text{ by the claim above.}\end{aligned}$$

(iii) $0 \le z_1 \le \delta$, and $z_2 > 2\delta$. Here,

$$\begin{aligned}\left\|g^{\delta}(\vec{n},\vec{v},\vec{w}) - g^{\delta}(\vec{n},\vec{y},\vec{w})\right\| &= 1 - 0 = \frac{1}{\delta}(2\delta - \delta)\\ &\le \frac{1}{\delta}(z_2 - z_1) \le \frac{2}{\delta}\|\vec{v} - \vec{y}\|.\end{aligned}$$

(iv) $\delta < z_1 \le z_2 \le 2\delta$. Then

$$\begin{aligned}\left\|g^{\delta}(\vec{n},\vec{v},\vec{w}) - g^{\delta}(\vec{n},\vec{y},\vec{w})\right\| &= \left|\frac{1}{\delta}(z_1 - \delta) - \frac{1}{\delta}(z_2 - \delta)\right|\\ &= \frac{1}{\delta}(z_2 - z_1) \le \frac{2}{\delta}\|\vec{v} - \vec{y}\| \text{ by the claim.}\end{aligned}$$

(v) $\delta < z_1 \le 2\delta$, and $z_2 > 2\delta$. Similarly,

$$\begin{aligned}\left\|g^{\delta}(\vec{n},\vec{v},\vec{w}) - g^{\delta}(\vec{n},\vec{y},\vec{w})\right\| &= \left|\frac{1}{\delta}(z_1 - \delta) - 1\right|\\ &= \frac{1}{\delta}(2\delta - z_1) \le \frac{1}{\delta}(z_2 - z_1) \le \frac{2}{\delta}\|\vec{v} - \vec{y}\|.\end{aligned}$$

(vi) $z_1 > 2\delta$, and $z_2 > 2\delta$. Here,

$$\left\|g^{\delta}(\vec{n},\vec{v},\vec{w}) - g^{\delta}(\vec{n},\vec{y},\vec{w})\right\| = |1 - 1| = 0 \le \frac{2}{\delta}\|\vec{v} - \vec{y}\|.$$

Without loss of generality, cases (i)-(vi) are exhaustive by the relabeling of z_1 and z_2. This proves inequality (1). For the second inequality, notice that $g^{\delta}(\vec{n},\vec{v},\vec{w}) = g^{\delta}(\vec{n},\vec{w},\vec{v})$ for all $\vec{v},\vec{w} \in \mathbf{R}^3$ and apply inequality (1). This proves the proposition.

Proposition 3.1.2 Let $\varphi \in D$. For fixed $\vec{n} \in S^2$ and $\vec{w} \in \mathbf{R}^3$ the function in $\vec{v}$, $\alpha \frac{\varphi(\vec{v})}{\xi} \chi_{\{\xi>0\}}(\vec{v}) B^\delta(\vec{n},\vec{v},\vec{w})$, is in D for $0 < \alpha = \frac{\delta^2}{b(3+2\delta)} \in \mathbf{R}$.

The constant α is fixed at a predetermined value chosen to simplify expressions which will be derived in section 3.2.

Proof: Measurability is clear by multiplication of measurable functions. Also, $0 \le \alpha \frac{\varphi(\vec{v})}{\xi} \chi_{\{\xi>0\}}(\vec{v}) B^\delta(\vec{n},\vec{v},\vec{w}) \le \alpha \frac{b}{\delta} = \frac{\delta}{(3+2\delta)} \le 1$. For the Lipschitz condition, let $\vec{v},\vec{y} \in \mathbf{R}^3$ with $x-$components $\xi_{\vec{v}}, \xi_{\vec{y}}$, and consider the following three cases.

(i) $\xi_{\vec{v}} \le \delta,\ \xi_{\vec{y}} \le \delta$. For this instance,

$$\left| \alpha \frac{\varphi(\vec{y})}{\xi_{\vec{y}}} \chi_{\{\xi_{\vec{y}}>0\}}(\vec{y}) B^\delta(\vec{n},\vec{y},\vec{w}) - \alpha \frac{\varphi(\vec{v})}{\xi_{\vec{v}}} \chi_{\{\xi_{\vec{v}}>0\}}(\vec{v}) B^\delta(\vec{n},\vec{v},\vec{w}) \right| = 0$$

since $B^\delta = 0$, $\chi_{\{\xi_{\vec{y}}>0\}}(\vec{y}) = 0$ if $\xi_{\vec{y}} \le 0$, and $\chi_{\{\xi_{\vec{v}}>0\}}(\vec{v}) = 0$ if $\xi_{\vec{v}} \le 0$.

(ii) $\xi_{\vec{v}} > \delta,\ \xi_{\vec{y}} > \delta$. Then

$$\left| \alpha \frac{\varphi(\vec{y})}{\xi_{\vec{y}}} \chi_{\{\xi_{\vec{y}}>0\}}(\vec{y}) B^\delta(\vec{n},\vec{y},\vec{w}) - \alpha \frac{\varphi(\vec{v})}{\xi_{\vec{v}}} \chi_{\{\xi_{\vec{v}}>0\}}(\vec{v}) B^\delta(\vec{n},\vec{v},\vec{w}) \right| =$$

$$= \alpha \left| \frac{\varphi(\vec{y})}{\xi_{\vec{y}}} B^\delta(\vec{n},\vec{y},\vec{w}) - \frac{\varphi(\vec{v})}{\xi_{\vec{v}}} B^\delta(\vec{n},\vec{v},\vec{w}) \right|$$

$$\le \alpha \left\{ \left| \varphi(\vec{y}) \left(\frac{B^\delta(\vec{n},\vec{y},\vec{w})}{\xi_{\vec{y}}} - \frac{B^\delta(\vec{n},\vec{v},\vec{w})}{\xi_{\vec{v}}} \right) \right| + \left| (\varphi(\vec{y}) - \varphi(\vec{v})) \frac{B^\delta(\vec{n},\vec{v},\vec{w})}{\xi_{\vec{v}}} \right| \right\}$$

$$\le \alpha \left\{ \left| B^\delta(\vec{n},\vec{y},\vec{w}) \left(\frac{1}{\xi_{\vec{y}}} - \frac{1}{\xi_{\vec{v}}} \right) \right| + \left| \frac{1}{\xi_{\vec{v}}} \left(B^\delta(\vec{n},\vec{y},\vec{w}) - B^\delta(\vec{n},\vec{v},\vec{w}) \right) \right| + \frac{b}{\delta} \|\vec{v} - \vec{y}\| \right\}$$

where the triangle inequality, the bounds on $B^\delta, \xi_{\vec{v}}$, and the properties of $\varphi \in D$ have been used. (Specifically, the Lipschitz continuity with constant 1, and the bound above by 1).

Applying the triangle inequality to the collision kernel yields the above

$$\le \alpha \left\{ b \frac{\left| \xi_{\vec{v}} - \xi_{\vec{y}} \right|}{\xi_{\vec{v}} \xi_{\vec{y}}} + \frac{1}{\delta} \left| B(\vec{n},\vec{y},\vec{w}) \left(g^\delta(\vec{n},\vec{y},\vec{w}) - g^\delta(\vec{n},\vec{v},\vec{w}) \right) \right| + \right.$$

$$+\frac{1}{\delta}\left|g^{\delta}(\vec{n},\vec{v},\vec{w})\big(B(\vec{n},\vec{y},\vec{w})-B(\vec{n},\vec{v},\vec{w})\big)\right|+\frac{b}{\delta}\|\vec{v}-\vec{y}\|\Big\}$$

$$\leq \alpha\left\{\frac{b}{\delta^2}\|\vec{v}-\vec{y}\|+\frac{2b}{\delta^2}\|\vec{v}-\vec{y}\|+\frac{b}{\delta}\|\vec{v}-\vec{y}\|+\frac{b}{\delta}\|\vec{v}-\vec{y}\|\right\}$$

using proposition 3.1.1 and the assumptions on B. Simplifying, the last equation is

$$=\alpha\left(\frac{3b}{\delta^2}+\frac{2b}{\delta}\right)\|v-y\|$$

$$=\alpha b\frac{(3+2\delta)}{\delta^2}\|\vec{v}-\vec{y}\|=\|\vec{v}-\vec{y}\|$$

(iii) $\xi_{\vec{v}}>\delta,\ \xi_{\vec{y}}\leq\delta$. Then the difference is given by

$$\left|\alpha\frac{\varphi(\vec{y})}{\xi_{\vec{y}}}\chi_{\{\xi_{\vec{y}}>0\}}(\vec{y})B^{\delta}(\vec{n},\vec{y},\vec{w})-\alpha\frac{\varphi(\vec{v})}{\xi_{\vec{v}}}\chi_{\{\xi_{\vec{v}}>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})\right|$$

$$=\alpha\left|\frac{\varphi(\vec{v})}{\xi_{\vec{v}}}B^{\delta}(\vec{n},\vec{v},\vec{w})\right|$$

$$\leq\alpha\frac{b}{\delta}\left|g^{\delta}(\vec{n},\vec{v},\vec{w})-g^{\delta}(\vec{n},\vec{y},\vec{w})\right|$$

since $g^{\delta}(\vec{n},\vec{y},\vec{w})=0$. By proposition 3.1.1, the above is

$$\leq\alpha\frac{2b}{\delta^2}\|\vec{v}-\vec{y}\|$$

$$=\frac{2}{3+2\delta}\|\vec{v}-\vec{y}\|<\|\vec{v}-\vec{y}\|.$$

Without loss of generality through relabeling, these cases are exhaustive and the Lipschitz condition holds with constant 1. This proves the proposition.

The final result of this section is similar to proposition 3.1.2, and also involves a constant $\beta\in\mathbf{R}$ which is fixed at a predetermined value for later use in section 3.2.

Proposition 3.1.3 Let $\varphi\in D$. For fixed $\vec{n}\in S^2$ and $\vec{v}\in\mathbf{R}^3$ the function in $\vec{w}$, $\beta\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})$ is in D for $0<\beta=\frac{\delta^2}{b(2+\delta)}\in\mathbf{R}$.

Proof: The proof is similar to that of proposition 3.1.2. Measurability and integrability are immediate by multiplication of measurable functions, and by using the properties of $\varphi \in D$. As well, $0 \le \beta \frac{\varphi(\vec{v})}{\xi} \chi_{\{\xi>0\}}(\vec{v}) B^{\delta}(\vec{n},\vec{v},\vec{w}) \le \beta \frac{b}{\delta} \le 1$. For the Lipschitz condition in $\vec{w}$, let $\vec{y} \in \mathbf{R}^3$. Then

$$\begin{aligned}
&\left|\beta \frac{\varphi(\vec{v})}{\xi} \chi_{\{\xi>0\}}(\vec{v})\left(B^{\delta}(\vec{n},\vec{v},\vec{w}) - B^{\delta}(\vec{n},\vec{v},\vec{y})\right)\right| \le \\
&\quad \le \beta \frac{1}{\delta}\left|B^{\delta}(\vec{n},\vec{v},\vec{w}) - B^{\delta}(\vec{n},\vec{v},\vec{y})\right| \\
&\quad \le \beta \frac{1}{\delta}\left\{\left|B(\vec{n},\vec{v},\vec{w})\left(g^{\delta}(\vec{n},\vec{v},\vec{w}) - g^{\delta}(\vec{n},\vec{v},\vec{y})\right)\right| + \right. \\
&\qquad\qquad \left. + \left|g^{\delta}(\vec{n},\vec{v},\vec{y})\left(B(\vec{n},\vec{v},\vec{w}) - B(\vec{n},\vec{v},\vec{y})\right)\right|\right\} \\
&\quad \le \beta \frac{1}{\delta}\left\{b\frac{2}{\delta}\|\vec{w}-\vec{y}\| + b\|\vec{w}-\vec{y}\|\right\} \\
&\quad = \beta \frac{b}{\delta^2}(2+\delta)\|\vec{w}-\vec{y}\| \le \|\vec{w}-\vec{y}\|.
\end{aligned}$$

Thus $\beta \frac{\varphi(\vec{v})}{\xi} \chi_{\{\xi>0\}}(\vec{v}) B^{\delta}(\vec{n},\vec{v},\vec{w})$ is in D with respect to $\vec{w} \in \mathbf{R}^3$ and the proof is complete.

Although propositions (3.1.1-3) may seem somewhat technical and tedious, they are necessary in the following sections. Additionally, the functions examined in (3.1.2-3) will need to be considered after the co-ordinate transformation J defined in section 1.1.

To this end, it is useful to note the following inequality. For any vectors $\vec{n} \in S^2$ and $\vec{v},\vec{y},\vec{w} \in \mathbf{R}^3$,

$$\begin{aligned}
\|\vec{v}' - \vec{y}'\| &= \left\|\left(\vec{v} - \vec{n}\langle\vec{n},\vec{v}-\vec{w}\rangle\right) - \left(\vec{y} - \vec{n}\langle\vec{n},\vec{y}-\vec{w}\rangle\right)\right\| \\
&= \left\|\vec{v} - \vec{y} + \vec{n}\langle\vec{n},\vec{y}-\vec{v}\rangle\right\| \le \|\vec{v}-\vec{y}\| + \|\vec{y}-\vec{v}\| = 2\|\vec{v}-\vec{y}\|.
\end{aligned}$$

A similar expression occurs in w', y', and can be written as

$$\|\vec{w}' - \vec{y}'\| = \left\|\left(\vec{v} + \vec{n}\langle\vec{n},\vec{v}-\vec{w}\rangle\right) - \left(\vec{v} + \vec{n}\langle\vec{n},\vec{v}-\vec{y}\rangle\right)\right\| \le \|\vec{y}-\vec{w}\|.$$

These inequalities allow propositions 3.1.2 and 3.1.3 to be reformulated for the appropriate functions with the collision transformation. Note that in a similar result to proposition 3.1.2, a factor of $\frac{1}{2}$ must be introduced so that the function $\frac{\alpha}{2}\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(\vec{n},\vec{v},\vec{w})$ satisfies the appropriate Lipschitz condition in $\vec{v}$.

3.2 The contraction inequality

In a similar style to chapter 2, a mapping on the space of measure solutions is defined, and then a contraction lemma proved to show uniqueness in section 3.3.

Define a function T on X by, for $\mu_{\bullet} \in X$ and $x \in [0,a]$,

$$T\mu_x(\vec{v}) = \left\{ \begin{array}{l} \int_0^x \frac{1}{\xi} C^{\delta}(\mu_{\tau},\mu_{\tau})d\tau + \mu_0^+(\vec{v}) \text{ if } \xi > 0, \\ \int_x^a \frac{1}{|\xi|} C^{\delta}(\mu_{\tau},\mu_{\tau})d\tau + \mu_a^-(\vec{v}) \text{ if } \xi < 0, \\ 0 \text{ if } \xi = 0 \end{array} \right\}.$$

Note that a solution of problem (1.4.5-6) is a fixed point of T, and also that from the form of C^{δ}, $T\mu_x$ may in fact be a signed measure. The following Lemma obtains a contraction inequality for T.

Lemma 3.2.1 There exists a positive constant $0 < k \in \mathbf{R}$ such that for any two elements $\mu_{\bullet},\tilde{\mu}_{\bullet} \in B_R(0)$ satisfying the boundary conditions $\mu_0|_{\{\xi>0\}} = \mu_0^+ = \tilde{\mu}_0|_{\{\xi>0\}}$ and $\mu_a|_{\{\xi<0\}} = \mu_a^- = \tilde{\mu}_a|_{\{\xi<0\}}$, the inequality $\rho(T\mu_{\bullet},T\tilde{\mu}_{\bullet}) \le ka\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})$ holds.

Proof: Let $\mu_{\bullet},\tilde{\mu}_{\bullet} \in X$ be two such measures. Then the definition of ρ is

$$\rho(T\mu_\bullet, T\tilde{\mu}_\bullet) = \sup_{x\in[0,a]}\sup_{\varphi\in D}\left|\int_{\mathbf{R}^3}\varphi(\vec{v})d(T\mu_x)(\vec{v}) - \int_{\mathbf{R}^3}\varphi(\vec{v})d(T\tilde{\mu}_x)(\vec{v})\right|.$$

Fixing $\varphi \in D$ and $x \in [0,a]$, writing out the full expression above yields

$$= \left|\int_{\xi>0}\varphi(\vec{v})\int_0^x \frac{1}{\xi}d\left[C^\delta(\mu_\tau,\mu_\tau)\right](\vec{v})d\tau + \int_{\xi>0}\varphi(\vec{v})d\mu_0^+(\vec{v}) + \int_{\xi<0}\varphi(\vec{v})\int_x^a \frac{1}{|\xi|}d\left[C^\delta(\mu_\tau,\mu_\tau)\right](\vec{v})d\tau + \right.$$

$$+\int_{\xi<0}\varphi(\vec{v})d\mu_a^-(\vec{v}) - \int_{\xi>0}\varphi(\vec{v})\int_0^x \frac{1}{\xi}d\left[C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})d\tau - \int_{\xi>0}\varphi(\vec{v})d\tilde{\mu}_0^+(\vec{v}) -$$

$$\left. - \int_{\xi<0}\varphi(\vec{v})\int_x^a \frac{1}{|\xi|}d\left[C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})d\tau - \int_{\xi<0}\varphi(\vec{v})d\tilde{\mu}_a^-(\vec{v})\right|$$

$$= \left|\int_{\xi>0}\varphi(\vec{v})\int_0^x \frac{1}{\xi}d\left[C^\delta(\mu_\tau,\mu_\tau) - C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})d\tau + \int_{\xi<0}\varphi(\vec{v})\int_x^a \frac{1}{|\xi|}d\left[C^\delta(\mu_\tau,\mu_\tau) - C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})d\tau\right|$$

$$= \left|\int_0^x\int_{\mathbf{R}^3}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})d\left[C^\delta(\mu_\tau,\mu_\tau) - C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})d\tau + \right.$$

$$\left.\int_x^a\int_{\mathbf{R}^3}\frac{\delta(\vec{v})}{|\xi|}\chi_{\{\xi<0\}}(\vec{v})d\left[C^\delta(\mu_\tau,\mu_\tau) - C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})d\tau\right|$$

$$\leq \int_0^a\left|\int_{\mathbf{R}^3}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})d\left[C^\delta(\mu_\tau,\mu_\tau) - C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})\right|d\tau + \tag{3.2.1}$$

$$+\int_0^a\left|\int_{\mathbf{R}^3}\frac{\varphi(\vec{v})}{|\xi|}\chi_{\{\xi<0\}}(\vec{v})d\left[C^\delta(\mu_\tau,\mu_\tau) - C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})\right|d\tau. \tag{3.2.2}$$

The only difference between the two expressions (3.2.1) and (3.2.2) are the characteristic functions $\chi_{\{\xi>0\}}$and $\chi_{\{\xi<0\}}$. Because of this similarity, (3.2.2) will be estimated later, and (3.2.1) is focused on here. Then,

$$\left|\int_{\mathbf{R}^3}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})d\left[C^\delta(\mu_\tau,\mu_\tau) - C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})\right| =$$

$$= \left|\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{S^2}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^\delta(\vec{n},\vec{v},\vec{w})\left[(dM_\tau\circ J - dM_\tau) - (d\tilde{M}_\tau\circ J - d\tilde{M}_\tau)\right]\right.$$

$$\leq \left|\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{S^2}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^\delta(\vec{n},\vec{v},\vec{w})\left[dM_\tau\circ J - d\tilde{M}_\tau\circ J\right]\right| +$$

$$+\left|\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\int_{S^2}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})\left[d\tilde{M}_{\tau}-dM_{\tau}\right]\right|$$

where $dM_{\tau}=d\vec{n}d\mu_{\tau}(\vec{w})d\mu_{\tau}(\vec{v})$, J is the collision transformation, and $d\tilde{M}_{\tau}=d\vec{n}d\tilde{\mu}_{\tau}(\vec{w})d\tilde{\mu}_{\tau}(\vec{v})$. Taking the inverse transformation, and changing the order of integration yields

$$=\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(-n,v',w')[d\mu_{\tau}(\vec{w})d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{w})d\tilde{\mu}_{\tau}(\vec{v})]d\vec{n}\right|+$$

$$+\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})[d\mu_{\tau}(\vec{w})d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{w})d\tilde{\mu}_{\tau}(\vec{v})]d\vec{n}\right|. \qquad (3.2.3)$$

Now, using the fact that

$$d\mu_{\tau}(\vec{w})d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{w})d\tilde{\mu}_{\tau}(\vec{v})=d\mu_{\tau}(\vec{w})[d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{v})]+[d\mu_{\tau}(\vec{w})-d\tilde{\mu}_{\tau}(\vec{w})]d\tilde{\mu}_{\tau}(\vec{v})$$

equation (3.2.3) can be estimated by

$$\leq\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(-\vec{n},\vec{v}',\vec{w}')[d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{v})]d\mu_{\tau}(\vec{w})d\vec{n}\right|+$$

$$+\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(-\vec{n},\vec{v}',\vec{w}')[d\mu_{\tau}(\vec{w})-d\tilde{\mu}_{\tau}(\vec{w})]d\tilde{\mu}_{\tau}(\vec{v})d\vec{n}\right|+$$

$$+\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})[d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{v})]d\mu_{\tau}(\vec{w})d\vec{n}\right|+$$

$$+\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})[d\mu_{\tau}(\vec{w})-d\tilde{\mu}_{\tau}(\vec{w})]d\tilde{\mu}_{\tau}(\vec{v})d\vec{n}\right|.$$

Using propositions 3.1.2 and 3.1.3 it is possible to bound the four integral equations above. Recall the constants α,β defined in these propositions. Then, it follows that

$$=\frac{2}{\alpha}\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\frac{\alpha}{2}\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(-\vec{n},\vec{v}',\vec{w}')[d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{v})]d\mu_{\tau}(\vec{w})d\vec{n}\right|+$$

$$+\frac{1}{\beta}\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\beta\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(-\vec{n},\vec{v}',\vec{w}')[d\mu_{\tau}(\vec{w})-d\tilde{\mu}_{\tau}(\vec{w})]d\tilde{\mu}_{\tau}(\vec{v})d\vec{n}\right|+$$

$$+\frac{1}{\alpha}\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\alpha\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})[d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{v})]d\mu_{\tau}(\vec{w})d\vec{n}\right|+$$

$$+\frac{1}{\beta}\left|\int_{S^2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3}\beta\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})[d\mu_{\tau}(\vec{w})-d\tilde{\mu}_{\tau}(\vec{w})]d\tilde{\mu}_{\tau}(\vec{v})d\vec{n}\right|$$

$$\leq\frac{2}{\alpha}\int_{S^2}\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3}\frac{\alpha}{2}\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(-\vec{n},\vec{v}',\vec{w}')[d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{v})]\right|d\mu_{\tau}(\vec{w})d\vec{n}+$$

$$+\frac{1}{\beta}\int_{S^2}\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3}\beta\frac{\varphi(\vec{v}')}{\xi'}\chi_{\{\xi'>0\}}(\vec{v}')B^{\delta}(-\vec{n},\vec{v}',\vec{w}')[d\mu_{\tau}(\vec{w})-d\tilde{\mu}_{\tau}(\vec{w})]\right|d\tilde{\mu}_{\tau}(\vec{v})d\vec{n}+$$

$$+\frac{1}{\alpha}\int_{S^2}\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3}\alpha\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})[d\mu_{\tau}(\vec{v})-d\tilde{\mu}_{\tau}(\vec{v})]\right|d\mu_{\tau}(\vec{w})d\vec{n}+$$

$$+\frac{1}{\beta}\int_{S^2}\int_{\mathbf{R}^3}\left|\int_{\mathbf{R}^3}\beta\frac{\varphi(\vec{v})}{\xi}\chi_{\{\xi>0\}}(\vec{v})B^{\delta}(\vec{n},\vec{v},\vec{w})[d\mu_{\tau}(\vec{w})-d\tilde{\mu}_{\tau}(\vec{w})]\right|d\tilde{\mu}_{\tau}(\vec{v})d\vec{n}. \tag{3.2.4}$$

Each of the integrals in equation (3.2.4) contains an expression equivalent to $\left|\int_{\mathbf{R}^3}\gamma(\vec{z})d\mu_{\tau}(\vec{z})-\int_{\mathbf{R}^3}\gamma(\vec{z})d\tilde{\mu}_{\tau}(\vec{z})\right|$, where $\gamma\in D$ by propositions (3.1.2) and (3.1.3) keeping in mind the remarks at the end of section 3.1. Now, using the definition of $\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})$, and the inequality $\left|\int_{\mathbf{R}^3}\gamma(\vec{z})d\mu_{\tau}(\vec{z})-\int_{\mathbf{R}^3}\gamma(\vec{z})d\tilde{\mu}_{\tau}(\vec{z})\right|\leq\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})$, the expression (3.2.4) above is bounded by

$$\leq\left(\frac{2}{\alpha}+\frac{1}{\alpha}\right)\int_{S^2}\int_{\mathbf{R}^3}\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})d\mu_{\tau}(\vec{w})d\vec{n}+\left(\frac{1}{\beta}+\frac{1}{\beta}\right)\int_{S^2}\int_{\mathbf{R}^3}\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})d\mu_{\tau}(\vec{v})d\vec{n}$$

$$=\frac{3}{\alpha}\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})\mu_{\tau}(\mathbf{R}^3)4\pi+\frac{2}{\beta}\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})\mu_{\tau}(\mathbf{R}^3)4\pi. \tag{3.2.5}$$

Since $\mu_{\bullet},\tilde{\mu}_{\bullet}\in B_R(0)\subseteq X$, the measures are bounded by R in the sense that $\|\mu_{\bullet}\|=\sup_{x\in[0,a]}\int_{\mathbf{R}^3}d\mu_x(\vec{v})\leq R$. Then substituting the values $\alpha=\frac{\delta^2}{b(3+2\delta)}$ and $\beta=\frac{\delta^2}{b(2+\delta)}$ the equation (3.2.5) is bounded above by

$$\leq 3b\frac{(3+2\delta)}{\delta^2}\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})\|\mu_{\bullet}\|4\pi+2b\frac{(2+\delta)}{\delta^2}\rho(\mu_{\bullet},\tilde{\mu}_{\bullet})\|\mu_{\bullet}\|4\pi$$

$$=4\pi bR\frac{(13+8\delta)}{\delta^2}\rho(\mu_{\bullet},\tilde{\mu}_{\bullet}).$$

Since this is a constant which is independent of τ for given measures, (3.2.1) can be estimated in its entirety by

$$(3.2.1) \le \int_0^a 4\pi bR \frac{(13+8\delta)}{\delta^2}\rho(\mu_\bullet,\tilde{\mu}_\bullet)d\tau = 4\pi bR \frac{(13+8\delta)}{\delta^2}\rho(\mu_\bullet,\tilde{\mu}_\bullet)a$$

To complete the lemma, this same argument of inequalities and facts needs to be applied to the integral in (3.2.2),

$$\left|\int_{\mathbf{R}^3} \frac{\varphi(\vec{v})}{|\xi|}\chi_{\{\xi<0\}}(\vec{v})d\left[C^\delta(\mu_\tau,\mu_\tau)-C^\delta(\tilde{\mu}_\tau,\tilde{\mu}_\tau)\right](\vec{v})\right|.$$

However, replacing ξ by $-\xi$, the integral becomes the same as (3.2.1) and the analysis is the same as before. Therefore, the numerical bounds are the same, and the above is

$$\le 4\pi bR \frac{(13+8\delta)}{\delta^2}\rho(\mu_\bullet,\tilde{\mu}_\bullet).$$

Now, returning to the original inequality to be proved,

$$\left|\int_{\mathbf{R}^3}\varphi(\vec{v})d(T\mu_x)(\vec{v}) - \int_{\mathbf{R}^3}\varphi(\vec{v})d(T\tilde{\mu}_x)(\vec{v})\right| \le 8\pi bR\frac{(13+8\delta)}{\delta^2}a\rho(\mu_\bullet,\tilde{\mu}_\bullet).$$

Taking the supremum over $x \in [0,a]$ and $\varphi \in D$, yields

$$\rho(T\mu_\bullet,T\tilde{\mu}_\bullet) \le 8\pi bR\frac{(13+8\delta)}{\delta^2}a\rho(\mu_\bullet,\tilde{\mu}_\bullet)$$

This proves the lemma by letting $k = 8\pi bR\frac{(13+8\delta)}{\delta^2}$.

3.3 Uniqueness for the measure problem

With the results of sections 3.1 and 3.2, the main theorem of this chapter can be formulated and proved quite succinctly. For convenience, the problem (1.4.5-6) is given in the statement of the theorem.

Theorem 3.3.1 Let $\mu_\bullet$ be any measure solution which satisfies

$$\xi \frac{d}{dx}\mu_x = C^\delta(\mu_x, \mu_x)$$

for all $x \in [0,a]$ with the boundary conditions

$$\mu_0\big|_{\{\xi>0\}} = \mu_0^+,\ \mu_a\big|_{\{\xi<0\}} = \mu_a^-.$$

For a fixed R large enough, and for small enough a, that solution is unique in $B_R(0)$.

Proof: R needs to be large enough to admit the boundary measures μ_0^+ and μ_a^-. Since existence in $B_R(0)$ is already done by theorem 1.4.3, let $\mu_\bullet, \tilde{\mu}_\bullet$ be two solutions in $B_R(0)$. Then $\mu_\bullet$ and $\tilde{\mu}_\bullet$ both satisfy the corresponding integral equation to the above problem, which is written out explicitly in the mapping T on X. Thus, $\mu_x = T\mu_x$, and $\tilde{\mu}_x = T\tilde{\mu}_x$ for $x \in [0,a]$. Now, applying lemma 3.2.1,

$$\rho(\mu_\bullet, \tilde{\mu}_\bullet) = \rho(T\mu_\bullet, T\tilde{\mu}_\bullet) \le ka\rho(\mu_\bullet, \tilde{\mu}_\bullet).$$

If a is chosen small enough such that $0 < a < \frac{1}{k}$, then

$$0 \le \rho(\mu_\bullet, \tilde{\mu}_\bullet) \le \tilde{k}\rho(\mu_\bullet, \tilde{\mu}_\bullet)$$

with $0 \le \tilde{k} < 1$. For this estimate to hold, $\rho(\mu_\bullet, \tilde{\mu}_\bullet) = 0$ must occur. Since ρ is a metric on X, for all $x \in [0,a]$ $\mu_x(\vec{v}) = \tilde{\mu}_x(\vec{v})$ Lebesgue almost everywhere. Hence $\mu_\bullet = \tilde{\mu}_\bullet$ since this equivalence is equality off a set of Lebegue measure zero. This proves the theorem.

In section 2.1, the similar contraction property was used to obtain existence through the Banach Fixed Point Theorem. This may not hold here since the metric space may be incomplete with respect to the Bounded Lipschitz Distance. However, due to the previous existence result [1] it is sufficient to prove uniqueness.

4 Conclusions

4.1 Limitations of the Approach

To prove uniqueness of a solution to the steady-state Boltzmann equation in one spatial dimension with initial boundary conditions, several assumptions and restrictions have been made. These are summarized here with some of the reasoning which led to their necessity.

The first assumption is an approximation concerning the applicability of the Boltzmann equation. It is assumed that the particle density function $f(\vec{r},\vec{v},t)$ has behaviour which is given approximately by the Boltzmann equation (1.1.1) and then later on by the steady-state problems (1.2.1) and (1.4.5). Although this is an important assumption, it is well justified [3] by taking the limit of the particle diameter $\sigma \to 0$ and the number of particles $N \to \infty$ such that $N\sigma^2 \to 1/\lambda$ which is finite. Since the number of particles which is typically studied is of the order of 10^{21} or 10^{22}, and the particles have small diameters, this is an accurate assumption.

Two assumptions on the collision kernel were also imposed to obtain the uniqueness result in chapter 2. The first was a truncation in section 1.1 to the collision kernel B such that it was zero for small particle velocities. This truncation was made to overcome the difficulties involved when taking into consideration small particle velocities, ξ. A slightly less severe truncation was used in [1] for the existence result of theorem 1.4.2, but this did not solve the problems of dealing with small values of ξ. Although this truncation cannot be justified physically, it was necessary to prove both existence in theorem 1.4.2 and uniqueness of solutions to the boundary value problem (1.2.1-2) in chapter 2.

The second assumption is that the collision kernel is assumed to be positive and bounded. Again, this is not a physically justifiable assumption, and it is stronger than the more common assumption of local boundedness assumed in [1].

For the uniqueness result to the measure problem, an additional assumption and truncation were made. The collision kernel is assumed to be Lipschitz continuous with respect to each of its variables in $\mathbf{R}^3$. This is a valid assumption since $B(\vec{n},\vec{v},\vec{w})$ generally depends only on $|(\vec{v}-\vec{w})\cdot\vec{n}|$ which is Lipschitz continuous. The truncation in the form of g^δ is introduced in section 1.4 and is again non-physical, but necessary to obtain the desired result.

The effects of the truncations and assumptions can be observed directly in the uniqueness results of theorems 2.2.1, 2.2.2 and 3.3.1, which rely on choosing a small enough interval to obtain a contractive mapping. The maximum size of the interval is proportional to the truncation size δ, and inversely proportional to the collision kernel bound, b. Hence, attempts to create less severe assumptions by letting $\delta \to 0$ and $b \to \infty$ have the effect of decreasing the maximum width of the interval to which a unique solution can be proved to exist.

4.2 The Linearized problem

In practice, it is difficult to compute possible solutions or approximate solutions to the full non-linear Boltzmann equation. For most cases in linear transport theory, it is necessary instead to solve the linearized or linear problem, considering the solutions obtained as approximations to the non-linear problem [3].

The time dependent linearized equation is arrived at through perturbation methods, and can be stated quite simply as

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = Lf,$$

where L is a linear operator on f. Computation is then done starting from an equilibrium Maxwellian solution, and computed iteratively to the desired accuracy. Much work has been done in this field, and references are plentiful (almost fifty are cited in [3] alone) with most of the work being published twenty to thirty years ago.

The non-linear problem can be examined by looking at the behaviours of the linearized version, and carefully identifying the approximations and limits of accuracy obtained. However, convergence in this setting is in general not attainable and only asymptotic approaches are realized [3]. This provides the motivation for examining the non-linear problem.

4.3 Results and open problems

There are two main results in this thesis. These are uniqueness of the steady state Boltzmann equation in a slab, and uniqueness of the measure solution to the steady state measure equation. Both of the proofs of these results are similar in method, using a contractive property on the function spaces involved.

The method of proof was arrived at while searching for a contractive property to which a fixed point theorem could be applied yielding not only uniqueness of a solution but also existence as is obtained in theorem 2.2.2. Several difficulties were encountered at this point, including completeness of the function spaces with respect to the various norms, and ensuring a non-negative solution.

To overcome these difficulties, the idea of the bounded Lipschitz distance was used to provide a contractive property. However, for this metric, completeness is unknown except for some specific measures such as probability measures [5]. Where completeness exists, in the case of problem (1.2.1-2), the Banach fixed point theorem was applied to provide a unique solution. However, there is no guarantee that the solution is positive in the function case as stated in theorem 2.2.2, and existence for the measure solution has been previously obtained via different methods [1].

With these results, the next logical step is to solve the question of existence and uniqueness of nonnegative solutions to the problem (1.2.1-2) without the truncations imposed. This problem is still open [1]. Also of interest is the steady-state Boltzmann equation with initial boundary conditions at one end of the interval only. This is called the half-space problem since a solution is sought after on the interval $[0, \infty)$. The asymptotic behaviour of a possible solution at large distances is another intriguing problem. Unfortunately, the methods involved in this paper do not seem to be readily applicable in these situations, yet may possibly serve as a stone in the path to their solutions.

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Appendix

The following definitions are taken from Folland [4].

A *metric* on a set Y is a function $\rho: Y \times Y \to [0,\infty)$ such that for all $x, y, z \in Y$ the following hold:

(i) $\rho(x,y) = 0$ if and only if $x = y$,

(ii) $\rho(x,y) = \rho(y,x)$, and

(iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

A set equipped with a metric is called a *metric space.*

If Y is a vector space over a field K (taken to be $\mathbf{R}$ in the preceding chapters) then a *norm* on Y is a function from Y into $[0,\infty)$, denoted by $x \to \|x\|$ such that for all $x, y \in Y$ and $\lambda \in K$ the following hold:

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$, and

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

A vector space equipped with a norm is called a *normed vector space.*

Theorem A.1 The function

$$\|f(x,\cdot)\| = \sup_{x \in [0,a]} \int_{\mathbf{R}^3} |f(x,\vec{v})| d\vec{v}$$

defines a norm on $X = C([0,a], L^1(\mathbf{R}^3))$.

Proof: Let $f, g \in X$ and $\lambda \in \mathbf{R}$ be arbitrary but fixed. The properties are checked by the definition given in the statement of the theorem, using the fact that $\int_{\mathbf{R}^3} |f(x,\vec{v})| d\vec{v}$ is a norm on $L^1(\mathbf{R}^3)$ [4].

(i) $\|f(x,\cdot)\| = 0 \Leftrightarrow \int_{\mathbf{R}^3} |f(x,\vec{v})| d\vec{v} = 0$ for all $x \in [0,a]$

$\Leftrightarrow |f(x,\vec{v})| = 0$ for all $x \in [0,a]$ and $\vec{v}$-almost everywhere on $\mathbf{R}^3$

$\Leftrightarrow f(x,\vec{v}) = 0$ for all $x \in [0,a]$ and $\vec{v}$-almost everywhere on $\mathbf{R}^3$, which is precisely the equivalence of $f = 0$.

(ii) $\|\lambda f\| = \sup_{x\in[0,a]} \int_{\mathbf{R}^3} |\lambda f(x,\vec{v})| d\vec{v} = \sup_{x\in[0,a]} |\lambda| \int_{\mathbf{R}^3} |\lambda f(x,\vec{v})| d\vec{v} = |\lambda| \cdot \|f\|$.

(iii) $$\|f+g\| = \sup_{x\in[0,a]} \int_{\mathbf{R}^3} |f(x,\vec{v}) + g(x,\vec{v})| d\vec{v} \le \sup_{x\in[0,a]} \left\{ \int_{\mathbf{R}^3} |f(x,\vec{v})| d\vec{v} + \int_{\mathbf{R}^3} |g(x,\vec{v})| d\vec{v} \right\}$$

$$\le \sup_{x\in[0,a]} \left\{ \int_{\mathbf{R}^3} |f(x,\vec{v})| d\vec{v} \right\} + \sup_{x\in[0,a]} \left\{ \int_{\mathbf{R}^3} |g(x,\vec{v})| d\vec{v} \right\} = \|f\| + \|g\|.$$

By properties (i)-(iii), the theorem is proved.

For the next theorem, recall the Bounded Lipschitz Distance function defined in section 1.4, and the set

$$D = \left\{ \varphi: \mathbf{R}^3 \to [0,1] \,\middle|\, \begin{array}{l} \varphi \text{ is Lebesgue measurable, and} \\ |\varphi(x) - \varphi(y)| \le \|x - y\| \text{ for all } x, y \in \mathbf{R}^3 \end{array} \right\}.$$

Theorem A.2: The function $\rho: Y \times Y \to \mathbf{R}$ given for $\mu_\bullet, \gamma_\bullet \in Y \subset X$, $X = C([0,a], M(\mathbf{R}^3))$, by

$$\rho(\mu_\bullet, \gamma_\bullet) = \sup_{x\in[0,a]} \sup_{\varphi\in D} \left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\mu_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) \right|$$

is a metric on a bounded subset Y of X.

Proof: Let $\mu_{\bullet}, \gamma_{\bullet}, \tau_{\bullet} \in Y \subset X$ where Y is a bounded subset of X in the sense that there exists a constant $0 < M \in \mathbf{R}$ such that $\|\mu_{\bullet}\| = \sup_{x\in[0,a]} \int_{\mathbf{R}^3} d\mu_x(\vec{v}) \le M$ for all $\mu_{\bullet} \in Y$. ρ is well-defined on $Y \times Y$ because

$$\rho(\mu_{\bullet}, \gamma_{\bullet}) \le \sup_{x\in[0,a]} \sup_{\varphi\in D} \left\{ \left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\mu_x(\vec{v}) \right| + \left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) \right| \right\}$$
$$\le \sup_{x\in[0,a]} \left\{ \int_{\mathbf{R}^3} d\mu_x(\vec{v}) + \int_{\mathbf{R}^3} d\gamma_x(\vec{v}) \right\} = \|\mu_{\bullet}\| + \|\gamma_{\bullet}\| \le 2M.$$

Clearly, $\rho(\mu_{\bullet}, \gamma_{\bullet}) = \rho(\gamma_{\bullet}, \mu_{\bullet})$ due to the absolute value of the integrals. Just as simply,

$$\rho(\mu_{\bullet}, \tau_{\bullet}) = \sup_{x\in[0,a]} \sup_{\varphi\in D} \left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\mu_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) + \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\tau_x(\vec{v}) \right|$$
$$\le \sup_{x\in[0,a]} \sup_{\varphi\in D} \left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\mu_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) \right| + \sup_{x\in[0,a]} \sup_{\varphi\in D} \left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\tau_x(\vec{v}) \right|$$
$$= \rho(\mu_{\bullet}, \tau_{\bullet}) + \rho(\tau_{\bullet}, \gamma_{\bullet}).$$

Since $\rho(\mu_{\bullet}, \mu_{\bullet}) = 0$ trivially, all that remains to be shown is the reverse direction. By the Riesz Representation theorem 1.3.1, the bounded measures in X are uniquely determined by integration over all functions $\varphi \in C_0(\mathbf{R}^3)$. In the present notation this is written as $\left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\mu_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) \right| = 0 \ \forall \varphi \in C_0(\mathbf{R}^3) \Leftrightarrow \mu_x = \gamma_x$. From the continuity of the integral and the absolute value, it is sufficient to consider a dense subspace of the non-negative functions of $C_0(\mathbf{R}^3)$. Specifically, the subspace

$$E = \left\{ \varphi \in C_0(\mathbf{R}^3) \,\middle|\, \begin{array}{l} \varphi \text{ is non-negative, has compact support, and} \\ \text{is Lipschitz continuous} \end{array} \right\}$$

is dense in the non-negative functions of $C_0(\mathbf{R}^3)$.

Let E_L be the subspace of E with positive Lipschitz constant L. Then the set E can be written as $E = \bigcup_{L>0} E_L$. Also, each function in E_L is equal to a function in E_1 multiplied by the constant L. Finally, each function in D is equal to a function in E_1 multiplied by a sufficiently small real constant. Therefore, it follows that

$$\left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\mu_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) \right| = 0 \ \forall \varphi \in D \Rightarrow \left| \int_{\mathbf{R}^3} \varphi(\vec{v}) d\mu_x(\vec{v}) - \int_{\mathbf{R}^3} \varphi(\vec{v}) d\gamma_x(\vec{v}) \right| = 0 \ \forall \varphi \in E$$

and this in turn implies that $\mu_\bullet = \gamma_\bullet$ since the above holds for all $x \in [0, a]$.

This proves the final property of the metric and completes the proof of theorem A.2.

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