NESTED SEQUENCES OF PRINCIPAL MINORS AND POTENTIAL STABILITY

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by

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Abstract

Sufficient conditions are determined for an n-by-n zero-nonzero pattern to allow a nested sequence of nonzero principal minors. A method is then given to sign such a pattern so that it allows a nested sequence of k-by-k principal minors with sign $(-1)^k$ for $1 \leq k \leq n$. It is a classical result that a sign pattern with such a sequence is potentially stable, thus classes of potentially stable sign patterns can be identified. A necessary and sufficient condition for a certain tree sign pattern class to be potentially stable is also given.

1. INTRODUCTION

An n-by-n *sign pattern (matrix)* $A = [a_{ij}]$ has $a_{ij} \in \{+,0,-\}$, and defines a *sign pattern class*

$$Q(A) = \{B \in M_n(\mathbb{R}) : \text{ sign } b_{ij} = a_{ij} \text{ for all } i,\, j\}\,.$$

Matrix B is *stable* if all its eigenvalues have negative real parts. Sign pattern A is *sign stable* if $Q(A)$ *requires* stability (*i.e.* every matrix $B \in Q(A)$ is stable), and is *potentially stable* if $Q(A)$ *allows* stability (*i.e.* some matrix $B \in Q(A)$ is stable). These stability types are preserved under permutation and signature similarity. Whereas sign stable patterns have been characterized (see [JKV] for details), the problem for potential stability remains unsolved even for general tree sign patterns. This is an important problem in qualitative matrix theory, and has applications in economics, biology and chemistry, where only the signs, rather than exact values for entries in a connection matrix, are known.

We approach the problem of potential stability by considering a nested sequence of principal minors. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be a rearrangement of the elements of $\{1, \cdots, n\}$, and $B\,[\{\alpha_1, \cdots, \alpha_k\}]$ denote the principal submatrix in rows and columns $\alpha_1, \cdots, \alpha_k$ of $B \in M_n(\mathbb{R})$. Sign pattern A allows a *nested sequence of properly signed principal minors* (abbreviated to a *properly signed nest*) if there exist $B \in Q(A)$ and $\alpha_1, \cdots, \alpha_n$ so that

$$\text{sign}\,(\det B\,[\{\alpha_1, \cdots, \alpha_k\}]) = (-1)^k \quad \text{for} \quad k = 1, \cdots, n.$$

Note that this is equivalent to the existence of $C \in Q(A)$ and a permutation matrix P such that $B = P^T C P$ has a *leading properly signed nest*, that is

$$\text{sign}\,(\det B\,[\{1, \cdots, k\}]) = (-1)^k \quad \text{for} \quad k = 1, \cdots, n.$$

It is important to emphasize here that this means that there is an actual matrix C with a properly signed nest; a nested sequence of index sets, each of which individually allows a properly signed minor, may not be sufficient.

Example 1.1 (see [JS]) Consider the sign pattern

$$A = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & + \end{bmatrix},$$

which allows a sequence of leading principal minors with proper signs. But such a sequence is not realizable by numbers, as can be seen by considering without loss of generality

$$C = \begin{bmatrix} -a & 1 & 0 \\ -d & b & 1 \\ 0 & e & c \end{bmatrix} \in Q(A),$$

where $a, b, c, d, e > 0$. Then a properly signed nest requires

$$-a < 0, \quad d - ab > 0, \quad c(d - ab) + ae < 0,$$

which is not possible. Note that A is not potentially stable [JS].

It is often useful to represent a sign pattern A by a signed directed graph $D(A)$ with vertex set $\{1, \cdots, n\}$, and edge set $\{(i,j) : a_{ij} \neq 0\}$ with (i,j) signed $+$ or $-$ according as a_{ij} is $+$ or $-$. A directed cycle has sign equal to the sign of the product of the entries of its edge set. If A is combinatorially symmetric (*i.e.* $a_{ij} \neq 0$ whenever $a_{ji} \neq 0$), then it can be represented by a signed (undirected) graph $G(A)$ together with loops. For $a_{ii} \neq 0$, the loop at vertex i is signed as a_{ii}. For the special case in which $G(A)$ is a tree, the (undirected) edge between i and $j \neq i$ is signed as the product $a_{ij}a_{ji}$; in this case eigenvalue information depends only on $\{a_{ii}, a_{ij}a_{ji}\}$.

An irreducible sign pattern with a tree graph is called a *tree sign pattern* as in [JJ, JS]. In Section 2 we give some background on potential stability, including results on tree sign patterns. In Section 3 we focus on principal minors and begin by considering a (zero-nonzero) *pattern* $A = [a_{ij}]$ with $a_{ij} \in \{*, 0\}$,where $*$ denotes any nonzero entry (either $+$ or $-$). As with sign patterns, an unsigned (undirected) graph $G(A)$ is used to represent a combinatorially symmetric pattern A. We find sufficient conditions for some B in the pattern class A to have a *nested sequence of nonzero principal minors* (abbreviated to a *nonzero nest*), *i.e.* for pattern A to allow such a sequence. Given the existence of a nonzero nest, we show how to sign the pattern so that it allows a properly signed nest. Using results of Sections 2 and 3, in Section 4 we identify classes of tree sign patterns that are potentially stable.

2. BACKGROUND ON POTENTIAL STABILITY

By considering the characteristic equation, obvious necessary conditions for potential stability are that $B \in Q(A)$ has at least one minor of order k signed as $(-1)^k$ for all $k = 1, \cdots, n$. Several authors (*e.g.* [Bo], [BS]) have recognized the importance in this context of a theorem proved by Fisher and Fuller [FF], and Ballantine [Ba]. We use this to give a sufficient condition for potential stability.

Theorem 2.1. *If A is an n-by-n sign pattern that allows a properly signed nest, then A is potentially stable. Moreover, A contains a nested sequence of potentially stable sign patterns of orders $1, 2, \cdots, n$.*

Proof. Consider $C \in Q(A)$ and a permutation matrix P so that $B = P^TCP$ has

$$\operatorname{sign}(\det B\,[\{1,2,\cdots,k\}]) = (-1)^k, \qquad k = 1,\cdots,n.$$

Thus the leading principal minors of B alternate in sign. From [FF], [Ba], there exists a positive diagonal matrix D so that DB is stable. Thus $PDBP^T \in Q(A)$ is stable, and so A is potentially stable. Moreover, according to the proof in [Ba], D may be chosen so that the sequence of submatrices $(PDBP^T)\,[\{1,2,\cdots,k\}]$, for $k = 1,\cdots,n$, is a nested sequence of stable principal submatrices. ▮

We note that the proof in [Ba] also indicates that all eigenvalues of each principal submatrix in the nest may be taken to be real and negative.

Example 2.2. To illustrate the above, consider the tree sign pattern

$$A = \begin{bmatrix} - & + & 0 \\ - & - & + \\ 0 & - & + \end{bmatrix}.$$

Then

$$B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -3 & 1 \end{bmatrix} \in Q(A)$$

has a leading properly signed nest, and the leading principal submatrices of B form a nested sequence of stable matrices. Note that $A[\{1,2\}]$ is in fact sign stable, but A is not.

The potentially stable tree sign pattern A in Example 2.2 gives additional potentially stable patterns, since the (1,3) or (3,1) entries can be arbitrarily signed + or − and (by continuity) potential stability is maintained. This is a special case of a general result proved in [JS, Th. 3]: if A is a potentially stable n-by-n sign pattern that is a subpattern of an n-by-n sign pattern $\hat{A}$, then $\hat{A}$ is also potentially stable.

In [JJ] two theorems on tree sign patterns are proved that are useful for identifying sign patterns that are ***not*** potentially stable. These results involve the ***inertia*** of a matrix B, that is the triple

$$i(B) = (i_+(B),\, i_-(B),\, i_0(B))$$

where $i_+(B)$ $\langle$resp. $i_-(B)$, $i_0(B)\rangle$ is the number of eigenvalues with positive $\langle$resp. negative, zero$\rangle$ real part, counting multiplicities. Matrix B is stable if and only if $i(B) = (0, n, 0)$. A tree sign pattern is *skew-symmetric* if each edge of its graph is negative. If A is a tree sign pattern, then the *skew-symmetric factorization* of A is $A = S\,A_1$ where S is a signature matrix with + as its (1,1) entry, and A_1 is a skew-symmetric tree sign pattern.

Theorem 2.3 [JJ, Th. 1]. *Let* $A = SA_1$ *be the skew-symmetric factorization of the tree sign pattern* A*. If no diagonal entry of* A_1 *is* +*, then* $i_+(B) \leq i_-(S)$ *and* $i_-(B) \leq i_+(S)$ *for all* $B \in Q(A)$.

3. EXISTENCE OF NESTED SEQUENCES

To identify classes of potentially stable matrices using Theorem 2.1, we need to know which sign patterns allow a properly signed nest. This question is open in general. However, we have the following result for patterns to allow a nonzero nest.

Theorem 3.1. *Let* A *be an* n*-by-*n *pattern that satisfies the following four conditions:*

1. *A has at least one nonzero diagonal entry.*
2. *A allows a nonzero determinant.*
3. *A is combinatorially symmetric.*
4. *A is irreducible.*

Then A *allows a nonzero nest.*

Proof. The proof is via induction on n, the number of vertices in the undirected graph $G(A)$. In the case $n = 1$, the theorem is trivially true. For $n \geq 2$, consider first the case in which $G(A)$ is a single (bidirectional) cycle. Choose some nonzero diagonal entry to be -1, and label this vertex 1. Choose a direction around the cycle, label the remaining vertices consecutively, and arbitrarily assign ± 1 to the two entries associated with the first $(n-1)$ edges. On the last edge between vertices 1 and n, sign one of the entries so that its directed cycle is negative, and assign it sufficiently large magnitude so that $\det A$ has sign $(-1)^n$ when the other entry is assigned the opposite sign and 1 in magnitude. Any other nonzero diagonal entries are assigned sufficiently small negative values. It is then easily checked that the nested sequence of leading principal minors is nonzero, and alternates in sign beginning with negative.

Now suppose that the assertion of the theorem is valid for each $n < k$ and consider a k-by-k pattern A meeting the assumptions. Since A allows a nonzero determinant, the k vertices of $G(A)$ may be partitioned into p cycles, each of which has one (in which case it corresponds to a nonzero diagonal entry of A) or more vertices. If $p = 1$, the theorem is true by the above discussion for a cycle; so assume $p > 1$. Since $G(A)$ is assumed connected, there exist $p - 1$ additional "bridge" edges of $G(A)$ that connect the p cycles in a tree-like manner, giving a subgraph G' of $G(A)$. It suffices to demonstrate a nonzero nest for G'. Designate a nonzero vertex v of G' corresponding to a nonzero diagonal entry, and pick cycle C that is connected by only one bridge to the rest of G' and that does not include v. Delete C and its bridge from G' to leave $\tilde{G}$. Since v lies in $\tilde{G}$, which is connected, and $\tilde{G}$ has $\ell < k$ vertices, the induction hypothesis applies to the pattern $\tilde{A}$ associated with $\tilde{G}$. Without loss of generality, suppose that the nested sequence for $\tilde{A}$ corresponds to the labelling $1, 2, \cdots, \ell$ of the vertices of $\tilde{G}$ and that specific values for the entries of $\tilde{A}$ are given. Let $x \in \tilde{G}$ and $y \in C$ be the vertices of the bridge connecting C to $\tilde{G}$ in G'. Assign labels to the vertices of C, beginning with $y = x + 1$ and values to the entries of A corresponding to C as though y were nonzero in the single cycle discussion above, and assign the entries on the edge between vertices x and y the values ± 1 arbitrarily. Increase the labels of any successors of x in $\tilde{G}$ by $k - \ell$. We then claim that the leading principal minors give a nonzero nest for A (with any nonzero entries not yet accounted for taken to be sufficiently small). The minors $\det A[\{1,\cdots,z\}]$ for $z \le x$ are inherited from $\tilde{A}$. The minors $\det[A\{1,\cdots,z\}]$ for $x < z < x+k-\ell$ are dominated by either the product of minors corresponding to $\{1,\cdots,x-1\}$, $\{x,y\}$, $\{y+1,\cdots,z\}$ in case $z-x$ is odd, or $\{1,\cdots,x\}$, $\{y,\cdots,z\}$ in case $z-x$ is even. The remaining minors are dominated by the product of the minor from C and the corresponding minor from A. This completes the induction step and proof. ▮

Note that if A is an n-by-n pattern satisfying the conditions of Theorem 3.1 and A is a subpattern of an n-by-n pattern $\hat{A}$, then $\hat{A}$ allows a nonzero nest. The result of Theorem 3.1 generalizes in a straightforward way to a reducible pattern.

Corollary 3.2. *Let A be an n-by-n pattern whose graph $G(A)$ has k connected components, so that A is permutation similar to $\bigoplus_{i=1}^{k} A_i$. If each A_i satisfies conditions* 1-4 *of Theorem* 3.1, *then A allows a nonzero nest.*

Note that condition 2 in Theorem 3.1 is equivalent to A having no p-by-q zero block with $p+q \ge n+1$. Conditions 1 and 2 in Theorem 3.1 are necessary for any irreducible pattern A to have a nonzero nest. In the presence of combinatorial symmetry, they become sufficient. However, the assumption of combinatorial symmetry cannot in general be relaxed, as the following example shows.

Example 3.3. The pattern

$$A = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix}$$

satisfies 1, 2 and 4 of Theorem 3.1, but has no nonzero nest.

As Example 3.3 illustrates, it is a classical result that if A allows a nonzero determinant, then there exists a permutation matrix P so that PA has leading principal minors nonzero. But, of course, PA does not in general preserve the eigenvalue properties of A. Some sufficient conditions for a not necessarily combinatorially symmetric pattern to have a nonzero nest are given in [A].

Suppose that pattern A satisfies the conditions of Theorem 3.1, and that the entries of matrix B with pattern A are assigned as in that theorem. Thus B has a leading nonzero nest, with b_{11} negative. Defining the signature matrix S by

$$s_{kk} = (-1)^k \text{ sign } \det B\left[\{1, \cdots, k\}\right],$$

gives SB with pattern A and a properly signed nest. This proves Theorem 3.4 below, where we give another proof that shows that each vertex in $G(A)$ corresponding to a nonzero diagonal entry may be chosen negative, and all edges may be chosen negative; however, the assignment of +/− to an entry associated with an edge is not always arbitrary. Such a signing is used for tree sign patterns in Section 4.

Theorem 3.4. *Let pattern A satisfy the conditions of Theorem* 3.1. *Then the entries of A may be signed so that the resulting sign pattern allows a properly signed nest.*

Proof. From the proof of Theorem 3.1, a single cycle may be signed as claimed. At the induction step in the proof of Theorem 3.1, if the nested sequence in $\tilde{A}$ displays the proper alternation, then the re-insertion of C preserves the alternation. Incorporate the properly signed nest alternation and the negative signing of vertices and edges into the induction hypothesis, and note that the proof is the same. ∎

This result also generalizes to a reducible pattern.

Corollary 3.5. *Let A satisfy the conditions of Corollary* 3.2. *Then the entries of A may be signed to allow a properly signed nest.*

Example 3.6. The proofs of Theorems 3.1 and 3.4 suggest an algorithm for assigning a sign and magnitude to each entry of a pattern (and simultaneously interchanging its rows and columns, *i.e.* relabelling the vertices of its associated graph) so that it has a leading properly signed nest. We now illustrate this algorithm on

$$A = \begin{bmatrix} 0 & * & * & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & * & * & * & 0 & * & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & * & 0 & 0 & * & 0 & 0 \\ * & 0 & * & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * & * & * \end{bmatrix},$$

for which $G(A)$ is as in Figure 1, where vertices corresponding to nonzero diagonal entries of A are colored black. Clearly A satisfies the conditions of Theorem 3.1.

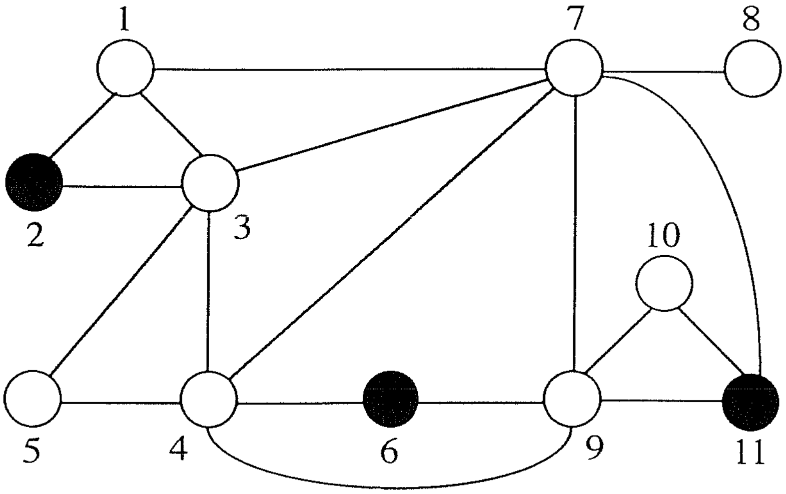


Fig. 1. The graph of the pattern A in Ex. 3.6

Choose any set of $p \geq 1$ disjoint, bidirectional cycles that cover the vertices of $G(A)$; see, for example, Figure 2, where $p = 5$. Since A is irreducible, there exists a set of $p - 1$ "bridge" edges in $G(A)$ that connect these cycles, giving a subgraph G' of $G(A)$; see, for example, Figure 3, where the bridge edges are shown by dashed lines and the labelling of the vertices has been omitted since our algorithm relabels them.

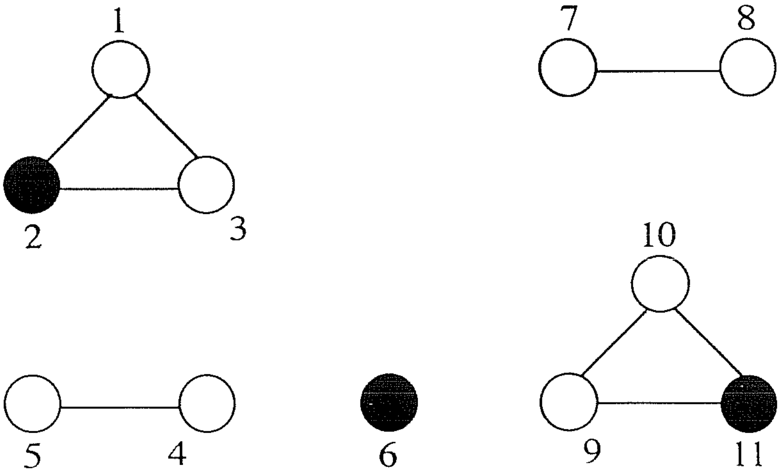


Fig. 2. A set of disjoint cycles of $G(A)$

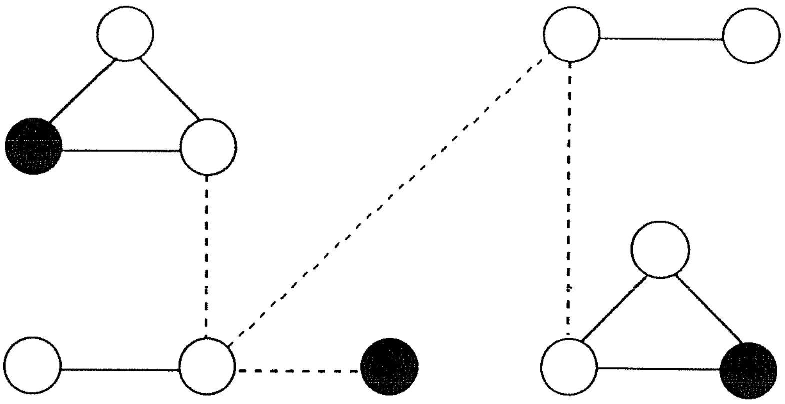


Fig. 3. An unsigned subgraph G' of $G(A)$

Starting with any black vertex in G', label it vertex 1 and assign the value -1 to the corresponding diagonal entry in A. Proceed in any direction around the cycle containing this vertex, labelling vertices consecutively and signing all edges negative (and arbitrarily assigning ± 1 to the two corresponding entries in A) until either a bridge is encountered or all but one of the edges of this cycle have been signed. In the latter case, the $\pm$ signs on the last edge signed can always be assigned (along with appropriate magnitudes) to the two entries of A so that one of the two directed cycles is negative, and so that sign $\det A[\alpha] = (-1)^{|\alpha|}$, where the indices of α correspond to the labels of this cycle. (Note that if the entire cycle is just a 2-cycle, then the assignment of ± 1 to the two entries of A is arbitrary.)

Whenever a bridge is encountered, it is signed negatively (and the two corresponding entries in A are arbitrarily assigned ± 1), and the labelling and signing of vertices and edges in the next cycle continue in the same manner as above. The entire subgraph G' (a "tree of cycles") is traversed in this "depth first" manner. When an entire pendant cycle (one that has only one bridge connected to it) has been labelled and signed, backtrack across its bridge, and continue as above until the entire graph has been traversed.

For the example in Figure 3, this procedure could result, for example, in the signed subgraph G' in Figure 4, and the corresponding matrix

$$B = \begin{bmatrix} -1 & 1 & \epsilon_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ -1 & 0 & 1 & \epsilon_{24} & \epsilon_{25} & 0 & 0 & 0 & 0 & 0 & 1 \\ \epsilon_{31} & -1 & 0 & 1 & 0 & \epsilon_{36} & 0 & \epsilon_{38} & 0 & 1 & 0 \\ 0 & \epsilon_{42} & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \epsilon_{52} & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_{63} & 1 & 0 & 0 & 1 & -2 & \epsilon_{69} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & \epsilon_{77} & 1 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_{83} & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \epsilon_{96} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where an (i,j) entry corresponding to a $*$ in A and not corresponding to a vertex or an edge in G' is denoted by ϵ_{ij}, which is assigned a value sufficiently small in magnitude to maintain the signs of the leading principal minors of B.

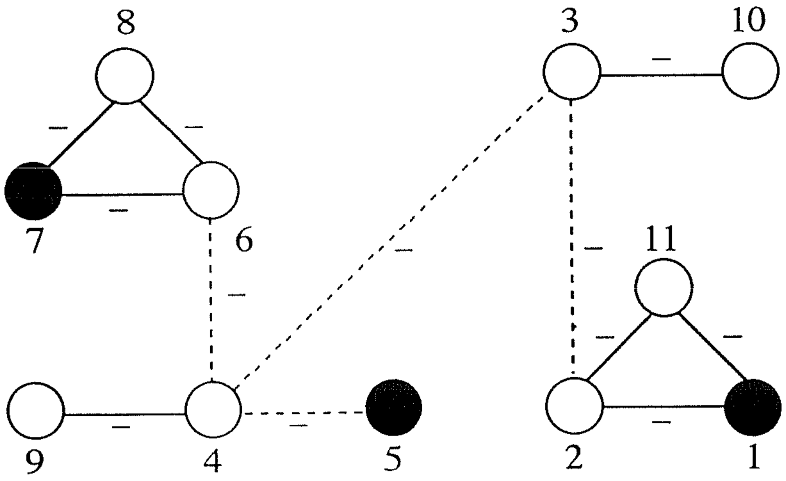


Fig. 4. The signed subgraph G' of $G(A)$

For the matrix B given above, the first four terms of the leading properly signed nest are

$$\det B[\{1\}] = -1,$$

$$\det B[\{1,2\}] = 1,$$

$$\det B[\{1,2,3\}] = -1 + \epsilon_{13} + \epsilon_{31} < 0.$$

$$\det B[\{1,2,3,4\}] = 1 - \epsilon_{24} - \epsilon_{42} + \epsilon_{24}\epsilon_{31} + \epsilon_{13}\epsilon_{42} + \epsilon_{13}\epsilon_{31}\epsilon_{24}\epsilon_{42} > 0.$$

By Theorem 2.1, the pattern A can be signed so that it is potentially stable.

We conclude this section with two results that follow from Theorem 3.4.

Corollary 3.7. *Suppose that A is a tree sign pattern in which at least one diagonal entry is negative, every edge is negative (except possibly those with both end vertices negative) and A allows a nonzero determinant. Then A allows a properly signed nest.*

Proof. First, note that the zero-nonzero pattern of A allows a nonzero nest by Theorem 3.1. But Theorem 3.4 signs all vertices and edges negative to allow a properly signed nest. If A contains no positive edge, then the result holds. On the other hand, if A contains k positive edges (between negative vertices), then set these edges to zero, giving a reducible sign pattern with $k+1$ components. As in the signing for Corollary 3.5, this sign pattern allows a properly signed nest. Restoring the positive edges with sufficiently small magnitude retains the properly signed nest for A. ▮

Corollary 3.8. *Suppose that A is a pattern with an irreducible, combinatorially symmetric subpattern $\hat{A}$ such that $G(\hat{A})$ has a spanning tree and the submatrix associated with this tree satisfies the conditions of Theorem* 3.1. *Then A allows a properly signed nest.*

Proof. Theorem 3.4 can be used to assign signs and magnitudes to entries in $\hat{A}$. The result follows by setting all other nonzero entries of A sufficiently small in magnitude. ▮

Note that the converse of Corollary 3.8 is not in general true, as the following shows.

Example 3.9. Consider the irreducible, combinatorially symmetric 7-by-7 pattern

$$A = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & * \\ * & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $G(A)$ as in Figure 5. This pattern satisfies the conditions of Theorem 3.1, in fact it allows a leading properly signed nest. But $G(A)$ has no spanning tree with nonzero determinant.

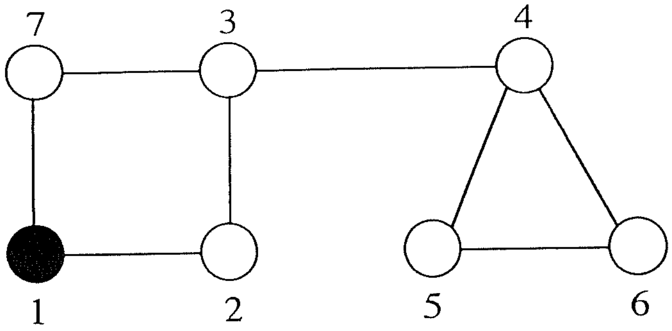


Fig. 5. The graph of the pattern A in Ex. 3.9

4. NESTED SEQUENCES AND POTENTIAL STABILITY

Results from Section 3 that give a properly signed nest can be combined with Theorem 2.1 to give potential stability; for example, a tree sign pattern satisfying the conditions of Corollary 3.7 is potentially stable. Without the assumption of combinatorial symmetry, the converse of Theorem 2.1 is not in general true, as the following example shows.

Example 4.1. Matrix

$$B = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -3 & 0 \end{bmatrix}$$

has -1 as a triple eigenvalue, and so is stable. Thus the sign pattern A such that $B \in Q(A)$ is potentially stable, but it does not have a properly signed nest (see Example 3.3).

We next turn to the possibility of a converse to Theorem 2.1 in the case of tree sign patterns. First, we demonstrate the converse within a restricted class.

Theorem 4.2. *Suppose that A is a tree sign pattern that has exactly one nonzero diagonal entry (that is negative). Then A is potentially stable if and only if A allows a properly signed nest.*

Proof. Sufficiency follows from Theorem 2.1. For necessity, assume that at least one edge of $G(A)$ is positive, and consider the skew-symmetric factorization of A. Then, by Theorem 2.3, $i_-(B) \le i_+(S)$ for all $B \in Q(A)$. But $i_+(S) \le n-1$, since S has at least one negative entry. Thus A is not potentially stable, which is a contradiction. Hence each edge of $G(A)$ is negative. Since A is potentially stable, it allows a nonzero determinant. Corollary 3.7 then shows that A allows a properly signed nest. ▮

Note that pattern A satisfying the conditions of Theorem 4.2 is sign semi-stable (*i.e.*, every matrix $B \in Q(A)$ has $i_+(B) = 0$).

If all diagonal entries of A are negative, then it is clearly potentially stable regardless of the signs of other entries. However, the following result shows that this is the only case for which a potentially stable tree sign pattern A can have all edges of $G(A)$ positive.

Theorem 4.3. *Let A be an n-by-n potentially stable tree sign pattern. If all edges in $G(A)$ are positive, then all a_{ii} are negative.*

Proof. For $B \in Q(A)$, there exists a positive diagonal matrix D such that $DBD^{-1} = F$ is a symmetric matrix, $F \in Q(A)$, and the diagonal of F is equal to the diagonal of B. Thus some $f_{ii} < 0$ as A is potentially stable. Suppose some $f_{jj} \geq 0$. If $n = 2$, then F is clearly unstable. If $n > 2$, then F has a 2-by-2 submatrix with a negative eigenvalue and a nonnegative eigenvalue. So by interlacing (see *e.g.* [HJ, p. 185], F (and thus B) is unstable. Thus all f_{jj} (and all a_{ii}) are negative. ▮

The converse of Theorem 2.1 is, however, not generally true for all tree sign patterns.

Example 4.4.

(a) The 4-by-4 matrix

$$B = \begin{bmatrix} -5 & 1 & 0 & 0 \\ 25 & 0 & 1 & 0 \\ 0 & -700 & 0 & 1 \\ 0 & 0 & 150 & 1 \end{bmatrix}$$

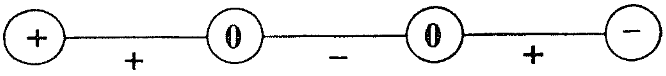
has eigenvalues -3.8699, -0.1244, and $-0.0029 \pm 22.7925i$, and thus is stable, but has no properly signed nest.

(b) The matrix B in (a) can be embedded in a 5-by-5 matrix to give

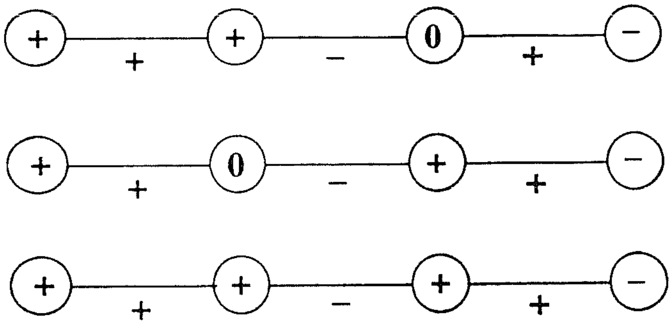
$$\begin{bmatrix} -5 & 1 & 0 & 0 & 0 \\ 25 & 0 & 1 & 0 & 1 \\ 0 & -700 & 0 & 1 & 0 \\ 0 & 0 & 150 & 1 & 0 \\ 0 & 5 & 0 & 0 & 0 \end{bmatrix},$$

which also is stable, but has no properly signed nest.

We note that, because of matrix B above,



and thus



must be added to correct the list of 4-by-4 potentially stable tree sign-pattern matrices (tridiagonal matrices) in [JS, Fig. 3]. In the above graphs, a +, − or 0 associated with a vertex specifies the sign of the corresponding diagonal entry, as in [JS]. All those previously in the list may be verified via the existence of a nested sequence as in Theorem 2.1.

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