NEUMANN EXPANSIONS FOR A CERTAIN CLASS OF GENERALIZED MULTIPLE HYPERGEOMETRIC SERIES ARISING IN PHYSICAL AND QUANTUM CHEMICAL APPLICATIONS

by

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ABSTRACT

The multivariable hypergeometric function

$$F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix},$$

which was studied recently by Niukkanen and Srivastava, provides an interesting and useful unification of the generalized hypergeometric ${}_pF_q$ function of one variable (with p numerator and q denominator parameters), Appell's and Kampé de Fériet's hypergeometric functions of two variables, and Lauricella's hypergeometric functions of n variables, as also of many other classes of hypergeometric series which arise naturally in various physical and quantum chemical applications. Indeed, as already observed by Srivastava, this multivariable hypergeometric function is an obvious special case of the generalized Lauricella hypergeometric function of n variables, which was first introduced and studied systematically by Srivastava and Daoust. By employing such fruitful connections of this function with much more general multiple hypergeometric functions studied in the literature rather systematically and widely, Srivastava presented several interesting and useful properties of this multivariable hypergeometric function, most of which did not appear in the work of Niukkanen. The object of this sequel to Srivastava's work is to derive a number of new Neumann expansions in series of Bessel functions for the multivariable hypergeometric function from substantially more general expansions involving, for example, multiple series with essentially arbitrary terms. Some interesting special cases of the Neumann expansions presented here are also indicated.

1. INTRODUCTION, NOTATIONS, AND DEFINITIONS

We begin by introducing a number of convenient notations and conventions which will be used throughout this paper. First of all, we put

$$\underset{\sim}{a} = (a^1,\ldots,a^p), \qquad \underset{\sim}{b} = (b^1,\ldots,b^q), \tag{1}$$

and

$$\underset{\sim}{a}_j = (a_j^1,\ldots,a_j^{p_j}), \qquad \underset{\sim}{b}_j = (b_j^1,\ldots,b_j^{q_j}), \tag{2}$$

so that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are <u>vectors</u> with dimensions p and q, respectively, and

$$\underset{\sim}{a}_j \quad \text{and} \quad \underset{\sim}{b}_j \qquad (j = 0,1,\ldots,n)$$

are <u>vectors</u> with dimensions p_j and q_j, respectively. Secondly, in terms of the Pochhammer symbol defined by

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad m = 0, \\ \lambda(\lambda+1)\ldots(\lambda+m-1), & \text{if} \quad m = 1,2,3,\ldots, \end{cases} \tag{3}$$

let

$$(\underset{\sim}{a})_m = \prod_{k=1}^{p} (a^k)_m, \qquad (\underset{\sim}{b})_m = \prod_{k=1}^{q} (b^k)_m, \tag{4}$$

and

$$(\underset{\sim}{a}_j)_m = \prod_{k=1}^{p_j} (a_j^k)_m, \qquad (\underset{\sim}{b}_j)_m = \prod_{k=1}^{q_j} (b_j^k)_m. \tag{5}$$

Next we define a generalized hypergeometric function of n variables by

$$F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$$

$$\equiv F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}\underset{\sim}{a}_0: & \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; & \\ & & x_1,\ldots,x_n\\ \underset{\sim}{b}_0: & \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n; & \end{matrix}\right]$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\underset{\sim}{a}_0)_{m_1+\ldots+m_n}}{(\underset{\sim}{b}_0)_{m_1+\ldots+m_n}} \prod_{j=1}^{n} \left\{\frac{(\underset{\sim}{a}_j)_{m_j}}{(\underset{\sim}{b}_j)_{m_j}} \frac{x_j^{m_j}}{m_j!}\right\}, \tag{6}$$

where, for (absolute) convergence of the multiple hypergeometric series,

$$1 + q_0 + q_k - p_0 - p_k \geqq 0 \qquad (k = 1,\ldots,n). \tag{7}$$

It should be remarked that the equality in (7) holds true provided that, in addition, we have <u>either</u>

$$p_0 > q_0 \quad \text{and} \quad |x_1|^{1/(p_0-q_0)} + \ldots + |x_n|^{1/(p_0-q_0)} < 1 \tag{8}$$

<u>or</u>

$$p_0 \leqq q_0 \quad \text{and} \quad \max\left\{|x_1|,\ldots,|x_n|\right\} < 1. \tag{9}$$

Furthermore, under certain parametric constraints, the multiple hypergeometric series in (6) converges also when

$$x_k = \pm 1 \qquad (k = 1,\ldots,n) \tag{10}$$

together, of course, with the equality in (7).

The recent studies by Niukkanen (1983, 1984) and Srivastava (1985a, b) on the multivariable hypergeometric function (6) are motivated by a remarkably vast field of physical and quantum chemical applications of such multiple hypergeometric series [see, for numerous other applications, Exton (1976, Chapters 7 and 8; 1978, Chapter 7), Carlson (1977), Srivastava and Kashyap (1982), and Srivastava and Karlsson (1985, Section 1.7)]. Indeed, as already observed by Srivastava (1985a), the multivariable hypergeometric function (6) is an obvious special case of the generalized Lauricella hypergeometric function of n variables, which was first introduced and studied by Srivastava and Daoust (1969, p. 454 et seq.), and this widely and systematically studied (Srivastava-Daoust) generalized Lauricella hypergeometric function has appeared in several subsequent works including, for example, two important books on the subject by Exton (1976, Section 3.7; 1978, Section 1.4), a book by Srivastava and Manocha (1984, p. 64 et seq.), and a book by Srivastava and Karlsson (1985, p. 37 et seq.); also, a further special case of the multivariable hypergeometric function (6) when

$$p_1 = \cdots = p_n \quad \text{and} \quad q_1 = \cdots = q_n \tag{11}$$

was considered earlier by Karlsson (1973). Srivastava (1985a, b) employed these fruitful connections of (6) with much more general multiple hypergeometric functions (studied in the literature rather systematically and widely) in order to present several interesting and useful properties of (6) (including, for example, regions of convergence, reduction and summation formulas, expansion and multiplication theorems, generating functions, and operational formulas), most of which did not appear in the work of Niukkanen (1983, 1984). In this sequel to Srivastava (1985a, b) we derive a number of new Neumann expansions in series of the Bessel functions (see, for hypergeometric notations, Slater 1966, Chapter 2):

$$J_\nu(z) = \frac{\left[\frac{1}{2} z\right]^\nu}{\Gamma(\nu+1)} \, {}_0F_1\left[\begin{matrix} \text{---}; \\ \nu+1; \end{matrix} \; -\frac{1}{4} z^2\right] \tag{12a}$$

$$= \frac{\left[\frac{1}{2} z\right]^\nu}{\Gamma(\nu+1)} \, e^{\pm iz} \; {}_1F_1\left[\begin{matrix} \nu + \frac{1}{2}; \\ 2\nu+1; \end{matrix} \; \mp 2iz\right] \tag{12b}$$

and

$$I_\nu(z) = \frac{\left[\frac{1}{2} z\right]^\nu}{\Gamma(\nu+1)} \, {}_0F_1\left[\begin{matrix} \text{---}; \\ \nu+1; \end{matrix} \; \frac{1}{4} z^2\right] \tag{13a}$$

$$= \frac{\left[\frac{1}{2} z\right]^\nu}{\Gamma(\nu+1)} \, e^{\pm z} \; {}_1F_1\left[\begin{matrix} \nu + \frac{1}{2}; \\ 2\nu+1; \end{matrix} \; \mp 2z\right] \tag{13b}$$

for the multivariable hypergeometric function (6) from substantially more general expansions involving, for example, multiple series with essentially arbitrary terms. We also consider several interesting special cases of the Neumann expansions presented here.

2. EXPANSIONS IN SERIES OF GENERALIZED HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS

For convenience, let $\Delta(\ell;\lambda)$ abbreviate the array of ℓ parameters:

$$\frac{\lambda}{\ell}, \frac{\lambda+1}{\ell}, \ldots, \frac{\lambda+\ell-1}{\ell} \qquad (\ell = 1,2,3,\ldots),$$

so that $\Delta(\ell;\underset{\sim}{a})$ abbreviates the array of ℓp parameters [cf. Equation (1)]:

$$\frac{a^i}{\ell}, \frac{a^i+1}{\ell}, \ldots, \frac{a^i+\ell-1}{\ell} \qquad (i = 1,\ldots,p;\ \ell = 1,2,3,\ldots).$$

Also let

$$r_m(\underset{\sim}{a},\underset{\sim}{c};\underset{\sim}{b},\underset{\sim}{d}) = \frac{(\underset{\sim}{a})_m\ (\underset{\sim}{c})_m}{(\underset{\sim}{b})_m\ (\underset{\sim}{d})_m} \qquad (m = 0,1,2,\ldots), \tag{14}$$

where, by analogy with the abbreviations introduced in (1) and (2),

$$\underset{\sim}{c} = (c^1,\ldots,c^r) \quad \text{and} \quad \underset{\sim}{d} = (d^1,\ldots,d^s), \tag{15}$$

so that $\underset{\sim}{c}$ and $\underset{\sim}{d}$ are <u>vectors</u> with dimensions r and s, respectively. Then, from the work of Srivastava (1981) containing several general classes of polynomial expansions for multivariable functions defined by multiple series with essentially arbitrary terms, it is not difficult to derive the following expansions for the generalized multiple hypergeometric function defined by (6) [*cf*. Srivastava and Karlsson (1985, p. 339 *et seq*.)]:

$$\mathcal{F}_\ell(\omega;x_1,\ldots,x_n) \equiv F^{\ell p+p_0:p_1;\ldots;p_n}_{\ell q+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;\underset{\sim}{a}),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \Delta(\ell;\underset{\sim}{b}),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\right.$$

$$\left.x_1\omega^\ell\ \ell^{\ell(p-q)},\ldots,x_n\omega^\ell\ \ell^{\ell(p-q)}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma_m(\underset{\sim}{a},\underset{\sim}{c};\underset{\sim}{b},\underset{\sim}{d})}{(\lambda+m)_m}\,\frac{(-\omega)^m}{m!}\ {}_{p+r}F_{q+s+1}\left[\begin{matrix}\underset{\sim}{a}+m,\ \underset{\sim}{c}+m;\\ \lambda+2m+1,\ \underset{\sim}{b}+m,\ \underset{\sim}{d}+m;\end{matrix}\ \omega\right]$$

$$\cdot\ F^{\ell(2+s)+p_0:p_1;\ldots;p_n}_{\ell r+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\lambda+m),\ \Delta(\ell;\underset{\sim}{d}),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \Delta(\ell;\underset{\sim}{c}),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\right.$$

$$\left.x_1\ell^{\ell(2-r+s)},\ldots,x_n\ell^{\ell(2-r+s)}\right], \qquad (16)$$

$p + r \leqq q + s + 2$ (the equality holds true when $|\omega| < 1$);

$$\mathcal{F}_\ell(\omega;x_1,\ldots,x_n) = \sum_{m=0}^{\infty} \Gamma_m(\underset{\sim}{a},\underset{\sim}{c};\underset{\sim}{b},\underset{\sim}{d})\,\frac{(-\omega)^m}{m!}\ {}_{p+r}F_{q+s}\left[\begin{matrix}\underset{\sim}{a}+m,\ \underset{\sim}{c}+m;\\ \underset{\sim}{b}+m,\ \underset{\sim}{d}+m;\end{matrix}\ \omega\right]$$

$$\cdot\ F^{\ell(1+s)+p_0:p_1;\ldots;p_n}_{\ell r+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\underset{\sim}{d}),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \Delta(\ell;\underset{\sim}{c}),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\right.$$

$$\left.x_1\ell^{\ell(1-r+s)},\ldots,x_n\ell^{\ell(1-r+s)}\right], \qquad (17)$$

$p + r \leqq q + s + 1$ (the equality holds true when $|\omega| < 1$);

$$\mathcal{F}_\ell(\omega;x_1,\ldots,x_n) = \beta \sum_{m=0}^{\infty} (1-\alpha m+\beta)_{m-1}\, \Gamma_m(\underset{\sim}{a},\underset{\sim}{c};\underset{\sim}{b},\underset{\sim}{d})\, \frac{(-\omega)^m}{m!}$$

$$\cdot\ {}_{p+r+1}F_{q+s}\left[\begin{matrix}(1-\alpha)m+\beta,\ \underset{\sim}{a}+m,\ \underset{\sim}{c}+m; \\ \underset{\sim}{b}+m,\ \underset{\sim}{d}+m;\end{matrix}\ \omega\right]$$

$$\cdot\ F^{\ell(2+s)+p_0:p_1;\ldots;p_n}_{\ell(2+r)+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;1+\beta/(1-\alpha)),\ \Delta(\ell;\underset{\sim}{d}),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \Delta(\ell;\beta/(1-\alpha)),\ \Delta(\ell;1-\alpha m+\beta),\ \Delta(\ell;\underset{\sim}{c}),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\ x_1\ell^{\ell(s-r)},\ldots,x_n\ell^{\ell(s-r)}\right],\quad \beta \neq 0, \tag{18}$$

$p + r \leqq q + s$ (the equality holds true when $|\omega| < 1$).

It is understood in every case that

$$1 + q_0 + q_k - p_0 - p_k \geqq \ell(p-q) \qquad (k = 1,\ldots,n), \tag{19}$$

where the equality holds true when the variables $|\omega|$ and $|x_1|, \ldots, |x_n|$ are appropriately constrained in accordance with (8) and (9). Furthermore, exceptional parameter values which would render either side invalid or undefined are tacitly excluded.

In view of the principle of confluence exhibited by

$$\lim_{\lambda\to\infty}\left\{(\lambda)_m\left[\frac{z}{\lambda}\right]^m\right\} = z^m = \lim_{\mu\to\infty}\left\{\frac{(\mu z)^m}{(\mu)_m}\right\}, \tag{20}$$

for bounded z and $m = 0, 1, 2, \ldots,$ the expansion formula (17) can easily be shown to be a limiting case of (16) when we replace ω by $\lambda\omega$ and x_k by x_k/λ^{ℓ} $(k = 1,\dots,n)$, and let $\lambda \to \infty$. On the other hand, the expansion formula (18) is not contained in (16); indeed, in its special case when $\alpha = 0$, (18) readily yields (17).

Making use of the relationships (12a, b) and (13a, b), each of the general expansions (16), (17) and (18) can be suitably applied to deduce for the multivariable hypergeometric function (6) a number of Neumann expansions in series of Bessel functions $J_{\nu}(z)$ and $I_{\nu}(z)$.

Our first set of Neumann expansions for the multivariable hypergeometric function (6) would result from (16) if we set

$$p = q = r = s = 0,\ \omega = \mp \frac{1}{4} z^2, \quad \text{and} \quad x_k = (w_k/\ell)^{2\ell} \qquad (k = 1,\dots,n),$$

and apply the definition (12a) or (13a). We thus obtain

$$\left[\frac{1}{2} z\right]^{\lambda} F^{p_0:p_1;\dots;p_n}_{q_0:q_1;\dots;q_n}\left[\begin{matrix}(-1)^{\ell}(w_1 z/2\ell)^{2\ell}\\ \vdots \\ (-1)^{\ell}(w_n z/2\ell)^{2\ell}\end{matrix}\right] = \sum_{m=0}^{\infty} \frac{(\lambda+2m)\Gamma(\lambda+m)}{m!} J_{\lambda+2m}(z)$$

$$\cdot\, F^{2\ell+p_0:p_1;\dots;p_n}_{q_0:q_1;\dots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\lambda+m),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\dots;\underset{\sim}{a}_n; & \\ & w_1^{2\ell},\dots,w_n^{2\ell} \\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\dots;\underset{\sim}{b}_n; & \end{matrix}\right] \tag{21}$$

and

$$\left[\frac{1}{2} z\right]^{\lambda} F^{p_0:p_1;\dots;p_n}_{q_0:q_1;\dots;q_n}\left[\begin{matrix}(w_1 z/2\ell)^{2\ell}\\ \vdots \\ (w_n z/2\ell)^{2\ell}\end{matrix}\right] = \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda+2m)\Gamma(\lambda+m)}{m!} I_{\lambda+2m}(z)$$

$$\cdot F^{2\ell+p_0:p_1;\dots;p_n}_{q_0:q_1;\dots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\lambda+m),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\dots;\underset{\sim}{a}_n; \\ \\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\dots;\underset{\sim}{b}_n;\end{matrix}\quad w_1^{2\ell},\dots,w_n^{2\ell}\right]. \tag{22}$$

If, in the expansion formula (16) (with λ replaced by 2λ), we set

$$p = q = r - 1 = s = 0,\ c^1 = \lambda + \tfrac{1}{2},\ \omega = z,\ \text{ and }\ x_k = (w_k/\ell)^{\ell} \qquad (k = 1,\dots,n),$$

and apply the definition (13b), we get

$$\left[\frac{1}{4}z\right]^{\lambda} F^{p_0:p_1;\dots;p_n}_{q_0:q_1;\dots;q_n}\left[\begin{matrix}(w_1z/\ell)^{\ell} \\ \vdots \\ (w_nz/\ell)^{\ell}\end{matrix}\right] = \frac{\Gamma(\lambda)}{\Gamma(2\lambda)}\, e^{z/2} \sum_{m=0}^{\infty} (-1)^m \frac{(\lambda+m)\Gamma(2\lambda+m)}{m!}\, I_{\lambda+m}\left[\frac{1}{2}z\right]$$

$$\cdot\ F^{2\ell+p_0:p_1;\dots;p_n}_{\ell+q_0:q_1;\dots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;2\lambda+m),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\dots;\underset{\sim}{a}_n; \\ \\ \Delta\left[\ell;\lambda+\frac{1}{2}\right],\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\dots;\underset{\sim}{b}_n;\end{matrix}\quad w_1^{\ell},\dots,w_n^{\ell}\right]. \tag{23}$$

Next we apply the general expansion (17) with

$$p = q = r = s - 1 = 0,\ d^1 = \lambda + 1,\ \omega = \mp\frac{1}{4}z^2,\ \text{ and }\ x_k = (w_k/\ell)^{2\ell} \qquad (k = 1,\dots,n).$$

Making use of the definition (12a) or (13a), we thus obtain the Neumann expansions:

$$\left[\frac{1}{2}z\right]^{\lambda} F^{p_0:p_1;\dots;p_n}_{q_0:q_1;\dots;q_n}\left[\begin{matrix}(-1)^{\ell}(w_1z/2\ell)^{2\ell} \\ \vdots \\ (-1)^{\ell}(w_nz/2\ell)^{2\ell}\end{matrix}\right] = \Gamma(\lambda+1) \sum_{m=0}^{\infty} \frac{\left[\frac{1}{2}z\right]^m}{m!}\, J_{\lambda+m}(z)$$

$$\cdot\, F^{2\ell+p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\lambda+1),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\ w_1^{2\ell},\ldots,w_n^{2\ell}\right] \tag{24}$$

and

$$\left[\frac{1}{2}z\right]^{\lambda} F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}(w_1z/2\ell)^{2\ell} \\ \vdots \\ (w_nz/2\ell)^{2\ell}\end{matrix}\right] = \Gamma(\lambda+1)\sum_{m=0}^{\infty}\frac{\left[-\frac{1}{2}z\right]^m}{m!}\, I_{\lambda+m}(z)$$

$$\cdot\, F^{2\ell+p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\lambda+1),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n; \\ \\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\ w_1^{2\ell},\ldots,w_n^{2\ell}\right]. \tag{25}$$

It should be remarked in passing that a very specialized version of the Neumann expansion (21) when $\ell = 1$ and $p_0 = q_0 = 0$ was given by Niukkanen (1983, p. 1823, Equation (45)). See also Srivastava (1985a, p. L230) for the special case $\ell = 1$ of each of the expansion formulas (16), (17), and (18).

3. FURTHER NEUMANN EXPANSIONS

Let us recall the familiar result (cf., e.g., Watson 1944, p. 147, Equation (1))

$$J_\mu(z)J_\nu(z) = \frac{\left[\frac{1}{2}z\right]^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\ {}_2F_3\left[\begin{matrix}\Delta(2;\mu+\nu+1); \\ \\ \mu+1,\ \nu+1,\ \mu+\nu+1;\end{matrix}\ -z^2\right] \tag{26}$$

or, equivalently,

$$I_\mu(z)I_\nu(z) = \frac{\left[\frac{1}{2}z\right]^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\,{}_2F_3\left[\begin{matrix}\Delta(2;\mu+\nu+1); \\ \mu+1,\ \nu+1,\ \mu+\nu+1;\end{matrix}\ z^2\right], \tag{27}$$

each of which incidentally is an immediate consequence of a well-known formula expressing the product (see, for example, Erdélyi et al. 1953, vol. 1, p. 185, Equation (2))

$${}_0F_1\left[\begin{matrix}-; \\ \rho;\end{matrix}\ z\right]\,{}_0F_1\left[\begin{matrix}-; \\ \sigma;\end{matrix}\ z\right]$$

as a hypergeometric ${}_2F_3$ function. In view of the relationships (26) and (27), we can apply the general expansion (16) in order to derive, for the multivariable hypergeometric function (6), expansions of the Neumann type in series of products of Bessel functions. Thus, if in (16), we write $\lambda = \mu + \nu$, and set

$$\begin{cases} p = q = 0,\ r = s = 2,\ c^1 = \frac{1}{2}(\mu+\nu+1),\ c^2 = \frac{1}{2}(\mu+\nu+2), \\ d^1 = \mu + 1,\ d^2 = \nu + 1,\ \omega = \mp z^2,\ \text{and}\ x_k = (w_k/\ell)^{2\ell} \qquad (k = 1,\dots,n), \end{cases}$$

we shall get the expansions

$$\left[\frac{1}{2}z\right]^{\mu+\nu} F^{p_0:p_1;\dots;p_n}_{q_0:q_1;\dots;q_n}\left[\begin{matrix}(-1)^\ell(w_1z/\ell)^{2\ell} \\ \vdots \\ (-1)^\ell(w_nz/\ell)^{2\ell}\end{matrix}\right]$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\mu+\nu}\sum_{m=0}^{\infty}\frac{(\mu+\nu+2m)(\mu+\nu)_m}{m!}\,J_{\mu+m}(z)J_{\nu+m}(z)$$

$$\cdot\; F^{4\ell+p_0:p_1;\ldots;p_n}_{2\ell+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\mu+\nu+m),\ \Delta(\ell;\mu+1),\ \Delta(\ell;\nu+1),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \Delta(\ell;(1+\mu+\nu)/2),\ \Delta(\ell;1+(\mu+\nu)/2),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\right.$$

$$\left.\begin{matrix}w_1^{2\ell},\ldots,w_n^{2\ell}\end{matrix}\right] \qquad (28)$$

and

$$\left[\frac{1}{2}z\right]^{\mu+\nu} F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}(w_1z/\ell)^{2\ell}\\ \vdots\\ (w_nz/\ell)^{2\ell}\end{matrix}\right]$$

$$= \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\mu+\nu}\sum_{m=0}^{\infty}(-1)^m\frac{(\mu+\nu+2m)(\mu+\nu)_m}{m!}I_{\mu+m}(z)I_{\nu+m}(z)$$

$$\cdot\; F^{4\ell+p_0:p_1;\ldots;p_n}_{2\ell+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\mu+\nu+m),\ \Delta(\ell;\mu+1),\ \Delta(\ell;\nu+1),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \Delta(\ell;(1+\mu+\nu)/2),\ \Delta(\ell;1+(\mu+\nu)/2),\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\right.$$

$$\left.\begin{matrix}w_1^{2\ell},\ldots,w_n^{2\ell}\end{matrix}\right]. \qquad (29)$$

In their special cases when $\mu = \nu$, the expansion formulas (28) and (29) simplify considerably, and we have

$$\left[\frac{1}{2}z\right]^{2\lambda} F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}(-1)^{\ell}(w_1z/\ell)^{2\ell}\\ \vdots\\ (-1)^{\ell}(w_nz/\ell)^{2\ell}\end{matrix}\right] = \frac{\lambda\{\Gamma(\lambda)\}^2}{\Gamma(2\lambda)}\sum_{m=0}^{\infty}\frac{(\lambda+m)\Gamma(2\lambda+m)}{m!}\left\{J_{\lambda+m}(z)\right\}^2$$

$$\cdot\ F^{3\ell+p_0:p_1;\ldots;p_n}_{\ell+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;2\lambda+m),\ \Delta(\ell;\lambda+1),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \Delta\left[\ell;\lambda+\frac{1}{2}\right],\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\ w_1^{2\ell},\ldots,w_n^{2\ell}\right] \tag{30}$$

and

$$\left[\frac{1}{2}z\right]^{2\lambda} F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}(w_1z/\ell)^{2\ell}\\ \vdots\\ (w_nz/\ell)^{2\ell}\end{matrix}\right] = \frac{\lambda\{\Gamma(\lambda)\}^2}{\Gamma(2\lambda)}\sum_{m=0}^{\infty}(-1)^m\frac{(\lambda+m)\Gamma(2\lambda+m)}{m!}\left\{I_{\lambda+m}(z)\right\}^2$$

$$\cdot\ F^{3\ell+p_0:p_1;\ldots;p_n}_{\ell+q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;2\lambda+m),\ \Delta(\ell;\lambda+1),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \Delta\left[\ell;\lambda+\frac{1}{2}\right],\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\ w_1^{2\ell},\ldots,w_n^{2\ell}\right]. \tag{31}$$

Finally, for the multivariable hypergeometric function (6) we give an expansion analogous to (30) and (31), but in series of mixed products of the type $J_\nu(z)I_\nu(z)$. Indeed, it is easily seen from the definitions (12a) and (13a) that (cf. Luke 1962, p. 25, Equation (20))

$$J_\nu(z)I_\nu(z) = \frac{\left[\frac{1}{2}z\right]^{2\nu}}{\{\Gamma(\nu+1)\}^2}\,{}_0F_3\left[\begin{matrix}\rule{5em}{0.4pt};\\ \Delta(2;\nu+1),\ \nu+1;\end{matrix}\ -\frac{z^4}{64}\right], \tag{32}$$

which incidentally is an immediate consequence of a well-known formula expressing the product (see, for example, Erdélyi et al. 1953, vol. 1, p. 186, Equation (3))

$${}_0F_1\left[\begin{matrix}-;\\ \rho;\end{matrix}\ z\right]{}_0F_1\left[\begin{matrix}-;\\ \rho;\end{matrix}\ -z\right]$$

as a hypergeometric ${}_0F_3$ function. In view of the relationship (32), we now apply the general result (16) with

$$p = q = r = s - 2 = 0,\ d^1 = \tfrac{1}{2}(\lambda+1),\ d^2 = \tfrac{1}{2}\lambda + 1,\ \omega = -\frac{z^4}{64},\ \text{and}$$
$$x_k = (w_k/\ell)^{4\ell} \qquad (k = 1,\ldots,n),$$

and we arrive at the desired expansion formula:

$$\left[\frac{1}{2}z\right]^{2\lambda} F^{p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}(-1)^{\ell}(w_1z/2\ell\sqrt{2})^{4\ell}\\ \vdots\\ (-1)^{\ell}(w_nz/2\ell\sqrt{2})^{4\ell}\end{matrix}\right]$$

$$= \Gamma(\lambda+1)\sum_{m=0}^{\infty}\frac{(\lambda+2m)\Gamma(\lambda+m)}{m!}J_{\lambda+2m}(z)I_{\lambda+2m}(z)$$

$$\cdot\ F^{4\ell+p_0:p_1;\ldots;p_n}_{q_0:q_1;\ldots;q_n}\left[\begin{matrix}\Delta(\ell;-m),\ \Delta(\ell;\lambda+m),\ \Delta(\ell;(1+\lambda)/2),\ \Delta(\ell;1+\lambda/2),\ \underset{\sim}{a}_0:\ \underset{\sim}{a}_1;\ldots;\underset{\sim}{a}_n;\\ \underset{\sim}{b}_0:\ \underset{\sim}{b}_1;\ldots;\underset{\sim}{b}_n;\end{matrix}\ w_1^{4\ell},\ldots,w_n^{4\ell}\right]. \qquad (33)$$

For several further applications of the various Neumann expansions presented here to simpler special functions in one and more variables, the interested users of such classes of multiple hypergeometric series as those considered in this paper should refer, for instance, to the works of Bailey (1935), Erdélyi _et al._

(1953, vol. 2, Chapter 7), Luke (1962, Chapters 1 and 7; 1969, Chapter 9; 1975, p. 223 *et seq*.), Slater (1960, Chapter 2), Srivastava (1965, 1966a, b, 1967, 1981), Srivastava and Daoust (1969), Srivastava and Karlsson (1985, Section 9.4), Srivastava and Panda (1976a, b), and Watson (1944, Chapters 5 and 11).

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