SPECTRA AND INVERSE SIGN PATTERNS OF NEARLY SIGN-NONSINGULAR MATRICES

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Abstract

A nearly sign-nonsingular (NSNS) matrix is a real $n \times n$ matrix having at least two nonzero terms in the expansion of its determinant with precisely one of these terms having opposite sign to all the other terms. Using graph-theoretic techniques, we study the spectra of irreducible NSNS matrices in normal form. Specifically, we show that such a matrix can have at most one nonnegative eigenvalue, and can have no nonreal eigenvalue z in the sector $\{z: |arg\ z| \le \pi/(n-1)\}$. We also derive results concerning the sign pattern of inverses of these matrices.

1. Introduction. An interesting class of matrices, which has only recently attracted attention, is the class of nearly sign-nonsingular matrices (see [5]). A *nearly sign-nonsingular (NSNS) matrix* is a real square matrix having at least two nonzero terms in the expansion of its determinant with precisely one of these terms having opposite sign to all the other terms. In contrast, *a sign-nonsingular matrix* has at least one nonzero term in the expansion of its determinant, and all terms in the determinantal expansion have the same sign. The well-known class of sign-nonsingular matrices was introduced by Bassett, Maybee, and Quirk [1].

In order to investigate the properties of NSNS matrices, we require the following notation and definitions. If A is an $n \times n$ matrix and $I \subseteq N \equiv \{1,2,3,\cdots,n\}$, then $A[I]$ denotes the principal submatrix in the rows and columns of I. A *directed graph (or digraph)* D is a pair (V, E) where V is a finite set and E is a set of ordered pairs of distinct elements of V, i.e., $E \subset V \times V$ such that E contains no elements of the form (v, v) for $v \in V$. The set V is called the *vertex set of* D and the set E is called the *arc set of* D. Let $A = [a_{ij}]$ be a real $n \times n$ matrix. The *digraph* of A is defined by $D(A) = (N, E)$ where the arc $(i, j) \in E$ if and only if $a_{ij} \neq 0$ for $i \neq j$. Thus $D(A)$ is always a loop-free, simple digraph.

Suppose that $D = (V, E)$ is a digraph. If the set $I = \{i_1, i_1, ..., i_q\}$ of vertices consists of q distinct elements of V, with $q > 1$, and each of the arcs $(i_k, i_{k+1}) \in E$ for $k = 1, 2, ..., q-1$, then the ordered set $p = (i_1, i_2, ..., i_q)$ is a *path* of D. If $p = (i_1, i_2, ..., i_q)$ is a path of $D(A)$, then the corresponding product $a_{i_1, i_2} a_{i_2, i_3} \cdots a_{i_{q-1}, i_q}$ of entries of A is a *path product* of A. We use the notation $p(i \to j)$ to denote a path from vertex i to vertex j, and $A[p]$ to designate the corresponding path product in A. A digraph is *strongly connected (strong)* if, given any two distinct vertices i and j, there exists a path $p(i \to j)$. We are concerned exclusively with irreducible matrices, so that for any matrix under consideration, $D(A)$ is assumed to be strongly connected.

The ordered set $(i_1, i_2, \ldots, i_q, i_1)$ is called a *cycle* of D if $(i_1, i_2, \ldots, i_q)$ is a path and $(i_q, i_1) \in E$. If $c = (i_1, i_2, \ldots, i_q, i_1)$ is a cycle, the *length* $\ell(c)$ of c is defined to be q (and to be consistent, we define the *length* $\ell(p)$ of path $p(i_1 \to i_q)$ to be $q - 1$). If c is a cycle of $D(A)$, we denote by $A[c]$ the corresponding cycle product of A, consisting of the product of the entries of A corresponding to the arcs of c. Two cycles are *disjoint* if they have no common vertex.

The following concepts are from [8]. A *minimal critical set* for D is a subset c_0 of V, of minimal cardinality, such that the subdigraph of D induced by $V \backslash c_0$ is acyclic. The unique number of elements in a minimal critical set is referred to as the *critical number* of D. If a digraph D has critical number r, then D is called *r-critical.* A (loop-free) digraph is called *intercyclic* if any two cycles share a common vertex. We state the following fundamental result, which follows from a theorem of McCuaig [8, Theorem 3.1].

Theorem 1.1 [8]. Let the digraph D be strong and intercyclic. Then a minimal critical set for D contains at most 3 vertices.

We also state the following characterization of NSNS matrices.

Theorem 1.2 [5, Section 2]. Suppose that B is an $n \times n$ irreducible NSNS matrix. Then there exist permutation matrices P, Q and signature matrices S_1, S_2 such that $A = S_1QBPS_2$ is a NSNS matrix satisfying the following four conditions:

i) $a_{ii} < 0$, for $i = 1, 2, \ldots, n$;

ii) $D(A)$ is intercyclic, with critical number $r \in \{1, 2, 3\}$;

iii) $a_{ij} \geq 0$ for every $i \neq j$;

iv) all of the nonzero entries above the main diagonal of A are in the last r columns.

It should be noted that if A is any matrix that satisfies conditions i), ii), and iii), then A must be a NSNS matrix (see [5]).

The NSNS matrix A of Theorem 1.2 is said to be in *normal form.* Specifically, if A is in normal form, then

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square matrices of orders $n - r$ and r, respectively, A_{11} is lower triangular, each diagonal entry of A_{11} and A_{22} is negative, and all other entries are nonnegative. A NSNS matrix in normal form is said to be a *1-spike* if $r = 1$, a *2-spike* if $r = 2$, or a *3-spike* if $r = 3$. Theorem 1.1 implies that every NSNS matrix in normal form is a 1-spike, a 2-spike, or a 3-spike. We will be concerned exclusively with irreducible NSNS matrices in normal form.

Example 1.3.

$$A = \begin{bmatrix} -0.9 & 0 & 0 & 0 & 1 & 1 \\ 1 & -0.9 & 0 & 0 & 0 & 1 \\ 1 & 0 & -0.9 & 0 & 1 & 0 \\ 0 & 1 & 1 & -0.9 & 0 & 0 \\ 0 & 1 & 0 & 1 & -0.9 & 1 \\ 0 & 0 & 1 & 1 & 1 & -0.9 \end{bmatrix}$$

is an irreducible 2-spike NSNS matrix in normal form. □

We note that if A is a reducible NSNS matrix in normal form, then $D(A)$ has exactly one strong component with more than one vertex.

If A is an $n \times n$ irreducible NSNS matrix in normal form, then we may write $A = T - \beta I_n$, where $T = [t_{ij}]$ is nonnegative and irreducible, $D(T)$ is strong and intercyclic, $\beta > \max_{1 \le i \le n} (t_{ii})$, and I_n is the $n \times n$ identity matrix. From this expression for A, it should come as no surprise that we make extensive use of the well-known Perron-Frobenius theory for irreducible nonnegative matrices. We call a vector *unisigned* if its entries are all positive, all negative or all zero. A vector that is not unisigned is called *non-unisigned.*

Theorem 1.4 [2]. If T is an irreducible nonnegative square matrix, then

i) $\rho(T)$, the spectral radius of T, is an eigenvalue of T;

ii) T has exactly one (up to scalar multiples) nonnegative eigenvector, this eigenvector is positive (i.e. unisigned), and it corresponds to the eigenvalue $\rho(T)$.

We conclude our introduction with a key result in our study of NSNS matrices that is implied by the intercyclic nature of their digraphs. Suppose that $D(A)$ has q cycles $\{c_1, c_2, ..., c_q\}$. Let $A[c_i]$, $i = 1, 2, ..., q$, be the corresponding cycle products of the matrix A, and let V_i be the set of vertices of N not belonging to the cycle c_i. Using this notation we have the following.

Theorem 1.5. Let $A = [a_{ij}]$ be an $n \times n$ irreducible NSNS matrix in normal form with $a_{ii} = -a_i < 0$. Then

i) $$\det A = (-1)^n \prod_{k=1}^{n} a_k + (-1)^{n-1} \sum_{i=1}^{q} A[c_i] \prod_{j \in V_i} a_j$$

and

ii) $$\det(A - \lambda I_n) = (-1)^n \prod_{k=1}^{n} (\lambda + a_k) + (-1)^{n-1} \sum_{i=1}^{q} A[c_i] \prod_{j \in V_i} (\lambda + a_j).$$

Note that the sign of the term in $\det A$ that consists of the product of the diagonal entries of A is $(-1)^n$, and the sign of all the other terms in $\det A$ is $(-1)^{n-1}$.

2. Spectral Properties of Irreducible NSNS Matrices in Normal Form. We have noted that if A is an $n \times n$ irreducible NSNS matrix in normal form, then we may write $A = T - \beta I_n$, with T nonnegative and irreducible, and $\beta > 0$. Thus, by Theorem 1.4, every irreducible NSNS matrix has at least one real eigenvalue. The matrix in Example 1.3 has precisely one nonnegative eigenvalue λ ($\lambda \cong 1.5002$). This observation begs the question of how many nonnegative eigenvalues an irreducible NSNS matrix in normal form can have. The answer is at most one (with algebraic multiplicity 1), as we prove in Theorem 2.3. We first prove this result (in Lemma 2.2) in the case when A has all diagonal entries equal.

We begin with an observation that is a consequence of Theorem 1.4.

Observation 2.1. Let A be an $n \times n$ irreducible NSNS matrix in normal form, and further suppose that $A = T - \beta I_n$, where T is an irreducible nonnegative matrix and $\beta > 0$. Then the following conditions are equivalent:

i) A has at most one nonnegative eigenvalue, which (if it exists) is given by $\rho(T) - \beta$;

ii) For any $\varepsilon \geq 0$, every nullvector of $A - \varepsilon I_n$ is unisigned.

Lemma 2.2. Let A be an $n \times n$ irreducible NSNS matrix in normal form and suppose that $a_{ii} = -\alpha < 0$, for $i = 1, 2, \ldots, n$. Then A can have at most one nonnegative eigenvalue, and for any $\varepsilon \geq 0$, every nullvector of $A - \varepsilon I_n$ is unisigned.

Proof. Consider the matrix $A_\lambda \equiv A - \lambda I_n$ and let r denote the critical number of $D(A)$. Then

$$A_\lambda = \begin{bmatrix} A_{11} - \lambda I_{n-r} & A_{12} \\ A_{21} & A_{22} - \lambda I_r \end{bmatrix}$$

where A_{11} and A_{22} are square submatrices of orders $n-r$ and r, respectively, and A_{11}

is lower triangular. Note that all diagonal entries of A_λ are equal to $-\lambda-\alpha$. We let $\lambda' = \lambda + \alpha$, and define $s(\lambda')$ to be the polynomial $\det A_\lambda$ expressed in terms of the variable λ'. Then Theorem 1.5 (ii) implies that $s(\lambda') = (-1)^n(\lambda')^n + (-1)^{n-1}\sum_{i=0}^{n-2} p_i\,(\lambda')^i$, with constants $p_i \geq 0$, $i = 0, 1, 2, \ldots, n-2$, and at least one p_i positive, as $D(A)$ is strong. By Descartes' rule of signs for real polynomials, the equation $s(\lambda') = 0$ has precisely one positive root. This in turn implies that A has at most one nonnegative eigenvalue, since all of the diagonal entries of A are negative. Writing $A = T - \beta I_n$ with $\beta > \max_{1 \leq i \leq n} (t_{ii})$, the only nonnegative eigenvalue that A can have is $\rho(T) - \beta$, and the second result follows from Observation 2.1.□

Theorem 2.3. Let A be an $n \times n$ irreducible NSNS matrix in normal form. Then A has at most one nonnegative eigenvalue.

Proof. Observation 2.1 implies that A has more than one nonnegative eigenvalue if and only if there is an $\varepsilon \geq 0$ such that $A - \varepsilon I_n$ has a non-unisigned nullvector, say v. Let $F = [f_{ij}]$ be the diagonal matrix with $f_{ii} = \dfrac{a_{11} - \varepsilon}{a_{ii} - \varepsilon}$. Then $F(A - \varepsilon I_n)$ is an irreducible NSNS matrix in normal form, all the diagonal entries of this matrix are equal, and $F(A - \varepsilon I_n)v = 0$, in contradiction to Lemma 2.2. Thus A can have at most one nonnegative eigenvalue. □

Writing $A = T - \beta I_n$ with T nonnegative and $\beta > 0$, the above results show that an irreducible NSNS matrix A has a unique positive eigenvalue if and only if $\rho(T) > \beta > \max_{1 \le i \le n} (t_{ii})$, has a zero eigenvalue if and only if $\rho(T) = \beta$, and has no nonnegative eigenvalue if and only if $\beta > \rho(T)$. This last inequality is true exactly when $-A$ is a nonsingular M-matrix (see [2]); the following result applies in this case.

Theorem 2.4 [9, Th. 2.5]. Suppose that A is an $n \times n$ irreducible NSNS matrix and that $A = T - \beta I_n$ with T nonnegative. Then the following conditions are equivalent.

i) $\beta > \rho(T)$;

ii) $\det(-A[I]) > 0$ for every $I \subseteq N$;

iii) A^{-1} is entrywise negative.

Note that this shows that $sign(\det A) = (-1)^n$ when $\beta > \rho(T)$. Obviously $\det A = 0$ in case $\beta = \rho(T)$, and we now determine the sign of $\det A$ in the remaining case.

Lemma 2.5. Suppose that A is an $n \times n$ irreducible NSNS matrix in normal form, $A = T - \beta I_n$ with T nonnegative and $\beta < \rho(T)$. Then $sign(\det A) = (-1)^{n-1}$.

Proof. For any matrix A, $\det A$ is equal to the product of the eigenvalues of A. For A satisfying the conditions of the lemma, there is exactly one positive eigenvalue, thus $sign(\det A) = (-1)^{n-1-2k}$ where $2k$ is the number of nonreal eigenvalues. □

Theorem 2.3 gives the following new result on the spectrum of an irreducible nonnegative matrix with an intercyclic digraph.

Corollary 2.6. Let T be an $n \times n$ irreducible nonnegative matrix and $\hat{t} = \max_{1 \le i \le n} (t_{ii})$, and suppose that $D(T)$ is intercyclic. Then T cannot possess a positive eigenvalue λ with $\hat{t} < \lambda < \rho(T)$.

Proof. For any $\varepsilon > 0$, $T - (\hat{t} + \varepsilon)I_n$ is an irreducible NSNS matrix in normal form, and its only possible nonnegative eigenvalue is $\rho(T) - (\hat{t} + \varepsilon)$. Thus T can have no eigenvalue in the specified range. □

We now investigate nonreal eigenvalues of $A = T - \beta I_n$, an $n \times n$ irreducible NSNS matrix in normal form. If $\beta > \rho(T)$, then, as noted previously, $-A$ is a nonsingular M-matrix and all of its eigenvalues lie in the open left half plane (see [2]). If $\beta = \rho(T)$, then $-A$ is a singular M-matrix and all of its eigenvalues lie in the closed left half plane. If $\rho(T) > \beta$, we prove that nonreal eigenvalues cannot lie in the closed sector with semiangle $\pi/(n-1)$ about the nonnegative real axis. For $n = 3$, it is easy to see (from the facts that A has a positive eigenvalue and trace $A < 0$) that A has no nonreal eigenvalue in the closed right-half plane. For general n, we develop some inequalities.

The characteristic polynomial for an $n \times n$ irreducible NSNS matrix $A = [a_{ij}]$ in normal form with $a_{ii} = -a_i < 0$ can be written as

$$\lambda^n + s_1\lambda^{n-1} + \sum_{k=2}^{n} (s_k - m_k)\lambda^{n-k}, \tag{1}$$

where

$$s_k = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1} a_{i_2} \cdots a_{i_k} > 0 \tag{2}$$

and

$$\begin{aligned} m_k &= \sum_{1 \le i_1 < \cdots < i_k \le n} (-1)^{k-1} \left(\det A[i_1, i_2, \cdots, i_k] - (-1)^k a_{i_1} a_{i_2} \cdots a_{i_k}\right) \\ &\equiv \sum_{1 \le i_1 < \cdots < i_k \le n} U[i_1, i_2, \cdots, i_k] \ge 0. \end{aligned} \tag{3}$$

Here $(-1)^k s_k$ is the k^{th} elementary symmetric function of the matrix $diag(-a_i)$, and $(-1)^{k-1} m_k$ is the sum of the k^{th} order principal minors of A with the product of the corresponding terms in $diag(-a_i)$ subtracted. For $2 \le k \le n$, each nonzero term in $U[i_1, i_2, \cdots, i_k]$ is positive, because A is a NSNS matrix in normal form.

Using the above notation, we have the following basic inequality.

Lemma 2.7. If A is an $n \times n$ irreducible NSNS matrix in normal form with characteristic polynomial (1), then

$$s_k m_{k-1} \le s_{k-1} m_k, \quad \text{for} \quad 3 \le k \le n,$$

where s_k, m_k are given by (2), (3).

Proof. Let k be fixed, $3 \le k \le n$. Then from (2), (3)

$$s_k m_{k-1} = \left(\sum_{1 \le j_1 < \cdots < j_k \le n} a_{j_1} a_{j_2} \cdots a_{j_k}\right)\left(\sum_{1 \le i_1 < \cdots < i_{k-1} \le n} U[i_1, i_2, \cdots, i_{k-1}]\right)$$

$$= \sum_{1 \le i_1 < \cdots < i_{k-1} \le n} \left\{\left(\sum_{\substack{1 \le j_1 < \cdots < j_k \le n \\ j_p \notin \{i_1, i_2, \cdots, i_{k-1}\},\ 1 \le p \le k}} a_{j_1} \cdots a_{j_{p-1}} a_{j_{p+1}} \cdots a_{j_k}\right)\left(a_{j_p} U[i_1, i_2, \cdots, i_{k-1}]\right)\right\}$$

$$\le \sum_{1 \le i_1 < \cdots i_{k-1} \le n} \left\{\left(\sum_{\substack{1 \le j_1 < \cdots < j_k \le n \\ j_p \notin \{i_1, i_2, \cdots, i_{k-1}\},\ 1 \le p \le k}} a_{j_1} \cdots a_{j_{p-1}} a_{j_{p+1}} \cdots a_{j_k}\right) U[i_1, \cdots, j_p, \cdots, i_{k-1}]\right\}$$

$$\le \left(\sum_{1 \le j_1 < \cdots < j_{k-1} \le n} a_{j_1} a_{j_2} \cdots a_{j_{k-1}}\right)\left(\sum_{1 \le i_1 < \cdots < i_k \le n} U[i_1, i_2, \cdots, i_k]\right)$$

$$= s_{k-1} m_k. \quad \square$$

Corollary 2.8. With the hypotheses of Lemma 2.7, if $s_{k-1} - m_{k-1} \le 0$ for some k, with $3 \le k \le n$, then $s_k - m_k \le 0$.

Proof. Assume that $s_{k-1} - m_{k-1} \le 0$. Then from Lemma 2.7

$$s_k m_{k-1} \le s_{k-1} m_k \le m_{k-1} m_k$$

which implies that $s_k \le m_k$, as $m_{k-1} \ge s_{k-1} > 0$. □

Since an $n \times n$ irreducible NSNS matrix $A = T - \beta I_n$ with $\rho(T) > \beta$ has one positive eigenvalue, its characteristic polynomial (1) must have at least one sign change in its coefficients. Note that $s_1 > 0$ and $s_n - m_n = (-1)^n \det A < 0$, by Lemma 2.5. The coefficient of λ^{n-1} is positive, and if λ^{n-r}, for $n - 1 \geq r \geq 2$, is the largest power of λ with a nonpositive coefficient, then by Corollary 2.8, all smaller powers also have nonpositive coefficients. Thus the characteristic polynomial (1) must have precisely one sign change in its coefficients. This property leads to our sector exclusion result. We always take $\arg z \in (-\pi, \pi]$.

Theorem 2.9. Let A be an $n \times n$ irreducible NSNS matrix in normal form. Then A has no nonreal eigenvalue z in the sector $\{z: |\arg z| \leq \pi/(n-1)\}$.

Proof. If $n = 2$, the result follows since both eigenvalues of A are real; so assume $n \geq 3$. If A has no positive eigenvalue, then as noted previously, the result is trivial and the sector can be extended to $\{z: |\arg z| \leq \pi/2\}$ for all n. Therefore, assume that A has a positive eigenvalue μ. The characteristic polynomial (1) can be factored using Horner's algorithm as

$$(\lambda - \mu)\left(\lambda^{n-1} + \sum_{k=2}^{n} b_k \lambda^{n-k}\right), \tag{4}$$

where

$$b_2 = s_1 + \mu > 0$$
$$b_k = s_{k-1} - m_{k-1} + \mu b_{k-1}, \qquad 3 \leq k \leq n,$$

and

$$0 = s_n - m_n + \mu b_n.$$

We claim that $b_k > 0$ for $2 \le k \le n$. To prove this claim, first note that the constant term $b_n > 0$, since $m_n > s_n$ and $b_n = (m_n - s_n)/\mu$. Assume that the coefficient $s_r - m_r$ of λ^{n-r} in (1), with $n - 1 \ge r \ge 2$, is nonpositive and that (when $r \ge 3$) $s_k > m_k$ for $2 \le k \le r - 1$. Then, by Corollary 2.8, $s_k \le m_k$ for $r \le k \le n$. Since $b_n > 0$ and $b_{k-1} = (b_k + m_{k-1} - s_{k-1})/\mu$, it follows that $b_{k-1} > 0$ for $k = n, n-1, \cdots, r+1$. And since $b_2 > 0$ and $b_k = s_{k-1} - m_{k-1} + \mu b_{k-1}$, it follows that $b_k > 0$ for $k = 3, 4, \cdots, r$. Thus the claim is proved and the polynomial of degree $(n-1)$ in (4) has all coefficients positive. By considering the imaginary part of this polynomial, it is easily shown (see, e.g. [4, Cor. 1]) that it can have no zero in $\{z: |\arg z| \le \pi/(n-1)\}$.□

We note that the existence of a *unique* positive eigenvalue for $A = T - \beta I$ with $\rho(T) > \beta$ can be seen from the form of the characteristic polynomial (4) with $b_k > 0$; see Theorem 2.3. The result of Theorem 2.9 can also be interpreted as a sector exclusion result for an intercyclic nonnegative matrix (cf. Corollary 2.6). The results of Theorems 2.3 and 2.9 apply to a reducible NSNS matrix where n denotes the order of its unique strong component.

The following result shows how a point outside the forbidden sector of Theorem 2.9 can be realized as an eigenvalue of some NSNS matrix.

Theorem 2.10. Let A be an $n \times n$ irreducible NSNS matrix in normal form. Then for $d > 0$, dA and $A - dI_n$ are NSNS matrices in normal form. If z is a nonreal eigenvalue of A and $w \ne z$ is a complex number such that $|\arg w| \ge \arg z$, then there exists a NSNS matrix B in normal form such that w is an eigenvalue of B.

Proof. The statement concerning dA and $A - dI_n$ follows from Theorem 1.2 and the definition of normal form. Since it is clear (by continuity) that any negative number is an eigenvalue of some NSNS matrix, assume that $w = u + iv$ with $v > 0$, and that $z = x + iy$ with $y > 0$. Set $d = v/y$ and $g = xv/y - u$. Then $d > 0$, $g \geq 0$ and w is an eigenvalue of the NSNS matrix $dA - gI_n$, which is in normal form.□

By the result of Theorem 2.10, if we could find an $n \times n$ NSNS matrix with an eigenvalue arbitrarily close to the boundary of the sector given in Theorem 2.9, then every point outside the forbidden sector would be an eigenvalue of some $n \times n$ NSNS matrix. However, the following shows that the result of Theorem 2.9 is not sharp for $n = 3$.

Theorem 2.11. Let A be an $n \times n$ irreducible NSNS matrix in normal form with a (unique) positive eigenvalue μ. Then, for $n = 3$, A has no nonreal eigenvalue z in the sector $\{z: \ |arg\ z| < 2\pi/n\}$.

Proof. For $n = 3$, the characteristic polynomial of A is given by (1) and (4), thus

$$\lambda^3 + s_1\lambda^2 + (s_2 - m_2)\lambda + s_3 - m_3 = (\lambda - \mu)\left(\lambda^2 + b_2\lambda + b_3\right)$$

where $b_2 = s_1 + \mu$, $b_3 = s_2 - m_2 + \mu b_2$, $0 = s_3 - m_3 + \mu b_3$. We claim that the quadratic above has no zero in the sector $\{z: \ |arg\ z| < 2\pi/3\}$. To prove this claim, write $\lambda = re^{i\theta}$ with $r > 0$ and $\theta = 2\pi/3$. Then

$$\lambda^2 + b_2\lambda + b_3 = U(r) + iV(r),$$

where

$$U(r) = r^2 \cos 2\theta + b_2 r \cos \theta + b_3$$

and

$$V(r) = r^2 \sin 2\theta + b_2 r \sin \theta .$$

Let $I_0^\infty(V(r)/U(r))$ denote the Cauchy index of the real rational function $V(r)/U(r)$. Then (see Section 41 of [6] and Theorem 2.1 of [3]) the quadratic has no zero in the sector exactly when $I_0^\infty(V(r)/U(r)) = -1$; that is, when $V(r)/U(r)$ has a unique jump from $+\infty$ to $-\infty$ as r increases from 0 to $+\infty$. The function $V(r)$ has a unique positive zero at $r = b_2$, and $U(b_2) = -b_2^2 + b_3$. Thus $I_0^\infty(V(r)/U(r)) = -1$ if and only if $b_2^2 > b_3$, which is equivalent to $b_2(s_1 + \mu) > (s_2 - m_2 + \mu b_2)$, or $s_1(s_1 + \mu) > s_2 - m_2$. But this last inequality is true, since $s_1^2 > s_2$ by (2) and $\mu > 0$, $m_2 \geq 0$. Thus the claim is proved. □

For $n = 3$, A does not have an eigenvalue on the boundary of the sector given in Theorem 2.11, since $U(r)$ and $V(r)$ do not vanish together. However,

$$A = \begin{bmatrix} -\beta & 0 & 1 \\ 1 & -\beta & 0 \\ 0 & 1 & -\beta \end{bmatrix} \quad \text{with} \quad 1 > \beta > 0,$$

is a 1-spike NSNS matrix in normal form with $\mu = 1 - \beta$, and an eigenvalue $e^{2\pi i/3} - \beta$, which is arbitrarily close to the boundary of the sector of Theorem 2.11 as $\beta \to 0^+$.

Note that for $n = 3$, we have the following spectral result. Let A be an irreducible 3×3 matrix with negative diagonal entries and nonnegative off diagonal entries. Then A has at most one nonnegative eigenvalue and the other eigenvalues lie in an open sector of semi-angle $\pi/3$ about the negative real axis. In the case that $-A$ is an M-matrix, then all eigenvalues lie in a sector of semi-angle $\pi/6$ about the negative real axis [4].

We conjecture that the result of Theorem 2.11 is also correct for $n = 4$; that is, all eigenvalues except μ lie in the open left-half plane. By the Routh-Hurwitz conditions on the cubic in (4), this is true if and only if $b_2 b_3 > b_4$, which is equivalent to $s_1(s_2 - m_2 + \mu s_1 + \mu^2) > s_3 - m_3$. We have not been able to verify this, although numerical results support our conjecture. The result of Theorem 2.11 is *not* true for $n = 5$. The matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 10^4 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 50 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 10^4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} - 0.001\, I_5$$

is a 1-spike NSNS matrix with an eigenvalue approximately $1.84 + 3.19i$, which has argument $\theta = .33\pi < 2\pi/5$. For $n \geq 5$, we do not know how close an eigenvalue can come to the boundary of the sector in Theorem 2.9.

3. Sign Patterns of Inverses of NSNS Matrices. If $A = [a_{ij}]$ is a real $n \times n$ matrix, the *sign pattern* of A is the matrix $sgn\ A = [sign\ a_{ij}]$. Thus $sgn\ A$ is a matrix with entries from $\{+, 0, -\}$. If A and B are real $n \times n$ matrices, they have the same sign pattern if either $a_{ij}\, b_{ij} > 0$ or both entries are zero for all i, j. If $A = T - \beta I_n$ is an $n \times n$ irreducible NSNS matrix, then A is singular if and only if $\beta = \rho(T)$. If $\beta > \rho(T)$, then $-A$ is a nonsingular M-matrix and by Theorem 2.4, the matrix A^{-1} is entrywise negative. Thus, as far as the sign pattern of A^{-1} is concerned, we need only consider the case $\max_{1 \leq i \leq n} t_{ii} < \beta < \rho(T)$. In the following, let $A^{-1} = [\bar{a}_{ij}]$.

Theorem 3.1. Let $A = T - \beta I_n$ be an $n \times n$ irreducible NSNS matrix in normal form with $\beta < \rho(T)$. If $sign\ a_{ij} = +$, then $sign\ \bar{a}_{ji} = +$.

Proof. We appeal to Corollary 9.1 of [7] to obtain

$$\bar{a}_{ji} = \frac{1}{\det A} \sum (-1)^{\ell_k} A[p_k(j \to i)] \det A[V(p_k)], \tag{5}$$

where the sum is over all paths p_k from j to i, ℓ_k is the length of p_k and $V(p_k)$ is the set of vertices not on p_k. Since the path $p_k(j \to i)$ and the arc (i,j) form a cycle of $D(A)$, the digraph of $A[V(p_k)]$ is acyclic (since $D(A)$ is intercyclic), so that $sign(\det A[V(p_k)]) = (-1)^{n-\ell_k-1}$. By Lemma 2.5 $sign(\det A) = (-1)^{n-1}$, and (5) therefore gives $sign(\bar{a}_{ji}) = sign(-1)^{n-1}(-1)^{n-1} = +$, since $A[p_k(j \to i)] > 0$. □

Let A be an irreducible NSNS matrix in normal form, and suppose that $a_{ij} = 0$ (note that $i \neq j$). We call a_{ij} an *essential zero* of A if replacing it with a *positive* entry creates a matrix that is not NSNS. If a_{ij} is not an essential zero, then we call it a *non-essential* zero of A. If replacing a_{ij} with a positive entry creates a NSNS matrix that is not in normal form, then a_{ij} is still considered to be a non-essential zero of A. Note that if A is an irreducible NSNS matrix in normal form, then there are exactly two terms in the expansion of $\det A$ if and only if $D(A)$ consists of a single Hamilton cycle. In this case, if $a_{ij} = 0$ is replaced by a negative (or positive) entry, then the resulting matrix remains NSNS (but not necessarily in normal form). However, for any other irreducible NSNS matrix A in normal form, if $a_{ij} = 0$ is replaced by a negative entry, the matrix created is not NSNS. Thus, our definition of an essential zero is not unduly restrictive.

We now examine the connection between essential zeros of an irreducible NSNS matrix A in normal form and paths in $D(A)$. Suppose that D is a digraph and that $p(i \to j)$ is a path in D. The path $p(i \to j)$ is a *critical path* if it meets every cycle of D.

Theorem 3.2. Suppose that A is an $n \times n$ irreducible NSNS matrix in normal form, and that $a_{ij} = 0$. Then a_{ij} is a non-essential zero of A if and only if every path $p(j \to i)$ is critical. Furthermore, if $A = T - \beta I_n$ with T nonnegative, $\beta < \rho(T)$, and a_{ij} is a non-essential zero of A, then *sign* $\bar{a}_{ji} = +$.

Proof. It is easy to verify that adding the arc (i,j) to $D(A)$ cannot produce a digraph with a pair of disjoint cycles if every path $p(j \to i)$ is critical. Thus, under these circumstances, if A' is obtained from A by replacing a_{ij} with a positive entry, then A' is also a NSNS matrix. The fact that *sign* $\bar{a}_{ji} = +$ follows immediately Theorem 3.1. □

Example 3.3. Let A be the matrix in Example 1.3. Then

$$sgn\ A^{-1} = \begin{bmatrix} - & + & + & + & + & + \\ - & - & + & + & + & - \\ - & + & - & + & - & + \\ - & - & - & - & + & + \\ + & - & + & + & - & + \\ + & + & - & + & + & - \end{bmatrix}.$$

Let B denote any NSNS matrix with the same sign pattern as A and a positive eigenvalue. Then by Theorems 3.1 and 3.2, seventeen entries of $sgn\ B^{-1}$ are known to be positive. Letting $B^{-1} = [\bar{b}_{ij}]$, fourteen positive entries of B^{-1} are identified by Theorem 3.1; for example, $sgn\ \bar{b}_{12} = +$ as $sgn\ b_{21} = +$. Three positive entries of B^{-1} are identified by

Theorem 3.2; for example, $\bar{b}_{14} = +$ as the (4, 1) entry of A is a non-essential zero. The entry a_{13}, on the other hand, is an essential zero of A since adding the arc (1, 3) creates a cycle (1, 3, 1) that is disjoint from (5, 6, 5). That a_{13} is an essential zero can also be seen from the contrapositive of Theorem 3.2 since $sgn\ \bar{a}_{31} = -$. However, not all the essential zeros of A can be identified in this way.

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