A FAMILY OF MEROMORPHICALLY UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract

In this paper, we introduce and study a new subclass $\mathbb{M}(p,\alpha,\beta)$ of meromorphically univalent functions with positive coefficients. We first obtain a necessary and sufficient condition for a function to be in the class $\mathbb{M}(p,\alpha,\beta)$. We then investigate the meromorphically starlikeness and convexity of functions in the class $\mathbb{M}(p,\alpha,\beta)$. Several other properties of functions in the class $\mathbb{M}(p,\alpha,\beta)$ are also derived.

1. Introduction

Let $\mathbb{M}(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) := \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \qquad (a_n \geq 0; \quad p \in \mathbb{N} = \{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic and univalent in the *punctured* unit disk

$$\mathcal{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\},$$

and which have a simple pole at the origin $(z = 0)$ with residue 1 there. A function $f(z) \in \mathbb{M}(p)$ is said to be in the class $\mathbb{M}(p,\alpha,\beta)$ if it also satisfies the inequality:

$$\Re\{zf(z) - \alpha z^2 f'(z)\} > \beta \tag{1.2}$$

for some α $(\alpha > 1)$ and β $(0 \leq \beta < 1)$, and for all $z \in \mathcal{D}$. Other subclasses of the class $\mathbb{M}(p)$ were studied recently by (for example) Cho *et al.* ([1], [2]).

The main purpose of this paper is to investigate various interesting properties of functions belonging to the class $\mathbb{M}(p,\alpha,\beta)$.

2. Distortion Inequalities

Theorem 1. *Let a function* $f(z)$ *be in the class* $\mathbb{M}(p)$. *Then* $f(z)$ *belongs to the class* $\mathbb{M}(p,\alpha,\beta)$ *if and only if*

$$\sum_{n=p}^{\infty} (n\alpha - 1)\, a_n \leq 1 + \alpha - \beta \qquad (\alpha > 1; \quad 0 \leq \beta < 1). \tag{2.1}$$

The result is sharp.

Proof. Suppose that $f(z) \in \mathbb{M}(p,\alpha,\beta)$. Then we find from (1.2) that

$$\Re\left\{z\left(\frac{1}{z}+\sum_{n=p}^{\infty} a_n\, z^n\right) - \alpha z^2\left(-\frac{1}{z^2}+\sum_{n=p}^{\infty} n a_n\, z^{n-1}\right)\right\}$$

$$= \Re\left\{1+\alpha-\sum_{n=p}^{\infty}(n\alpha-1)\, a_n\, z^{n+1}\right\} > \beta \qquad (z \in \mathbb{D};\quad \alpha > 1;\quad 0 \leq \beta < 1).$$

If we choose z to be real and let $z \to 1-$, we get

$$1+\alpha-\sum_{n=p}^{\infty}(n\alpha-1)\, a_n \leq \beta \qquad (\alpha > 1;\quad 0 \leq \beta < 1),$$

which is equivalent to (2.1).

Conversely, let us suppose that the inequality (2.1) holds true. Then we have

$$\begin{aligned}\left|zf(z)-\alpha z^2 f'(z)-1-\alpha\right| &= \left|-\sum_{n=p}^{\infty}(n\alpha-1)\, a_n\, z^{n+1}\right| \\ &\leq \sum_{n=p}^{\infty}(n\alpha-1)\, a_n\, |z|^{n+1} \\ &\leq 1+\alpha-\beta \qquad (z \in \mathbb{D};\quad \alpha > 1;\quad 0 \leq \beta < 1),\end{aligned}$$

which implies that $f(z) \in \mathbb{M}(p,\alpha,\beta)$.

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z) = \frac{1}{z} + \frac{1+\alpha-\beta}{p\alpha-1}\, z^p \qquad (p \in \mathbb{N}). \tag{2.2}$$

Corollary 1. *If* $f(z) \in \mathbb{M}(p,\alpha,\beta)$, *then*

$$a_n \leq \frac{1+\alpha-\beta}{n\alpha-1} \qquad (n \geq p;\quad p \in \mathbb{N}). \tag{2.3}$$

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *and the function* $g(z)$ *defined by*

$$g(z) = \frac{1}{z} + \sum_{n=p}^{\infty} b_n\, z^n \qquad (b_n \geq 0;\quad p \in \mathbb{N}) \tag{2.4}$$

be in the same class $\mathbb{M}(p,\alpha,\beta)$. *Then the function* $h(z)$ *defined by*

$$h(z) = (1-\lambda)\,f(z) + \lambda\,g(z) = \frac{1}{z} + \sum_{n=p}^{\infty} c_n\,z^n \qquad (c_n := (1-\lambda)\,a_n + \lambda\,b_n \geqq 0; \quad 0 \leqq \lambda \leqq 1) \tag{2.5}$$

is also in the class $\mathbb{M}(p,\alpha,\beta)$.

Proof. Suppose that each of the functions $f(z)$ and $g(z)$ is in the class $\mathbb{M}(p,\alpha,\beta)$. Then, making use of (2.1), we see that

$$\begin{aligned}\sum_{n=p}^{\infty}(n\alpha-1)\,c_n &= \sum_{n=p}^{\infty}(n\alpha-1)\big[(1-\lambda)\,a_n + \lambda\,b_n\big] \\ &= (1-\lambda)\sum_{n=p}^{\infty}(n\alpha-1)\,a_n + \lambda\sum_{n=p}^{\infty}(n\alpha-1)\,b_n \\ &\leqq (1-\lambda)(1+\alpha-\beta) + \lambda(1+\alpha-\beta) \\ &= 1+\alpha-\beta \qquad (\alpha > 1; \quad 0 \leqq \beta < 1; \quad 0 \leqq \lambda \leqq 1),\end{aligned} \tag{2.6}$$

which completes the proof of Theorem 2.

3. Distortion Theorems

Theorem 3. *If* $f(z) \in \mathbb{M}(p,\alpha,\beta)$, *then*

$$\frac{1}{|z|} - \frac{1+\alpha-\beta}{p\alpha-1}\,|z|^p \leqq |f(z)| \leqq \frac{1}{|z|} + \frac{1+\alpha-\beta}{p\alpha-1}\,|z|^p \qquad (z \in \mathbb{D}; \quad p \in \mathbb{N}). \tag{3.1}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = \frac{1}{z} + \frac{1+\alpha-\beta}{p\alpha-1}\,z^p \qquad (\alpha > 1; \quad 0 \leqq \beta < 1; \quad p \in \mathbb{N}). \tag{3.2}$$

Proof. Since

$$\sum_{n=p}^{\infty} a_n \leqq \frac{1+\alpha-\beta}{p\alpha-1} \qquad (\alpha > 1; \quad 0 \leqq \beta < 1; \quad p \in \mathbb{N}) \tag{3.3}$$

and

$$\sum_{n=p}^{\infty} n a_n \leq \frac{p(1+\alpha-\beta)}{p\alpha-1} \qquad (\alpha > 1; \quad 0 \leq \beta < 1; \quad p \in \mathbb{N}) \tag{3.4}$$

for $f(z) \in \mathbb{M}(p, \alpha, \beta)$, we have

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|} - |z|^p \sum_{n=p}^{\infty} a_n \\ &\geq \frac{1}{|z|} - \frac{1+\alpha-\beta}{p\alpha-1} |z|^p \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |f(z)| &\leq \frac{1}{|z|} + |z|^p \sum_{n=p}^{\infty} a_n \\ &\leq \frac{1}{|z|} + \frac{1+\alpha-\beta}{p\alpha-1} |z|^p, \end{aligned} \tag{3.6}$$

which complete the proof of Theorem 3.

Theorem 4. *If $f(z) \in \mathbb{M}(p, \alpha, \beta)$, then*

$$\frac{1}{|z|^2} - \frac{p(1+\alpha-\beta)}{p\alpha-1} |z|^{p-1} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{p(1+\alpha-\beta)}{p\alpha-1} |z|^{p-1} \qquad (z \in \mathbb{D}; \quad p \in \mathbb{N}). \tag{3.7}$$

Equalities in (3.7) *are attained by the function $f(z)$ given by* (3.2).

Proof. We find from (1.1) and (3.4) that

$$\begin{aligned} |f'(z)| &\geq \frac{1}{|z|^2} - |z|^{p-1} \sum_{n=p}^{\infty} n a_n \\ &\geq \frac{1}{|z|^2} - \frac{p(1+\alpha-\beta)}{p\alpha-1} |z|^{p-1} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} |f'(z)| &\leq \frac{1}{|z|^2} + |z|^{p-1} \sum_{n=p}^{\infty} n a_n \\ &\leq \frac{1}{|z|^2} + \frac{p(1+\alpha-\beta)}{p\alpha-1} |z|^{p-1}, \end{aligned} \tag{3.9}$$

which complete the proof of Theorem 4.

4. Meromorphically Starlikeness and Convexity of Functions in the Class $\mathbb{M}(p, \alpha, \beta)$

A function $f(z) \in \mathbb{M}(p)$ is said to be *meromorphically starlike of order* δ $(0 \leq \delta < 1)$ if it satisfies the inequality:

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \delta \qquad (z \in \mathbb{D}; \quad 0 \leq \delta < 1). \tag{4.1}$$

Further, a function $f(z) \in \mathbb{M}(p)$ is said to be *meromorphically convex of order* δ $(0 \leq \delta < 1)$ if and only if $zf'(z)$ is meromorphically starlike of order δ (*cf.* Duren [3]), that is, if it satisfies the inequality:

$$\Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \delta \qquad (z \in \mathbb{D}; \quad 0 \leq \delta < 1). \tag{4.2}$$

Theorem 5. *If* $f(z) \in \mathbb{M}(p, \alpha, \beta)$, *then* $f(z)$ *is meromorphically starlike of order* δ $(0 \leq \delta < 1)$ *in* $0 < |z| < r_1(p, \alpha, \beta, \delta)$, *where*

$$r_1(p, \alpha, \beta, \delta) = \inf_{n \geq p} \left\{\frac{(1-\delta)(n\alpha - 1)}{(1+\alpha-\beta)(n+2-\delta)}\right\}^{1/(n+1)} \qquad (p \in \mathbb{N}). \tag{4.3}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = \frac{1}{z} + \frac{1+\alpha-\beta}{n\alpha-1} z^n \qquad (\alpha > 1; \quad 0 \leq \beta < 1; \quad n \geq p; \quad p \in \mathbb{N}). \tag{4.4}$$

Proof. We must show that

$$\left|-\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \delta \qquad \text{for} \qquad |z| < r_1(p, \alpha, \beta, \delta).$$

Indeed we have

$$\left|-\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=p}^{\infty} (n+1)\, a_n |z|^{n+1}}{1 - \sum\limits_{n=p}^{\infty} a_n |z|^{n+1}} \leq 1 - \delta \qquad (0 \leq \delta < 1), \tag{4.5}$$

if

$$\frac{(n+2-\delta)|z|^{n+1}}{1-\delta} \leq \frac{n\alpha - 1}{1+\alpha-\beta} \qquad (z \in \mathbb{D}; \quad \alpha > 1; \quad 0 \leq \beta < 1; \quad 0 \leq \delta < 1). \tag{4.6}$$

Solving (4.6) for $|z|$, we obtain

$$|z| \leq r_1(p,\alpha,\beta,\delta) = \left[\frac{(1-\delta)(n\alpha-1)}{(1+\alpha-\beta)(n+2-\delta)}\right]^{1/(n+1)} \qquad (n \geq p; \quad p \in \mathbb{N}), \tag{4.7}$$

which evidently proves Theorem 5.

Theorem 6. *If* $f(z) \in \mathbb{M}(p,\alpha,\beta)$, *then* $f(z)$ *is meromorphically convex of order* δ $(0 \leq \delta < 1)$ *in* $0 < |z| < r_2(p,\alpha,\beta,\delta)$, *where*

$$r_2(p,\alpha,\beta,\delta) = \inf_{n \geq p} \left\{\frac{(1-\delta)(n\alpha-1)}{n(1+\alpha-\beta)(n+2-\delta)}\right\}^{1/(n+1)} \qquad (p \in \mathbb{N}). \tag{4.8}$$

The result is sharp for the function $f(z)$ *given by* (4.4).

Acknowledgments

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References

[1] N.E. Cho, S. Owa, S.H. Lee, and O. Altintaş, Generalization class of certain meromorphic univalent functions with positive coefficients, *Kyungpook Math. J.* **29**(1989), 133-139.

[2] N.E. Cho, S.H. Lee, and S. Owa, A class of meromorphic univalent functions with positive coefficients, *Kôbe J. Math.* 4(1987), 43-50.

[3] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.