Stability of Deterministic Signals in Stochastic Financial Stock Price Data

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Abstract

The focus of this project is to quantify the persistence of deterministic trends within stock price paths from the S&P500. We start by considering "synthetic assets" – linear combinations of held and shorted stocks, which are designed based on underlying financial theory to exhibit predictable behaviour. We then model the synthetic assets out of sample by mapping their price paths to a non euclidean space, where their evolution can be described approximately linearly using a best fit Koopman operator $\hat{K}$. Then using this linear approximation, we asses the persistence of the fitted operator $\hat{K}$ by the use of spectral analysis and a stochastic adaptation of Lyapunov stability theory.

Creating Synthetic Assets

Predictable synthetic assets can be created by taking advantage of correlations between individual stock price movements, with a technique analogues to common pairs trading strategies. The key idea is to buy and short various stocks such that any correlated components within the individual stock prices are cancelled. Hence their summed value –the synthetic asset– is a process of random noise around a fixed point. This constraint gives the synthetic asset the important property of "mean reversion", which allows for it to be easily traded by selling the synthetic asset when it's above it's mean, and buying it when it's below.

The resulting objective is to optimize our weights for each stock in the synthetic asset. Let $X \in \mathbb{R}^{n \times T}$ be a matrix of n stock price paths over time $t \in [1, T]$ and let $y \in \mathbb{R}^n$ be a weight vector, hence our synthetic asset at time t is

$$S_t = y^\top X_t.$$

(Sung Min Yoon, 2024) proposed a solution to solve for y such that S_t exhibits maximum mean reversion, by proposing the following semi-definite programming problem

$$\boxed{\min_{Y \in \mathbb{S}^n} \quad \mathrm{Tr}(MY) + \rho \|Y\|_1 \quad \text{s.t.} \quad \mathrm{Tr}(AY) \geq \nu,\ Y \succeq 0}$$

Where $\rho > 0$ and $\nu > 0$ are hyper-parameters controlling sparsity and minimum variance respectively.

$$\tilde{X}_t = X_t - \frac{1}{T}\sum_{s=1}^{T} X_s, \qquad A_k = \frac{1}{T-k}\sum_{t=1}^{T-k} \tilde{X}_t \tilde{X}_{t+k}^\top, \qquad A := A_0, \qquad M := A_1 A^{-1} A_1^\top.$$

Dropping the non-convex rank-one constraint $Y = yy^\top$ and enforcing only $Y \succeq 0$ yields a convex SDP while still encouraging approximately rank-one solutions.

Main Empirical Motivation

Using a semidefinite programming (SDP) approach, we construct synthetic assets as linear combinations of stocks that exhibit strong mean-reverting behaviour in sample. While this structure persists out of sample, it gradually weakens and evolves toward dynamics consistent with a random walk. **Figure 1** shows this transition, with pronounced mean reversion during the in-sample period (blue) and a transient but diminishing effect out of sample (red). **Figure 2** further illustrates this behaviour using 970 out-of-sample realizations, separated by whether they begin above or below their mean; the corresponding averages show a clear tendency to move toward the mean, highlighting short-term persistence that eventually dissipates.

Supervisor: Dr. Heath Emerson

Main Empirical Motivation Cont.

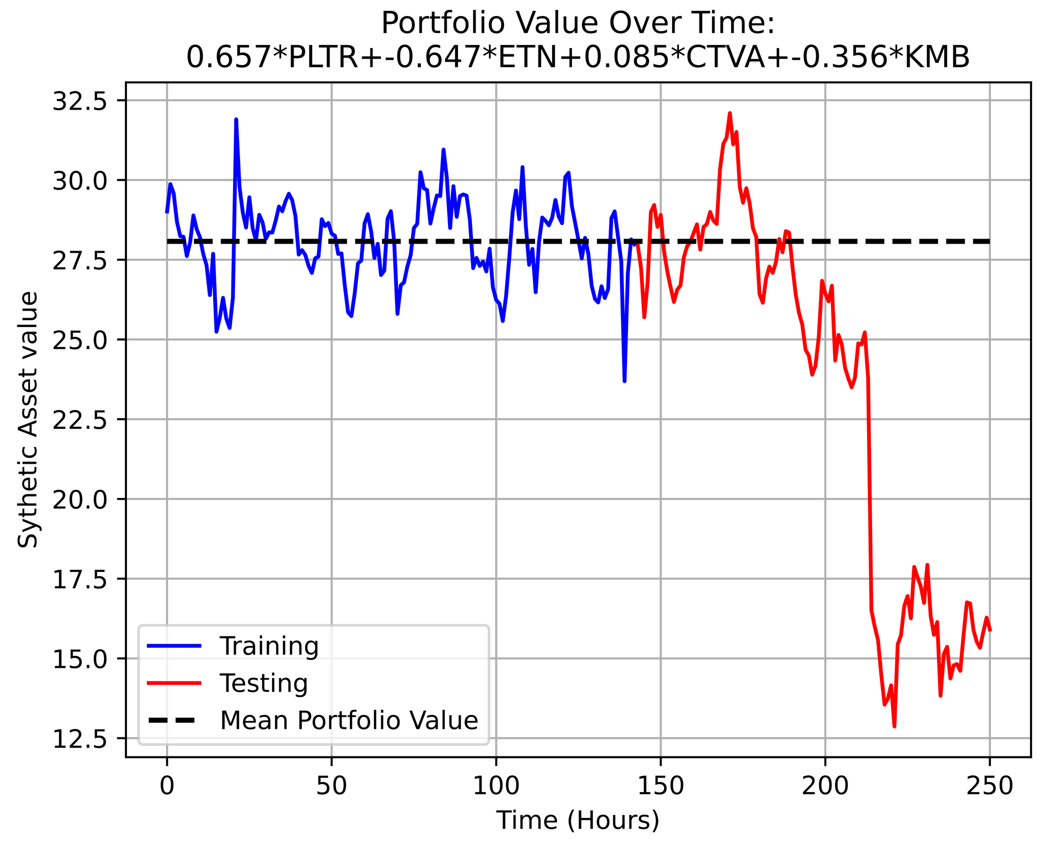

Figure 1: Single synthetic asset with training (blue) and out-of-sample (red) regions. (Dec 7-31, 2024)

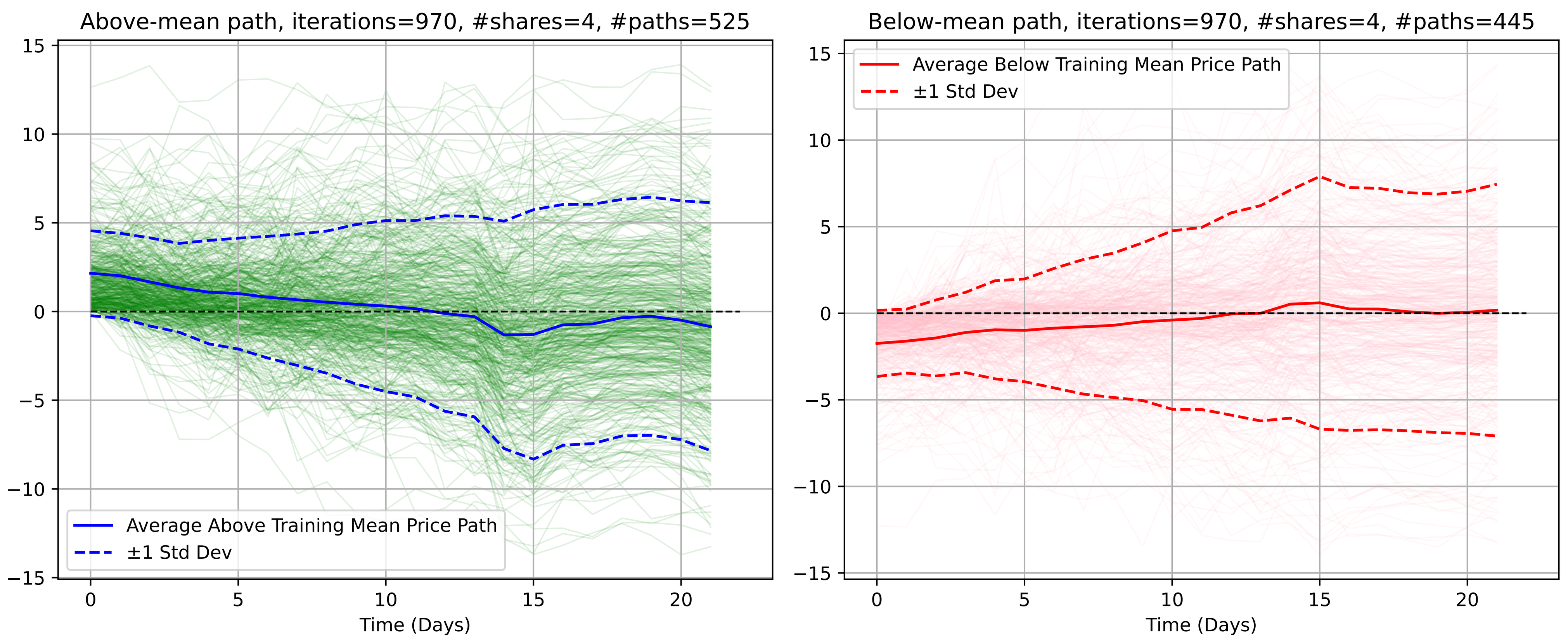

Figure 2: Simulation of 970 out-of-sample synthetic assets centered at zero, separated by initial sign. (Dec 9-31, 2024)

Mathematical Approach

The goal of this section is to capture the deterministic dynamics present in the out of sample data. For this, we use Koopman operator theory, followed by a stochastic adaptation of Lyapunov stability theory.

Proposition. (Koopman, 1931) For a nonlinear dynamical system $x_{t+1} = f(x_t)$, there exists a linear, generally infinite-dimensional Koopman operator $\mathcal{K}$ acting on observables g such that

$$g(x_{t+1}) = (\mathcal{K}g)(x_t),$$

and the spectral properties of $\mathcal{K}$ characterize the evolution of the nonlinear dynamics. Since an infinite dimensional operator is not feasible for this applied problem, we will map the synthetic asset's price path to a non Euclidean vector space and find a best fit linear operator $\hat{K}$ using robust Extended Dynamic Mode Decomposition (EDMD) to describe the price paths evolution through this lifted space.

$$\psi_{t+1} = \hat{K}\psi_t + \varepsilon_t, \qquad \psi_t = g(S_t), \qquad g : \mathbb{R} \to \mathbb{R}^n.$$

Now, assuming a good fit, $\hat{K}$'s spectral and dynamical properties provide insight to the deterministic behaviour.

Proposition. We can conclude that $\hat{K}$ contracts exponentially and quantify its contraction rate if there exists a Lyapunov function V for $\hat{K}$ such that

$$\mathbb{E}[V(\psi_{t+1}) \mid \psi_t] \ \leq\ \gamma V(\psi_t) + \beta_t.$$

$$V(\psi_{t+1}) \ =\ \psi_{t+1}^\top P \psi_{t+1} \ =\ (K\psi_t)^\top P (K\psi_t).$$

Where β_t is the accumulated noise at time t, γ is the contraction factor, and P is an optimal metric for the measuring $\hat{K}$'s persistence. Lastly, if Exponential contraction holds, we can solve for γ with

$$\boxed{\min_{P \succ 0,\ \gamma} \quad \gamma + \mathrm{tr}(P\Sigma) \ \text{ s.t. } \ \hat{K}^\top P \hat{K} \preceq \gamma P, \quad \mathrm{tr}(P) = 1}$$

Where $\mathrm{tr}(P\Sigma) = \varepsilon_t^\top P \varepsilon_t$ is treated as a noise penalty. This problem is solved via semidefinite programming using a bisection search on the contraction rate.

Results

The following results were obtained from

- Estimating $\hat{K}$ via pooled robust EDMD
- Lifting S_t to observables $(\mathbf{x}_t,\ \Delta\mathbf{x}_t,\ |\mathbf{x}_t|,\ |\mathbf{x}_t|\,\Delta\mathbf{x}_t,\ \mathbf{x}_t\,\Delta\mathbf{x}_t)$
- 100 synnthetic assets from 4 random S&P 500 stocks (Dec 9–31, 2024)

Diagnostic	**Key numerical result(s)**	**Interpretation**
Contraction	$\rho_{\mathrm{opt}} \approx 0.928174$	Certified contraction in a quadratic metric
$\hat{K}$ eigenvalues	$\lambda(K) \approx \{0.9636,\ 0.9580,\ -0.0528,\ -0.0469,\ -0.0375\}$	All dynamical modes strictly stable; no marginal or unstable directions
Singular values / operator norm	$\sigma(K) \approx \{0.9638,\ 0.9584,\ 0.0541,\ 0.0459,\ 0.0374\}$ $\|K\|_2 \approx 0.963842 < 1$	Contraction holds in the standard Euclidean norm; no one-step transient amplification
Conditioning / non-normality	Eigenvector condition number $\kappa(V) \approx 2.523$	Moderately non-normal but well-conditioned; stability is not due to fragile cancellations
Powers of $\hat{K}$	Monotone decay: ≈ 0.9638 at $t = 1$, ≈ 0.6906 at $t = 10$	Uniform multi-step decay with a clear persistence timescale; deterministic structure fades steadily

Conclusion and Future Directions

Overall, the results demonstrate that the learned Koopman operator is strongly contractive in a suitable quadratic metric, with stability in both spectral and Lyapunov analysis. Mathematically, this confirms uniform exponential decay of deterministic structure and rules out transient amplification or marginal modes. Practically, the contraction rate provides a quantitative measure of how long predictable behavior persists in the synthetic asset. These results naturally motivate further applications, such as fitting autoregressive models to individual synthetic asset paths and using the estimated contraction rates as persistence metrics. Such metrics can inform trading policies that adapt position sizing, holding periods, or entry and exit rules based on the expected lifetime of deterministic signals.

References

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