

A NEW LOOK AT WHOLE-FOREST MODELING

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WILLIAM J. REED and DARREL ERRICO

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WILLIAM J. REED²

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DARRELL ERRICO³

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²Associate Professor, Department of Mathematics, University of Victoria, P.O. Box 1700, Victoria, B.C. V8W 2Y2.

³Technical Adviser, Research Branch, B.C. Forest Service, Parliament Buildings, Victoria, B.C. V8W 3E7.

ABSTRACT

The harvest scheduling problem customarily known as Model II can be reformulated using a dynamic equation, and solved in two ways using linear programming. Viewing the problem in this way offers many insights and is convenient for deriving extensions to the basic problem. Extensions include the risk of catastrophic loss through fire; the problem of a changing land base; and the imposition of area constraints on the forest. An economic interpretation of certain dual variables in one of the solution methods is given. The mathematical equivalence of the standard Model II linear program, and the dynamic equation formulation is demonstrated and the numerical efficiency of the various methods is examined for simple problems.

Additional Key Words: Harvest scheduling, linear programming, fire risk.

1. INTRODUCTION The problem of the scheduling of harvests from a forest comprising many even-aged stands, and the related problem of making long-run projections of timber supply from such a forest, are of central importance in forest management. Both simulation and optimization techniques have been developed to address these problems, and large-scale packaged routines using both methods are widely employed.

Optimization routines (e.g. Timber RAM (Navon, 1971), MUSYC (Johnson and Jones, 1979), FORPLAN (Johnson, Jones and Kent, 1980)) offer the choice of two alternative linear programming formulations termed Model I and Model II by Johnson and Scheurman (1977). Simulation routines (e.g. TREES (Tedder,

Schmidt and Gourley, 1980) are usually based on a procedure which numerically updates the state of the forest at the end of each period, after management options have been exercised. The updating takes into account the aging of the forest as well as harvesting and other management decisions. (See Johnson and Tedder (1983) for a more complete discussion of simulation techniques.)

The procedure of using a dynamic equation to update the state of the forest, has been used as the basis of optimization analysis by McDonough and Park (1975) and Lyon and Sedjo (1983). The earlier of the above papers describes how the techniques of Optimal Control Theory (specifically the Discrete Maximum Principle) can be used to obtain an optimal solution to the scheduling problem. The latter paper uses the Maximum Principle to obtain, numerically, a long-run projection of the regional supply of timber under optimal economic management. Because the objective functions considered in both papers are non-linear the results obtained are not directly comparable to the results of an analysis using Model I or Model II linear programming (LP) formulations.

In this paper we consider the problem solved by the Model II formulation and show how it can be conveniently formulated by using dynamic equations to describe the period-to-period transitions of the forest, and by considering a linear objective function to be maximized. The resulting optimization problem is linear in both its objective and its constraints. It can therefore be solved by linear programming techniques. Although the linear program which arises does not look the same as the Model II form of Johnson and Scheurman (1977) (which we shall call the "standard Model II form"), it is in fact a different form of the same Model II problem. This is proved formally in the paper. It will be shown that an advantage of the dynamic equation formulation is that it provides an intuitive understanding of the transition of a forest through time. A second advantage is that many extensions of the basic Model II problem can be easily derived within this

framework. These extensions include: (i) the possibility of random catastrophic loss by fire or other loss agent, (ii) the problem of a changing land base, (iii) the inclusion of multiple timber types and regeneration options, and (iv) the inclusion of constraints which control the age-class distribution of the forest. While these extensions can be handled in the standard Model II form (see Johnson, Jones and Kent, 1980, and Johnson and Stuart, 1985) their derivation using that form may not be as convenient. In fact an easy way to determine the Model II constraints is to derive them from the dynamic equation formulation.

Computational aspects of various methods of solving the optimal dynamic control problem and of solving the standard Model II form problem are discussed, for small problems. It is also shown how certain dual LP variables obtained from the solution of the dynamic equation problem provide marginal values for the net present value of standing timber at any age and in any period, in a forest-level (or "whole forest") context.

2. MODEL FORMULATION AND SOLUTION In the Model I scheduling problem (Johnson and Scheurman, 1977), each timber type/age-class combination which contains hectares at the start of the problem forms a management unit, whose integrity is retained over the whole of the planning horizon. Each activity in the linear program represents a sequence of management activities which take place over the planning horizon for a single set of hectares in a management unit.

In contrast, in the Model II formulation hectares do not necessarily remain in the same management unit over the planning horizon. Rather each activity in the linear program may represent the total area of a forest type which undergoes an identical time sequence of management activities (e.g. regeneration, treatment, and harvest) for a single rotation. In this section we shall formulate the basic Model II problem with a dynamic equation, governing the evolution of the forest. For

simplicity we shall use matrix notation. Specifically, let \tilde{x}_t be a column vector, with transpose

$$(1) \quad \tilde{x}_t' = (x_1^t, x_2^t, \dots, x_k^t)$$

where x_i^t ($i = 1, 2, \dots, k-1$) denotes the area of the forest with even-age stands in age-class i at the beginning of period t (i.e. with stands of age between $(i-1)$ periods and i periods) and x_k^t is the area of the forest with even-age stands in age-class k , which we shall assume includes all stands of age $(k-1)$ and older. It will be assumed that k is chosen sufficiently large so that all stands in age-class k have approximately the same volume and value per unit area.

Also we let \tilde{h}_t be a column vector with transpose

$$(2) \quad \tilde{h}_t' = (h_1^t, h_2^t, \dots, h_k^t)$$

where h_i^t is the area clear-cut harvested in age-class i ($i = 1, \dots, k$) in period t .

To begin with we shall assume that all areas harvested in a period are regenerated and return to age-class 1 at the start of the following period. The explicit form of the dynamic equation is thus:

$$(3) \quad \tilde{x}_{t+1} = R\tilde{x}_t - S\tilde{h}_t$$

where

$$R = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} -1 & -1 & \dots & -1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & 1 \end{bmatrix}$$

We shall assume a revenue function g of the linear form

$$(4) \quad g(\underline{h}_t) = \underline{V}' \underline{h}_t$$

where $\underline{V}' = (V_1, V_2, \dots, V_k)$ is a vector of values per unit area for ages 1 to k . The volume harvested H_t in period t is given by

$$(5) \quad H_t = \underline{v}' \underline{h}_t$$

where $\underline{v}' = (v_1, v_2, \dots, v_k)$ is a volume-at-age vector.

In order to ensure a controlled flow of timber from the forest, constraints are usually imposed on the sequence of volumes $\{H_t\}$. These are usually in the form of sequential flow constraints (see e.g. Johnson and Scheurman, 1977) in which the maximum proportional increase or decrease in volume harvest is specified, i.e.

$$(6) \quad (1-\gamma_1)H_{t-1} \leq H_t \leq (1+\gamma_2)H_{t-1} \quad t = 2, 3, \dots$$

where γ_1 ($0 \leq \gamma_1 \leq 1$) is the maximum proportional decrease permitted, and γ_2 (≥ 0) is the maximum proportional increase permitted.

Flow constraints of other forms are sometimes employed. While we shall not treat them separately, the modifications necessary to impose them, in what follows, are obvious and will not be mentioned explicitly.

The problem of maximizing present discounted revenue from the forest, over a finite time horizon subject to sequential flow constraints can thus be expressed as:

Maximize

$$(7) \quad J = \sum_{t=1}^N \alpha^t \mathbf{v}' \mathbf{h}_t + \alpha^{N+1} \mathbf{r}' \mathbf{x}_{N+1}$$

subject to

$$(8) \quad \mathbf{x}_{t+1} = \mathbf{R} \mathbf{x}_t - \mathbf{S} \mathbf{h}_t, \quad t = 1, 2, \dots, N$$

$$(9) \quad 0 \leq \mathbf{h}_t \leq \mathbf{x}_t, \quad t = 1, 2, \dots, N$$

and

$$(10) \quad (1-\gamma_1) \mathbf{v}' \mathbf{h}_{t-1} \leq \mathbf{v}' \mathbf{h}_t \leq (1+\gamma_2) \mathbf{v}' \mathbf{h}_{t-1}, \quad t = 2, 3, \dots, N$$

and given an initial state \mathbf{x}_1 , where $\mathbf{r}' = (r_1, r_2, \dots, r_k)$ is a vector of per-hectare net present values for areas in age classes $1, 2, \dots, k$ at time $N+1$. The product $\mathbf{r}' \mathbf{x}_{N+1}$ thus represents the value of inventory remaining in period $N+1$. By incorporating a terminal payoff in the objective in this way many of the problems associated with using a finite planning horizon can be avoided.

If there is existing harvesting activity taking place, with the total volume cut in the most recent period being H_0 , and it is desirable to ensure continuity of supply, then there would be an additional constraint corresponding to (10) with $t = 1$.

In the above optimization problem both the objective and constraints are linear in the control variables \mathbf{h}_t and in the state variables \mathbf{x}_t . The problem can

thus be solved by standard LP techniques. Put in standard LP terms the problem is:

$$\begin{aligned} \text{(LP1)} \quad & \text{maximize } \underline{c}' \underline{u} \\ & \text{subject to:} \end{aligned}$$

$$A \underline{u} \begin{bmatrix} = \\ \leq \end{bmatrix} \underline{b}$$

and

$$(12) \quad \underline{u} \geq 0,$$

where \underline{u} is the $(2N+1)k$ -dimensional column vector of "activities" formed by vertically concatenating the $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{N+1}$ and the $\underline{h}_1, \underline{h}_2, \dots, \underline{h}_N$ vectors, and \underline{c}' is the $(2N+1)k$ -dimensional row vector

$$\underline{c}' = [\underline{0}', \underline{0}', \dots, \alpha^{N+1} \underline{x}', \alpha \underline{V}', \alpha^2 \underline{V}', \dots, \alpha^N \underline{V}']$$

(where $\underline{0}'$ is a k -dimensional row vector of zeros). The constraint matrix A is of dimension $[(2N+1)k+2(N-1)] \times [(2N+1)k]$ and is of the form

Using the dynamic equation (3) iteratively⁴ leads to

$$(13) \quad \underline{x}_t = R^{t-1} \underline{x}_1 - (R^{t-2} S \underline{h}_1 + R^{t-3} S \underline{h}_2 + \dots + S \underline{h}_{t-1}) \quad t = 2, \dots, N.$$

Using this, the constraints (10), ($\underline{h}_t \leq \underline{x}_t$), can be expressed as:

$$(14) \quad R^{t-2} S \underline{h}_1 + R^{t-3} S \underline{h}_2 + \dots + S \underline{h}_{t-1} + \underline{h}_t \leq R^{t-1} \underline{x}_1$$

which does not involve the state vectors \underline{x}_t , other than the initial state \underline{x}_1 . Also \underline{x}_{N+1} can be eliminated from the objective (7) by using (13) to give

$$(15) \quad J = \alpha^{N+1} \underline{r}' R^N \underline{x}_1 + \sum_{t=1}^N \left[\alpha^t \underline{v}' - \alpha^{N+1} \underline{r}' R^{N-t} S \right] \underline{h}_t.$$

The first term, $\alpha^{N+1} \underline{r}' R^N \underline{x}_1$, does not depend on the harvest vectors, and thus the maximization problem can be solved with the LP

$$(16) \quad \text{(LP2)} \quad \begin{array}{ll} \text{maximize} & \underline{c}^* \underline{u}^* \\ \text{subject to:} & \\ & A \underline{u}^* \leq \underline{b}^* \\ \text{and} & \underline{u}^* \geq 0 \end{array}$$

⁴Specifically from (6):

$$\begin{aligned} \underline{x}_2 &= R \underline{x}_1 - S \underline{h}_1 \\ \text{and } \underline{x}_3 &= R \underline{x}_2 - S \underline{h}_2 \\ &= R(R \underline{x}_1 - S \underline{h}_1) - S \underline{h}_2 \\ &= R^2 \underline{x}_1 - (R S \underline{h}_1 + S \underline{h}_2) \end{aligned}$$

etc.

where now \underline{y}^* is an Nk -dimensional column vector of activities formed by vertically concatenating the harvest vectors $\underline{h}_1, \dots, \underline{h}_N$, and $\underline{c}^{*'} is the Nk -dimensional row vector formed by horizontally concatenating the N k -dimensional components $[\alpha^t \underline{y}' - \alpha^{N+1} \underline{r}' R^{N-t} S]$ for $t = 1, \dots, N$. The constraint matrix A^* is of dimension $[Nk + 2(N-1)] \times Nk$ and is of the form$

$$(17) \quad \left[\begin{array}{cccccccc} \text{I} & & & & & & & \\ \text{S} & & & & & & & \\ \text{RS} & & & & & & & \\ & \text{I} & & & & & & \\ & \text{S} & & & & & & \\ & & \text{I} & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ \text{R}^{N-2}\text{S} & \text{R}^{N-3}\text{S} & \text{R}^{N-4}\text{S} & \cdot & \cdot & \cdot & \text{RS} & \text{S} & \text{I} \\ \hline \underline{\ell}'_1 & -\underline{y}' & & & & & & & \\ \underline{\ell}'_2 & \underline{y} & & & & & & & \\ & \underline{\ell}'_1 & -\underline{y}' & & & & & & \\ & -\underline{\ell}'_2 & \underline{y}' & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & & & & & & \underline{\ell}'_1 & -\underline{y}' & \\ & & & & & & -\underline{\ell}'_2 & \underline{y}' & \end{array} \right] \cdot$$

The right-hand side vector \underline{b}^* is of dimension $Nk + 2(N-1)$ and is of the form

$$(18) \quad \underline{b}^* = \begin{bmatrix} \underline{x}_1 \\ \text{R}\underline{x}_1 \\ \text{R}^2\underline{x}_1 \\ \vdots \\ \text{R}^{N-1}\underline{x}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The constraints in (16) contain redundant rows which can be removed, reducing problem size⁵. Specifically one can remove the first $(k-2)$ rows from each of the first $(N-1)$ partition-rows in (17) and (18). This reduces the row dimension of (17) and (18) to $4(N-1) + k$.

It is shown in Appendix 1, that the form LP2 is essentially equivalent to the linear program of the standard Model II form. The only difference lies in the fact that in LP2 all areas above a certain age k are classified as belonging to one "collector" age-class. In the standard Model II form there is no such collector age-class. If we set k equal to the oldest possible age that trees could reach over the planning horizon, i.e. $k = M' + N$, where M' is the age of the oldest trees in the initial inventory, then this difference vanishes and the two linear programs are identical.

The fact that the optimization problem which we have formulated here, in terms of a dynamic equation, can be solved by a linear program (LP2) which is essentially the same as the standard Model II form should come as no surprise. There are usually several ways of formulating a given optimization problem. We feel that the dynamic equation LP1 formulation provides a useful alternative to the standard Model II form for some situations because of certain of its features.

One such feature of the dynamic-equation LP1 formulation is that it gives an

⁵An inequality (row) will be redundant if it can be derived from the remaining inequalities (rows). In this case, the redundancy is of the following form: For two $[\leq]$ inequalities (a) and (b) each with at least one non-negative coefficient (a) is redundant if

- (i) inequality (a) has non-zero coefficients only for a subset of those variables having non-zero coefficients in (b);
- (ii) a non-zero coefficient in (a) is less than or equal to the coefficient for the same variable in (b);
- (iii) the right hand side of (b) is less than or equal to that of (a).

intuitively appealing sense of the evolution of a forest. The dynamic equation (3) which forms the main part of the constraint matrix of the LP formulation, describes explicitly the transition of the forest from one period to the next. The area constraints (b) of the standard form (Johnson and Scheurman, 1977, p. 5), while they essentially serve the same purpose as the dynamic equation, do not provide an obvious description of this period to period transition of the forest. The importance of the ease of understanding of the model to the user should not be underestimated.

Coupled with the ease of understanding which arises with the use of the dynamic equations is the ease of constructing the LP formulation for a particular problem. To illustrate this, we use the example provided by Johnson and Tedder (1983) (their Table 1). This is a two-period problem with no terminal payoff for forested land remaining after two periods. Moreover the discount rate is zero, and it is required that the same volume be harvested in each period. The problem requires five age classes with $\underline{x}'_1 = (0, 0, 5, 3, 0)$, $\underline{v}' = (0, .5, 1.0, 2.0, 2.1)$,

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

The objective coefficient vector \underline{c}' for the LP1 problem is

$$[\underline{0} \ \underline{0} \ \underline{0} \ \underline{v}' \ \underline{v}'];$$

the constraint matrix A is

$$\left[\begin{array}{ccc|cc} \text{I} & & & \text{S} & \\ -\text{R} & \text{I} & & & \text{S} \\ & -\text{R} & \text{I} & \text{I} & \\ -\text{I} & & & & \text{I} \\ & -\text{I} & & \tilde{\text{y}}' & -\tilde{\text{y}}' \end{array} \right];$$

and the right-hand-side vector $\tilde{\text{b}}$ is

$$\begin{bmatrix} \tilde{\text{x}}_1 \\ \tilde{\text{0}} \\ \tilde{\text{0}} \\ \tilde{\text{0}} \\ \tilde{\text{0}} \\ \tilde{\text{0}} \\ \tilde{\text{0}} \end{bmatrix}.$$

Incorporating the activity variable names the LP1 formulation is:

$$\begin{array}{ll} \text{maximize} & \tilde{\text{y}}' \tilde{\text{h}}_1 + \tilde{\text{y}}' \tilde{\text{h}}_2 \\ \text{subject to} & \text{I} \tilde{\text{x}}_1 = \tilde{\text{x}}_1 \\ & -\text{R} \tilde{\text{x}}_1 + \text{I} \tilde{\text{x}}_2 + \text{S} \tilde{\text{h}}_1 = \tilde{\text{0}} \\ & \quad -\text{R} \tilde{\text{x}}_2 + \text{I} \tilde{\text{x}}_3 + \text{S} \tilde{\text{h}}_2 = \tilde{\text{0}} \\ & -\text{I} \tilde{\text{x}}_1 + \text{I} \tilde{\text{h}}_1 \leq \tilde{\text{0}} \\ & \quad -\text{I} \tilde{\text{x}}_2 + \text{I} \tilde{\text{h}}_2 \leq \tilde{\text{0}} \\ & \tilde{\text{y}}' \tilde{\text{h}}_1 - \tilde{\text{y}}' \tilde{\text{h}}_2 = 0 \end{array}$$

Note that up to this point one need not be concerned with the individual rows and columns of the model. This allows for more rapid and concise generation of the LP problem. The complete detailed linear program is obtained by replacing each

matrix symbol with its expanded elemental form. This is given in Table 1. This tableau represents a full model having no restrictions on harvest ages. It includes some harvest activities which cannot logically take place (for example, harvest activities in the first period for age classes which have no areas). The dynamic equations however prevent these activities from having non-zero values in the solution. If desired, such activities can be removed from the problem. When there are minimum harvest age restrictions to be included, these can be handled either by including extra constraints setting the corresponding activities to zero, or by removing the activities.

Using the dynamic equation LP1 formulation also provides a convenient means to make extensions to the basic model. For example, the problem of random catastrophic losses, such as destruction of areas through fire, can be managed by a relatively simple modification of the matrices R and S which occur in the dynamic equation (3). Thus, if we assume that during period t random proportions $p_1^t, p_2^t, \dots, p_k^t$ of the areas in age-classes $1, \dots, k$ are destroyed by fire and that subsequent to regeneration these areas move to age-class 1 in period $t+1$ then we can use the dynamic equation (3) in slightly modified form

$$(19) \quad \tilde{x}_{t+1} = R_t \tilde{x}_t - S_t h_t$$

where R_t and S_t are random matrices of the form

$$R_t = \begin{bmatrix} p_1^t & p_2^t & \cdots & p_k^t \\ 1-p_1^t & 1-p_2^t & & \\ & \ddots & \ddots & \\ & & 1-p_{k-1}^t & 1-p_k^t \end{bmatrix}$$

and

$$S_t = \begin{bmatrix} -1+p_1^t & -1+p_2^t & \cdots & -1+p_k^t \\ 1-p_1^t & & & \\ & 1-p_2^t & & \\ & & \ddots & \\ & & & 1-p_{k-1}^t & 1-p_k^t \end{bmatrix}$$

Since equation (19) involves random elements it is a stochastic difference equation. Exact solution of the corresponding stochastic maximization problem could be achieved in principle through the use of techniques such as Dynamic Programming. A more practical solution technique, however, is to use a Certainty Equivalence procedure (see e.g. Chow, 1975). This involves substituting the expected values of the random matrices R_t and S_t in equation (3) and proceeding to a linear program in the fashion already described. If this technique is used in a feedback fashion (that is with an optimization problem being solved in each period after updating the forest inventory to take account of the actual fires that have occurred) the resulting solution should provide a good approximation to the optimal solution for the stochastic maximization problem. For more details on (19) and aspects of modelling the forest where risk of fire is present see Reed and Errico (1986).

Another extension which is relatively easy to derive with the dynamic equation LP1 formulation is the inclusion of a changing land base. For example, if it is anticipated that in period 2, areas $\tilde{E}' = (E_1, E_2, \dots, E_k)$ in the various age-classes are to be removed from the land base, then the dynamic equation for period $t = 2$ would be modified to read

$$(20) \quad \tilde{x}_3 = R\tilde{x}_2 - S\tilde{h}_2 - \tilde{E},$$

with all the other equations remaining unchanged. For the LP1 form, this translates to simply changing the right-hand-side vector to $-\underline{E}$ for those rows corresponding to the appropriate dynamic equation. As Johnson and Tedder (1983) point out, such a change in the standard Model II form is "difficult and cumbersome".

Area constraints which control the age-class distribution of the forest (e.g. for non-timber values such as water, wildlife, or recreation) can be added to the LP1 formulation by imposing constraints on the state variable (\underline{x}_t) activities. To incorporate this capability in the standard Model II form additional "accounting" activities are set up (see, for example, Johnson, Jones, and Kent, 1980). These activities are nothing more than a form of \underline{x}_t variables of the LP1 formulation.

Another extension considered by Allard, Errico and Reed (1988), covers the situation in which the only limitation on harvest activity comes about through the amount of available logging and milling capacity. Such capacity can be increased through investment. By including current capacity and investment as state variables the problem of determining the optimal harvest pattern, along with the pattern of optimal investment can be solved using a program of the LP1 form.

Returning now to the basic LP1 problem, we point out some useful features of its solution. Firstly, the dual variables at the optimum for the LP1 problem, corresponding to the various constraints, have an interpretation not found in the LP2 or the standard Model II forms. The dual variables corresponding to the dynamic equation constraints (8) provide the marginal value of land containing trees of each age, and at each time period throughout the planning horizon. Thus the marginal net present value of a hectare of standing timber can be computed in the forest-level context at any period. This marginal forest-level net present value may be very different from the stand-level net present value. It will depend not only on the age (volume) of timber in a given hectare, but also on the

age-distribution of stands throughout the forest, on the harvest history, and on other constraints to the problem. The dual variables corresponding to the availability constraints (9) represent the shadow prices of land temporarily (for a single period) removed from or added to the harvestable land base.

If there are area constraints controlling the age-class distribution (e.g. for non-timber purposes) included in the problem, the corresponding dual variables provide the marginal opportunity costs of these constraints.

A second, although perhaps less important, characteristic of the LP1 form is that the solution provides directly the age-class distribution of areas, \bar{x}_t , at every period up to the planning horizon, under optimal management. Of course this information can be recovered from the optimal harvest sequence in the LP2 form, but it requires extra computational steps in the form of a report writer.

We now turn to a discussion of the relative computational efficiencies of the LP1, LP2, and standard Model II forms.

3. SOLUTION CHARACTERISTICS. Solution of a linear programming problem using most solution software packages is mainly sensitive to the number of rows (constraints) and the number of non-zero coefficients (density) in the constraint matrix. An increase in the magnitude of either results in an increase in computing time necessary for successful solution. Solution time is less affected by the number of columns (activities) in a problem (Beale, 1968). In this section we present some results for the model forms discussed previously. The examples given here represent small problems relative to those of typical applications in Canada and the U.S. and more thorough testing is required, for larger more typical problems. Such testing is beyond the scope of this study. We speculate later, however, what these results might yield.

Comparisons were made for three problems. In the first it was assumed that there was no risk of fire. The second and third correspond to two different fire scenarios, one with a constant hazard and the other with an age-dependent hazard. The first (no fire hazard) problem was solved in three ways: (a) by the standard Model II method, using the US Forest Service model MUSYC (Johnson and Jones, 1979), (b) using the LP1 form; and (c) using the LP2 form. The second and third problems were solved only by methods (b) and (c), since MUSYC is unable to accommodate catastrophic losses. Thus in all, seven linear programs were solved.

In all cases the same single-type forest was assumed, it being based on that of white spruce (*Picea glauca* (Moench) Voss and *Picea engelmannii* Parry) on sites of site index 20+ m (reference age 100) of medium accessibility to mills in the Fort Nelson Timber Supply Area of north-eastern British Columbia. This is the same as in the example given by Reed and Errico (1986(a)) and details may be found there. The following items were identical for the seven problems:

- (i) Objective (maximize harvested volume)
- (ii) Initial age distribution of timber (see Table 2)
- (iii) Discount rate (0%)
- (iv) Period length (20 years)
- (v) Length of planning horizon (36 periods)
- (vi) Harvest flow constraints (sequential flow: $\pm 10\%$ from period to period)
- (vii) Accessibility restrictions (none)
- (viii) Regeneration delay (none)
- (ix) Timber age restrictions (none)

Table 3 gives details of the problem dimensions, matrix density, and solution time for the solution step using linear programming solution software MPS III (Ketrion, 1984), for the seven problems tested.

From the number of columns (activities) it is seen that LP1 has more than twice the number of activities than LP2 due to the inclusion of the state variables (\underline{x}_t) as activities. The reason why LP2 has fewer activities than the standard Model II form is because the latter does not allow a single age-class for timber above a particular age and thus requires separate activities for each age up to the maximum that could be attained over the 36 period planning horizon (i.e. up to $44 = 8 + 36$ periods).

Table 3 also demonstrates that LP1 has many more constraints (rows) than LP2. This is because the dynamic equations (16) are included as separate constraints in LP1, and because some row redundancies have been removed in LP2.

The matrix density figures of Table 3 show that while density remains approximately the same in LP1 for all fire hazard scenarios, in LP2 it increases dramatically for an age-dependent fire hazard. The reason for this lies in the fact that the matrix $R^{\ell}S$ in (13) and (17) reduces to a simple sparse form for large ℓ when the p_i are constant, but does not reduce to a sparse form when the p_i are age-dependent. The standard Model II form has a higher density than the comparable LP2 problem largely due to the additional coefficients associated with the additional activities which handle older-aged timber.

The ultimate effect of model size and density is perhaps best evaluated through comparison of a single statistic — solution time. From the solution-time statistics of Table 3 it can be seen that the standard Model II form performs relatively well, despite its high density. This is probably due to its small number of rows. The LP2 form outperforms the LP1 form for constant (age-independent) fire probabilities, and also performs slightly better than the standard Model II form for the case of no fire hazard. However for the case of age-dependent fire hazard the LP1 form performs better than the LP2 form. This can be attributed to the former maintaining low density in this case, while the latter is much higher in density.

These results agree with Beale (1968, p. 83) who states that for improving solution time, reductions in row dimensions of a problem are not usually worthwhile if, for each row reduced, more than about half a dozen non-zero matrix elements are added. For the examples given here, when fire probability is constant, there is actually a reduction in the total number of non-zero elements in the LP2 problem compared with the LP1 problem. Hence, the reduced computation time. In the case of non-constant fire hazard however, the LP2 problem has an additional (approximately) twelve non-zero elements for every row eliminated from the LP1 problem. In this case, we observe an increase in computation time.

To get some idea of the relative efficiency of LP1 to LP2 for larger problems, the row and column dimensions along with the number of non-zero elements and matrix density have been computed for various numbers of age classes ($k = 8, 15, 20$) and various numbers of periods in the planning horizon ($N = 10, 20, 30, 40$). The results are given in Table 4. It can be seen that for LP2 in all cases the matrix density is much larger when using an age-dependent hazard (non-constant p_i) than when using an age-independent hazard (constant p_i). From the figures in Table 4 for age-dependent hazard it can be determined that for $N = 10, 20$, the number of non-zero elements per row added in reducing the problem from LP1 to LP2, is less six while for $N = 30, 40$ it is greater than six. Thus, applying Beale's rule of thumb given above, one would expect that for an age-dependent hazard, the LP1 form could be solved faster than the LP2 form, when the number of periods, N , in the planning horizon is large. In other cases the LP2 form would probably be computationally more efficient.

We add two final notes regarding the LP1 and LP2 formulations. Firstly, it is easier to generate the linear programming tableau for LP1 than for LP2 since fewer calculations are required to derive the coefficients and since the constraint matrix has a more regular form. This results in reduced programming effort as well

as lower computing costs for the generation step. Secondly, we believe that for more typical large-scale problems formulated in practice, where there are various constraints on the area states, the LP1 form will show a less rapid increase in density than the LP2 or standard Model II forms, when similar constraints are implemented through the control of harvest activities only. This maintenance of sparsity is important for very large problems.

4. DISCUSSION In this paper we have presented two alternative formulations, LP1 and LP2, for the harvest scheduling problem addressed by the Model II linear programming procedure. These formulations were derived from a dynamic equation which described the transition of the forest from one period to the next. We claim that the dynamic equation, LP1 form presents an attractive alternative to the standard Model II form for certain situations.

One of the main advantages which the dynamic equation formulation possesses is that it provides a more intuitively appealing description of the evolution of a forest. As well, the dynamic equation provides a convenient means for deriving extensions. In the paper the following extensions are discussed: (i) the problem of a changing land base, (ii) the inclusion of constraints which control the age-class distribution of the forest, (iii) the possibility of random catastrophic losses through fire or other loss agent. In another paper (Reed and Errico, 1986(b)) further extensions are considered. These include the possibility of partial salvage of volumes burnt, the inclusion of multiple timber types and the problems of accessibility and variable recovery costs. While all of the above extensions can be handled ultimately in the standard Model II formulation, the derivation may not be straightforward. Indeed, a convenient way to derive the appropriate tableau is to use an indirect approach, by starting with the dynamic equations and corresponding

constraints, and then eliminating the state variables recursively, in the manner discussed in Section 2.

In simple problems, when there is age-independent risk of catastrophic loss present (including the case of no risk) LP2 performs better than LP1 (and indeed slightly better than the standard Model II linear program, where there is no risk present — this being due to the use of a collector age-class), in terms of computation time. This is not the case for more realistic problems when fire destruction probabilities are age-dependent. In such situations, LP1 is likely to outperform LP2, at least when there is a long planning horizon. This is because of the fact that the constraint matrix for LP1 remains sparse, whereas for LP2 it becomes very dense. Typically problems with many constraints and a high density are slow to solve (Beale, 1968).

There are other features of the LP1 method. Firstly, coding a matrix generator program is easier, and secondly generation of the linear program is quicker than with the LP2 form. Thirdly, the values of the dual variables at the optimum provide some economic information not found in the standard Model II or LP2 forms. Finally, the output of LP1 provides directly the age-class distribution of the forest at every period, under optimal management. To obtain this using LP2 or the standard form, either extra computational steps in the form of a report writer are required, or accounting variables (which are really like the state variables in the LP1 formulation) must be added.

While recent developments in FORPLAN (Johnson and Stuart, 1985) allow for the inclusion of catastrophic mortality, using the standard Model II procedure, the relative computational performance of that method, and of the LP1 method have yet to be investigated. Aside from computational aspects we feel that the dynamic equation formulation presented here, with the LP1 method of solution,

merits attention as a useful way of approaching the forest-level harvest scheduling problem. At the very least it provides insight into the Model II problem, especially with respect to extensions and how they may be incorporated into the standard Model II linear programming procedure.

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Appendix I

Equivalence of LP2 with the Standard Model II Form

Model II, Form II of Johnson and Scheurman (1977) is

$$\text{maximize } \sum_{j=1}^N \sum_{i=-M}^{j-1} D_{ij} y_{ij} + \sum_{i=-M}^N E_{iN} z_{iN}$$

subject to:-

1. area constraints

$$(a) \quad \sum_{j=1}^N y_{ij} + z_{iN} = A_i \quad i = -M, \dots, 0$$

$$(b) \quad \sum_{k=j+2}^N y_{jk} + z_{jN} = \sum_{i=-M}^{j-1} y_{ij} \quad j = 1, \dots, N$$

2. harvest flow constraints

$$(1-\gamma_1)H_j \leq H_{j+1} \leq (1+\gamma_2)H_j \quad j = 1, \dots, N-1$$

where

D_{ij} = discounted net revenue per hectare of areas regenerated in period i and harvested in period j (equivalent to $\alpha^j V_{j-1}$ in our notation)

E_{iN} = discounted value per hectare of areas regenerated in period i and left as ending inventory at the end of period N (equivalent to $\alpha^{N+1} r_{N+1-i}$ in our notation)

y_{ij} = area regenerated in period i and harvested in period j (equivalent to h_{j-i}^j in our notation)

A_i = area present in period one that was regenerated in period i ($i = -M, \dots, 0$), with $M+1$ being the age of the oldest timber in the initial (period 1) inventory, (A_{-j} is equivalent to x_1^{j+1} in our notation)

z_{jN} = area regenerated in period j and left as ending inventory after the harvest in period N (equivalent to x_{N-j+1}^{N+1} in our notation)

In general the standard Model II form allows for the specification of a minimum cutting age. For the sake of simplicity we have set this equal to one, in establishing the equivalence with our model. To establish a minimum cutting age in our model we would simply constrain some of the components of the harvest vectors \underline{h}_t to be equal to zero or delete them from the formulation.

The standard Model II form, unlike the model of this paper, does not classify all areas with stands above a certain age, as a single age-class. To accommodate for this difference in models we shall modify our model slightly by allowing for stands of age up to $M + N + 1$ and modifying the matrices R and S to be $M + N + 1$ dimensional square matrices

$$R = \begin{bmatrix} 0, & 0, & \cdots & 0 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1, & 0 \end{bmatrix}, \quad S = \begin{bmatrix} -1, & -1, & & -1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & 0 \end{bmatrix}$$

The initial and terminal vectors of our model expressed in terms of standard Model II form parameters are the $M + N + 1$ dimensional vectors

$$\tilde{x}_1 = \begin{bmatrix} A_0 \\ A_{-1} \\ \vdots \\ A_M \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \tilde{x}_{N+1} = \begin{bmatrix} z_{NN} \\ z_{N-1,N} \\ \vdots \\ z_{-M,N} \end{bmatrix}.$$

We now derive the standard Model II form from LP2. When there is no upper age class, all of the constraints (14) in LP2 can be derived from (13) with $t = N + 1$, i.e. from

$$(A1.2) \quad \tilde{x}_{N+1} = R^N \tilde{x}_1 - \left[R^{N-1} Sh_1 + R^{N-2} Sh_2 + \cdots + RS h_{N-1} + Sh_N \right]$$

This equation represents $M + N + 1$ equality constraints. The first N of these are equivalent to the area constraints (b), while the last $M + 1$ are equivalent to the area constraints (a) of the standard Model II form. This can be established by multiplying out the right hand side of (A1.2), and using the equivalences in (A1.1) and the equivalence

$$(A1.3) \quad h_i^j = x_{j-i,j} \quad i = 1, \cdots, M, \quad j = 1, \cdots, M + N + 1$$

To do this for general M and N is quite tedious. For the sake of illustration we do it here for the case $N = 2$, $M = 3$. (A1.2) gives in this case

$$\underline{x}_3 = R^2 \underline{x}_1 - RS \underline{h}_1 - S\underline{h}_2$$

i.e. using (A1.1),

$$\begin{bmatrix} z_{22} \\ z_{12} \\ z_{02} \\ z_{-12} \\ z_{-22} \\ z_{-32} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ A_{-1} \\ A_{-2} \\ A_{-3} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \\ h_4^1 \\ h_5^1 \\ h_6^1 \end{bmatrix}$$

$$- \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \\ h_4^2 \\ h_5^2 \\ h_6^2 \end{bmatrix}$$

Using (A1.3) this gives six equations

$$\begin{aligned} z_{22} &= y_{12} + y_{02} + y_{-12} + y_{-22} + y_{-32} + y_{-42} \\ z_{12} &= y_{01} + y_{-11} + y_{-21} + y_{-31} + y_{-41} + y_{-51} - y_{12} \\ z_{02} &= A_0 - y_{01} - y_{02} \\ z_{-12} &= A_{-1} - y_{-11} - y_{-12} \\ z_{-22} &= A_{-2} - y_{-21} - y_{-22} \\ z_{-32} &= A_{-3} - y_{-31} - y_{-32} \end{aligned}$$

The last four equations can be written

$$\sum_{j=1}^2 y_{ij} + z_{iN} = A_i \quad i = -3, -2, -1, 0$$

which are the area constraints (a) of Model II. The first two equations are

$$\sum_{k=2}^2 y_{1k} + z_{12} = \sum_{i=-5}^0 y_{i1} \quad \text{and} \quad z_{22} = \sum_{i=-4}^1 y_{i2}$$

which are seen to be the two area constraints (b) once we recognize that

$$y_{-51} = y_{-41} = y_{-42} = 0,$$

since there were no hectares regenerated in periods -5 and -4 available for harvest

The sequential flow constraints, and the non-negativity constraints are the same in both models. The objective of LP2 is

$$J = \sum_{t=1}^N \alpha^t v_s h_s^t + \alpha^{N+1} r' x_{N+1}$$

The first term can be re-expressed as

$$\sum_{t=1}^N \sum_{s=1}^{M+N+1} \alpha^t v_s h_s^t = \sum_{t=1}^N \sum_{s=1}^{M+N+1} \alpha^t v_s y_{t-s,t}$$

By a change of summation variables

$$i = t - s, \quad j = t$$

the sum can be written as

$$\sum_{j=1}^N \sum_{i=j-(M+N+1)}^{j-1} a^j V_{j-i} y_{ij} = \sum_{j=1}^N \sum_{i=-M}^{j-1} D_{ij} y_{ij},$$

since $i \geq -M$ ($-M$ is the time of generation of the oldest trees in the initial inventory). This is the first term of the objective function of the standard Model II form. Similarly the second term of our model, corresponding to terminal payoffs for areas left standing at the end of the planning horizon, is equivalent to the second term in the objective of the standard Model II form.

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Table 2: Initial age distribution of timber for harvest scheduling problem used in solution tests.

<u>Age-Class</u>	<u>Area (hectares)</u>
1	241
2	125
3	1 404
4	2 004
5	9 768
6	16 385
7	2 815
8	61 995

Table 3: Size and solution statistics for the Standard Model II, LP1 and LP2 linear programming formulations.

Formulation	Per annum probability of destruction through fire \bar{p}_i	No. of Columns	No. of Rows	No. of non-zero elements	Density %	Solution time (sec)
Standard Model II	0 all ages	674	116	3810	4.87	3.97
LP1	0 all ages	1239	656	3951	0.49	6.61
LP1	.01 all ages	1239	656	4239	0.52	5.80
LP1	.015 for age < 70 .005 for age \geq 70	1239	656	4239	0.52	6.65
LP2	0 all ages	437	149	2865	4.40	3.10
LP2	.01 all ages	437	149	2865	4.40	3.73
LP2	.015 for age < 70 .005 for age \geq 70	437	149	10514	16.15	6.91

Notes:

- Column dimensions include slack and surplus variables.
Row dimensions include objective row.
- Density is defined as the percentage of non-zero entries in the tableau.
- Solution times are for the MPS III (Ketron, 1984) execution step only operating on an IBM 3081 Model K processor.

No. of planning horizon (N)	No. of age-classes (k)	LP2				LP1					
		No. of rows	No. of columns	Age-independent hazard (p constant) i	No. of non-zero elements	Density %	Age-dependent hazard (p non-constant) i	No. of non-zero elements	Density %	No. of non-zero elements	Density %
10	8	45	125	10.56	594	906	16.11	187	335	1031	1.55
10	15	52	202	8.32	874	1564	14.89	334	640	1997	0.92
10	20	57	257		514	2134	14.57	439	859	2687	0.71
20	8	85	245	6.65	1384	3376	16.21	367	695	2071	0.81
20	15	92	392	8.11	2924	5714	15.84	654	1269	4017	0.48
20	20	97	497	7.81	3764	7184	14.90	859	1679	5407	0.37
30	8	125	365	4.76	2174	7446	16.32	547	1035	3111	0.55
30	15	137	582	6.24	4974	12864	16.13	974	1889	6037	0.33
30	20	137	737	6.95	7014	16234	16.08	1279	1499	8127	0.25
40	8	165	485	3.70	2964	13116	16.39	727	1375	4151	0.42
40	15	172	772	5.29	7024	23014	17.33	1294	2509	8057	0.25
40	20	177	977	5.94	10264	29284	16.93	1699	3319	10847	0.19

Table 4: Size and density statistics for the LP1 and LP2 formulations, for different planning horizons (N) and numbers of timber types (k). Column dimensions include slack and surplus variables. Row dimensions include the objective row. It is assumed that unit volumes in all k age-classes are non-zero. If some of these unit volumes were zero, the density figures would be slightly different.