Analysis of a Two fluid model and its comparison with MHD system

by

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B.Sc., Sichuan University, 2012
M.Sc., University of Victoria, 2014

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ABSTRACT

In this thesis, we study a two fluid system which describes the motion of two charged particles in a strict neutral incompressible plasma. We study the well-posdness of the system in both space dimensions two and three. Regardless of the size of the initial data, we prove the global well-posedness of the Cauchy problem when the space dimension is two. In space dimension three, we construct global weak-solutions, and we prove the local well-posedness of Kato-type solutions. These solutions turn out to be global when the initial data are sufficiently small. We also study the stability of the solution around zero given that the initial data is small and has sufficient regularity. It turns out that our system is a system of regularity-loss and the L^2 norm of lower derivatives of the solution decays. At last, this two fluid system can be used to derive the classic MHD at least formally. Arsenio, Ibrahim and Masmoudi (2015) proved that the two fluid system converges to MHD under some constraints. We showed numerically that the two fluid system converges to MHD with no such constraint and found the approximate converge rate.

Contents

List of Figures

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Chapter 1

Introduction

1.1 Background

Plasma which was first introduced by the chemist Irving Langmuir in 1920s (see [17]), is sometimes called the fourth state of the matter. Plasma is usually formed by keeping heating the gas so that some of the electrons gain enough energy and run away from the atoms which become the positive charged ions. This phenomena is called ionization. Plasmas are easy to see in our nature. For example the lightning, outer sphere of the sun and outer core of the earth are plasmas. These plasmas usually have high temperature since the amount of heat added on them to guarantee ionization is extremely high. For instance the temperature of outer sphere of the sun is around $5500K$. These 'hot' plasmas are called thermal plasma. Meanwhile, there is another kind of plasma called cold plasma or nonthermal plasma (see [26]). In microscopic view, a cold plasma is similar to a thermal plasma: part of the electrons leave their atoms and move freely. However, the difference is that speed of the heavy ions are extremely slower than the speed of the electrons. In other words, the temperature of the electrons can be very high but the temperature of the heavy ions is just a little bit higher than the room temperature. For example, neon lights and it can be touched by hand.

Since plasma comes from the ionization of neutral atoms, a plasma system should be nearly neutral. The important thing here is that the moving electrons and atoms can create electromagnetic field. For instance, in each planet, the outer core is a moving plasma due to the high temperature and pressure thus the planet has electromagnetic field around it. This is especially meaningful to our earth. You can image that the earth would be a dead planet if there was no electromagnetic field preventing the high energy particles from the sun.

In a plasma, not all atoms become ions. Typically in the lightning, neon light, only small amount of total atoms are ionized. In extremely high temperature and pressure environment, all atoms can be ionized such as outer sphere of the sun. Especially, in controlled nuclear fusion (ref. [26]), the motion of plasma can be predicted under some suitable models like fluid or kinetic model. Meanwhile, the structure of the fully ionized plasma is simple. There are only two types of particles: negative charged electrons and positive charged ions. Furthermore, the distribution of particles are Maxwellian (see [26]) due to the high temperature and therefore the fluid model is good enough to simulate the fully ionized plasma.

1.2 Macroscopic equations for a plasma

The plasma dynamics was first given by Braginskii [5] 1965. The following modern version of derivation refers to [26]. Assume that the plasma is fully ionized, isotropic and consider the Boltzmann's equation

$$\partial_t f + v \cdot \nabla_x f + \frac{F}{m} \nabla_v f = (\frac{\delta f}{\delta t})_{coll}, \tag{1.1}$$

where $f = f(t, x, v)$ is the phase distribution function and $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$; $(\frac{\delta f}{\delta t})_{coll}$ is the collision term; m is the mass of the particle and F is the lorentz force $F = q(E + v \times B)$ with q the charge carried by each particle. To derive the equations of density or momentum, one can multiply (1.1) by the function $g = 1$ and $g = mv$ respectively and integrate over v. Assume that the weighted averages of g is $\langle g \rangle$, i.e.,

$$\langle g \rangle = \frac{\int g(t, x, v) f(t, x, v) dv}{\int f(t, x, v) dv} = \frac{\int g(t, x, v) f(t, x, v) dv}{n(t, x)},$$

since the number density $n(t, x) = \int f(t, x, v) dv$.

Multiplying (1.1) by g and integrating over v, the first three terms on left hand side are

$$\int g\partial_t f dv = \int \partial_t(gf) - (\partial_t g)f dv = \partial_t\langle ng\rangle - n\langle\partial_t g\rangle,$$
$$\int gv\cdot\nabla_x f dv = \int \nabla_x\cdot(gvf)dv - \int \nabla_x g\cdot vf dv = \nabla_x\cdot(n\langle gv\rangle) - n\langle\nabla_x\cdot(vg)\rangle,$$
$$\int g\frac{F}{m}\nabla_v f dv = -\int f\nabla_v\cdot(\frac{gF}{m})dv = -\frac{n}{m}\langle\nabla_v\cdot(gF)\rangle = -\frac{n}{m}\langle F\cdot\nabla_v g\rangle.$$

The last step is true because $F = q(E + v\times B)$ so that $\nabla_v\cdot F = 0$. Putting them together yields

$$\partial_t(n\langle g\rangle) - n\langle\partial_t g\rangle + \nabla_x\cdot(n\langle gv\rangle) - n\langle\nabla_x\cdot(gv)\rangle - \frac{n}{m}\langle F\cdot\nabla_v g\rangle = \int g(\frac{\delta f}{\delta t})_{coll}dv. \quad (1.2)$$

For continuity equation, letting $g = 1$ gives

$$\partial_t n + \nabla_x\cdot(n\langle v\rangle) = \int(\frac{\delta f}{\delta t})_{coll}dv. \quad (1.3)$$

For momentum equation, letting $g = mv$ in (1.2) yields (noting that density $\rho = mn$)

$$\partial_t(\rho\langle v\rangle) + \sum_{i=1}^{3}\partial_{x_i}(\rho\langle v_i v\rangle) - n\langle F\rangle = \int mv(\frac{\delta f}{\delta t})_{coll}dv. \quad (1.4)$$

Introducing the random velocity v_r(see [26]) such that $v = \langle v\rangle + v_r$, then it holds that

$$\langle v_i v_j\rangle = \langle v_i\rangle\langle v_j\rangle + \langle v_{ri}v_{rj}\rangle,$$

where v_i is the i-th component of velocity, $i = 1, 2, 3$.
Using the above equation, the j-th component of nonlinear term in (1.4) can be expanded in the following way:

$$\partial_{x_i}(\rho\langle v_i v_j\rangle) = \partial_{x_i}(\rho\langle v_i\rangle)\langle v_j\rangle + \rho\langle v_i\rangle\partial_{x_i}\langle v_j\rangle + \partial_{x_i}(\rho\langle v_{ri}v_{rj}\rangle).$$

Therefore

$$\sum_{i=1}^{3}\partial_{x_i}(\rho\langle v_i v\rangle) = \nabla_x\cdot(\rho\langle v\rangle)\langle v\rangle + \rho\langle v\rangle\cdot\nabla_x\langle v\rangle + \sum_{i=1}^{3}(\rho\langle v_{ri}v_r\rangle). \quad (1.5)$$

The random velocity term actually gives the gradient of pressure. Indeed, since the flow is isotropic, $\langle v_{ri}v_{rj}\rangle = \frac{1}{3}\langle|v_r|^2\rangle\delta_{ij}$. Let $p = \frac{1}{3}\rho\langle|v_r|^2\rangle$, then

$$\sum_{i=1}^{3}\partial_{x_i}(\rho\langle v_{ri}v_{rj}\rangle = \nabla_x p. \tag{1.6}$$

Next we expand the time derivative term of (1.4). Recall that $\rho = mn$, so by the continuity equation (1.3), we have

$$\partial_t(\rho\langle v\rangle) = -\nabla_x\cdot(\rho\langle v\rangle)\langle v\rangle + \rho\partial_t\langle v\rangle + \int m\langle v\rangle(\frac{\delta f}{\delta t})_{coll}dv. \tag{1.7}$$

Plugging (1.5), (1.6) and (1.7) into (1.4) yields

$$\rho\partial_t\langle v\rangle + \rho\langle v\rangle\cdot\nabla_x\langle v\rangle - nq\langle E + v\times B\rangle + \nabla_x p = \int mv_r(\frac{\delta f}{\delta t})_{coll}dv := -R. \tag{1.8}$$

The right hand side term R measures the change of momentum due to the collision with particles of other kind. Actually, (1.8) can simplify furthermore. Noting that E, B depend only on the average velocity (ref. [26]), so

$$\langle E + v\times B\rangle = E + \langle v\rangle\times B.$$

Then (1.8) becomes

$$\rho\partial_t\langle v\rangle + \rho\langle v\rangle\cdot\nabla_x\langle v\rangle - nq(E + \langle v\rangle\times B) + \nabla_x p = -R. \tag{1.9}$$

In a plasma, we denote v_+, v_- as the average velocity of ions and electrons. Then the momentum transfer term R can be specified:

$$R = -m_-n_-\nu_{ei}(v_+ - v_-) := -\alpha(v_+ - v_-), \tag{1.10}$$

where $m_\pm, n_\pm$ are mass and number density of ions and electrons, ν_{ei} is the collision frequency between ions and electrons which is a constant. A more physical interpretation is introducing the conductivity $\sigma = \frac{n_-^2e^2}{m_-\nu_{ei}}$ where e is the elementary charge. Then

$$R = -\frac{n_-^2e^2}{\sigma}(v_+ - v_-). \tag{1.11}$$

In a plasma setting (1.9) becomes two equations for ions and electrons

$$\begin{aligned} \rho_-\partial_t v_- + \rho_- v_- \cdot \nabla v_- + n_- e(E + v_- \times B) + \nabla p_- &= -R, \\ \rho_+\partial_t v_+ + \rho_+ v_+ \cdot \nabla v_+ - Z n_+ e(E + v_+ \times B) + \nabla p_+ &= R, \end{aligned} \tag{1.12}$$

where $R = -\alpha(v_+ - v_-)$, Z is the charge number for ions, $\nabla = \nabla_x$ for simplicity. For continuity equations, since the plasma is fully ionized, the ionization and recombination are neglected. So the right hand side of (1.3) is 0. Thus the continuity equations are

$$\begin{aligned} \partial_t \rho_- + \nabla \cdot (\rho_- v_-) &= 0, \\ \partial_t \rho_+ + \nabla \cdot (\rho_+ v_+) &= 0. \end{aligned} \tag{1.13}$$

To complete the whole system, we need Maxwell equation and two state equation. Noting that actually (1.12) is for ideal flow, one can add the viscosity terms. Therefore, after this completion, the whole system for fully ionized plasma is

$$\begin{cases} \partial_t \rho_- + \nabla \cdot (\rho_- v_-) = 0 \\ \partial_t \rho_+ + \nabla \cdot (\rho_+ v_+) = 0 \\ \rho_- \partial_t v_- = \mu_- \Delta v_- - \rho_- v_- \cdot \nabla v_- - n_- e(E + v_- \times B) - R - \nabla p_- \\ \rho_+ \partial_t v_+ = \mu_+ \Delta v_+ - \rho_+ v_+ \cdot \nabla v_+ + Z n_+ e(E + v_+ \times B) + R - \nabla p_+ \\ \varepsilon_0 \partial_t E = \dfrac{1}{\mu_0} \nabla \times B - j, \quad \partial_t B = -\nabla \times E, \\ j = Z n_+ e v_+ - n_- e v_-, \\ p_\pm n_\pm^{-\gamma} = constant \\ R := -\alpha(v_+ - v_-) \\ \mathrm{div} B = 0, \quad \mathrm{div} E = Z n_+ e - n_- e, \end{cases} \tag{1.14}$$

where $\rho_\pm = m_\pm n_\pm$. The physical meaning of these parameters are

- $m_\pm$: mass of ion and electron;
- $n_\pm$: number density of ion and electron;
- e: the elementary charge;
- Z: charge number of ion;

- $\mu_\pm$: dynamic viscosities of ion and electron;
- ε_0, μ_0: vacuum dielectric constant and permeability;
- α: a positive coefficient for momentum change between ions and electrons. More specifically, $\alpha = \nu_{ei} n_- m_- = \frac{n_-^2 e^2}{\sigma}$;
- γ: a constant depends on the heat flux assumption and the isotropy of the energy distribution. For example, for isothermal plasma, the temperature is fixed and $\gamma = 1$.

Remark 1.2.1. Due to the quasi-neutral property of plasma, $Zn_+e - n_-e \approx 0$. One can use this approximation in other equations in (1.14) except $\text{div}E = Zn_+e - n_-e$ which is kept due to the fact that in experiment even very small changes in $\text{div}E$ will lead to an observable changes in electromagnetic field. This is the so-called plasma approximation (see [17]).

In this thesis, we mainly consider the incompressible isothermal neutral plasma. In this plasma setting, (1.14) can be greatly simplified:

$$\begin{cases} \rho_-\partial_t v_- = \mu_-\Delta v_- - \rho_- v_- \cdot \nabla v_- - \beta(E + v_- \times B) - R - \nabla p_- \\ \rho_+\partial_t v_+ = \mu_+\Delta v_+ - \rho_+ v_+ \cdot \nabla v_+ + \beta(E + v_+ \times B) + R - \nabla p_+ \\ \partial_t E = \dfrac{1}{\varepsilon_0\mu_0}\nabla \times B - \dfrac{\beta}{\varepsilon_0}(v_+ - v_-) \\ \partial_t B = -\nabla \times E \\ R := -\alpha(v_+ - v_-) \\ \text{div}v_- = \text{div}v_+ = \text{div}B = \text{div}E = 0, \end{cases} \tag{1.15}$$

where $\beta = en_- = en_+Z$ meaning that the plasma is strictly neutral.

Remark 1.2.2. The assumption that the plasma is incompressible and isothermal makes sure two densities and the temperature of the system are constants. So that (1.15) is a closed system.

A typical example of incompressible plasma is the outer core of the earth. The motion of ions and electrons around the core of the earth creates the strong magnetic field around the earth, protecting lives from high energy particles coming from the sun.

1.3 Relation to magnetohydrodynamics equations

The two-fluid system has a strong relation with magnetohydrodynamics equations(MHD).

$$\begin{cases} \partial_t v = \nu \Delta v - v \cdot \nabla v - \dfrac{1}{\rho \mu_0} B \times (\nabla \times B) - \dfrac{1}{\rho} \nabla p, \\ \partial_t B = \dfrac{1}{\sigma \mu_0} \Delta B + \nabla \times (v \times B), \\ \nabla \cdot B = \nabla \cdot v = 0, \end{cases} \tag{1.16}$$

where ρ, ν are the density and kinematic viscosity of the fluid, σ is the conductivity. If we treat the plasma as a single fluid, for example, considering one ion and several electron as one neutral particle, the resulting governing equations should be MHD. Indeed, MHD can be derived formally from (1.14)(see [26]). Here, we use the incompressible neutral plasma system (1.15) to show the derivation for simplicity.

Remark 1.3.1. MHD are valid under the low frequency case which means the characteristic time T is much bigger than the characteristic length L divided by the speed of the light c(see [26]):

$$T \gg \frac{L}{c}.$$

In other words, the motion of the particles in plasma is much smaller than the propagation speed of EM field.

To derive MHD, we define the following bulk variables

- Density: $\rho = \rho_- + \rho_+$;
- Velocity: $v = \frac{\rho_- v_- + \rho_+ v_+}{\rho}$;
- Current Density: $j = \beta(v_+ - v_-)$;
- Fluid pressure: $p = p_- + p_+$;
- Current pressure: $\tilde{p} = \frac{p_+}{\rho_+} - \frac{p_-}{\rho_-}$.

Furthermore, let us assume that the kinematic viscosity of two fluid is the same: $\frac{\mu_-}{\rho_-} = \frac{\mu_+}{\rho_+} := \nu$.

The sum and difference of first two equations of (1.15) give

$$\partial_t v = \nu\Delta v - v\cdot\nabla v + \frac{1}{\rho} j\times B - \frac{1}{\rho}\nabla p - \frac{\rho_+\rho_-}{\beta^2\rho^2} j\cdot\nabla j, \tag{1.17}$$

$$E + v\times B - \frac{1}{\sigma} j = \frac{\rho_+\rho_-}{\beta^2\rho}(\partial_t j - \nu\Delta j + j\cdot\nabla v + v\cdot\nabla j) + \frac{\rho_+\rho_-(\rho_- - \rho_+)}{\beta^2\rho^2}\frac{1}{\beta} j\cdot\nabla j + \frac{\rho_- - \rho_+}{\beta^2}\frac{\beta}{\rho} j\times B + \frac{\rho_+\rho_-}{\rho\beta^2}\beta\nabla\tilde{p}. \tag{1.18}$$

A typical plasma in nuclear fusion has the electro number density $n_- \approx 10^{20}\ meter^{-3}$, so $\beta \approx 16\ coulombs/meter^3$. One can calculate the ratio $\frac{\rho_+\rho_-}{\beta^2\rho^2} \approx 10^{-6}$. So we neglect the last term in (1.17) and obtain the momentum equation of MHD:

$$\partial_t v = \nu\Delta v - v\cdot\nabla v + \frac{1}{\rho} j\times B - \frac{1}{\rho}\nabla p. \tag{1.19}$$

Similarly, in (1.18),

$$\frac{\rho_+\rho_-}{\beta^2\rho} \approx 10^{-13}, \quad \frac{\rho_+\rho_-(\rho_- - \rho_+)}{\beta^2\rho^2} \approx 10^{-14}, \quad \frac{\rho_- - \rho_+}{\beta^2} \approx 10^{-10}, \quad \frac{\rho_+\rho_-}{\rho\beta^2} \approx 10^{-13}.$$

These constants are in the same unit and if other terms are in a comparable size then we can set the right hand side of (1.18) to be zero and get Ohm's law:

$$E + v\times B = \frac{1}{\sigma} j. \tag{1.20}$$

Next, we derive simplified Ampère's law. Rewrite the third equation of (1.15) as follows

$$\varepsilon_0\partial_t E = \frac{1}{\mu_0}\nabla\times B - j. \tag{1.21}$$

Neglecting the left hand side gives the simplified Ampère's law:

$$\nabla\times B = \mu_0 j. \tag{1.22}$$

Now (1.19), (1.20), (1.22) and the fourth equation of (1.15) together, one can eliminate E and j and obtain the MHD (1.16).

1.4 Previous work

Mathematical analysis about the well-posedness of system (1.15) goes back to the work of Giga-Yoshida [16]. They considered the system in a three-dimensional bounded domain with no-slip and perfectly conductive boundary condition and prove (unique) local solvability as well as global-in-time solvability for a small initial data whose magnetic effect is small compared with velocity. Their method is based on nonlinear semigroup theory initiated by Kōmura [21] which was applied to the Navier-Stokes system [35].

We also emphasize that the system (1.15) are in striking difference with the following slightly modified Navier-Stokes-Maxwell one fluid model studied in [19], [20] and [12].

$$\begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla p &= j\times B \\ \partial_t E - \nabla\times B &= -j \\ \partial_t B + \nabla\times E &= 0 \\ \mathrm{div} v = \mathrm{div} B &= 0 \\ \sigma(E + v{\times}B) &= j. \end{cases} \tag{1.23}$$

Assuming that the electromagnetic field E, B is just in L^2, the term $j\times B$ in velocity equation contains an L^1 function. This does not allow us to gain any regularity using the parabolic estimate. That is why the known results about the existence of weak solution to (1.23) require extra regularity on E, B fields. The existence of global weak solutions(with regularities) of (1.23) in space dimension three was recently solved by D. Arsénio and I. Gallagher [1]. The global well-posedness in 2D is treated in [25]. The local well-posedness and the existence of global small solutions were studied in [19] and [12] for initial data in $u_0, E_0, B_0 \in \dot{H}^{\frac{1}{2}}\times\dot{H}^{\frac{1}{2}}\times\dot{H}^{\frac{1}{2}}$.

For the stability part, we refer to the pioneer works of Schonbek [28] and [29] where she derived the optimal decay rate of solutions to 2D and 3D Navier-Stokes system. In [28], she established that for 2D Navier-Stokes equation, if the initial data $u_0 \in H^s\cap L^1$ (no smallness condition), then the solution u satisfies $\|D^k u\|_{L^2}^2 \lesssim (1+t)^{-1/2-k/2}$, where D^k is ∂^κ with some multi index κ satisfying $|\kappa| = k$. Furthermoer if the average of u_0 is zero, i.e. $\int u_0 dx = 0$ then the lower bound holds $\|u_0\|_{L^2} \gtrsim (1+t)^{-1/2}$. In her later work [29] a better result is given: if $\int u_0 dx = 0$ and $u_0 \in L^1\cap H^1$ then it holds that $\|u_0\|_{L^2}^2 \approx (1+t)^{-d/2+1},\ d = 2, 3$. When $d = 2$, u is the classic solution. For $d = 3$, u is a suitable Leray-Hopf solution in the sense of Caffarell, Kohn, and Nirenberg [6].

The idea is to decomposed the frequency space into two time depended subsets, then obtain a first order differential inequality for H^k norm of the solution. The difficulty here is mainly the low frequency part which was overcame by taking advantages of the linear system of Navier-Stokes equation.
For our system, one can observe that (1.15) is damped Navier-Stokes equations coupled with Maxwell equations. Due to the coupling, the whole linear system requires more regularity on initial data to get the desired decay (see Lemma 3.2.1). Briefly speaking, the solution of the linear system in Fourier side satisfies

$$\hat{U}(t,\xi) \lesssim e^{-\rho(\xi)t}\hat{U}_0(\xi),$$

where $U = (v_-, v_+, E, B)$ and $\rho \approx |\xi|^2$ for $|\xi| \leq 1$, $\rho \approx \frac{1}{|\xi|^2}$ for $|\xi| \geq 1$. So that at the linear level one has

$$\|D^k U\|_{L^2} \lesssim (1+t)^{-3/4-k/2}\|U_0\|_{L^2} + (1+t)^{-l/2}\|D^{k+l}U_0\|_{L^2}, \quad \text{for any integers } k, l.$$

The bad behavior of the solution in high frequency part requires the extra regularity on initial data to get the time decay. Thus our model is a system of regularity-loss type. There are plenty of works studying the decay property of equations of regularity-loss type, for example the work of Hosono and Kawashima [18] on some nonlinear hyperbolic-elliptic equation, Houari [27] on a nonlinear Bresse system. Another well-know system of regularity-loss type is one fluid compressible Euler-Maxwell system. We refer [32] and [33] for details. Recently Xu and Cao [34] proved the decay of one fluid compressible Navier-Stokes-Maxwell system.

Applying the scaling $\tilde{E} = \sqrt{\frac{\varepsilon_0}{2}}E, \tilde{B} = \sqrt{\frac{1}{2\mu_0}}B$ and setting $\varepsilon = \frac{1}{\beta\sqrt{2\mu_0}}, \nu_\pm = \nu, \rho_\pm =$

1 our system becomes

$$
\begin{cases}
\partial_t v_- = \nu\Delta v_- - v_-\cdot\nabla v_- - \dfrac{1}{\varepsilon}(c\tilde{E} + v_-\times\tilde{B}) - R - \nabla p_- \\
\partial_t v_+ = \nu\Delta v_+ - v_+\cdot\nabla v_+ + \dfrac{1}{\varepsilon}(c\tilde{E} + v_+\times\tilde{B}) + R - \nabla p_+ \\
\dfrac{1}{c}\partial_t\tilde{E} = \nabla\times\tilde{B} - \dfrac{1}{2\varepsilon}(v_+ - v_-) \\
\dfrac{1}{c}\partial_t\tilde{B} = -\nabla\times\tilde{E} \\
R := -\dfrac{1}{2\sigma\varepsilon^2}(v_+ - v_-) \\
\mathrm{div} v_- = \mathrm{div} v_+ = \mathrm{div}\tilde{B} = 0, \quad \mathrm{div}\tilde{E} = 0,
\end{cases}
\tag{1.24}
$$

where $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ is the speed of light, σ is the electrical conductivity.
In 2015, Arsénio, Ibrahim and Masmoudi [2] proved that the solution of (1.24) actually converges to the standard MHD system under some relaxing limits both in 2D and 3D. For example in 2D, if $\lim_{c\to\infty} e^{c^2}\varepsilon = \infty$, the two-fluid system converges to MHD. We would like to point it out that when taking the limit c and ε is not independent with each other. It is still open that whether the above system (1.24) converges to MHD when $\varepsilon \to 0, c \to \infty$ independently.
The one fluid Navier-Stokes-Maxwell system can converges to MHD as well. For example, the following one fluid Navier-Stokes-Maxwell system in 2D

$$
\begin{cases}
\partial_t u + u\cdot\nabla u - \nu\Delta u = -\nabla p + j\times B, \\
\dfrac{1}{c}\partial_t B + \nabla\times E = 0, \\
\dfrac{1}{c}\partial_t E - \nabla\times B = -j, \\
j = \sigma(cE + u\times B), \\
\mathrm{div} u = \mathrm{div} B = 0,
\end{cases}
\tag{1.25}
$$

converges to MHD when $c \to \infty$. The result was recently proved by D. Arsénio and I. Gallagher [1].

Remark 1.4.1. There are two one-fluid Navier-Stokes-Maxwell systems in the above statement. Actually the two one-fluid systems (1.25) and (1.23) are exactly the same. The difference comes from using different unit systems. System (1.23) is the dimensionless version of one fluid Navier-Stokes-Maxwell equation using the SI system

(International System of Units), meanwhile system (1.25) uses Gaussian units. The benefit of applying Gaussian units is that the speed of light will be explicitly seen in the formulation.

Indeed, the two-fluid system (1.24) is also in Gaussian units. The scaling on E, B to get (1.24) from our two-fluid system (1.15) shows translating formulas from SI dimensionless system to Gaussian units. For more details about units on measuring, we refer to [23].

1.5 Main Results and Ideas

Before we state the results, we would introduce the short-hand notation throughout the paper $L^p_T X = L^p(0,T;X)$, and we also use the notation $A \lesssim B$ which means $A \leq CB$, where $C > 0$ is a universal constant. Also we define the weak solution of our system:

Definition 1.5.1. A time-dependent vector field (v_-, v_+, E, B) with components in $L^2_{loc}((0,T] \times \mathbb{R}^d), d = 2,3$ is a weak solution to (1.15) if for any $t < T$ and any smooth, compactly supported, divergence-free test function $\phi(t,x)$, the vector field (v_-, v_+, E, B) solves

$$
\begin{cases}
\quad \rho_- \int_{\mathbb{R}^d} (v_- \cdot \phi)(t,x) - (v_- \cdot \phi)(0,x) dx - \rho_- \int_0^t \int_{\mathbb{R}^d} (v_- \cdot \partial_t \phi)(t',x) dx dt' \\
= \int_0^t \int_{\mathbb{R}^d} \nu_- v_- \cdot \Delta \phi + \rho_- v_- \otimes v_- : \nabla \phi - \beta (E + v_- \times B) \cdot \phi - R \cdot \phi dx dt', \\
\\
\quad \rho_+ \int_{\mathbb{R}^d} (v_+ \cdot \phi)(t,x) - (v_+ \cdot \phi)(0,x) dx - \rho_+ \int_0^t \int_{\mathbb{R}^d} (v_+ \cdot \partial_t \phi)(t',x) dx dt' \\
= \int_0^t \int_{\mathbb{R}^d} \nu_+ v_+ \cdot \Delta \phi + \rho_+ v_+ \otimes v_+ : \nabla \phi + \beta (E + v_+ \times B) \cdot \phi + R \cdot \phi dx dt', \\
\\
\quad \int_{\mathbb{R}^d} (E \cdot \phi)(t,x) - (E \cdot \phi)(0,x) dx - \int_0^t \int_{\mathbb{R}^d} (E \cdot \partial_t \phi)(t',x) dx dt' \\
= \int_0^t \int_{\mathbb{R}^d} \frac{1}{\varepsilon_0 \mu_0} B \cdot (\nabla \times \phi) - \frac{\beta}{\varepsilon_0} (v_+ - v_-) \cdot \phi dx dt', \\
\\
\quad \int_{\mathbb{R}^d} (B \cdot \phi)(t,x) - (B \cdot \phi)(0,x) dx - \int_0^t \int_{\mathbb{R}^d} (B \cdot \partial_t \phi)(t',x) dx dt' \\
= \int_0^t \int_{\mathbb{R}^d} -E \cdot (\nabla \times \phi) dx dt', \\
\\
R := -\alpha (v_+ - v_-).
\end{cases}
$$

1.5.1 Well-posedness of The Cauchy problem

The first part of the results is about the well-posdness. In the 2D case, we basically use the classical compactness argument (c.f. [25], [22]) to prove our result. For the 3D case, the proof of the existence of global weak solutions goes along the same lines as for the incompressible Navier-Stokes equations. For the sake of completeness, we outline it in this thesis. Our results extend those of [16] to the space dimension two, and improve them in terms of requiring less regularity on the velocity fields. These results already published in [15].
Recall that our system is

$$\begin{cases} \rho_-\partial_t v_- = \mu_-\Delta v_- - \rho_- v_- \cdot \nabla v_- - \beta(E + v_- \times B) - R - \nabla p_- \\ \rho_+\partial_t v_+ = \mu_+\Delta v_+ - \rho_+ v_+ \cdot \nabla v_+ + \beta(E + v_+ \times B) + R - \nabla p_+ \\ \partial_t E = \dfrac{1}{\varepsilon_0\mu_0}\nabla \times B - \dfrac{\beta}{\varepsilon_0}(v_+ - v_-) \\ \partial_t B = -\nabla \times E \\ R := -\alpha(v_+ - v_-) \\ \mathrm{div} v_- = \mathrm{div} v_+ = \mathrm{div} B = \mathrm{div} E = 0, \end{cases} \tag{1.15}$$

Theorem 1.5.1 (Global well-posedness for 2D)**.** Assume that $v_+(t=0), v_-(t=0) \in L^2$ and $E(t=0), B(t=0) \in L^2$. Then for any $0 < s_1' < 1$ and any $T > 0$, there exists a unique weak solution of (1.15) such that $v_+, v_- \in L^1(0,T; H^{s_1'+1}) \cap C([0,T]; L^2)$, $E, B \in C([0,T]; L^2)$. Furthermore, the solution satisfies the following estimate,

$$\|E\|_{C([0,T];L^2)} + \|B\|_{C([0,T];L^2)} \lesssim C_0 C_T$$

and

$$\|v\|_{L^1_T H^{s_1'+1}} \lesssim C_0 C_T^2,$$

where $C_0 = \|v_-\|_{L^2} + \|v_+\|_{L^2} + \|E_0\|_{L^2} + \|B_0\|_{L^2}$, $C_T = C\max(1,T)$ with C a universal constant, and $v = v_\pm$.

Next we concern the existence of global weak solution to (1.15) in 3D case.

Theorem 1.5.2. For initial data $v_+(t=0), v_-(t=0) \in L^2$ and $E(t=0), B(t=0) \in$

L^2 with div $v_{-,0} =$ div $v_{+,0} = 0$, there exists a weak solution

$$\begin{cases} v_- \in L^\infty(0,\infty;L^2) \cap L^2(0,\infty;\dot{H}^1) \\ v_+ \in L^\infty(0,\infty;L^2) \cap L^2(0,\infty;\dot{H}^1) \\ B \in C([0,\infty);L^2) \\ E \in C([0,\infty);L^2) \\ R := -\alpha(v_+ - v_-) \in L^2(0,\infty;L^2), \end{cases}$$

satisfying the following energy inequality

$$\begin{aligned} &\frac{\rho_-}{2\varepsilon_0}\|v_-\|_{L^2}^2 + \frac{\rho_+}{2\varepsilon_0}\|v_+\|_{L^2}^2 + \frac{1}{2}\|E\|_{L^2}^2 + \frac{1}{2\varepsilon_0\mu_0}\|B\|_{L^2}^2 \\ &+\frac{\mu_-}{\varepsilon_0}\|v_-\|_{L_t^2\dot{H}^1}^2 + \frac{\mu_+}{\varepsilon_0}\|v_+\|_{L_t^2\dot{H}^1}^2 + \frac{\alpha}{\varepsilon_0}\|v_- - v_+\|_{L_t^2L^2}^2 \\ &\le \\ \frac{\rho_-}{2\varepsilon_0}\|v_-(0)\|_{L^2}^2 + \frac{\rho_+}{2\varepsilon_0}\|v_+(0)\|_{L^2}^2 &+ \frac{1}{2}\|E(0)\|_{L^2}^2 + \frac{1}{2\varepsilon_0\mu_0}\|B(0)\|_{L^2}^2. \end{aligned} \tag{1.26}$$

Next, we move to the problem of global existence. Before going any further, we first need to rewrite the system as follows and define some constants.

$$\begin{cases} \partial_t v_- - \nu_-\Delta v_- + v_-\cdot\nabla v_- = -a_-(E + v_-\times B) + b_-(v_+ - v_-) - \frac{1}{\rho_-}\nabla p_- \\ \partial_t v_+ - \nu_+\Delta v_+ + v_+\cdot\nabla v_+ = a_+(E + v_+\times B) - b_+(v_+ - v_-) - \frac{1}{\rho_+}\nabla p_+ \\ \partial_t E = \frac{1}{\varepsilon_0\mu_0}\nabla\times B - \frac{\beta}{\varepsilon_0}(v_+ - v_-) \\ \partial_t B = -\nabla\times E \\ \mathrm{div} v_- = \mathrm{div} v_+ = \mathrm{div} B = \mathrm{div} E = 0, \end{cases} \tag{1.27}$$

where we set $\nu_\pm = \frac{\mu_\pm}{nm_\pm}$, $a_\pm = \frac{\beta}{\rho_\pm}$, $b_\pm = \frac{\alpha}{\rho_\pm}$. Moreover, we define $\nu = \min(\nu_-, \nu_+)$. We construct local-in-time mild-type solution to (1.27) for the 3D case. The proof combines *a priori* estimate techniques with the Banach fixed point theorem.

Theorem 1.5.3. If the initial conditions of the physical model (1.15)(or (1.27)) are such that $(v_{-,0}, v_{+,0}, E_0, B_0) \in H^{1/2}\times H^{1/2}\times L^2\times L^2$, then there exists $T > 0$ and a unique solution (v_-, v_+, E, B) of system (1.15) such that

$$\begin{aligned} v_\pm &\in C([0,T]; H^{\frac{1}{2}}) \cap L^2(0,T; \dot{H}^{\frac{3}{2}}) \\ E,\ B &\in C([0,T]; L^2). \end{aligned}$$

Furthermore, if $\|v_\pm(0)\|_{\dot{H}^{1/2}} < \frac{\nu}{2c_1}$, then the unique solution is global, satisfies the energy estimate and for any $T > 0$

$$\|v_\pm\|_{C(0,T;\dot{H}^{\frac{1}{2}})} < \frac{\nu}{2c_1},$$

where c_1 is a constant only depending on dimension.

Remark 1.5.1. Since the Rayleigh friction term R does not improve the regularity of our system (See Lemma 2.1.1), the results we get here are the same as for the classical Navier-Stokes equations.

The following result is about the existence of smooth solutions of system (1.15). The proof is based on time-weighted energy method which will be briefly introduced in next subsection.

Theorem 1.5.4. Let $s \geq 3$ be an integer and the initial data of system (1.15) $U_0 = (v_{-,0}, v_{+,0}, E_0, B_0) \in H^s$. Then there exists a constant $\delta > 0$ such that if $\|U_0\|_{H^s} < \delta$, the Cauchy problem of (1.15) has a unique solution and satisfies that for any $T > 0, U \in C([0,T]; H^s) \cap C^1([0,T); H^{s-1})$.

Remark 1.5.2. The above theorem can be considered as a bonus of time-weighted energy method which is the key method to prove decay of the solution. The above result and the decay result of next subsection is collected in [14].

1.5.2 Stability and decay of the solution

The second part of results is about the time decay of small solution and the stability around zero solution.

The difficult is that our system is actually a regularity-loss type system. Time weighted energy method is a key to prove the decay for a regularity-loss type system. Here we choose a nonlinear hyperbolic-elliptic system from [18] to briefly introduce the time weighted energy method. The system reads

$$\begin{cases} \partial_t u + \partial_x(u^2/2) + \partial_x q = 0, \\ \partial_x^4 q - \partial_x^2 q + q + \partial_x u = 0, \end{cases} \tag{1.28}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. One can easily solve the linearized system in Fourier side for u: $\hat{u}(t,\xi) = e^{-\rho(\xi)t}\hat{u}_0$, where $\rho(\xi) = \frac{\xi^2}{1+\xi^2+\xi^4}$. Then the linear solution u satisfies

$$\|D^k u\|_{L^2} \lesssim (1+t)^{-1/4-k/2}\|u_0\|_{L^2} + (1+t)^{-l/2}\|D^{k+l}u_0\|_{L^2}, \quad \text{for any integers } k, l.$$

Hence system (1.28) is also a system of regularity-loss type. Now back to the nonlinear system, we want to obtain the decay of k-th derivative of the nonlinear solution u. If we apply k-th spatial derivative D^k to (1.28) and the classic energy method, one gets with initial data $u_0 \in H^s$

$$\|u\|_{H^s}^2 + 2\int_0^t \|q\|_{H^{s+2}}^2 d\tau \lesssim \|U_0\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{L^\infty}\|\partial_x u(\tau)\|_{H^{s-1}}^2 d\tau.$$

To control the nonlinearity, we need to control the term $\int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 d\tau$. Usually this can be done by the help of dissipative term in the second equation of (1.28). However, the dissipative term only gives us

$$\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \lesssim \int_0^t \|q(\tau)\|_{H^{s+2}}^2 d\tau,$$

which can not control the nonlinearity due to the loss of regularity.
To overcome this difficult of regularity-loss, when applying the classic H^k energy method, instead of multiplying by $D^k u$, we multiply by $(1+t)^\alpha D^k u$. This will give us

$$\begin{aligned} (1+t)^\alpha\|\partial_x^k u\|_{L^2}^2 &+ 2\int_0^t (1+\tau)^\alpha \|\partial_x^k q(\tau)\|_{H^2}^2 d\tau \\ &\lesssim \|\partial_x^k u_0\|_{L^2}^2 + \alpha\int_0^t (1+\tau)^{\alpha-1}\|\partial_x^k u(\tau)\|_{L^2}^2 d\tau \\ &+ \int_0^t (1+\tau)^\alpha \|\partial_x u(\tau)\|_{L^\infty}\|\partial_x^k u(\tau)\|_{L^2}^2 d\tau. \end{aligned}$$

If we choose $\alpha < 0$, then the term $\alpha\int_0^t (1+\tau)^{\alpha-1}\|\partial_x^k u(\tau)\|_{L^2}^2 d\tau$ is like an artificial dissipative term and is good enough to control the nonlinearity if $(1+t)\|\partial_x u\|_{L^\infty}$ is small. And usually, if the dissipation in system is strong enough, one can choose $\alpha = k$ to get the time decay. For system (1.28), we refer to [18] for the detailed discussion.

After applying the time-weighted energy method, one can obtain the existence of smooth solution to our system (Theorem 1.5.4) and the following decay result.

Theorem 1.5.5. Let $s \geq 7$ and $U_0 \in L^1 \cap H^s$ the initial data of system (1.15). Then there exists a constant $\delta > 0$ such that if $\|U_0\|_{L^1 \cap H^s} \leq \delta$ the unique solution given by Theorem 1.5.4 satisfies the following decay property,

$$\|D^k U\|_{H^{s-2k-3}} \lesssim (1+t)^{-3/4-k/2},$$

for all integers $0 \leq k \leq [(s-1)/2] - 1$.

Remark 1.5.3. Taking $v_{\pm,0} = v_0, E = B = 0$ then the above decay result reads

$$\|D^k v\|_{H^{s-2k-3}} \lesssim (1+t)^{-3/4-k/2},$$

and thus one recovers Schonbek's result (see [28] and [29]) for small k.

Remark 1.5.4. Unlike the Euler-Maxwell system (see [32]), there is no uniformly time decay if the initial data is only in H^s. The presence of the dissipation term requires that $U_0 \in L^1$ to get the uniform decay. See for example [28].

Remark 1.5.5. This remark compares our result with the decay result of classic one fluid Navier-Stokes-Maxwell system (1.23). Ghoul, Ibrahim and Said-Houari [13] showed that for s big enough and small initial data $U_0 \in L^1 \cap H^s$, it holds that $\|D^k U\|_{L^2} \lesssim (1+t)^{-3/4-k/2}$ for $0 \leq k \leq s$. They use Lyapunov functional method to prove the decay of (E, B) in linear level which actually behaves better than the solutions to hyperbolic system. So that the whole system (1.23) is not regularity-loss type.

1.5.3 Numerical study of the convergence of the two fluid model to the MHD solutions

Recall that the following system is the special case of our two-fluid model (1.15).

$$\begin{cases} \partial_t v_- = \nu \Delta v_- - v_- \cdot \nabla v_- - \frac{1}{\varepsilon}(cE + v_- \times B) - R - \nabla p_- \\ \partial_t v_+ = \nu \Delta v_+ - v_+ \cdot \nabla v_+ + \frac{1}{\varepsilon}(cE + v_+ \times B) + R - \nabla p_+ \\ \frac{1}{c}\partial_t E = \nabla \times B - \frac{1}{2\varepsilon}(v_+ - v_-) \\ \frac{1}{c}\partial_t B = -\nabla \times E \\ R := -\frac{1}{2\sigma\varepsilon^2}(v_+ - v_-) \\ \mathrm{div} v_- = \mathrm{div} v_+ = \mathrm{div} B = 0, \quad \mathrm{div} E = 0, \end{cases} \tag{1.29}$$

The above system is obtained by applying the transformation from Gaussian units to SI units and normalizing the two densities in (1.15) to be 1. As we mentioned before, Arsénio, Ibrahim and Masmoudi [2] proved that system (1.29) converges to MHD under some special relaxing limits, and ε is required as a function of c . However, for the most general case: $c \to \infty, \varepsilon \to 0$ and c, ε are independent of each other, there is no rigorous proof that two-fluid system converges to MHD. Fortunately, we can still show this convergent phenomena numerically in 2D periodic case. The basic scheme we use is Crank-Nicolson method and pseudo spectral method(see [8] and [11]). The numerical result shows that

$$\|\omega_{TF} - \omega_{MHD}\|_{L^2} + \|A_{TF} - A_{MHD}\|_{L^2} \approx \max\left\{10^{C_\varepsilon}\varepsilon^2, 10^{C_c}c^{-2}\right\},$$

where C_c, C_ε are two constants only depending on the fluid and magnetic viscosities $\mu, \frac{1}{\sigma}$.
This numerical convergence rate shows no additional relations required between c, ε. In this work we did not simulate the 3D case due to the huge cost. The meshgrid of x, y, z coordinate leads to the 3D matrix calculation when the Crank-Nicolson method is applied.

Much of the original material in the following document is adapted from two of the author's research preprints:

[15] (the research is joint work with his thesis supervisor, Dr. Slim Ibrahim, and co-author Dr. Yoshizaki Giga and Dr. Tsuyoshi Yoneda) and
[14] (the research is joint work with his thesis supervisor, Dr. Slim Ibrahim, and co-author Tej Eddine Ghoul).
In particular, all of Chapter 3, which evolves around the proof of Theorem 1.5.5, along with section 2.5, where the proof of Theorem 1.5.4 is presented, from the main content of [14], "Long time behavior of a two fluid model". Chapter 2 is adapted from [15], "Global well posedness for a two-fluid model". The manuscripts have been accepted in "Advances in Mathematical Science and Applications" and "Differential Integral Equations", respectively. Chapter 4 will be submitted for publication soon.

1.6 Agenda

The thesis is organized as following. The first chapter includes the introduction of a two-fluid system, related works, the relation with MHD and the main results. The second chapter shows the result and proofs of local and global well-posedness. The third chapter contains the proof of the stability result. Then the fourth chapter presents the simulations of two-fluid system and MHD, including the result comparison and scheme analysis. At last Appendices contains the introduction of basic tools used in this thesis, for example, Littlewood-Paley decomposition, Sobolev spaces and Lyapunov method, etc.

Chapter 2

Local and global wellposedness: Results and their proofs

In this chapter, we focus on the wellposedness results about our two-fluid model.

2.1 Parabolic regularization, product estimates and energy estimates

Here, we collect the main tools that will help us in the proofs of our results. The first one concerns a parabolic regularization with or without friction term. Please see Appendix for the definition of Sobolev spaces ($\dot{H}^s$), space-time Sobolev spaces ($L^p_T H^s$) and Chemin−Lerner spaces ($\tilde{L}^p_T H^s$).

Lemma 2.1.1 (parabolic regularization)**.** Let u be a smooth divergence-free vector field which solves

$$\begin{aligned} \partial_t u - \mu\Delta u + b\nabla p &= f_1 + f_2, \quad \mathrm{div} u = 0 \\ u|_{t=0} &= u_0 \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} \partial_t u + au - \mu\Delta u + b\nabla p &= f_1 + f_2, \quad \mathrm{div} u = 0 \\ u|_{t=0} &= u_0 \end{aligned} \tag{2.2}$$

on some interval $[0, T]$, where a and b are nonnegative constants. Then, for every

$p \geq r_1 \geq 1$, $p \geq r_2 \geq 1$ and $s \in \mathbb{R}$,

$$\begin{aligned}\|u\|_{C([0,T];\dot{H}^s)\cap \tilde{L}^p_T \dot{H}^{s+2/p}} &\lesssim (1+\mu^{-\frac{1}{p}})\|u_0\|_{\dot{H}^s} + (1+\mu^{-\frac{1}{p}})\mu^{-1+\frac{1}{r_1}}\|f_1\|_{\tilde{L}^{r_1}_T \dot{H}^{s-2+2/r_1}} \\ &\quad +(1+\mu^{-\frac{1}{p}})\mu^{-1+\frac{1}{r_2}}\|f_2\|_{\tilde{L}^{r_2}_T \dot{H}^{s-2+2/r_2}}. \qquad (2.3)\end{aligned}$$

We also have a similar result in nonhomogeneous spaces but with T-dependent constants. More specifically,

$$\begin{aligned}\|u\|_{\tilde{L}^p_T H^{s+2/p}} &\lesssim C_T\Big(\mu^{-\frac{1}{p}}\|u_0\|_{H^s} + \mu^{-1-\frac{1}{p}+\frac{1}{r_1}}\|f_1\|_{\tilde{L}^{r_1}_T H^{s-2+2/r_1}} \\ &\quad +\mu^{-1-\frac{1}{p}+\frac{1}{r_2}}\|f_2\|_{\tilde{L}^{r_2}_T H^{s-2+2/r_2}}\Big). \qquad (2.4)\end{aligned}$$

where $C_T = C\max\{1,T\}$ with C a universal constant.

Proof. We only give a sketch of the proof in the case (2.2) with $f_1 = f$ and $f_2 = 0$. For more details of the proof, we refer to [25]. The equation (2.2) can be written as the following:

$$\partial_t u - (\mu\Delta - aI)u + b\nabla\pi = f.$$

By Duhamel's formula, we have

$$u(t) = e^{t(\mu\Delta - aI)}u_0 + \int_0^t e^{(t-t')(\mu\Delta-aI)}\mathbb{P}f(t')dt'. \qquad (2.5)$$

Applying Δ_q, the frequency localization operator(see Appendix A.2), to (2.5), taking L^2 norm in space, and using the standard estimate for Δ_q(see for instance [25] and Appendix A.2), we get

$$\begin{aligned}\|\Delta_q u(t)\|_{L^2} &\leq \|e^{-at}\Delta_q u_0\|_{L^2}e^{-c2^{2q}\mu t} + \int_0^t e^{-c2^{2q}\mu(t-t')}\|e^{-a(t-t')}\Delta_q\mathbb{P}f(t')\|_{L^2}dt' \\ &\leq \|\Delta_q u_0\|_{L^2}e^{-c2^{2q}\mu t} + \int_0^t e^{-c2^{2q}\mu(t-t')}\|\Delta_q\mathbb{P}f(t')\|_{L^2}dt'.\end{aligned}$$

Then we can follow the same method which is used to get the estimate for (2.1)(See [25]). Taking L^p norm in time and using Young's inequality (in time) we obtain

$$\begin{aligned}\|\Delta_q u\|_{L^p_T L^2} &\lesssim \mu^{-\frac{1}{p}}2^{-\frac{2q}{p}}\|\Delta_q u_0\|_{L^2} + \|e^{-c2^{2q}\mu t}1_{t>0}\|_{L^\alpha}\|\Delta_q f\|_{L^r_T L^2} \\ &\lesssim \mu^{-\frac{1}{p}}2^{-\frac{2q}{p}}\|\Delta_q u_0\|_{L^2} + \mu^{-\frac{1}{\alpha}}2^{-\frac{2q}{\alpha}}\|\Delta_q f\|_{L^r_T L^2},\end{aligned}$$

where $\frac{1}{p}+1=\frac{1}{\alpha}+\frac{1}{r}$. At last, multiplying by $2^{q(s+\frac{2}{p})}$ and taking l^2 norm over $q\in\mathbb{Z}$, we get the desired result. $\square$

Remark 2.1.1. We should note the universal constant in the estimate is independent of a. So in the proof of local existence, the small existence time T is independent of α which appears in R. While applying the lemma to physical model, we put R to the right hand side so that the universal constant will still be independent of α.

The next Lemma is a standard energy estimate for the Maxwell's system.

Lemma 2.1.2. Let f be in $L^1((0,T);H^s)$ for $s\in\mathbb{R}$, and $a>0$ be a constant. Then, Maxwell's equations

$$\begin{cases} \partial_t E - a\nabla\times B &= f\\ \partial_t B+\nabla\times E &= 0\\ E(t=0) &= E_0\\ B(t=0) &= B_0\end{cases}$$

has a unique solution $(E,B)\in C([0,T];H^s)$, and it satisfies

$$\|E\|_{C([0,T];H^s)}+\sqrt{a}\|B\|_{C([0,T];H^s)}\le\|E_0\|_{H^s}+\sqrt{a}\|B_0\|_{H^s}+\|f\|_{L^1_TH^s}.$$

Proof. The proof is straightforward (see for example [25]). Applying H^s inner product with E to the first equation and with aB to the second equation, adding them implies

$$\frac{1}{2}\frac{d}{dt}\|E\|^2_{H^s}+\frac{a}{2}\frac{d}{dt}\|B\|^2_{H^s}=(f,E)_{H^s},$$

where $(\cdot,\cdot)_{H^s}$ stands for H^s inner product. Let $F=(E,\sqrt{a}B)$, $g=(f,0)$, the above identity becomes

$$\frac{1}{2}\frac{d}{dt}\|F\|^2_{H^s}=(g,F)_{H^s}.$$

Applying Cauchy Schwarz inequality yields

$$\frac{1}{2}\frac{d}{dt}\|F\|^2_{H^s}\le\|g\|_{H^s}\|F\|_{H^s},$$

which is equivalent to

$$\frac{d}{dt}\|F\|_{H^s}\le\|g\|_{H^s}.$$

Then integrating in time proves the inequality. For the time continuity part, we rewrite the system as the following equivalent form

$$\begin{cases} \partial_t E - \sqrt{a}\nabla\times\sqrt{a}B &= f \\ \partial_t\sqrt{a}B + \sqrt{a}\nabla\times E &= 0 \\ E(t=0) &= E_0 \\ B(t=0) &= B_0. \end{cases}$$

Then Duhamel's formula gives us

$$F(t) = e^{tL}F(0) + \int_0^t e^{(t-\tau)L}g(\tau)d\tau,$$

with $L = \begin{pmatrix} 0 & -\sqrt{a}\nabla\times \\ \sqrt{a}\nabla\times & 0 \end{pmatrix}$. Since the semigroup generated by a skew-symmetric operator is a continuous semigroup, then the solution $F(t)$ is continuous in time. □

Next, we set the nonlinear estimates necessary to derive the *a priori* bounds.

Lemma 2.1.3 (products estimate)**.** For $0 < s < d/2$, and u, v, B are functions of x, we have

$$\|uv\|_{\dot{H}^{s-d/2}} \lesssim \|u\|_{H^s}\|v\|_{L^2}. \tag{2.6}$$

$$\|uv\|_{H^{2s-d/2}} \lesssim \|u\|_{H^s}\|v\|_{H^s}. \tag{2.7}$$

In particular, when $d = 2$, it holds that

$$\|u\nabla v\|_{H^{s-1}} \lesssim \|u\|_{L^2}\|v\|_{H^1} + \|u\|_{H^1}\|v\|_{\dot{H}^1}. \tag{2.8}$$

While $d = 3$, one has

$$\|u\cdot\nabla v\|_{L^2_T H^{-\frac{1}{2}}} \lesssim \|u\|_{L^\infty_T H^{\frac{1}{2}}}\|v\|_{L^2_T H^{\frac{3}{2}}} \tag{2.9}$$

$$\|u\cdot\nabla v\|_{\tilde{L}^{\frac{4}{3}}_T L^2} \lesssim \|u\|^{\frac{1}{2}}_{L^\infty_T H^{\frac{1}{2}}}\|u\|^{\frac{1}{2}}_{L^2_T H^{\frac{3}{2}}}\|v\|_{L^2_T H^{\frac{3}{2}}} \tag{2.10}$$

$$\|u\times B\|_{L^2_T H^{-\frac{1}{2}}} \lesssim T^{\frac{1}{4}}\|u\|^{\frac{1}{2}}_{L^\infty_T H^{\frac{1}{2}}}\|u\|^{\frac{1}{2}}_{L^2_T H^{\frac{3}{2}}}\|B\|_{L^\infty_T L^2}. \tag{2.11}$$

Proof. For (2.6), Hölder's inequality and Sobolev embedding(see Appendix A.1) give that

$$\begin{aligned}\|uv\|_{\dot{H}^{s-d/2}} &\leq \|uv\|_{L^{\frac{d}{d-s}}}\\ &\lesssim \|u\|_{L^{\frac{2d}{d-2s}}}\|v\|_{L^2}\\ &\lesssim \|u\|_{H^s}\|v\|_{L^2}.\end{aligned}$$

Estimate (2.7) is classic and we refer for example to [3], Corollary 2.55. For (2.8), estimating the low and high frequencies separately yields

$$\begin{aligned}\|u\nabla v\|_{H^{s-1}}^2 &= \sum_q (1+2^q)^{2(s-1)}\|(u\nabla v)\|_{L^2}^2\\ &\lesssim \sum_{q\leq 0}\|(u\nabla v)\|_{L^2}^2 + \sum_{q\geq 1} 2^{2q(s-1)}\|(u\nabla v)\|_{L^2}^2\\ &\lesssim \sum_{q\leq 0}\|(u\nabla v)\|_{L^2}^2 + \|u\nabla v\|_{\dot{H}^{s-1}}^2\\ &\lesssim \sum_{q\leq 0}\|(u\nabla v)\|_{L^2}^2 + \|u\|_{H^s}^2\|v\|_{\dot{H}^1}^2,\end{aligned}$$

where (2.6) is applied in the last step.
Hence we only need to estimate $\sum_{q\leq 0}\|(u\nabla v)\|_{L^2}^2$. Thanks to Bony decomposition (see for example [3]), we have

$$\begin{aligned}\|(u\nabla v)\|_{L^2}^2 \lesssim \sum_{|k-q|\leq 2}\|S_{k-1}u\Delta_k(\nabla v)\|_{L^2}^2 + \sum_{|k-q|\leq 2}\|S_{k-1}(\nabla v)\Delta_k u\|_{L^2}^2\\ + \|\sum_{\substack{k\geq q+3\\|k-l|\leq 1}}(\Delta_k u\Delta_l(\nabla v))\|_{L^2}^2,\end{aligned}$$

where $S_q = \sum_{k\leq q-1}\Delta_k$. Applying Bernstein's lemma (noting that $d=2$) and using $q\leq 0$ gives

$$\sum_{|k-q|\leq 2}\|S_{k-1}u(\nabla v)\|_{L^2}^2 \lesssim \sum_{|k-q|\leq 2}\|S_{k-1}u\|_{L^2}^2 2^{4k}\|v\|_{L^2}^2 \lesssim \|u\|_{L^2}^2\sum_{|k-q|\leq 2}\|\Delta_k v\|_{L^2}^2,$$

and therefore,

$$\sum_{q\leq 0}\sum_{|k-q|\leq 2}\|S_{k-1}u(\nabla v)\|_{L^2}^2 \lesssim \|u\|_{L^2}^2\|v\|_{L^2}^2. \tag{2.12}$$

Again applying Bernstein's lemma and the fact that $q \le 0$ gives

$$\begin{aligned}\sum_{|k-q|\le 2} \|S_{k-1}\nabla v u\|_{L^2}^2 &\lesssim \sum_{|k-q|\le 2} \|S_{k-1}(\nabla v)\|_{L^2}^2 2^{2k}\|u\|_{L^2}^2 \\ &\lesssim \|v\|_{\dot{H}^1}^2 \sum_{|k-q|\le 2} \|\Delta_k u\|_{L^2}^2,\end{aligned}$$

yielding

$$\sum_{q\le 0}\sum_{|k-q|\le 2} \|S_{k-1}\nabla v u\|_{L^2}^2 \lesssim \|u\|_{L^2}^2 \|v\|_{\dot{H}^1}^2, \tag{2.13}$$

as desired. For the last term, applying Bernstein's lemma and the Hölder inequality implies

$$\begin{aligned}&\|\sum_{\substack{k\ge q+3\\|k-l|\le 1}} (\Delta_k u \Delta_l(\nabla v))\|_{L^2} \\ &\le \sum_{\substack{k\ge q+3\\|k-l|\le 1}} 2^q \|\Delta_k u \Delta_l(\nabla v)\|_{L^1} \\ &\lesssim \sum_{\substack{k\ge q+3\\|k-l|\le 1}} 2^{q+l} \|\Delta_k u\|_{L^2} \|\Delta_l v\|_{L^2} \\ &\lesssim \sum_{\substack{k\ge q+3\\|k-l|\le 1}} 2^{q+k} \|\Delta_k u\|_{L^2} \|\Delta_l v\|_{L^2}.\end{aligned}$$

Therefore, Young's and Hölder's inequalities give

$$\begin{aligned}\sum_{q\le 0}\|\sum_{\substack{k\ge q+3\\|k-l|\le 1}} (\Delta_k u \Delta_l(\nabla v))\|_{L^2}^2 &= \left(\|\sum_{\substack{k\ge q+3\\|k-l|\le 1}} (\Delta_k u \Delta_l(\nabla v))\|_{L^2}\right)_{l^2(q\le 0)}^2 \\ &\lesssim \left(\sum_{\substack{k\ge q+3\\|k-l|\le 1}} 2^{2k}\|\Delta_k u\|_{L^2}\|\Delta_l v\|_{L^2} 2^{q-k}\right)_{l^2(q\le 0)}^2 \\ &\lesssim \left(2^{2q}\|u\|_{L^2} \sum_{|q-l|\le 1} \|\Delta_l v\|_{L^2}\right)_{l^1}^2 (2^q)_{l^2(q\le 0)}^2 \\ &\lesssim (2^q\|u\|_{L^2} 2^q \sum_{|q-l|\le 1} \|\Delta_l v\|_{L^2})_{l^1}^2 \\ &\lesssim \|u\|_{\dot{H}^1}^2 \|v\|_{\dot{H}^1}^2.\end{aligned} \tag{2.14}$$

(2.12), (2.13) and (2.14) together yields

$$\sum_{q\leq 0}\|(u\nabla v)\|_{L^2}^2 \lesssim \|u\|_{L^2}^2\|v\|_{H^1}^2 + \|u\|_{\dot{H}^1}^2\|v\|_{\dot{H}^1}^2.$$

Hence

$$\begin{aligned}\|u\nabla v\|_{H^{s-1}} &\lesssim \|u\|_{L^2}\|v\|_{H^1} + \|u\|_{H^s}\|v\|_{\dot{H}^1} + \|u\|_{\dot{H}^1}\|v\|_{\dot{H}^1}\\ &\lesssim \|u\|_{L^2}\|v\|_{H^1} + \|u\|_{H^1}\|v\|_{\dot{H}^1}.\end{aligned}$$

For the last four estimates, we only show the proof of (2.10) as the others are similar. Noting that $d = 3$, we have

$$\begin{aligned}\|u\cdot\nabla v\|_{\tilde{L}_T^{\frac{4}{3}}L^2} &\lesssim \|u\cdot\nabla v\|_{L_T^{\frac{4}{3}}L^2}\\ &\lesssim \|u\|_{L_T^4L^6}\|\nabla v\|_{L_T^2L^3}\\ &\lesssim \|u\|_{L_T^4H^1}\|\nabla v\|_{L_T^2H^{\frac{1}{2}}}.\end{aligned}$$

Interpolation(see Appendix A.1) gives us

$$\begin{aligned}\|u\cdot\nabla v\|_{\tilde{L}_T^{\frac{4}{3}}L^2} &\lesssim \left(\int_0^T\|u(t)\|_{H^1}^4dt\right)^{\frac{1}{4}}\|\nabla v\|_{L_T^2H^{\frac{1}{2}}}\\ &\lesssim \left(\int_0^T\|u(t)\|_{H^{\frac{1}{2}}}^2\|u(t)\|_{H^{\frac{3}{2}}}^2dt\right)^{\frac{1}{4}}\|\nabla v\|_{L_T^2H^{\frac{1}{2}}},\end{aligned}$$

and using Hölder inequality, we obtain

$$\begin{aligned}\|u\cdot\nabla v\|_{\tilde{L}_T^{\frac{4}{3}}L^2} &\lesssim \left(\|u\|_{L_T^\infty H^{\frac{1}{2}}}^2\|u\|_{L_T^2H^{\frac{3}{2}}}^2\right)^{\frac{1}{4}}\|v\|_{L_T^2H^{\frac{3}{2}}}\\ &\lesssim \|u\|_{L_T^\infty H^{\frac{1}{2}}}^{\frac{1}{2}}\|u\|_{L_T^2H^{\frac{3}{2}}}^{\frac{1}{2}}\|v\|_{L_T^2H^{\frac{3}{2}}}.\end{aligned}$$

□

The next lemma is a standard energy identity for the whole two-fluid system.

Lemma 2.1.4. For system (1.15), we have the following energy identity.

$$\begin{aligned} \frac{\rho_-}{2\varepsilon_0}\|v_-\|_{L^2}^2 + \frac{\rho_+}{2\varepsilon_0}\|v_+\|_{L^2}^2 + \frac{1}{2}\|E\|_{L^2}^2 + \frac{1}{2\varepsilon_0\mu_0}\|B\|_{L^2}^2 \\ + \frac{\mu_-}{\varepsilon_0}\|v_-\|_{L_t^2\dot{H}^1}^2 + \frac{\mu_+}{\varepsilon_0}\|v_+\|_{L_t^2\dot{H}^1}^2 + \frac{\alpha}{\varepsilon_0}\|v_- - v_+\|_{L_t^2L^2}^2 \\ = \\ \frac{\rho_-}{2\varepsilon_0}\|v_-(0)\|_{L^2}^2 + \frac{\rho_+}{2\varepsilon_0}\|v_+(0)\|_{L^2}^2 + \frac{1}{2}\|E(0)\|_{L^2}^2 + \frac{1}{2\varepsilon_0\mu_0}\|B(0)\|_{L^2}^2. \end{aligned} \tag{2.15}$$

Proof. The proof is standard. Multiply $\frac{v_-}{\varepsilon_0}$, $\frac{v_+}{\varepsilon_0}$, E, $\frac{B}{\varepsilon_0\mu_0}$ to the first four equations of (1.15) respectively, integrate over space and add them together. With the divergence free condition, we can get the desired result. □

2.2 Global Well-Posedness in the 2D Case

In this section we prove Theorem 1.5.1. Recall the Theorem is

Theorem 1.5.1 (Global well-posedness for 2D)**.** Assuming $v_+(t=0), v_-(t=0) \in L^2$ and $E(t=0), B(t=0) \in L^2$. Then for any $0 < s_1' < 1$ and any $T > 0$, there exists a unique weak solution of (1.15) such that $v_+, v_- \in L^1(0,T; H^{s_1'+1}) \cap C([0,T]; L^2)$, $E, B \in C([0,T]; L^2)$. Furthermore, the solution satisfies the following estimate,

$$\|E\|_{C([0,T];L^2)} + \|B\|_{C([0,T];L^2)} \lesssim C_0 C_T$$

and

$$\|v\|_{L_T^1 H^{s_1'+1}} \lesssim C_0 C_T^2,$$

where $C_0 = \|v_-\|_{L^2} + \|v_+\|_{L^2} + \|E_0\|_{L^2} + \|B_0\|_{L^2}$, $C_T = C\max(1,T)$ with C a universal constant, and $v = v_\pm$.

The main idea is to apply the classical compactness method to prove the existence of a global weak solution then prove the uniqueness of the weak solution so that we obtain the unique strong solution (c.f. [25]). The whole proof goes into three steps. Firstly, we provide an *a priori* estimate.

Step 1: A priori estimate.

The *a priori* estimate is given by the following lemma.

Lemma 2.2.1. If (v_-, v_+, E, B) solves (1.15) on $[0,T]$, then for any $0 < s_1 < 1$ the following estimate holds:

$$\|v\|_{L^1_T H^{s_1+1}} \lesssim C_0 C_T^2.$$

Here $C_0 = \|v_-\|_{L^2} + \|v_+\|_{L^2} + \|E_0\|_{L^2} + \|B_0\|_{L^2}$, and $v = v_\pm$.

Proof. For simplicity, let $v = v_\pm$, $C_T = C\max\{1,T\}$ with C a universal constant and $v' = v_\mp$. By the energy identity,

$$\|v\|_{L^\infty_T L^2} + \|v\|_{L^2_T \dot{H}^1} \lesssim \|v_0\|_{L^2} + \|v_0'\|_{L^2} + \|E_0\|_{L^2} + \|B_0\|_{L^2} \lesssim C_0.$$

Now fix s_1, there exists s such that $0 < s_1 < s < 1$. Thanks to estimate (2.4) in Lemma 2.1.1 (replacing s by $s-1$, let $p_1 = p_2 = r_1 = r_2 = 1$),

$$\|v\|_{\tilde{L}^1_T H^{s+1}} \leq C_T(\|v_0\|_{H^{s-1}} + \|v'\|_{\tilde{L}^1_T H^{s-1}} \\ + \|E\|_{\tilde{L}^1_T H^{s-1}} + \|v \times B\|_{\tilde{L}^1_T H^{s-1}} + \|v\nabla v\|_{\tilde{L}^1_T H^{s-1}}).$$

The fact that $0 < s_1 < s$, $\|v\|_{L^1_T H^{s_1+1}} \leq \|v\|_{\tilde{L}^1_T H^{s+1}}$(see Appendix A.3) together with energy identity implies

$$\begin{aligned} \|v\|_{L^1_T H^{s_1+1}} &\leq C_T(\|v_0\|_{L^2} + T\|v'\|_{L^\infty_T L^2} \\ &\qquad + T\|E\|_{L^\infty_T L^2} + \|v \times B\|_{L^1_T H^{s-1}} + \|v\nabla v\|_{L^1_T H^{s-1}}) \\ &\leq C_T(C_0 + 2TC_0 + \|v \times B\|_{L^1_T H^{s-1}} + \|v\nabla v\|_{L^1_T H^{s-1}}). \end{aligned}$$

Estimates (2.6) and (2.8) in Lemma 2.1.3 give the following estimates on these nonlinear terms,

$$\|v \times B\|_{L^1_T H^{s-1}} \lesssim \|v\|_{L^1_T H^1} \|B\|_{L^\infty_T L^2} \\ \lesssim (T\|v\|_{L^\infty_T L^2} + T^{1/2}\|v\|_{L^2 \dot{H}^1})\|B\|_{L^\infty_T L^2} \lesssim (T + T^{1/2})C_0^2,$$

$$\begin{aligned} \|v\nabla v\|_{L^1_T H^{s-1}} &\lesssim \|v\|_{L^\infty_T L^2}\|v\|_{L^1_T H^1} + \|v\|_{L^2 H^1}\|v\|_{L^2 \dot{H}^1} \\ &\lesssim (T\|v\|_{L^\infty_T L^2} + T^{1/2}\|v\|_{L^2 \dot{H}^1})\|v\|_{L^\infty_T L^2} + \|v\|_{L^2 H^1}\|v\|_{L^2 \dot{H}^1} \\ &\lesssim (T\|v\|_{L^\infty_T L^2} + T^{1/2}\|v\|_{L^2 \dot{H}^1})\|v\|_{L^\infty_T L^2} \\ &\qquad + (T^{1/2}\|v\|_{L^\infty_T L^2} + \|v\|_{L^2 \dot{H}^1})\|v\|_{L^2 \dot{H}^1} \\ &\lesssim (1 + 2T^{1/2} + T)C_0^2. \end{aligned}$$

Hence

$$\|v\|_{L^1_T H^{s_1+1}} \lesssim C_T(C_0 + 2C_0T + C_0^2(1 + 3T^{1/2} + 2T)) \lesssim C_0C_T^2.$$

□

STEP 2: A COMPACTNESS ARGUMENT.

We will approximate the original system by a frequency cutoff system and apply the classical compactness argument to pass the limit (c.f [25]). Let us define a frequency cutoff operator J_k by

$$J_k u := \mathcal{F}^{-1}(1_{B(0,k)}(\xi)\hat{u}(\xi)),$$

where $\mathcal{F}$ and $\hat{\cdot}$ are the Fourier transform in the space variable. We consider the following approximating system

$$\begin{cases} \rho_-\partial_t v_-^k = \mu_-\Delta J_k v_-^k - \rho_- J_k(J_k v_-^k \cdot \nabla J_k v_-^k) - en(E^k + J_k(J_k v_-^k \times J_k B^k)) \\ \qquad\qquad - R^k - \nabla p_-^k \\ \rho_+\partial_t v_+^k = \mu_+\Delta J_k v_+^k - \rho_+ J_k(J_k v_+^k \cdot \nabla J_k v_+^k) + eZn(E^k + J_k(J_k v_+^k \times J_k B^k)) \\ \qquad\qquad + R^k - \nabla p_+^k \\ \partial_t E^k = \dfrac{1}{\varepsilon_0\mu_0}\nabla \times J_k B^k - \dfrac{ne}{\varepsilon_0}(Zv_+^k - v_-^k) \\ \partial_t B^k = -\nabla \times J_k E^k \\ R^k := -\alpha(v_+^k - v_-^k) \\ \mathrm{div} v_-^k = \mathrm{div} v_+^k = \mathrm{div} B^k = \mathrm{div} E^k = 0 \end{cases} \tag{2.16}$$

with initial data,

$$v_-^k(t=0) = J_k(v_-(t=0)), v_+^k(t=0) = J_k(v_+(t=0)),$$

$$E^k(t=0) = J_k(E(t=0)), B^k(t=0) = J_k(B(t=0)).$$

The above system is now an ODE that has a unique solution $(v_-^k, v_+^k, E^k, B^k) \in C([0,T_k]; L^2)$ with a positive maximal time of existence T_k. Since $J_k^2 = J_k$, $J_k(v_-^k, v_+^k, E^k, B^k)$ is also a solution. Hence uniqueness implies $J_k(v_-^k, v_+^k, E^k, B^k) = (v_-^k, v_+^k, E^k, B^k)$ and therefore we can get rid of J_k in front of (v_-^k, v_+^k, E^k, B^k) and only leave J_k in front of nonlinear terms:

$$\begin{cases} \rho_-\partial_t v_-^k = \mu_-\Delta v_-^k - \rho_- J_k(v_-^k\cdot\nabla v_-^k) - en(E^k + J_k(v_-^k\times B^k)) - R^k - \nabla p_-^k \\ \rho_+\partial_t v_+^k = \mu_+\Delta v_+^k - \rho_+ J_k(v_+^k\cdot\nabla v_+^k) + eZn(E^k + J_k(v_+^k\times B^k)) + R^k - \nabla p_+^k \\ \partial_t E^k = \frac{1}{\varepsilon_0\mu_0}\nabla\times B^k - \frac{ne}{\varepsilon_0}(Zv_+^k - v_-^k) \\ \partial_t B^k = -\nabla\times E^k \\ R^k := -\alpha(v_+^k - v_-^k) \\ \mathrm{div}\ v_-^k = \mathrm{div}\ v_+^k = \mathrm{div} B^k = \mathrm{div} E^k = 0. \end{cases} \tag{2.17}$$

Now we prove that actually $T_k = \infty$. We will see that the energy identity still holds for (2.17),

$$\begin{aligned} &\frac{\rho_-}{2\varepsilon_0}\|v_-^k\|_{L^2}^2 + \frac{\rho_+}{2\varepsilon_0}\|v_+^k\|_{L^2}^2 + \frac{1}{2}\|E^k\|_{L^2}^2 + \frac{1}{2\varepsilon_0\mu_0}\|B^k\|_{L^2}^2 \\ &+\frac{\mu_-}{\varepsilon_0}\|v_-^k\|_{L_t^2\dot H^1}^2 + \frac{\mu_+}{\varepsilon_0}\|v_+^k\|_{L_t^2\dot H^1}^2 + \frac{\alpha}{\varepsilon_0}\|v_-^k - v_+^k\|_{L_t^2L^2}^2 \\ &= \\ &\frac{\rho_-}{2\varepsilon_0}\|J_kv_-(0)\|_{L^2}^2 + \frac{\rho_+}{2\varepsilon_0}\|J_kv_+(0)\|_{L^2}^2 + \frac{1}{2}\|J_kE(0)\|_{L^2}^2 + \frac{1}{2\varepsilon_0\mu_0}\|J_kB(0)\|_{L^2}^2. \end{aligned} \tag{2.18}$$

Hence the L^2 norm of (v_-^k, v_+^k, E^k, B^k) is bounded uniformly in time and we get $T_k = \infty$. Moreover, a priori estimate we get in the previous section also holds, i.e, for any $T > 0$

$$\|v_\pm^k\|_{C([0,T];L^2)\cap L_T^2\dot H^1} + \|E^k\|_{C([0,T];L^2)} + \|B^k\|_{C([0,T];L^2)} \lesssim C_0,$$

and

$$\|v^k\|_{L_T^1H^{s_1+1}} \lesssim C_0C_T^2.$$

where $C_0 = \|v_-\|_{L^2} + \|v_+\|_{L^2} + \|E_0\|_{L^2} + \|B_0\|_{L^2}$, $v = v_\pm$.

To apply the compactness argument, we need to bound $\partial_t(v_-^k, v_+^k)$ in $L_T^2H^{-3/2}$ uniformly in k. Applying Lemma 2.1.3 we obtain

$$\begin{aligned} \|\partial_t v^k\|_{L_T^2H^{-3/2}} &\lesssim \|\Delta v^k\|_{L_T^2H^{-3/2}} + \|E^k + v^k + v'^k\|_{L_T^2H^{-3/2}} \\ &\quad+\|v^k\cdot\nabla v^k\|_{L_T^2H^{-3/2}} + \|v^k\times B^k\|_{L_T^2H^{-3/2}} \\ &\lesssim \|v^k\|_{L_T^2\dot H^1} + T^{1/2}C_0 + \|v^k\otimes v^k\|_{L_T^2H^{-1/2}} + \|v^k\times B^k\|_{L_T^2H^{s-1}} \\ &\lesssim C_0 + T^{1/2}C_0 + \|v^k\|_{L_T^\infty L^2}\|v^k\|_{L_T^2H^{1/2}} + \|v^k\|_{L_T^2H^1}\|B^k\|_{L_T^\infty L^2}. \end{aligned}$$

By energy estimate, we have

$$\|v^k\|_{L^2_T H^{1/2}} \le \|v^k\|_{L^2_T H^1} \le (T^{1/2}\|v^k\|_{L^\infty_T L^2} + \|v^k\|_{L^2_T \dot{H}^1}) \lesssim (T^{1/2}+1)C_0,$$

leading to the bound on $\|\partial_t v^k\|_{L^2_T H^{-3/2}}$:

$$\|\partial_t v^k\|_{L^2_T H^{-3/2}} \lesssim (1+T^{1/2})(2C_0^2 + C_0).$$

Next we introduce Aubin-Lions-Simon Lemma (see for example [4], [30], [22] and [31]). The proof is given in [4].

Lemma 2.2.2 (Aubin-Lions-Simon)**.** Let $X \subset Y \subset Z$ be three Banach spaces. We assume that the embedding of Y in Z is continuous and that the embedding of X in Y is compact. Let p, r be such that $1 \le p, r \le \infty$. For $T > 0$, we define

$$E_{p,r} = \left\{u \in L^p_T X, \partial_t v \in L^r_T Z\right\}.$$

i) If $p < \infty$, the embedding of $E_{p,r}$ in $L^p_T Y$ is compact.
ii) If $p = \infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C([0,T];Y)$ is compact.

Before applying this lemma, we summarize the work above. Actually, we have shown

$$\|v^k\|_{L^1_T H^{s_1+1} \cap C([0,T];L^2) \cap L^2_T \dot{H}^1} \lesssim \max(C_0 C_T^2, C_0),$$
$$\|\partial_t v^k\|_{L^2_T H^{-3/2}} \lesssim (1+T^{1/2})(2C_0^2 + C_0),$$

and

$$\|E^k\|_{C([0,T];L^2)} + \|B^k\|_{C([0,T];L^2)} \lesssim C_0.$$

Let $B_n := \{x : |x| \le n\}$. Then applying Aubin-Lions-Simon Lemma gives us that $\{v^k\}$ is compact in $L^1(0,T;H^{s_1'+1}(B_n)) \cap L^2(0,T;L^2(B_n))$ with $0 \le s_1' < s_1$. Hence, there exist

$$v_\pm \in L^1(0,T;H^{s_1'+1}(B_n)) \cap L^2(0,T;L^2(B_n)), \quad E, B \in L^\infty(0,T;L^2(B_n))$$

and a subsequence k_m such that as $m \to \infty$,

$$v_\pm^{k_m} \to v_\pm \text{ strongly in } L^1(0,T;H^{s_1'+1}(B_n)) \cap L^2(0,T;L^2(B_n)),$$

$$(E^{k_m}, B^{k_m}) \rightharpoonup (E, B) \text{ weakly-* in } L^\infty(0,T;L^2(B_n)).$$

Indeed, by a diagonal extraction argument the above strong convergences are true for all $B_n, n > 0$. Therefore we can pass the limit in (2.17) and obtain a weak solution. The continuity of $v_\pm$ is obtained with the fact that $v_\pm$ is unique and solves Navier-Stokes equations. Consider $-en(E + v_- \times B) - R$ as a body force of the following Navier-Stokes equation

$$\rho_-\partial_t v_- - \mu_-\Delta v_- + \rho_-(v_- \cdot \nabla)v_- + \nabla p_- = -en(E + v_\times B) - R.$$

Since v_- is unique in energy space $L^\infty_T L^2 \cap L^2_T \dot{H}^1$ which will be proved in next step, v_- also satisfies that $v_- \in C([0,T]; L^2)$. The proof of continuity of v_+ is the same. The continuity of (E, B) is obtained after applying Lemma 2.1.2.

Step 3: Uniqueness of solutions.

Here we prove the uniqueness of solutions to (1.15) (v_+, v_-, E, B) in

$$L^\infty_T L^2 \cap L^2_T H^1 \times L^\infty_T L^2 \cap L^2_T H^1 \times L^\infty_T L^2 \times L^\infty_T L^2.$$

Take (v^1_+, v^1_-, E^1, B^1) and (v^2_+, v^2_-, E^2, B^2) are two solutions of (1.15). Letting $(v_+, v_-, E, B) = (v^1_+, v^1_-, E^1, B^1) - (v^2_+, v^2_-, E^2, B^2)$, then (v_+, v_-, E, B) satisfies the following system with zero initial datum:

$$\begin{cases} \rho_-\partial_t v_- = \mu_-\Delta v_- - \rho_- v^2_- \cdot \nabla v_- - \rho_- v_- \cdot \nabla v^1_- \\ \qquad\qquad -en(E + v^2_- \times B + v_- \times B^1) - R - \nabla p_- \\ \rho_+\partial_t v_+ = \mu_+\Delta v_+ - \rho_+ v^2_+ \cdot \nabla v_+ - \rho_+ v_+ \cdot \nabla v^1_+ \\ \qquad\qquad +eZn(E + v^2_+ \times B + v_+ \times B^1) + R - \nabla p_+ \\ \partial_t E = \frac{1}{\varepsilon_0\mu_0}\nabla \times B - \frac{ne}{\varepsilon_0}(Zv_+ - v_-) \\ \partial_t B = -\nabla \times E \\ R := -\alpha(v_+ - v_-) \\ \mathrm{div} v_- = \mathrm{div} v_+ = \mathrm{div} B = \mathrm{div} E = 0. \end{cases} \tag{2.19}$$

Recall that $v = v_\pm$, $v' = v_\mp$. Let $X = L^\infty_T L^2 \cap L^2_T H^1$ so that $v_\pm, v^i_\pm \in X$ and let $q_1 > 0$, $0 < s' < 1$ be such that $\frac{1}{q_1} = \frac{1+s'}{2}$. Next we apply Lemma 2.1.1 to estimate $\|v\|_X$. For $\|v\|_{L^\infty_T L^2}$ we choose $p = \infty, s = 0, r_1 = r_2 = q_1$ in the lemma, for $\|v\|_{L^2_T H^1}$ we choose $p = 2, s = 0, r_1 = r_2 = q_1$. Finally one obtains

$$\|v\|_X \lesssim C_T\|E + v' + v^2 \times B + v \times B^1 + v^2\nabla v + v\nabla v^1\|_{L^{q_1} H^{s'-1}}, \tag{2.20}$$

where $C_T = C\max(1,T)$. Since $s'-1<0$, by (2.6) one has

$$\begin{aligned}\|v^2\times B\|_{L_T^{q_1}H^{s'-1}} &\le \|v^2\times B\|_{L_T^{q_1}\dot H^{s'-1}}\\ &\lesssim \|v^2\|_{L_T^{q_1}H^{s'}}\|B\|_{L_T^\infty L^2}\\ &\lesssim T^{\frac{1}{q_1}-\frac12}\|v^2\|_{L_T^2H^1}\|B\|_{L_T^\infty L^2}\\ &\lesssim T^{\frac{1}{q_1}-\frac12}\|v^2\|_X\|B\|_{L_T^\infty L^2}.\end{aligned}\tag{2.21}$$

Similarly it holds that

$$\|v\times B^1\|_{L_T^{q_1}H^{s'-1}}\lesssim T^{\frac{1}{q_1}-\frac12}\|B^1\|_{L_T^\infty L^2}\|v\|_X.\tag{2.22}$$

With the help of (2.7) we have

$$\|v^2\nabla v\|_{L_T^{q_1}H^{s'-1}}\lesssim \|v^2\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\|v\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\lesssim \|v^2\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\|v\|_X,\tag{2.23}$$

and

$$\|v\nabla v^1\|_{L^{q_1}H^{s'-1}}\lesssim \|v^1\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\|v\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\lesssim \|v^1\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\|v\|_X,\tag{2.24}$$

where in last step we use the embedding $X\subset L_T^{2q_1}H^{\frac{s'+1}{2}}$: for any $u\in X$,

$$\begin{aligned}\|u\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}} &= \left(\int_0^T\|u\|_{H^{\frac{s'+1}{2}}}^{2q_1}(t)dt\right)^{\frac{1}{2q_1}}\\ &\le\left(\int_0^T\left(\|u\|_{H^1}^{\frac{s'+1}{2}}\|u\|_{L^2}^{\frac{1-s'}{2}}\right)^{2q_1}dt\right)^{\frac{1}{2q_1}}\\ &\le \|u\|_{L_T^\infty L^2}^{\frac{1-s'}{2}}\|u\|_{L_T^2H^1}^{\frac{1}{q_1}}\\ &\le \|u\|_X.\end{aligned}$$

Substituting (2.21) $\sim$ (2.24) back into (2.20) yields

$$\begin{aligned}\|v\|_X \lesssim\ & T^{\frac{1}{q_1}}C_T\left(\|E\|_{L_T^\infty L^2}+\|v'\|_{L_T^\infty L^2}\right)\\ &+T^{\frac{1}{q_1}-\frac12}\left(\|v^2\|_X\|B\|_{L_T^\infty L^2}+\|B^1\|_{L_T^\infty L^2}\|v\|_X\right)\\ &+C_T\left(\|v^2\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\|v\|_X+\|v^1\|_{L_T^{2q_1}H^{\frac{s'+1}{2}}}\|v\|_X\right).\end{aligned}$$

Moreover, thanks to Lemma 2.1.2, we have

$$\|E\|_{L^\infty_T L^2} + \sqrt{\frac{1}{\varepsilon_0\mu_0}}\|B\|_{L^\infty_T L^2} \lesssim \|v\|_{L^1_T L^2} + \|v'\|_{L^1_T L^2} \lesssim T(\|v\|_X + \|v'\|_X).$$

Finally, choosing T small enough, we obtain

$$\begin{aligned}\|v\|_X + \|v'\|_X + \|E\|_{L^\infty_T L^2} + \sqrt{\frac{1}{\varepsilon_0\mu_0}}\|B\|_{L^\infty_T L^2}\\ \leq 1/2(\|v\|_X + \|v'\|_X + \|E\|_{L^\infty_T L^2} + \sqrt{\frac{1}{\varepsilon_0\mu_0}}\|B\|_{L^\infty_T L^2}),\end{aligned}$$

implying that $v_+ = v_- = 0$ and $E = B = 0$ which yields the uniqueness of the solution on a small time interval. We can repeat this argument and get the global uniqueness.

2.3 Global Weak Solutions in the 3D Case

In this section we prove Theorem 1.5.2. Recall that Theorem 1.5.2 is

Theorem 1.5.2. For initial data $v_+(t=0), v_-(t=0) \in L^2$ and $E(t=0), B(t=0) \in L^2$ with div $v_{-,0} =$ div $v_{+,0} = 0$, there exists a weak solution

$$\begin{cases} v_- \in L^\infty(0,\infty;L^2) \cap L^2(0,\infty;\dot{H}^1) \\ v_+ \in L^\infty(0,\infty;L^2) \cap L^2(0,\infty;\dot{H}^1) \\ B \in C([0,\infty);L^2) \\ E \in C([0,\infty);L^2) \\ R := -\alpha(v_+ - v_-) \in L^2(0,\infty;L^2), \end{cases}$$

satisfying the following energy inequality

$$\begin{aligned} \frac{\rho_-}{2\varepsilon_0}\|v_-\|^2_{L^2} + \frac{\rho_+}{2\varepsilon_0}\|v_+\|^2_{L^2} + \frac{1}{2}\|E\|^2_{L^2} + \frac{1}{2\varepsilon_0\mu_0}\|B\|^2_{L^2} \\ + \frac{\mu_-}{\varepsilon_0}\|v_-\|^2_{L^2_t\dot{H}^1} + \frac{\mu_+}{\varepsilon_0}\|v_+\|^2_{L^2_t\dot{H}^1} + \frac{\alpha}{\varepsilon_0}\|v_- - v_+\|^2_{L^2_t L^2} \\ \leq \\ \frac{\rho_-}{2\varepsilon_0}\|v_-(0)\|^2_{L^2} + \frac{\rho_+}{2\varepsilon_0}\|v_+(0)\|^2_{L^2} + \frac{1}{2}\|E(0)\|^2_{L^2} + \frac{1}{2\varepsilon_0\mu_0}\|B(0)\|^2_{L^2}. \end{aligned} \tag{2.25}$$

The proof is similar to the incompressible Navier-Stokes equations. Again, we consider the frequency cut off system in previous section (2.17) and use the Aubin-Lions-Simon Lemma (see for example [30] and [31]) to prove that the solutions of cut off system converges to a weak solution of the original system. We know (2.17) has a unique solution $(v_-^k, v_+^k, E^k, B^k) \in C([0,\infty); L^2)$. According to the energy estimate, (v_-^k, v_+^k, E^k, B^k) is bounded in

$$L_T^\infty L^2 \cap L_T^2 \dot{H}^1 \times L_T^\infty L^2 \cap L_T^2 \dot{H}^1 \times L_T^\infty L^2 \times L_T^\infty L^2.$$

It is sufficient to prove $\partial_t v^k$ is bounded in $L_T^2 H^{-3/2}$. Then following the compactness argument in previous section finishes the proof.
By (2.17), we obtain

$$\begin{aligned}
\|\partial_t v^k\|_{L_T^2 H^{-3/2}} &\lesssim \|\Delta v^k\|_{L_T^2 H^{-3/2}} + \|E^k\|_{L_T^2 H^{-3/2}} + \|v^k\|_{L_T^2 H^{-3/2}} + \|v'^k\|_{L_T^2 H^{-3/2}} \\
&\quad + \|v^k \nabla v^k\|_{L_T^2 H^{-3/2}} + \|v^k \times B^k\|_{L_T^2 H^{-3/2}} \\
&\lesssim \|v^k\|_{L_T^2 H^{1/2}} + T^{1/2}(\|E^k\|_{L_T^\infty L^2} + \|v^k\|_{L_T^\infty L^2} + \|v'^k\|_{L_T^\infty L^2}) \\
&\quad + \|v^k \nabla v^k\|_{L_T^2 H^{-3/2}} + \|v^k \times B^k\|_{L_T^2 H^{-3/2}}.
\end{aligned}$$

By interpolation,

$$\|v^k\|_{L_T^2 H^{1/2}} \lesssim \|v^k\|_{L_T^2 L^2} + \|v^k\|_{L_T^2 \dot{H}^1} \lesssim T^{1/2}\|v^k\|_{L_T^\infty L^2} + \|v^k\|_{L_T^2 \dot{H}^1}.$$

Hence

$$\|\partial_t v^k\|_{L_T^2 H^{-3/2}} \lesssim C_0 + 4T^{1/2} C_0 + \|v^k \nabla v^k\|_{L_T^2 H^{-3/2}} + \|v^k \times B^k\|_{L_T^2 H^{-3/2}}.$$

Moreover, by Sobolev embedding and Hölder's inequality,

$$\begin{aligned}
\|v^k \nabla v^k\|_{L_T^2 H^{-3/2}} &= \|\mathrm{div}(v^k \otimes v^k)\|_{L_T^2 H^{-3/2}} \\
&\leq \|v^k \otimes v^k\|_{L_T^2 H^{-1/2}} \\
&\lesssim \|v^k \otimes v^k\|_{L_T^2 L^{3/2}} \\
&\lesssim \|v^k\|_{L_T^\infty L^2} \|v^k\|_{L_T^2 L^6} \\
&\lesssim \|v^k\|_{L_T^\infty L^2} \|v^k\|_{L_T^2 \dot{H}^1} \lesssim C_0^2.
\end{aligned}$$

With the help of (2.6), one gets

$$\begin{aligned}\|v^k \times B^k\|_{L^2_T H^{-3/2}} &\leq \|v^k \times B^k\|_{L^2_T H^{s-3/2}} \\ &\lesssim \|v^k\|_{L^2_T H^s} \|B^k\|_{L^\infty_T L^2} \\ &\lesssim \|v^k\|_{L^2_T H^1} \|B^k\|_{L^\infty_T L^2} \\ &\lesssim (T^{1/2}\|v^k\|_{L^\infty_T L^2} + \|v^k\|_{L^2_T \dot{H}^1})\|B^k\|_{L^\infty_T L^2} \\ &\lesssim C_0^2(T^{1/2}+1).\end{aligned}$$

Therefore, we get the bound for $\|\partial_t v\|_{L^2_T H^{-3/2}}$,

$$\|\partial_t v\|_{L^2_T H^{-3/2}} \lesssim C_0 + 4T^{1/2}C_0 + C_0^2(T^{1/2}+2).$$

This completes the proof.

2.4 Local well-Posedness in the 3D Case

In this section, we prove Theorem 1.5.3:

Theorem 1.5.3. If the initial conditions of the physical model (1.15)(or (1.27)) are such that $(v_{-,0}, v_{+,0}, E_0, B_0) \in H^{1/2} \times H^{1/2} \times L^2 \times L^2$, then there exists $T > 0$ and a unique solution (v_-, v_+, E, B) of system (1.15) such that

$$\begin{aligned} v_\pm &\in C([0,T]; H^{\frac{1}{2}}) \cap L^2(0,T; \dot{H}^{\frac{3}{2}}) \\ E,\ B &\in C([0,T]; L^2).\end{aligned}$$

Furthermore, if $\|v_\pm(0)\|_{\dot{H}^{1/2}} < \frac{\nu}{2c_1}$, then the unique solution is global, satisfies the energy estimate and for any $T > 0$

$$\|v_\pm\|_{C(0,T;\dot{H}^{\frac{1}{2}})} < \frac{\nu}{2c_1},$$

where c_1 is a constant only depending on dimension.

The idea is to apply the fixed point argument.

Step 1: A priori estimate.

By Lemma 2.1.1, for any (v_-, v_+, E, B) solution of (1.15), we have

$$\|v\|_{L^\infty_T H^{\frac{1}{2}} \cap L^2_T H^{\frac{3}{2}}} \leq C_T(\|v(0)\|_{H^{\frac{1}{2}}} + \|v \times B\|_{L^2_T H^{-\frac{1}{2}}} + \|R\|_{\tilde{L}^1_T H^{\frac{1}{2}}} + \|E + v \cdot \nabla v\|_{\tilde{L}^{\frac{4}{3}}_T L^2}), \tag{2.26}$$

where v stands for v_- or v_+. And by lemma 2.1.2, we obtain:

$$\|E\|_{L^\infty_T L^2} + \|B\|_{L^\infty_T L^2} \leq \gamma(\|E(0)\|_{L^2} + \|B(0)\|_{L^2} + \|v_-\|_{L^1_T L^2} + \|v_+\|_{L^1_T L^2}), \tag{2.27}$$

where $\gamma = \max(1, \sqrt{\frac{1}{\varepsilon_0 \mu_0}})/\min(1, \sqrt{\frac{1}{\varepsilon_0 \mu_0}})$.

Step 2: Contraction argument.

Let $\Gamma := (v_-, v_+, E, B)^T$ be such that

$$\begin{aligned} v_- &\in X^{v_-} := L^\infty_T H^{\frac{1}{2}} \cap L^2_T H^{\frac{3}{2}} \\ v_+ &\in X^{v_+} := L^\infty_T H^{\frac{1}{2}} \cap L^2_T H^{\frac{3}{2}} \\ E &\in X^E := L^\infty_T L^2 \\ B &\in X^B := L^\infty_T L^2, \end{aligned}$$

and set $X = X^v_- \times X^v_+ \times X^E \times X^B$. Then the norm of Γ can be defined as $\|\Gamma\|_X := \|v_-\|_{X^v_-} + \|v_+\|_{X^v_+} + \|E\|_{X^E} + \|B\|_{X^B}$. We look for a solution in the following integral form

$$\Gamma(t) = e^{tA}\Gamma(0) + \int_0^t e^{(t-s)A} f(\Gamma(s)) ds.$$

The operator A and function $f(\Gamma)$ are defined as

$$A = \begin{pmatrix} \frac{\nu_-}{nm_-}\Delta & 0 & 0 & 0 \\ 0 & \frac{\nu_+}{nm_+}\Delta & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon_0\mu_0}\nabla\times \\ 0 & 0 & -\nabla\times & 0 \end{pmatrix},$$

$$f(\Gamma) = \begin{pmatrix} \mathcal{P}(-v_- \cdot \nabla v_- - \frac{e}{m_-}(E + v_- \times B) - \frac{R}{nm_-}) \\ \mathcal{P}(-v_+ \cdot \nabla v_+ + \frac{eZ}{m_+}(E + v_+ \times B) + \frac{R}{nm_+}) \\ -\frac{ne}{\varepsilon_0}(Zv_+ - v_-) \\ 0 \end{pmatrix}.$$

Define a map $\Phi : X \to X$ as

$$\Phi(\Gamma) := \int_0^t e^{(t-s)A} f(e^{sA}\Gamma_0 + \Gamma(s))ds$$

where $\Gamma_0 = (v_{-0}, v_{+0}, E_0, B_0)^T$. We denote the components

$$\Phi(\Gamma) = (\Phi(\Gamma)_-^v, \Phi(\Gamma)_+^v, \Phi(\Gamma)^E, \Phi(\Gamma)^B).$$

Note that $\Phi(-e^{tA}\Gamma_0) = 0$, and by Lemma 2.1.1,

$$\|e^{tA}\Gamma_0\|_X \leq C\|\Gamma_0\|_{H^{\frac{1}{2}}\times H^{\frac{1}{2}}\times L^2\times L^2},$$

where C is a universal constant. Moreover, setting $r = C\|\Gamma_0\|_{H^{\frac{1}{2}}\times H^{\frac{1}{2}}\times L^2\times L^2}$ and denoting by B_r the ball centered at 0 with radius r in the space X. Our goal is to prove if T is small enough then $\Phi(B_r) \subset B_r$.

Assume $\Gamma \in B_r$ and set $\bar{\Gamma} := e^{tA}\Gamma_0 + \Gamma$ (also define $\bar{v}_-, \bar{v}_+, \bar{E}, \bar{B}, \bar{R}$ in the same manner). Then by (2.26) and Lemma 2.1.3,

$$\begin{aligned}
\|\Phi(\Gamma)\|_{X^v} &\leq C_T(\|\bar{v}\times\bar{B} + \bar{v}\cdot\nabla\bar{v}\|_{L_T^2 H^{-\frac{1}{2}}} + \|\bar{E}\|_{\tilde{L}_T^{\frac{4}{3}} L^2} + \|\bar{R}\|_{\tilde{L}_T^1 H^{\frac{1}{2}}}) \\
&\leq C_T\left(\|\bar{v}\|_{L_T^2 H^1} + \|\bar{v}\|_{L_T^2 H^{\frac{3}{2}}} + T^{\frac{3}{4}} + 2T\right)\|\bar{\Gamma}\|_X \\
&\leq 2C_T\left(\|\bar{v}\|_{L_T^2 H^1} + \|\bar{v}\|_{L_T^2 H^{\frac{3}{2}}} + T^{\frac{3}{4}} + 2T\right) r.
\end{aligned}$$

Thus we can choose T small such that

$$2C_T\left(\|\bar{v}\|_{L_T^2 H^1} + \|\bar{v}\|_{L_T^2 H^{\frac{3}{2}}} + T^{\frac{3}{4}} + 2T\right) < \frac{1}{4}.$$

So

$$\|\Phi(\Gamma)\|_{X^{v_-}} < \frac{1}{4}r, \quad \|\Phi(\Gamma)\|_{X^{v_+}} < \frac{1}{4}r.$$

Similarly, by (2.27) we can also get

$$\begin{aligned}
&\|\Phi(\Gamma)\|_{X^E} + \|\Phi(\Gamma)\|_{X^B} \\
\leq\ & \gamma\|\bar{v}_-\|_{L_T^1 L^2} + \gamma\|\bar{v}_+\|_{L_T^1 L^2} \\
\leq\ & \gamma T\|\bar{v}_-\|_{L_T^\infty H^{\frac{1}{2}}} + \gamma T\|\bar{v}_+\|_{L_T^\infty H^{\frac{1}{2}}} \\
\leq\ & 4\gamma T r.
\end{aligned}$$

By choosing T small enough, one obtains

$$\|\Phi(\Gamma)\|_{X^E}+\|\Phi(\Gamma)\|_{X^B}<\frac{1}{2}r.$$

Thus

$$\|\Phi(\Gamma)\|_X<r.$$

And then $\Phi(\Gamma)\in B_r$. Let v be v_- or v_+, $\Gamma_1,\Gamma_2\in B_r$ and $\bar{\Gamma}_i=e^{tA}\Gamma_0+\Gamma_i$, $i=1,2$. By the *a priori estimate* (2.26):

$$\begin{aligned}\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_{X^v} &\leq C_T\|(\bar{v}_1-\bar{v}_2)\times\bar{B}_2+\bar{v}_1\times(\bar{B}_1-\bar{B}_2)\|_{L^2_TH^{-\frac{1}{2}}}\\ &\quad +C_T\|(\bar{v}_1-\bar{v}_2)\cdot\nabla\bar{v}_2+\bar{v}_1\cdot\nabla(\bar{v}_1-\bar{v}_2)\|_{\tilde{L}^{\frac{4}{3}}_TL^2}\\ &\quad +C_T\|\bar{E}_1-\bar{E}_2\|_{\tilde{L}^{\frac{4}{3}}_TL^2}+C_T\|\bar{R}_1-\bar{R}_2\|_{L^1_TH^{\frac{1}{2}}}.\end{aligned}\tag{2.28}$$

The fact that

$$\begin{aligned}\|\bar{R}_1-\bar{R}_2\|_{L^1_TH^{\frac{1}{2}}} &\leq \alpha T\|v_{+,1}-v_{+,2}+v_{-,2}-v_{-,1}\|_{L^\infty_TH^{\frac{1}{2}}}\\ &\leq 2\alpha T\|\Gamma_1-\Gamma_2\|_X\end{aligned}\tag{2.29}$$

together with (2.10) and (2.11) in Lemma 2.1.3, (2.28) becomes

$$\begin{aligned}\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_{X^v} &\leq C_TT^{\frac{1}{4}}\left(\|\bar{B}_2\|_{L^\infty_TL^2}+\|\bar{v}_1\|^{\frac{1}{2}}_{L^\infty_TH^{\frac{1}{2}}}\|\bar{v}_1\|^{\frac{1}{2}}_{L^2_TH^{\frac{3}{2}}}\right)\|\Gamma_1-\Gamma_2\|_X\\ &\quad +C_T\left(\|\bar{v}_2\|_{L^2_TH^{\frac{3}{2}}}+\|\bar{v}_1\|^{\frac{1}{2}}_{L^\infty_TH^{\frac{1}{2}}}\|\bar{v}_1\|^{\frac{1}{2}}_{L^2_TH^{\frac{3}{2}}}\right)\|\Gamma_1-\Gamma_2\|_X\\ &\quad +C_T(T^{\frac{3}{4}}+2\alpha T)\|\Gamma_1-\Gamma_2\|_X\\ &\leq C_TT^{\frac{1}{4}}(4r)\|\Gamma_1-\Gamma_2\|_X\\ &\quad +C_T\left(\|\bar{v}_2\|_{L^2_TH^{\frac{3}{2}}}+(2r)^{\frac{1}{2}}\|\bar{v}_1\|^{\frac{1}{2}}_{L^2_TH^{\frac{3}{2}}}\right)\|\Gamma_1-\Gamma_2\|_X\\ &\quad +C_T(T^{\frac{3}{4}}+2\alpha T)\|\Gamma_1-\Gamma_2\|_X.\end{aligned}$$

We choose T small enough so that

$$\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_{X^v}\leq\frac{1}{8}\|\Gamma_1-\Gamma_2\|_X.$$

Similarly, by (2.27) we can estimate $\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_{X^E}+\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_{X^B}$ when

T is small:

$$\begin{aligned}
&\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_{X^E}+\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_{X^B}\\
\leq\ &\gamma C_T(\|\bar v_{-,1}-\bar v_{-,2}\|_{L^1_TL^2}+\|\bar v_{+,1}-\bar v_{+,2}\|_{L^1_TL^2})\\
\leq\ &\gamma C_T T(\|\bar v_{-,1}-\bar v_{-,2}\|_{L^\infty_T H^{\frac12}}+\|\bar v_{+,1}-\bar v_{+,2}\|_{L^\infty_T H^{\frac12}})\\
\leq\ &\frac14\|\Gamma_1-\Gamma_2\|_X,
\end{aligned}$$

where $v=v_\pm$.

Finally, together with the above estimates

$$\|\Phi(\Gamma_1)-\Phi(\Gamma_2)\|_X\leq\frac12\|\Gamma_1-\Gamma_2\|_X.$$

Thus a fixed point argument gets the local existence.

Step 3: Global existence for small initial data.

In this step, the universal constant will be written explicitly in order to see how the physical parameters affect the estimates and how the solutions depend upon them. This will be needed in a forthcoming work about relaxation limits. We rewrite $X^v_T=L^\infty_T H^{\frac12}\cap L^2_T H^{\frac32}$ to emphasize the dependence upon time T and recall the system (1.27),

$$\begin{cases}
\partial_t v_- -\nu_-\Delta v_- + v_-\cdot\nabla v_- = -a_-(E+v_-\times B)+b_-(v_+-v_-)-\frac{1}{\rho_-}\nabla p_-\\
\partial_t v_+ -\nu_+\Delta v_+ + v_+\cdot\nabla v_+ = a_+(E+v_+\times B)-b_+(v_+-v_-)-\frac{1}{\rho_+}\nabla p_+\\
\partial_t E=\frac{1}{\varepsilon_0\mu_0}\nabla\times B-\frac{ne}{\varepsilon_0}(Zv_+-v_-)\\
\partial_t B=-\nabla\times E\\
\mathrm{div}v_-=\mathrm{div}v_+=\mathrm{div}B=\mathrm{div}E=0,
\end{cases}$$

where $\nu_\pm=\frac{\mu_\pm}{\rho_\pm}$, $a_+=\frac{eZ}{m_+}$, $a_-=\frac{e}{m_-}$, $b_\pm=\frac{\alpha}{\rho_\pm}$.

After setting

$$\lambda_1=\max\{\frac{\rho_-}{2\varepsilon_0},\frac{\rho_+}{2\varepsilon_0},\frac12,\frac{1}{2\varepsilon_0\mu_0}\},$$

and

$$\lambda_2=\min\{\frac{\rho_-}{2\varepsilon_0},\frac{\rho_+}{2\varepsilon_0},\frac12,\frac{1}{2\varepsilon_0\mu_0},\frac{\mu_-}{\varepsilon_0},\frac{\mu_+}{\varepsilon_0},\frac{\alpha}{\varepsilon_0}\},$$

Energy estimate given by Lemma 2.1.4 gives us the bounds

$$\|v_\pm\|_{L^1_T\dot{H}^1}, \|v_\pm\|_{L^\infty_T L^2}, \|E\|_{L^\infty_T L^2} \leq \sqrt{\frac{\lambda_1}{\lambda_2}} B_0, \tag{2.30}$$

where $B_0 = \|v_-(0)\|_{L^2} + \|v_+(0)\|_{L^2} + \|E(0)\|_{L^2} + \|B(0)\|_{L^2}$.
Next we do energy estimate in $\dot{H}^{\frac{1}{2}}$, and prove that if $\|v_\pm(0)\|_{\dot{H}^{\frac{1}{2}}}$ is small enough, the $\|v_\pm(t)\|_{\dot{H}^{\frac{1}{2}}}$ remains small after some time and $\|v_\pm\|_{X_T^{v_\pm}}$ remains bounded so that one can extend the time of existence to infinity. Multiplying the first and second equations of (1.27) by $|\nabla|v_-$ and $|\nabla|v_+$ respectively and integrating over $\mathbb{R}^3$ one obtains

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v_\pm\|^2_{\dot{H}^{1/2}} + \mu_\pm\|v_\pm\|^2_{\dot{H}^{3/2}} \quad \leq \quad & (a_\pm\|E\|_{L^2} + b_\pm\|v_+ - v_-\|)\|v_\pm\|_{\dot{H}^1} \\
& + a_\pm c_1\|B\|_{L^2}\|v_\pm\|_{\dot{H}^1}\|v_\pm\|_{\dot{H}^{3/2}} \\
& + c_1\|v_\pm\|^2_{\dot{H}^1}\|v_\pm\|_{\dot{H}^{3/2}} \\
\leq \quad & (a_\pm + 2b_\pm)\sqrt{\lambda_1/\lambda_2}B_0\|v_\pm\|_{\dot{H}^1} \\
& + a_\pm c_1\sqrt{\lambda_1/\lambda_2}B_0\|v_\pm\|_{\dot{H}^1}\|v_\pm\|_{\dot{H}^{3/2}} \\
& + c_1\|v_\pm\|^2_{\dot{H}^1}\|v_\pm\|_{\dot{H}^{3/2}},
\end{aligned} \tag{2.31}$$

where we use the following inequality for nonlinear terms,

$$|(a \times b, |\nabla|c)| \leq \|b\|_{L^2}\|a\|_{L^6}\||\nabla|c\|_{L^3} \leq c_1\|b\|_{L^2}\|a\|_{\dot{H}^1}\|c\|_{\dot{H}^{3/2}},$$

and

$$|(a\nabla b, |\nabla|c)| \leq \|a\|_{L^6}\|\nabla b\|_{L^3}\||\nabla|c\|_{L^2} \leq c_1\|a\|_{\dot{H}^1}\|c\|_{\dot{H}^1}\|b\|_{\dot{H}^{3/2}},$$

and c_1 here is a constant only depending on dimension. Letting $\mu = \min(\mu_-, \mu_+)$ and $v = v_\pm$ we rewrite (2.31) by

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2_{\dot{H}^{1/2}} + \mu\|v\|^2_{\dot{H}^{3/2}} \leq (a_\pm + 2b_\pm)\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\|v\|_{\dot{H}^1} + a_\pm c_1\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\|v\|_{\dot{H}^1}\|v\|_{\dot{H}^{3/2}} \\
+ c_1\|v\|^2_{\dot{H}^1}\|v\|_{\dot{H}^{3/2}}.
\end{aligned} \tag{2.32}$$

Assuming that $\|v_\pm\|_{\dot{H}^{1/2}}(0) \leq A_0 < \frac{\mu}{2c_1}$, by continuity, there exists time $T^*(A_0)$ such that for all $0 \leq t \leq T^*(A_0)$, $\|v_\pm\|_{\dot{H}^{1/2}}(t) \leq \frac{\mu}{2c_1}$. We consider (2.32) for $t \leq T^*(A_0)$. After using the interpolation $\|v\|_{\dot{H}^1} \leq \|v\|^{1/2}_{\dot{H}^{1/2}}\|v\|^{1/2}_{\dot{H}^{3/2}}$ for every terms on the right hand side one can get rid of the last term in (2.32) and obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{\dot H^{1/2}}^2+\frac{\mu}{2}\|v\|_{\dot H^{3/2}}^2\le&(a_\pm+2b_\pm)\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\|v\|_{\dot H^{1/2}}^{1/2}\|v\|_{\dot H^{3/2}}^{1/2}\\&+a_\pm c_1\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\|v\|_{\dot H^{1/2}}^{1/2}\|v\|_{\dot H^{3/2}}^{3/2}.\end{aligned}\tag{2.33}$$

By Young's inequality, there exists a universal constant c such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{\dot H^{1/2}}^2+\frac{\mu}{2}\|v\|_{\dot H^{3/2}}^2\ \le\ & c\mu^{-1/3}\left((a_\pm+2b_\pm)\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\right)^{4/3}\|v\|_{\dot H^{1/2}}^{2/3}\\&+c\mu^{-3}\left(a_\pm c_1\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\right)^{4}\|v\|_{\dot H^{1/2}}^{2}+\frac{\mu}{4}\|v\|_{\dot H^{3/2}}^2.\end{aligned}$$

Since $\|v\|_{\dot H^{1/2}}<\mu/(2c_1)$ for $t\le T^*(A_0)$, we have

$$\begin{aligned}\frac{d}{dt}\|v\|_{\dot H^{1/2}}^2\le&\,2c\mu^{-1/3}\left((a_\pm+2b_\pm)\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\right)^{4/3}\left(\frac{\mu}{2c_1}\right)^{2/3}\\&+2c\mu^{-3}\left(a_\pm c_1\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\right)^{4}\left(\frac{\mu}{2c_1}\right)^{2}.\end{aligned}$$

Let $C=c^{1/2}\max(\mu^{1/6}\left((a_\pm+2b_\pm)\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\right)^{2/3}\left(\frac{1}{2c_1}\right)^{1/3},\mu^{-1/2}\left(a_\pm c_1\sqrt{\frac{\lambda_1}{\lambda_2}}B_0\right)^{2}\left(\frac{1}{2c_1}\right))$, we end up with

$$\|v\|_{C([0,t];\dot H^{1/2})}\le A_0+2Ct.$$

Therefore $T^*(A_0)\ge\frac{\frac{\mu}{2c_1}-A_0}{2C}$.

To prove $\|v\|_{X_T^v}$ is bounded, we go back to (2.32). For the right hand side of (2.32), apply $\dot H^1$ interpolation only for the last term to get rid of it and Young's inequality for the first and second terms,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{\dot H^{1/2}}^2+\frac{\mu}{2}\|v\|_{\dot H^{3/2}}^2\ \le\ &\frac{1}{2}\frac{\lambda_1}{\lambda_2}(a_\pm+2b_\pm)^2B_0^2+\frac{1}{2}\|v\|_{\dot H^1}^2\\&+C_\mu\frac{\lambda_1}{\lambda_2}a_\pm^2c_1^2B_0^2\|v\|_{\dot H^1}^2+\frac{\mu}{4}\|v\|_{\dot H^{3/2}}^2.\end{aligned}$$

Noticing that $\|v\|^2_{\dot{H}^{3/2}}(t) = \frac{d}{dt}\|v\|^2_{L^2_t\dot{H}^{3/2}}$, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|v\|^2_{\dot{H}^{1/2}} + \frac{\mu}{4}\|v\|^2_{L^2_t\dot{H}^{3/2}}\right) \leq \frac{1}{2}\frac{\lambda_1}{\lambda_2}(a_\pm + 2b_\pm)^2 B_0^2 + (\frac{1}{2} + c\mu^{-1}\frac{\lambda_1}{\lambda_2}a_\pm^2 c_1^2 B_0^2)\|v\|^2_{\dot{H}^1}.$$

Therefore for $T < T^*(A_0)$,

$$\frac{1}{2}\|v\|^2_{C([0,T];\dot{H}^{1/2})} + \frac{\mu}{4}\|v\|^2_{L^2_T\dot{H}^{3/2}} \leq \frac{1}{2}A_0^2 + \frac{1}{2}\frac{\lambda_1}{\lambda_2}(a_\pm + 2b_\pm)^2 B_0^2 T + (\frac{1}{2} + c\mu^{-1}\frac{\lambda_1}{\lambda_2}a_\pm^2 c_1^2 B_0^2)B_0^2 T,$$

which yields the boundedness of $\|v\|_{X_T^v}$ since $\|v(t)\|_{L^2}$ is always bounded by energy estimate. Therefore according to the local existence proof at Step 2, the solution exists up to the time T and $\frac{\frac{\mu}{2c_1} - A_0}{2C} \leq T < T^*(A_0)$. Furthermore, $\|v(t)\|_{\dot{H}^{\frac{1}{2}}} < \frac{\mu}{2c_1}$ for all $t < T$. One can repeat the extension argument above starting at time T and hence the time of existence can be extended to infinity.

2.5 Existence of smooth solution in the 3D case

In this section, we will prove the last theorem about the wellposedness:

Theorem 1.5.4. Let $s \geq 3$ be an integer and the initial data of system (1.15) $U_0 = (v_{-,0}, v_{+,0}, E_0, B_0) \in H^s$. Then there exists a constant $\delta > 0$ such that if $\|U_0\|_{H^s} < \delta$, the Cauchy problem of (1.15) has a unique solution and satisfies that for any $T > 0, U \in C([0,T]; H^s) \cap C^1([0,T); H^{s-1})$.

Here starts the proof.

By the similar fixed point argument as in the previous section, one can easily construct a unique solution in $U \in C([0,T]; H^s)$ with T small enough. We will prove that $U \in C([0,T]; H^s)$ for any $T > 0$ so that the global existence holds true. Then using the equation, it is clear that

$$\partial_t v_\pm \in C([0,T]; H^{s-2}), \qquad \partial_t E, \partial_t B \in C([0,T]; H^{s-1}).$$

Now, we prove that $U \in C([0,T]; H^s)$ for any $T > 0$.

Recall that $U = (v_-, v_+, E, B), v = (v_-, v_+)$. Fixing $s > 0$, we define the following norms:

$$\begin{aligned} N_0(t) &= \sup_{0\le\tau\le t} \|U\|_{H^s} \\ D_0^2(t) &= \int_0^t \|Dv\|_{H^s}^2 + \|v_- - v_+\|_{H^s}^2 + \|E\|_{H^{s-1}}^2 + \|DB\|_{H^{s-2}}^2 d\tau \\ W_0(t) &= \sup_{0\le\tau\le t} \|U\|_{W^{1,\infty}}, \quad J_0^2(t) = \int_0^t \|v\|_{L^\infty}^2 d\tau. \end{aligned} \tag{2.34}$$

It is sufficient to prove the uniform bound of $N_0(t)$ and $D_0(t)$. Recall the notation D^k is ∂^κ with some multi index κ satisfying $|\kappa| = k$. Applying D^l to (1.15) we have

$$\begin{cases} \rho_-\partial_t D^l v_- - \nu_-\Delta D^l v_- + \rho_- v_- \cdot \nabla D^l v_- + \beta(D^l E + D^l v_- \times B) + \alpha(D^l v_- - D^l v_+) \\ \qquad = -\rho_-[D^l(v_-\cdot\nabla v_-) - v_-\cdot\nabla D^l v_-] - \beta[D^l(v_-\times B) - D^l v_- \times B] + \nabla D^l p_-, \\ \rho_+\partial_t D^l v_+ - \nu_+\Delta D^l v_+ - \rho_+ v_+ \cdot \nabla D^l v_+ - \beta(D^l E + D^l v_+ \times B) - \alpha(D^l v_- - D^l v_+) \\ \qquad = -\rho_+[D^l(v_+\cdot\nabla v_+) - v_+\cdot\nabla D^l v_+] + \beta[D^l(v_+\times B) - D^l v_+ \times B] + \nabla D^l p_-, \\ \partial_t D^l E = \dfrac{1}{\varepsilon_0\mu_0}\nabla\times D^l B - \dfrac{\beta}{\varepsilon_0} D^l(v_+ - v_-), \\ \partial_t D^l B = -\nabla\times D^l E. \end{cases} \tag{2.35}$$

Next, we will apply standard energy estimate in H^l. Multiplying the first equation by $D^l v_-$, second equation by $D^l v_+$, third equation by $\varepsilon_0 D^l E$, and the last equation by $\frac{1}{\mu_0} D^l B$ then integrating in space and adding together yields.

$$\begin{aligned} &\frac{\rho_-}{2}\frac{d}{dt}\|D^l v_-\|_{L^2}^2 + \frac{\rho_+}{2}\frac{d}{dt}\|D^l v_+\|_{L^2}^2 + \frac{\varepsilon_0}{2}\frac{d}{dt}\|D^l E\|_{L^2}^2 + \frac{d}{dt}\frac{1}{2\mu_0}\|D^l B\|_{L^2}^2 \\ &\quad + \nu_-\|\nabla D^l v_-\|_{L^2}^2 + \nu_+\|\nabla D^l v_+\|_{L^2}^2 + \alpha\|D^l v_- - D^l v_+\|_{L^2}^2 \\ &= -\rho_-\left(D^l(v_-\cdot\nabla v_-) - v_-\cdot\nabla D^l v_-, D^l v_-\right) - \rho_+\left(D^l(v_+\cdot\nabla v_+) - v_+\cdot\nabla D^l v_+, D^l v_+\right) \\ &\quad - \beta\left(D^l(v_-\times B) - D^l v_-\times B, D^l v_-\right) + \beta\left(D^l(v_+\times B) - D^l v_+\times B, D^l v_+\right), \end{aligned} \tag{2.36}$$

where $(\cdot,\cdot)$ is the L^2 inner product.
Applying the following lemma (we refer to Lemma 3.4 in [24]) to the right hand side of (2.36),

Lemma 2.5.1. For $l > 0$, it holds that

$$\begin{aligned} &\|D^l(ab)\|_{L^2} \lesssim \|a\|_{L^\infty}\|D^l b\|_{L^2} + \|D^l a\|_{L^2}\|b\|_{L^\infty} \\ &\|D^l(a\cdot\nabla b) - a\cdot\nabla D^l b\|_{L^2} \lesssim \|D^l a\|_{L^2}\|\nabla b\|_{L^\infty} + \|\nabla a\|_{L^\infty}\|D^l b\|_{L^2} \end{aligned}$$

we have the classic H^l energy estimate,

$$\begin{aligned}&\frac{\rho_-}{2}\frac{d}{dt}\|D^l v_-\|_{L^2}^2+\frac{\rho_+}{2}\frac{d}{dt}\|D^l v_+\|_{L^2}^2+\frac{\varepsilon_0}{2}\frac{d}{dt}\|D^l E\|_{L^2}^2+\frac{d}{dt}\frac{1}{2\mu_0}\|D^l B\|_{L^2}^2\\&\quad+\nu_-\|\nabla D^l v_-\|_{L^2}^2+\nu_+\|\nabla D^l v_+\|_{L^2}^2+\alpha\|D^l v_- - D^l v_+\|_{L^2}^2\\&\lesssim\|\nabla v_\pm\|_{L^\infty}\|D^l v_\pm\|_{L^2}^2+\|B\|_{L^\infty}\|D^l v_\pm\|_{L^2}^2+\|v_\pm\|_{L^\infty}\|D^l v_\pm\|_{L^2}\|D^l B\|_{L^2},\end{aligned}\tag{2.37}$$

where $1\le l\le s$. When $l=0$, (2.37) becomes the classic energy estimate in which the right hand side of (2.36) vanishes.
Therefore summing (2.37) from $l=0$ to s then integrating in time give us the first ápriori energy estimate

$$N_0(t)+\int_0^t\|Dv\|_{H^s}^2+\|v_- - v_+\|_{H^s}^2d\tau\lesssim\|U_0\|_{H^s}^2+W_0(t)D_0^2(t)+N_0(t)J_0(t)D_0(t).\tag{2.38}$$

Now we linearize (1.15) around the trivial solution $U=0$. Namely, rewrite this system as

$$\begin{cases}\rho_-\partial_t v_- - \nu_-\Delta v_- + \beta E+\alpha(v_- - v_+)+\nabla p_- = -\rho_- v_-\cdot\nabla v_- - v_-\times B,\\ \rho_+\partial_t v_+ - \nu_+\Delta v_+ - \beta E-\alpha(v_- - v_+)+\nabla p_+ = -\rho_+ v_+\cdot\nabla v_+ + v_+\times B,\\ \partial_t E-\dfrac{1}{\varepsilon_0\mu_0}\nabla\times B+\dfrac{\beta}{\varepsilon_0}(v_+ - v_-)=0,\\ \partial_t B+\nabla\times E=0,\end{cases}\tag{2.39}$$

To estimate the dissipation term $D_0(t)$, apply D^l to (2.39), multiply the first three equations by $\frac{1}{\rho_-}D^lE, -\frac{1}{\rho_+}D^lE, D^l(v_- - v_+)$ respectively and summing them yields

$$\begin{aligned}&\partial_t\left(D^l(v_- - v_+),D^lE\right)+\beta(\frac{1}{\rho_-}+\frac{1}{\rho_+})\|D^lE\|_{L^2}^2=\\&(\frac{\nu_-}{\rho_-}\Delta D^l v_- - \frac{\nu_+}{\rho_+}\Delta D^l v_+, D^lE)\\&-\left(\frac{\alpha}{\rho_-}+\frac{\alpha}{\rho_+}\right)\left(D^l(v_- - v_+),D^lE\right)+\frac{\beta}{\varepsilon_0}\|D^l(v_- - v_+)\|_{L^2}^2\\&+\frac{1}{\varepsilon_0\mu_0}\left(\nabla\times D^lB,D^l(v_- - v_+)\right)+f_1+f_2,\end{aligned}\tag{2.40}$$

where,

$$\begin{aligned}f_1&=-\left(D^l(\mathrm{div}(v_-\otimes v_-)-\mathrm{div}(v_+\otimes v_+)),D^lE\right),\\ f_2&=-\left(\frac{1}{\rho_-}D^l(v_-\times B)+\frac{1}{\rho_+}D^l(v_+\times B),D^lE\right).\end{aligned}$$

By Lemma 2.5.1, f_1, f_2 can be estimated as the following,

$$
\begin{aligned}
&f_1 \lesssim \|v_\pm\|_{L^\infty}\|D^{l+1}v_\pm\|_{L^2}\|D^lE\|_{L^2},\\
&f_2 \lesssim (\|D^lv_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|D^lB\|_{L^2})\|D^lE\|_{L^2}.
\end{aligned}
$$

Applying Cauchy-Schwarz and Young's inequality to the other terms on the right hand side of (2.40), then plugging the estimates of f_1, f_2 into (2.40) yields that for $l = 0$,

$$
\begin{aligned}
&\partial_t\left((v_- - v_+), E\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|E\|_{L^2}^2\\
&\lesssim C_\varepsilon\|D_0^2v_\pm\|_{L^2}^2 + C_\varepsilon\|(v_- - v_+)\|_{L^2}^2 + \varepsilon\|E\|_{L^2}^2 + \varepsilon\|DB\|_{L^2}^2\\
&\quad + \|v_\pm\|_{L^\infty}\|Dv_\pm\|_{L^2}\|E\|_{L^2}\\
&\quad + (\|v_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|B\|_{L^2})\|E\|_{L^2}.
\end{aligned}
\tag{2.41}
$$

For $l \geq 1$, using $\big(\nabla\times D^lB, D^l(v_- - v_+)\big) = \big(D^lB, \nabla\times D^l(v_- - v_+)\big)$, we get

$$
\begin{aligned}
&\partial_t\left(D^l(v_- - v_+), D^lE\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|D^lE\|_{L^2}^2\\
&\lesssim C_\varepsilon\|D^{l+2}v_\pm\|_{L^2}^2 + C_\varepsilon\|D^l(v_- - v_+)\|_{H^1}^2 + \varepsilon\|D^lE\|_{L^2}^2 + \varepsilon\|D^lB\|_{L^2}^2\\
&\quad + \|v_\pm\|_{L^\infty}\|D^{l+1}v_\pm\|_{L^2}\|D^lE\|_{L^2}\\
&\quad + (\|D^lv_\pm\|_{L^2}\|B\|_{L^\infty} + \|v_\pm\|_{L^\infty}\|D^lB\|_{L^2})\|D^lE\|_{L^2}.
\end{aligned}
\tag{2.42}
$$

Remark 2.5.1. Here we deal with the case $l = 0$ and $l > 0$ in sightly different ways. Noting that we will integrate in time at last and the L^2 norms of derivatives of B should be controlled by the dissipation norm $D_0(t)$, the terms $\|B\|_{L^2}$ and $\|D^sB\|_{L^2}$ can not appear on the right hand side of (2.41) and (2.42).

By choosing $\varepsilon < \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})$, integrating (2.41), (2.42) in time and summing from

$l = 0$ to $l = s - 1$ yields

$$\begin{aligned}\int_0^t \|E\|_{H^{s-1}}^2 d\tau \lesssim \|U\|_{H^s}^2 &+ \int_0^t \|Dv\|_{H^s}^2 + \|v_- - v_+\|_{H^s}^2 d\tau \\ &+ \|U_0\|_{H^s}^2 + \varepsilon \int_0^t \|DB\|_{H^{s-2}}^2 d\tau \\ &+ \int_0^t \sum_{l=0}^{s-1} \|v_\pm\|_{L^\infty} \|D^{l+1} v_\pm\|_{L^2} \|D^l E\|_{L^2} d\tau \\ &+ \int_0^t \sum_{l=0}^{s-1} (\|D^l v_\pm\|_{L^2} \|B\|_{L^\infty} + \|v_\pm\|_{L^\infty} \|D^l B\|_{L^2}) \|D^l E\|_{L^2} d\tau\end{aligned} \tag{2.43}$$

By (2.38) and

$$\begin{aligned}&\int_0^t \sum_{l=0}^{s-1} (\|v_\pm\|_{L^\infty} \|D^{l+1} v_\pm\|_{L^2} + \|D^l v_\pm\|_{L^2} \|B\|_{L^\infty} + \|v_\pm\|_{L^\infty} \|D^l B\|_{L^2}) \|D^l E\|_{L^2} \\ &\lesssim N_0(t) J_0(t) D_0(t),\end{aligned}$$

the inequality (2.43) becomes

$$\int_0^t \|E\|_{H^{s-1}}^2 d\tau \lesssim \|U_0\|_{H^s}^2 + \varepsilon \int_0^t \|DB\|_{H^{s-2}}^2 d\tau + W_0(t) D_0^2(t) + N_0(t) J_0(t) D_0(t). \tag{2.44}$$

For the dissipation estimate of B we only use maxwell equations in (2.39). Applying $\nabla \times D^l$ to third and fourth equations in (2.39) yields

$$\begin{cases} \partial_t \nabla \times D^l E + \dfrac{1}{\varepsilon_0 \mu_0} \Delta D^l B + \dfrac{\beta}{\varepsilon_0} \nabla \times D^l (v_- - v_+) = 0, \\ \partial_t \nabla \times D^l B - \Delta D^l E = 0. \end{cases} \tag{2.45}$$

Multiply the first equation by $D^l B$ and the second one by $D^l E$ then integrate in space and add together yielding

$$\begin{aligned}&\partial_t (\nabla \times D^l E, D^l B) - \frac{1}{\varepsilon_0 \mu_0} \|D^{l+1} B\|_{L^2}^2 + \\ &\|D^{l+1} E\|_{L^2}^2 + \frac{\beta}{\varepsilon_0} \left(D^l (v_- - v_+), \nabla \times D^l B \right) = 0.\end{aligned}$$

Applying Cauchy-Schwarz inequality the above inequality becomes

$$\begin{aligned} \|D^{l+1}B\|_{L^2}^2 \lesssim \varepsilon\|D^{l+1}B\|_{L^2}^2 + C_\varepsilon\|D^l(v_- - v_+)\|_{L^2}^2 + \|D^{l+1}E\|_{L^2}^2 \\ + \partial_t(\nabla\times D^lE, D^lB). \end{aligned} \tag{2.46}$$

Choosing ε small, integrating (2.46) in time and summing from $l = 0$ to $l = s-2$ give

$$\begin{aligned} \int_0^t \|DB\|_{H^{s-2}}^2 d\tau \lesssim \|U\|_{H^s}^2 + \|U_0\|_{H^s}^2 + \int_0^t \|E\|_{H^{s-1}}^2 d\tau \\ + \int_0^t \|v_- - v_+\|_{H^s}^2 d\tau. \end{aligned} \tag{2.47}$$

Again using (2.38) and (2.44), (2.47) becomes

$$\int_0^t \|DB\|_{H^{s-2}}^2 d\tau \lesssim \|U_0\|_{H^s}^2 + W_0(t)D_0^2(t) + N_0(t)J_0(t)D_0(t). \tag{2.48}$$

By a suitable linear combination of (2.38),(2.44) and (2.48) and choosing small ε in (2.44) we have

$$N_0^2(t) + D_0^2(t) \lesssim \|U_0\|_{L^2}^2 + W_0(t)D_0^2(t) + N_0(t)J_0(t)D_0(t). \tag{2.49}$$

For $s \geq 3$, it holds that $W_0(t) \lesssim N_0(t)$, $J_0(t) \lesssim D_0(t)$. Therefore, together with (2.49) there exists a small constant $\delta > 0$ such that if $\|U_0\|_{H^s} \leq \delta$, then $N_0(t) + D_0(t) \lesssim \delta$ uniformly in time. This ends the proof of Theorem 1.5.4.

Remark 2.5.2. For $s \geq 3$, it is clear that $W_0(t) \lesssim N_0(t)$. To prove $J_0(t) \lesssim D_0(t)$, it is sufficient to prove $\|v\|_{L^\infty}^2 \lesssim \|v\|_{\dot{H}^1}\|v\|_{\dot{H}^2}$. Indeed, we have

$$\begin{aligned} |v| = &\left|\int_{\xi\in\mathbb{R}^3} e^{ix\cdot\xi}\hat{v}d\xi\right| = \left|\int_{|\xi|\leq\lambda} + \int_{|\xi|\geq\lambda} e^{ix\cdot\xi}\hat{v}d\xi\right| \\ \leq &\int_{|\xi|\leq\lambda} |\xi||\hat{v}|\frac{d\xi}{|\xi|} + \int_{|\xi|\geq\lambda} |\xi|^2|\hat{v}|\frac{d\xi}{|\xi|^2} \\ \leq &\left(\int_{|\xi|\leq\lambda} |\xi|^2|\hat{v}|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|\leq\lambda} \frac{1}{|\xi|^2} d\xi\right)^{\frac{1}{2}} \\ + &\left(\int_{|\xi|\geq\lambda} |\xi|^4|\hat{v}|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|\geq\lambda} \frac{1}{|\xi|^4} d\xi\right)^{\frac{1}{2}} \\ \lesssim &\lambda^{1/2}\|v\|_{\dot{H}^1} + \lambda^{-1/2}\|v\|_{\dot{H}^2}. \end{aligned}$$

Optimizing in λ leads to $\lambda = \frac{\|v\|_{\dot{H}^2}}{\|v\|_{\dot{H}^1}}$. Therefore the desired estimate holds.

Chapter 3

Asymptotic stability

In this Chapter, we mainly discuss the asymptotic stability result in 3D case including a more convienent way to develop the linear estimate: Lyapunov method. Furthermoer, the general way of ernergy estimate fails because the system is a regularity-loss type. Then the time-weighted energy method will be used to overcome this difficult.

3.1 An introduction to equations of regularity-loss type and time-weighted energy method

The equations of regularity-loss type means that for the linearized model, to obtain the time decay of H^k control of the solution, one actually need H^{k+1} or more regularity on initial data. For example, the following 1D nonlinear Bresse system from [27]:

$$\begin{cases} \phi_{tt} - (\phi_x - \psi - l\omega)_x - k_0^2 l(\omega_x - l\phi) = 0, \\ \psi_{tt} - \sigma\psi_{xx} - (\phi_x - \psi - l\omega) + \gamma_1\psi_t = 0, \\ \omega_{tt} - k_0^2(\omega_x - l\phi)_x - l(\phi_x - \psi - l\omega) + \gamma_2\omega_t = 0, \end{cases}$$

where $l, k_0, \gamma_1, \gamma_2$ are constants and σ is a smooth function such that $\sigma'(v) > 0$ for $v > 0$. Furthermore, $\sigma'(0) = a^2$ where a is a constant. The linear estimate of this system reads:

$$\|\partial_x^k U(t)\|_{L^2} \lesssim (1+t)^{-1/4-k/2}\|U_0\|_{L^1} + (1+t)^{-l/2}\|\partial_x^{k+l}U_0\|_{L^2}, \quad a = 1$$
$$\|\partial_x^k U(t)\|_{L^2} \lesssim (1+t)^{-1/4-k/2}\|U_0\|_{L^1} + (1+t)^{-l/4}\|\partial_x^{k+l}U_0\|_{L^2}, \quad a \neq 1,$$

where $U = (\phi, \psi, \omega)$. One can see that if we want the time decay control of $\|\partial_x^k U\|_{L^2}$, more regularity is required for the initial data.
Another example is a nonlinear hyperbolic-elliptic system from [18] which we will use to briefly introduce the time-weighted energy method:

$$\begin{cases} \partial_t u + \partial_x(u^2/2) + \partial_x q = 0, \\ \partial_x^4 q - \partial_x^2 q + q + \partial_x u = 0. \end{cases} \tag{3.1}$$

The linear system is easy to solve in Fourier side for u: $\hat{u}(t,\xi) = e^{-\rho(\xi)t}\hat{u}_0$, where $\rho(\xi) = \frac{\xi^2}{1+\xi^2+\xi^4}$. Then the linear solution u satisfies

$$\|D^k u\|_{L^2} \lesssim (1+t)^{-1/4-k/2}\|u_0\|_{L^2} + (1+t)^{-l/2}\|D^{k+l}u_0\|_{L^2}.$$

Hence system (1.28) is also a system of regularity-loss type.
Back to the nonlinear system, the goal is to obtain the decay of k-th derivative of nonlinear solution u. If we only apply D^k to (3.1) and the classic energy method, one gets with initial data $u_0 \in H^s$

$$\|u\|_{H^s}^2 + 2\int_0^t \|q\|_{H^{s+2}}^2 d\tau \lesssim \|U_0\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{L^\infty}\|\partial_x u(\tau)\|_{H^{s-1}}^2 d\tau.$$

To control the nonlinearity, we need to control the term $\int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 d\tau$. Usually this can be done by the help of dissipative term in second equation of (3.1). However, the dissipative term only gives us

$$\int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 d\tau \lesssim \int_0^t \|q(\tau)\|_{H^{s+2}}^2 d\tau,$$

which can not control the nonlinearity due to the loss of regularity. So the classic energy method does not help here.
To overcome this difficult of regularity-loss, when applying the classic H^k energy method, instead of multiplying by $D^k u$, we multiply by $(1+t)^\alpha D^k u$. This will give

us

$$
\begin{aligned}
&(1+t)^{\alpha}\|\partial_x^k u\|_{L^2}^2 + 2\int_0^t (1+\tau)^{\alpha}\|\partial_x^k q(\tau)\|_{H^2}^2 d\tau \\
&\lesssim \|\partial_x^k u_0\|_{L^2}^2 + \alpha\int_0^t (1+\tau)^{\alpha-1}\|\partial_x^k u(\tau)\|_{L^2}^2 d\tau \\
&+ \int_0^t (1+\tau)^{\alpha}\|\partial_x u(\tau)\|_{L^\infty}\|\partial_x^k u(\tau)\|_{L^2}^2 d\tau.
\end{aligned}
$$

If we choose $\alpha < 0$, then the term $\alpha\int_0^t (1+\tau)^{\alpha-1}\|\partial_x^t u(\tau)\|_{L^2}^2 d\tau$ is like an artificial dissipative term and is good enough to control the nonlinearity if $(1+t)\|\partial_x u\|_{L^\infty}$ is small. And usually, if the dissipation in system is strong enough, one can choose $\alpha = k$ to get the time decay. For system (1.28), we refer to [18] for the detailed discussion.

3.2 Linear estimate

Back to our system, we state the linear decay first. We can rewrite system (1.15) as $\partial_t U = \mathcal{L}U + \mathcal{N}(U)$, where $\mathcal{L}$ is the linear operator. However, in our case, the two velocity equations and the non-normalized physical constants make the classic method for linear estimate very complicated and hard. To overcome such complex computation, we introduce a Lyapunov function that gives in a more systematic way the linear decay. Such an idea can be implicitly found, for example, in [9] and [10], for other models.

The linear estimate is the following.

Lemma 3.2.1. The solution to linear system of (1.15): $\partial_t U = \mathcal{L}U$, where

$$
\mathcal{L} = \left\{ \begin{array}{cccc}
\nu_-\Delta - \frac{\alpha}{\rho_-} & \frac{\alpha}{\rho_-} & -\frac{\beta}{\rho_-} & 0 \\
\frac{\alpha}{\rho_+} & \nu_+\Delta - \frac{\alpha}{\rho_+} & \frac{\beta}{\rho_+} & 0 \\
\frac{\beta}{\varepsilon_0} & -\frac{\beta}{\varepsilon_0} & 0 & \frac{1}{\varepsilon_0\mu_0}\nabla\times \\
0 & 0 & -\nabla\times & 0,
\end{array} \right\}
$$

satisfies that for all $k \geq 0, l \geq 0$

$$
\|D^k(e^{t\mathcal{L}}U_0)\|_{L^2}^2 \lesssim (1+t)^{-3/2-k}\|U_0\|_{L^1}^2 + (1+t)^{-l}\|D^{k+l}U_0\|_{L^2}^2.
$$

Proof. Rewrite the linear system of (1.15) in Fourier side,

$$\begin{cases} \rho_-\partial_t\hat{v}_- = -\nu_-|\xi|^2\hat{v}_- - \beta\hat{E} + \alpha(\hat{v}_+ - \hat{v}_-) \\ \rho_+\partial_t\hat{v}_+ = -\nu_+|\xi|^2\hat{v}_+ + \beta\hat{E} - \alpha(\hat{v}_+ - \hat{v}_-) \\ \partial_t\hat{E} = c_1 i\xi \times \hat{B} - c_2\beta(\hat{v}_+ - \hat{v}_-) \\ \partial_t\hat{B} = -i\xi \times \hat{E} \\ \xi \cdot \hat{v}_\pm = \xi \cdot \hat{B} = \xi \cdot \hat{E} = 0, \end{cases} \tag{3.2}$$

where $c_1 = \frac{1}{\varepsilon_0\mu_0}$, $c_2 = 1/\varepsilon_0$ and define the energy

$$\hat{\mathcal{E}} := \frac{1}{2}\rho_-|\hat{v}_-|^2 + \frac{1}{2}\rho_+|\hat{v}_+|^2 + \frac{1}{2c_2}|\hat{E}|^2 + \frac{c_1}{2c_2}|\hat{B}|^2.$$

We immediately have the following energy balance.

$$\frac{d}{dt}\hat{\mathcal{E}} = -|\xi|^2(\nu_-|\hat{v}_-|^2 + \nu_+|\hat{v}_+|^2) - \alpha|\hat{v}_- - \hat{v}_+|^2. \tag{3.3}$$

Define the Lyapunov function

$$L(\xi, t) := \gamma(1 + |\xi|^2)\hat{\mathcal{E}} + F,$$

where

$$F := -\frac{1}{1 + |\xi|^2} < i\xi \times \hat{B}, \hat{E} > + < \hat{v}_- - \hat{v}_+, \hat{E} >,$$

$< a, b >:= \Re(a \cdot \bar{b})$ for $a, b \in \mathbb{C}^3$ and γ is a constant which will be determined later. Now we calculate time derivative of L. Noting that $i\xi \times (i\xi \times \hat{E}) = |\xi|^2\hat{E}$ and $|\xi \times \hat{B}|^2 = |\xi|^2|\hat{B}|^2$ under the divergence free condition of E, B, we obtain

$$\begin{aligned} &\frac{d}{dt}(-\frac{1}{1 + |\xi|^2} < i\xi \times \hat{B}, \hat{E} >) \\ =& \frac{|\xi|^2}{1 + |\xi|^2}|\hat{E}|^2 - \gamma_2 c_1 \frac{|\xi|^2}{1 + |\xi|^2}|\hat{B}|^2 \\ &+ c_2\frac{1}{1 + |\xi|^2} < i\xi \times \hat{B}, \beta(\hat{v}_+ - \hat{v}_-) >, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&\frac{d}{dt}(< \hat{v}_- - \hat{v}_+, \hat{E} >) \\
= &|\xi|^2 < \frac{\nu_+}{\rho_+}\hat{v}_+ - \frac{\nu_-}{\rho_-}\hat{v}_-, \hat{E} > -\beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})|\hat{E}|^2 \\
&+ \alpha(\frac{1}{\rho_-} + \frac{1}{\rho_+}) < \hat{v}_+ - \hat{v}_-, \hat{E} > \\
&- c_1 < \hat{v}_+ - \hat{v}_-, i\xi \times \hat{B} > +c_2 < \hat{v}_+ - \hat{v}_-, \beta(\hat{v}_+ - \hat{v}_-) >,
\end{aligned} \tag{3.5}$$

where c_1, c_2 are constants only depending on physical parameters. Taking (3.3) into account we have

$$\begin{aligned}
\frac{d}{dt}L = &- \gamma|\xi|^2(1 + |\xi|^2)(\nu_-|\hat{v}_-|^2 + \nu_+|\hat{v}_+|^2) - \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})|\hat{E}|^2 \\
&- c_1\frac{|\xi|^2}{1 + |\xi|^2}|\hat{B}|^2 - \alpha\gamma(1 + |\xi|^2)|\hat{v}_- - \hat{v}_+|^2 \\
&+ \frac{|\xi|^2}{1 + |\xi|^2}|\hat{E}|^2 + c_2\frac{1}{1 + |\xi|^2} < i\xi \times \hat{B}, \beta(\hat{v}_+ - \hat{v}_-) > \\
&+ |\xi|^2 < \frac{\nu_+}{\rho_+}\hat{v}_+ - \frac{\nu_-}{\rho_-}\hat{v}_-, \hat{E} > +\alpha(\frac{1}{\rho_-} + \frac{1}{\rho_+}) < \hat{v}_+ - \hat{v}_-, \hat{E} > \\
&- c_1 < \hat{v}_+ - \hat{v}_-, i\xi \times \hat{B} > +c_2 < \hat{v}_+ - \hat{v}_-, \beta(\hat{v}_+ - \hat{v}_-) > .
\end{aligned}$$

Cauchy-Schwarz inequality and Young's inequality yields

$$
\begin{aligned}
\frac{d}{dt}L \leq & -\gamma|\xi|^2(1+|\xi|^2)(\nu_-|\hat{v}_-|^2+\nu_+|\hat{v}_+|^2) - \beta(\frac{1}{\rho_-}+\frac{1}{\rho_+})|\hat{E}|^2 \\
& - c_1\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2 - \alpha\gamma(1+|\xi|^2)|\hat{v}_- - \hat{v}_+|^2 \\
& + |\hat{E}|^2 + \varepsilon c_2\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2 + c(\varepsilon)c_2\beta^2\frac{1}{1+|\xi|^2}|\hat{v}_+ - \hat{v}_-|^2 \\
& + c(\varepsilon)|\xi|^4\left(\left(\frac{\nu_+}{\rho_+}\right)^2|\hat{v}_+|^2 + \left(\frac{\nu_-}{\rho_-}\right)^2|\hat{v}_-|^2\right) + \varepsilon|\hat{E}|^2 \\
& + c(\varepsilon)\alpha^2\left(\frac{1}{\rho_-}+\frac{1}{\rho_+}\right)^2|\hat{v}_+ - \hat{v}_-|^2 + \varepsilon|\hat{E}|^2 \\
& + c(\varepsilon)c_1^2(1+|\xi|^2)|\hat{v}_+ - \hat{v}_-|^2 + \varepsilon\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2 + c_2\beta|\hat{v}_+ - \hat{v}_-|^2 \\
\leq & -\left(\gamma - c(\varepsilon, c_2, \beta, \rho_\pm, \nu_\pm)\right)|\xi|^2(1+|\xi|^2)(\nu_-|\hat{v}_-|^2+\nu_+|\hat{v}_+|^2) \\
& - \left(\frac{\beta}{\rho_-}+\frac{\beta}{\rho_+} - 1 - 2\varepsilon\right)|\hat{E}|^2 - (c_1 - \varepsilon c_2 - \varepsilon)\frac{|\xi|^2}{1+|\xi|^2}|\hat{B}|^2 \\
& - \left(\alpha\gamma - c(\varepsilon, \alpha, \rho_\pm, c_1, c_2)\right)(1+|\xi|^2)|\hat{v}_- - \hat{v}_+|^2.
\end{aligned}
\tag{3.6}
$$

By choosing ε small enough, we have

$$
\left(\frac{\beta}{\rho_-}+\frac{\beta}{\rho_+} - 1 - 2\varepsilon\right) > 0, \quad (c_1 - \varepsilon c_2 - \varepsilon) > 0.
$$

After ε is fixed, we choose γ big enough so that

$$
(\gamma - c(\varepsilon, c_2, \beta_\pm, \rho_\pm, \nu_\pm)) > 0, \quad (\alpha\gamma - c(\varepsilon, \alpha, \rho_\pm, c_1, c_2)) > 0.
$$

Therefore, there exists a positive constant d_1 such that

$$
\begin{aligned}
\frac{d}{dt}L \leq & -d_1|\xi|^2(1+|\xi|^2)(\frac{1}{2}\rho_-|\hat{v}_-|^2 + \frac{1}{2}\rho_+|\hat{v}_+|^2) \\
& - d_1\frac{3}{4c_2}|\hat{E}|^2 - d_1\frac{|\xi|^2}{1+|\xi|^2}\frac{c_1}{c_2}|\hat{B}|^2.
\end{aligned}
\tag{3.7}
$$

Because

$$
\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4} \leq |\xi|^2(1+|\xi|^2), \quad \frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4} \leq \frac{3}{2}, \quad \frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4} \leq \frac{2|\xi|^2}{1+|\xi|^2},
$$

(3.7) implies

$$\begin{aligned}\frac{d}{dt}L = & -d_1\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}(\frac{1}{2}\rho_-|\hat{v}_-|^2+\frac{1}{2}\rho_+|\hat{v}_+|^2)\\ & -d_1\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}(\frac{1}{2c_2}|\hat{E}|^2+\frac{c_1}{2c_2}|\hat{B}|^2)\\ \leq & -d_1\frac{|\xi|^2(1+|\xi|^2)}{1+|\xi|^4}\hat{\mathcal{E}}.\end{aligned} \tag{3.8}$$

On the other hand, since

$$|F| \lesssim (|\hat{v}_-|^2+|\hat{v}_+|^2+|\hat{E}|^2+|\hat{B}|^2),$$

there exists d_2, d_3 such that

$$d_2(1+|\xi|^2)\hat{\mathcal{E}} \leq L \leq d_3(1+|\xi|^2)\hat{\mathcal{E}}. \tag{3.9}$$

Furthermore, we can notice that d_3 can be chosen as large as it can be, Plugging (3.9) into (3.8) implies

$$\frac{d}{dt}L \leq -\frac{d_1}{d_3}\frac{|\xi|^2}{(1+|\xi|^4)}L.$$

Integrating the differential inequality gives us, for $t \geq 0$,

$$L(\xi,t) \leq L(\xi,0)e^{-\frac{d_1}{d_3}\frac{|\xi|^2}{(1+|\xi|^4)}t}.$$

Again, thanks to (3.9), we end up with

$$\hat{\mathcal{E}}(\xi,t) \leq \frac{d_3}{d_2}\hat{\mathcal{E}}(\xi,0)e^{-\frac{d_1}{d_3}\frac{|\xi|^2}{(1+|\xi|^4)}t}. \tag{3.10}$$

By (3.10), for any $k \geq 0$, it holds that

$$\begin{aligned}\|D^kU\|_{L^2}^2 & \lesssim \int_{\mathbb{R}^3}|\hat{U}_0|^2|\xi|^{2k}e^{-\frac{d_1}{d_3}\rho(|\xi|)t}d\xi\\ & \lesssim \int_{|\xi|\leq 1}+\int_{|\xi|>1}|\hat{U}_0|^2|\xi|^k e^{-\frac{d_1}{d_3}\rho(|\xi|)t}d\xi.\end{aligned}$$

Estimating the low and high frequency parts of the above inequality will prove our lemma.

For $|\xi| \leq 1$, $\rho(|\xi|) \geq \frac{|\xi|^2}{2}$. Therefore, for any $k > 0$

$$\begin{aligned} \int_{|\xi|\leq 1} |\hat{U}_0|^2 |\xi|^{2k} e^{-\frac{d_1}{d_3}\rho(|\xi|)t} d\xi &\leq \|\hat{U}_0\|_{L^\infty}^2 \int_{|\xi|\leq 1} |\xi|^{2k} e^{-\frac{d_1}{d_3}\rho(|\xi|)t} d\xi \\ &\lesssim c(k)(1+t)^{-3/2-k} \|U_0\|_{L^1}^2 \end{aligned} \tag{3.11}$$

For $|\xi| > 1$, $\rho(|\xi|) \geq \frac{1}{2|\xi|^2}$. Therefore, for any $l > 0$,

$$\begin{aligned} \int_{|\xi|>1} |\hat{U}|^2 |\xi|^{2k} e^{-\frac{d_1}{d_3}\rho(|\xi|)t} d\xi &\leq \int_{|\xi|\geq 1} |\hat{U}|^2 |\xi|^{2k} e^{-\frac{d_1}{d_3}\frac{1}{2|\xi|^2}t} d\xi \\ &\leq \sup_{|\xi|\geq 1} (e^{-\frac{d_1}{d_3}} \frac{t}{2|\xi|^2} |\xi|^{-2l}) \int_{|\xi|>1} |\hat{U}|^2 |\xi|^{2k+2l} d\xi \\ &\leq c(l)(1+t)^{-l} \|U\|_{\dot{H}^{k+l}}^2 \end{aligned} \tag{3.12}$$

Put (3.12) and (3.11) together and we finish the proof. □

3.3 Proof of Theorem 1.5.5

Recall the stability theorem is

Theorem 1.5.5. Let $s \geq 7$ and $U_0 \in L^1 \cap H^s$ be the initial data of system (1.15). Then there exists a constant $\delta > 0$ such that if $\|U_0\|_{L^1 \cap H^s} \leq \delta$ the unique solution given by Theorem 1.5.4 satisfies the following decay property,

$$\|D^k U\|_{H^{s-2k-3}} \lesssim (1+t)^{-3/4-k/2},$$

for all integers $0 \leq k \leq [(s-1)/2] - 1$.

To prove the decay result, we introduce the following time weighted norm and the corresponding dissipation norm.

$$
\begin{aligned}
M(t) &= \sum_{k=0}^{[(s-1)/2]-1} \sup_{0\le\tau\le t}(1+\tau)^{3/4+k/2}\|D^kU\|_{H^{s-2k-3}},\\
N^2(t) &= \sum_{k=0}^{[s/2]} \sup_{0\le\tau\le t}(1+\tau)^{k}\|D^kU\|^2_{H^{s-2k}},\\
D^2(t) &= \sum_{k=0}^{[s/2]} \int_0^t (1+\tau)^k \left(\|D^{k+1}v\|^2_{H^{s-2k}} + \|D^k(v_- - v_+)\|^2_{H^{s-2k}}\right) d\tau\\
&\quad + \sum_{k=0}^{[s/2]-1} \int_0^t (1+\tau)^k \left(\|D^kE\|^2_{H^{s-2k-1}} + \|D^{k+1}B\|^2_{H^{s-2k-2}}\right) d\tau,
\end{aligned}
\tag{3.13}
$$

where $U = (v_-, v_+, E, B), v = (v_-, v_+)$.
The goal is to bound $M(t)$ uniformly in time when the initial data is small enough. Using Duhamel principle and linear estimate, one can have a self control estimate of $M(t)$ provided that $N(t)$ is bounded (see Lemma 3.3.4). The boundedness of $N(t)$ can be derived through time weighted energy method as well as $D(t)$.
The proof is based on several lemmas.

Lemma 3.3.1. Let $s \ge 7$, $0 \le k \le [s/2]$. We have

$$
\begin{aligned}
&(1+t)^k\|D^kU\|^2_{H^{s-2k}} + \int_0^t (1+\tau)^k \left(\|D^{k+1}v\|^2_{H^{s-2k}} + \|D^k(v_- - v_+)\|^2_{H^{s-2k}}\right) d\tau\\
&\lesssim \|U_0\|^2_{H^s} + k\int_0^t (1+\tau)^{k-1}\|D^kU\|^2_{H^{s-2k}} d\tau + (N(t)+M(t))D^2(t).
\end{aligned}
$$

Proof. Multiplying (2.37) by $(1+t)^k$ and integrating in time then summing over $k \le l \le s-k$ yields

$$
\begin{aligned}
&(1+t)^k\|D^kU\|^2_{H^{s-2k}} + \int_0^t (1+\tau)^k \left(\|D^{k+1}v\|^2_{H^{s-2k}} + \|D^k(v_- - v_+)\|^2_{H^{s-2k}}\right) d\tau\\
&\lesssim \|U_0\|^2_{H^s} + k\int_0^t (1+\tau)^{k-1}\|D^kU\|^2_{H^{s-2k}} d\tau + T_1 + T_2,
\end{aligned}
\tag{3.14}
$$

where

$$T_1 = \int_0^t (1+\tau)^k \|Dv\|_{L^\infty} \sum_{l=k}^{s-k} \|D^l v\|_{L^2}^2 d\tau,$$

$$T_2 = \int_0^t (1+\tau)^k \sum_{l=k}^{s-k} \left(\|B\|_{L^\infty} \|D^l v\|_{L^2}^2 + \|v\|_{L^\infty} \|D^l v\|_{L^2} \|D^l B\|_{L^2} \right) d\tau.$$

Noting that when $l = 0$, (2.37) is a consequence of the classic energy identity

$$\frac{\rho_-}{2} \frac{d}{dt} \|v_-\|_{L^2}^2 + \frac{\rho_+}{2} \frac{d}{dt} \|v_+\|_{L^2}^2 + \frac{\varepsilon_0}{2} \frac{d}{dt} \|E\|_{L^2}^2 + \frac{d}{dt} \frac{1}{2\mu_0} \|B\|_{L^2}^2$$
$$+ \nu_- \|\nabla v_-\|_{L^2}^2 + \nu_+ \|\nabla v_+\|_{L^2}^2 + \alpha \|v_- - v_+\|_{L^2}^2 = 0.$$

So that when $k = 0$,

$$T_1 = \int_0^t \|Dv\|_{L^\infty} \|Dv\|_{H^{s-1}}^2 d\tau \lesssim W_0(t) D_0^2(t) \lesssim N(t) D^2(t),$$

$$T_2 = \int_0^t \|B\|_{L^\infty} \|Dv\|_{H^{s-1}}^2 + \|v\|_{L^\infty} \|Dv\|_{H^{s-1}} \|DB\|_{H^{s-1}} d\tau$$
$$\lesssim W_0(t) D_0^2(t) + N_0(t) J_0(t) D_0(t) \lesssim N(t) D^2(t).$$

For $1 \le k \le [s/2]$, we estimate T_1, T_2 as

$$\begin{aligned}
T_1 &= \int_0^t (1+\tau)^k \|Dv\|_{L^\infty} \|D^k v\|_{H^{s-2k}}^2 d\tau \\
&\lesssim \sup_{0 \le \tau \le t} \{(1+\tau) \|Dv\|_{L^\infty}\} \int_0^t (1+\tau)^{k-1} \|D^k v\|_{H^{s-2k}}^2 d\tau \\
&\lesssim M(t) D^2(t), \\
T_2 &\lesssim \int_0^t (1+\tau)^k \left(\|B\|_{L^\infty} \|D^k v\|_{H^{s-2k}}^2 + \|v\|_{L^\infty} \|D^k v\|_{H^{s-2k}} \|D^l B\|_{H^{s-2k}} \right) d\tau \\
&\lesssim M(t) D^2(t) + M(t) \int_0^t (1+\tau)^{k-1} \left(\|D^k v\|_{H^{s-2k}} \|D^k B\|_{H^{s-2k}} \right) d\tau \\
&\lesssim M(t) D^2(t),
\end{aligned}$$

where we use $\sup_{0 \le \tau \le t} (1+\tau) \|U\|_{W^{1,\infty}} \lesssim M(t)$ for $s \ge 7$.
Substituting the estimates of T_1 and T_2 into (3.14) proves our lemma. □

Now comes the estimate of the dissipation of E and B. More precisely, we have

Lemma 3.3.2. Let $s \geq 7, 0 \leq k \leq [s/2]-1$. It holds that

$$\begin{aligned}&\int_0^t (1+\tau)^k \left(\|D^k E\|^2_{H^{s-2k-1}} + \|D^{k+1}B\|^2_{H^{s-2k-2}}\right) d\tau \\ \lesssim &\|U_0\|^2_{H^s} + k\int_0^t (1+\tau)^{k-1}\|D^k U\|^2_{H^{s-2k}} d\tau + (N(t)+M(t))D^2(t).\end{aligned}$$

Proof. The dissipation estimate for $v_\pm, v_- - v_+$ are done in the previous lemma. Let us first do estimate on E. Like the proof of Theorem 1.5.4 (see Remark 2.5.1), we deal with the case $l=k$ and $l>k$ in different ways. Fixing $0 \leq k \leq [s/2]-1$, by (2.40) for $l=k$, we have

$$\begin{aligned}&\partial_t \left(D^k(v_- - v_+), D^k E\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|D^k E\|^2_{L^2} \\ &\lesssim C_\varepsilon \|D^{k+2}v\|^2_{L^2} + C_\varepsilon\|D^k(v_- - v_+)\|^2_{L^2} + \varepsilon\|D^k E\|^2_{L^2} + \varepsilon\|D^{k+1}B\|^2_{L^2} \\ &+ \|v\|_{L^\infty}\|D^{k+1}v\|_{L^2}\|D^k E\|_{L^2} + \|B\|_{L^\infty}\|D^k v\|_{L^2}\|D^k E\|_{L^2} \\ &+ \|v\|_{L^\infty}\|D^k B\|_{L^2}\|D^k E\|_{L^2}.\end{aligned} \tag{3.15}$$

For $l>k$ we have

$$\begin{aligned}&\partial_t \left(D^l(v_- - v_+), D^l E\right) + \beta(\frac{1}{\rho_-} + \frac{1}{\rho_+})\|D^l E\|^2_{L^2} \\ &\lesssim C_\varepsilon \|D^{l+2}v\|^2_{L^2} + C_\varepsilon\|D^l(v_- - v_+)\|^2_{H^1} + \varepsilon\|D^l E\|^2_{L^2} + \varepsilon\|D^l B\|^2_{L^2} \\ &+ \|v\|_{L^\infty}\|D^{l+1}v\|_{L^2}\|D^l E\|_{L^2} + \|B\|_{L^\infty}\|D^l v\|_{L^2}\|D^l E\|_{L^2} \\ &+ \|v\|_{L^\infty}\|D^l B\|_{L^2}\|D^l E\|_{L^2}.\end{aligned} \tag{3.16}$$

Multiplying (3.15) and (3.16) by $(1+t)^k$, integrating in time and summing over $k \leq l \leq s-k-1$ yields

$$\begin{aligned}&\int_0^t (1+\tau)^k\|D^k E\|^2_{H^{s-2k-1}} d\tau \\ \lesssim &\|U_0\|^2_{H^s} + (1+t)^k\|D^k U\|^2_{H^{s-2k-1}} + k\int_0^t (1+\tau)^{k-1}\|D^k U\|^2_{H^{s-2k-1}} d\tau \\ &+ C_\varepsilon \int_0^t (1+\tau)^k \left(\|D^{k+2}v\|^2_{H^{2-2k-1}} + \|D^k(v_- - v_+)\|^2_{H^{s-2k}}\right) d\tau \\ &+ \varepsilon\int_0^t (1+t)^k\|D^{k+1}B\|^2_{H^{s-2k-2}} d\tau + S,\end{aligned} \tag{3.17}$$

where

$$
\begin{aligned}
S &= \sum_{l=k}^{s-k-1} \int_0^t (1+\tau)^k \left(\|v\|_{L^\infty} \|D^{l+1} v\|_{L^2} \|D^l E\|_{L^2} + \|B\|_{L^\infty} \|D^l v\|_{L^2} \|D^l E\|_{L^2} \right. \\
&\quad \left. + \|v\|_{L^\infty} \|D^l B\|_{L^2} \|D^l E\|_{L^2} \right) d\tau.
\end{aligned}
$$

By Lemma 3.3.1, the inequality (3.17) becomes

$$
\begin{aligned}
\int_0^t (1+\tau)^k \|D^k E\|^2_{H^{s-2k-1}} d\tau \lesssim & \|U_0\|^2_{H^s} + k \int_0^t (1+\tau)^{k-1} \|D^k U\|^2_{H^{s-2k}} d\tau \\
& + \varepsilon \int_0^t (1+t)^k \|D^{k+1} B\|^2_{H^{s-2k-2}} d\tau + S \\
& + (N(t) + M(t)) D^2(t),
\end{aligned}
\tag{3.18}
$$

Like the proof of previous lemma, we estimate the remaining term S by the case $k = 0$ and $k \geq 1$. When $k = 0$, one has

$$
\begin{aligned}
S \lesssim & \int_0^t \|v\|_{L^\infty} \|Dv\|_{H^{s-1}} \|E\|_{H^{s-1}} + \|B\|_{L^\infty} \|v\|_{H^{s-1}} \|E\|_{H^{s-1}} \\
& + \|v\|_{L^\infty} \|B\|_{H^{s-1}} \|E\|_{H^{s-1}} d\tau \\
\lesssim & N_0(t) J_0(t) D_0(t) \lesssim N(t) D^2(t).
\end{aligned}
$$

For $1 \leq k \leq [s/2] - 1$, we have

$$
\begin{aligned}
S &\lesssim \int_0^t (1+\tau)^k \left(\|v\|_{L^\infty} \|D^{k+1} v\|_{H^{s-2k-1}} \|D^k E\|_{H^{s-2k-1}} \right. \\
&\quad + \|B\|_\infty \|D^k v\|_{H^{s-2k-1}} \|D^k E\|_{H^{s-2k-1}} \\
&\quad \left. + \|v\|_{L^\infty} \|D^k B\|_{H^{s-2k-1}} \|D^k E\|_{H^{s-2k-1}} \right) d\tau \\
&\lesssim M(t) \int_0^t (1+\tau)^{k-1} \|D^k E\|_{H^{s-2k-1}} \left(\|D^k v\|_{H^{s-2k}} + \|D^k B\|_{H^{s-2k-1}} \right) d\tau \\
&\lesssim M(t) D^2(t).
\end{aligned}
$$

Therefore, plugging the estimate of S into (3.18) yields for $0 \leq k \leq [s/2]-1$

$$\begin{aligned}\int_0^t (1+\tau)^k \|D^k E\|^2_{H^{s-2k-1}} d\tau \lesssim & \|U_0\|^2_{H^s} + k\int_0^t (1+\tau)^{k-1}\|D^k U\|^2_{H^{s-2k}} d\tau \\ & + \varepsilon \int_0^t (1+t)^k \|D^{k+1} B\|^2_{H^{s-2k-2}} d\tau \\ & + (N(t)+M(t))D^2(t).\end{aligned} \tag{3.19}$$

For dissipation estimate on B, multiplying (2.46) by $(1+t)^k$, integrating in time and summing l from k to $s-k-2$ yield

$$\begin{aligned}&\int_0^t \|D^{k+1} B\|^2_{H^{s-2k-1}} d\tau \\ \lesssim & \|U_0\|^2_{H^s} + (1+t)^k \|D^k U\|^2_{H^{s-2k}} \\ & +k\int_0^t (1+\tau)^{k-1}\left(\|D^{k+1}E\|^2_{H^{s-2k-2}} + \|D^k B\|^2_{H^{s-2k-2}}\right) d\tau \\ & + \int_0^t (1+\tau)^k \left(C_\varepsilon \|D^k(v_+ - v_+)\|^2_{H^{s-2k-2}} + \|D^{k+1}E\|^2_{H^{s-2k-2}} d\tau\right).\end{aligned} \tag{3.20}$$

By Lemma 3.3.1 inequality (3.20) becomes

$$\begin{aligned}\int_0^t \|D^{k+1} B\|^2_{H^{s-2k-1}} d\tau \lesssim & \|U_0\|^2_{H^s} + k\int_0^t (1+\tau)^{k-1}\|D^k U\|^2_{H^{s-2k}} d\tau \\ & +(N(t)+M(t))D^2(t).\end{aligned} \tag{3.21}$$

Choosing ε small enough and a suitable linear combination of (3.19) and (3.21) finishes the proof. □

Using Lemma 3.3.1 and Lemma 3.3.2 we can derive the following key inequality that shows the self control of $N(t)$ and $D(t)$.

Lemma 3.3.3. Let $s \geq 7$. Then it holds that

$$N^2(t) + D^2(t) \lesssim \|U_0\|^2_{H^s} + (N(t)+M(t))D^2(t).$$

Proof. It is sufficient to prove that for $0 \le k \le [s/2]-1$,

$$\begin{aligned}(1+t)^k\|D^kU\|^2_{H^{s-2k}} + \int_0^t (1+\tau)^k \big(\|D^{k+1}v\|^2_{H^{s-2k}} + \|D^k(v_+ - v_+)\|^2_{H^{s-2k}} \\ \|D^kE\|^2_{H^{s-2k-1}} + \|D^{k+1}B\|^2_{H^{s-2k-2}}\big)\, d\tau \\ \lesssim \|U_0\|^2_{H^s} + (N(t)+D(t))\, D^2(t),\end{aligned} \tag{3.22}$$

and for $k=[s/2]$,

$$\begin{aligned}(1+t)^k\|D^kU\|^2_{H^{s-2k}} + \int_0^t (1+\tau)^k \big(\|D^{k+1}v\|^2_{H^{s-2k}} + \|D^k(v_+ - v_+)\|^2_{H^{s-2k}} \\ \lesssim \|U_0\|^2_{H^s} + (N(t)+D(t))\, D^2(t).\end{aligned} \tag{3.23}$$

The proof can be done easily using an induction argument. Clearly (3.22) is true for $k=0$. Assume that (3.22) is true for $k=l-1, 1 \le l \le [s/2]-1$. Then for $k=l$,

$$\begin{aligned}(1+t)^l\|D^lU\|^2_{H^{s-2l}} + \int_0^t (1+\tau)^l \big(\|D^{l+1}v\|^2_{H^{s-2l}} + \|D^l(v_+ - v_+)\|^2_{H^{s-2l}} \\ \|D^lE\|^2_{H^{s-2l-1}} + \|D^{l+1}B\|^2_{H^{s-2l-2}}\big)\, d\tau \\ \lesssim \|U_0\|^2_{H^s} + (N(t)+M(t))\, D^2(t) + l\int_0^t (1+\tau)^{l-1}\|D^lU\|^2_{H^{s-2l}} d\tau.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^t (1+\tau)^{l-1}\|D^lU\|^2_{H^{s-2l}} d\tau \\ \lesssim &\int_0^t (1+\tau)^{l-1} \big(\|D^lv\|^2_{H^{s-2(l-1)}} + \|D^{l-1}(v_- - v_+)\|^2_{H^{s-2(l-1)}} \\ &+\|D^{l-1}E\|^2_{H^{s-2(l-1)-1}} + \|D^lB\|^2_{H^{s-2l}}\big)\, d\tau \\ \lesssim &\|U_0\|^2_{H^s} + (N(t)+M(t))\, D^2(t),\end{aligned}$$

inequality (3.22) holds for $k=l$.
When $k=[s/2]$, inequality (3.23) holds for the same reason. □

To proof our main theorem, we still need another inequality that controls $M(t)$.

Lemma 3.3.4. Let $s \ge 3$. Then it holds that

$$M(t) \lesssim \|U_0\|_{L^1\cap H^s} + M^2(t) + M(t)N(t).$$

Proof. By Duhamel principle,

$$U = e^{t\mathcal{L}}U_0 + \int_0^t e^{(t-\tau)\mathcal{L}}\mathcal{P}\mathcal{N}(U(\tau))d\tau, \tag{3.24}$$

Where $\mathcal{P}$ is Leray projection. Fixing $0 \le k \le [\frac{s-1}{2}] - 1$, for $0 \le m \le s - 2k - 3$ applying D^{k+m} to (3.24) and taking L^2 norm yields

$$\|D^{k+m}U\|_{L^2} \lesssim \|D^{k+m}e^{t\mathcal{L}}U_0\|_{L^2} + R_1(m) + R_2(m), \tag{3.25}$$

where,

$$\begin{aligned}
R_1 = &\int_0^{t/2} \|D^{k+m+1}e^{(t-\tau)\mathcal{L}}(v_-\otimes v_-, v_+\otimes v_+, 0, 0)^T(\tau)\|_{L^2}d\tau \\
&+ \int_{t/2}^t \|D^{m+1}e^{(t-\tau)\mathcal{L}}D^k(v_-\otimes v_-, v_+\otimes v_+, 0, 0)^T(\tau)\|_{L^2}d\tau \\
&:= R_{11} + R_{12}, \\
R_2 = &\int_0^{t/2} \|D^{k+m}e^{(t-\tau)\mathcal{L}}(-v_-\times B, v_+\times B, 0, 0)^T(\tau)\|_{L^2}d\tau \\
&+ \int_{t/2}^t \|D^{m}e^{(t-\tau)\mathcal{L}}D^k(-v_-\times B, v_+\times B, 0, 0)^T(\tau)\|_{L^2}d\tau \\
&:= R_{21} + R_{22}.
\end{aligned}$$

For the estimate of the linear part, applying Lemma 3.2.1 by replacing k by $k+m$ and l by $k+2$ the summing in m over $0 \le m \le s-2k-3$ yields

$$\begin{aligned}
&\|D^k e^{t\mathcal{L}}U_0\|_{H^{s-2k-3}} \sim \sum_{m=0}^{s-2k-3} \|D^{k+m}e^{t\mathcal{L}}U_0\|_{L^2} \\
\lesssim &\sum_{m=0}^{s-2k-3} (1+t)^{-3/4-(k+m)/2}\|U_0\|_{L^1} + (1+t)^{-1-k/2}\|D^{2k+m+2}U\|_{L^2} \\
\lesssim &(1+t)^{-3/4-k/2}\|U_0\|_{L^1\cap H^s}.
\end{aligned} \tag{3.26}$$

To estimate R_1 and R_2, we separate the time integral into two parts: from 0 to $t/2$ and $t/2$ to t, as we showed in the definition of R_1 and R_2. The decay of the first part comes from the linear estimate while the decay of the second part is due to the definition of our weighted norms.

For R_{11}, applying the linear estimate Lemma 3.2.1 with $k = k+m+1, l = k+2$ and summing m over $0 \le m \le s-2k-3$ we have

$$\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{11}(m) \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-5/4-(k+m)/2} \|v \otimes v\|_{L^1} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+3}(v \otimes v)\|_{L^2} d\tau \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-5/4-(k+m)/2} \|v\|_{L^2}^2 d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+3} v\|_{L^2} \|v\|_{L^\infty} d\tau \\
\lesssim &M^2(t) \int_0^{t/2} (1+t-\tau)^{-5/4-k/2} (1+\tau)^{-3/2} d\tau \\
+ &\sup_{0 \le \tau \le t/2} \{(1+\tau)\|v\|_{L^\infty}\} N(t) \int_0^{t/2} (1+t-\tau)^{-1-k/2} (1+\tau)^{-1} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} M^2(t) + (1+t)^{-3/4-k/2} M(t) N(t) (1+t)^{-1/4} \ln(1+t) \\
\lesssim &(1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t) \right),
\end{aligned}$$

where we use Lemma 2.5.1 in the second step. Noting that in above inequality, we need $2k+m+3 \le s$ for all $0 \le m \le s-2k-3$ and this is where the restriction on k comes from.

For R_{12} we follow the similar procedure as we did for R_{11}. When applying the linear estimate, we set $k = m+1, l = 1$ in Lemma 3.2.1 which leads to

$$\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{12}(m) \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-5/4-m/2} \|D^k(v\otimes v)\|_{L^1} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+2}(v\otimes v)\|_{L^2} d\tau \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-5/4-m-2} \|D^k v\|_{L^2}\|v\|_{L^2} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+2} v\|_{L^2}\|v\|_{L^\infty} d\tau \\
\lesssim &M^2(t) \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-5/4-m/2}(1+\tau)^{-3/2-k/2} d\tau \\
+ &\sup_{t/2\le\tau\le t}\{(1+\tau)\|v\|_{L^\infty}\} \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-1}\|D^{k+m+1}v\|_{H^1} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} M^2(t) \\
&+ M(t)N(t) \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-1}(1+\tau)^{-(k+1)/2} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} M^2(t) \\
&+ M(t)N(t) \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-3/4}(1+\tau)^{-k/2-3/4} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right).
\end{aligned}$$

Together with the estimate of R_{11}, R_{12}, we have

$$\sum_{m=0}^{s-2k-3} R_1(m) \lesssim (1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right). \tag{3.27}$$

We estimate R_2 in a similar way. For R_{21} applying the linear estimate Lemma 3.2.1

with $k = k + m, l = k + 2$ and summing m over $0 \leq m \leq s - 2k - 3$ yields

$$\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{21}(m) \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-3/4-(k+m)/2} \|v \times B\|_{L^1} d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+2}(v \times B)\|_{L^2} d\tau \\
\lesssim &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-3/4-(k+m)/2} \|U\|_{L^2}^2 d\tau \\
+ &\sum_{m=0}^{s-2k-3} \int_0^{t/2} (1+t-\tau)^{-1-k/2} \|D^{2k+m+2}U\|_{L^2} \|U\|_{L^\infty} d\tau \\
\lesssim &M^2(t) \int_0^{t/2} (1+t-\tau)^{-3/4-k/2} (1+\tau)^{-3/2} d\tau \\
+ &\sup_{0 \leq \tau \leq t/2} \left\{(1+\tau)\|U\|_{L^\infty}\right\} N(t) \int_0^{t/2} (1+t-\tau)^{-1-k/2} (1+\tau)^{-1} d\tau \\
\lesssim &(1+t)^{-3/4-k/2} M^2(t) + (1+t)^{-3/4-k/2} M(t)N(t)(1+t)^{-1/4} \ln(1+t) \\
\lesssim &(1+t)^{-3/4-k/2} \left(M^2(t) + M(t)N(t)\right).
\end{aligned}$$

For R_{21}, just like the way we estimated R_{12} but now choosing $k = m, l = 1$. Applying

Lemma 3.2.1 we get

$$\begin{aligned}
&\sum_{m=0}^{s-2k-3} R_{22}(m)\\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-3/4-m/2} \|D^k(v\times B)\|_{L^1} d\tau\\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+1}(v\times B)\|_{L^2} d\tau\\
\lesssim &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-3/4-m-2} \|D^k U\|_{L^2}\|U\|_{L^2} d\tau\\
+ &\sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2} \|D^{k+m+1} U\|_{L^2}\|U\|_{L^\infty} d\tau\\
\lesssim &M^2(t) \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-3/4-m/2}(1+\tau)^{-3/2-k/2} d\tau\\
+ &\sup_{t/2\le\tau\le t}\{(1+\tau)\|U\|_{L^\infty}\} \sum_{m=0}^{s-2k-3} \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-1}\|D^{k+m+1}v\|_{H^1} d\tau\\
\lesssim &M^2(t) \int_{t/2}^{t} (1+t-\tau)^{-3/2}(1+\tau)^{-3/4-k/2} d\tau\\
+ &M(t)N(t) \int_{t/2}^{t} (1+t-\tau)^{-1/2}(1+\tau)^{-1}(1+\tau)^{-(k+1)/2} d\tau\\
\lesssim &(1+t)^{-3/4-k/2}\left(M^2(t)+M(t)N(t)\right).
\end{aligned}$$

So that we have

$$R_2 \lesssim (1+t)^{-3/4-k/2}\left(M^2(t)+M(t)N(t)\right). \tag{3.28}$$

The lemma is proved when we put (3.26), (3.27) and (3.28) together. □

Now we have all the ingredients to prove Theorem 1.5.5. By Lemma 3.3.3 and Lemma 3.3.4, we have for $s \geq 7$

$$\begin{aligned}
N^2(t)+D^2(t) &\lesssim \|U_0\|_{H^s}^2 + (N(t)+M(t))\,D^2(t),\\
M(t) &\lesssim \|U_0\|_{L^1\cap H^s} + M^2(t) + M(t)N(t).
\end{aligned}$$

Let $Y = N(t)+M(t)+D(t)$. So that the above inequalities imply that

$$Y^2(t) \lesssim \|U_0\|_{H^s}^2 + Y^3(t) + Y^4(t).$$

Thus by choosing $\|U_0\|_{L^1 \cap H^s} \leq \delta$ with δ small enough, one has

$$Y = M(t) + N(t) + D(t) \lesssim \delta,$$

for all $t \geq 0$ which proves the theorem.

Chapter 4

Numerical comparsion with MHD

As we mentioned in the introduction, the two-fluid model can be used to formally derive the classic Magnetohydrodynamics(MHD) under some relaxing limit. So the numerical results of two systems are expected to be close to each other when taking the relaxing limit. In this chapter, we show the numerical comparision between two-fluid model and MHD.

4.1 Numerical setting

The numerical test is done in space dimension two. Then, the velocity has two components $v(x, y) = (v_1(x, y), v_2(x, y), 0)$. Besides we can set the magnetic field having two components too $B(x, y) = (B_1(x, y), B_2(x, y), 0)$. Thus the vorticity $\omega = (0, 0, \omega(x, y)) = \nabla \times v$ and vector potential $A = (0, 0, A(x, y))$ (defined by $B = \nabla \times A$ since B is divergence free) have only one component each. Therefore, it will be much easier if the two systems are written in vorticity and vector potential form and we compare velocities, vector potentials of these two systems. The one component vectors are considered as scaler functions below.

4.1.1 Two-fluid model reformulation

Here we use the two-fluid model given in [2] which is basically our two-fluid model written in Gaussian units after normalizing two densities to one and taking the same viscosities.

$$\begin{cases} \partial_t v_- = \nu\Delta v_- - v_- \cdot \nabla v_- - \dfrac{1}{\varepsilon}(cE + v_- \times B) - R - \nabla p_- \\ \partial_t v_+ = \nu\Delta v_+ - v_+ \cdot \nabla v_+ + \dfrac{1}{\varepsilon}(cE + v_+ \times B) + R - \nabla p_+ \\ \dfrac{1}{c}\partial_t E = \nabla \times B - \dfrac{1}{2\varepsilon}(v_+ - v_-) \\ \dfrac{1}{c}\partial_t B = -\nabla \times E \\ R := -\dfrac{1}{2\sigma\varepsilon^2}(v_+ - v_-) \\ \mathrm{div} v_- = \mathrm{div} v_+ = \mathrm{div} B = \mathrm{div} E = 0. \end{cases} \tag{4.1}$$

In order to compare (4.1) to MHD solution, we first rewrite the above system using the bulk velocity u and the current density j:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \varepsilon^2 j \cdot \nabla j - \nu\Delta u = -\nabla p + j \times B, \\ \varepsilon^2 \partial_t j + \varepsilon^2(u \cdot \nabla j + j \cdot \nabla u) - \varepsilon^2 \nu \Delta j + \dfrac{1}{\sigma} j = -\nabla \tilde{p} + cE + u \times B, \\ \dfrac{1}{c}\partial_t E - \nabla \times B = -j, \\ \dfrac{1}{c}\partial_t B + \nabla \times E = 0, \end{cases} \tag{4.2}$$

where $u = \frac{v_+ + v_-}{2}$, $j = \frac{v_+ - v_-}{2\varepsilon}$.

Next, applying curl to the first and second equations of (4.2), scaling the electric field $cE \to E$ and extracting the equation of A through the fourth equation of (4.2), we obtain

$$\begin{cases} \partial_t \omega - \nu\Delta\omega = B \cdot \nabla j - u \cdot \nabla\omega, \\ \varepsilon^2 \partial_t \Omega + \dfrac{1}{\sigma}\Omega - \varepsilon^2 \nu \Delta\Omega - \nabla \times E = \nabla \times (u \times B - \varepsilon^2 u \cdot \nabla j), \\ \dfrac{1}{c^2}\partial_t E + \Delta A - \Delta^{-1}(\nabla \times \Omega) = 0, \\ \partial_t A + E = 0, \end{cases} \tag{4.3}$$

where $\Omega = (\Omega_1, \Omega_2, 0) = \nabla \times j$. Notice that in 2D, only the second equation of the above system has two components while the other three equations have only one. System (4.3) is then good enough to do the numerical test.

4.1.2 MHD reformulation

Recall that the MHD system (after normalizing the density and permeability to be one) is

$$\begin{cases} \partial_t v = \nu \Delta v - v \cdot \nabla v - B \times (\nabla \times B) - \nabla p, \\ \partial_t B = \dfrac{1}{\sigma} \Delta B + \nabla \times (v \times B), \\ \nabla \cdot B = \nabla \cdot v = 0, \end{cases} \tag{4.4}$$

Applying the curl to first equation of (4.4) and extracting the equation of A from second equation of (4.4) yielding the right system for numerical test:

$$\begin{cases} \partial_t \omega - \nu \Delta \omega = B \cdot \nabla j - u \cdot \nabla \omega, \\ \partial_t A - \dfrac{1}{\sigma} \Delta A = -u \cdot \nabla A, \quad j = -\Delta A. \end{cases} \tag{4.5}$$

4.1.3 Initial data, domain and boundary conditions

We choose the domain $(x, y) \in [0, 2\pi) \times [0, 2\pi)$ and the two systems (4.3), (4.5) are subject to periodic boundary condition.
For MHD (4.5) the initial vorticity and vector potential are

$$\omega_0 = \cos(x) + \cos(y), \quad A_0 = \frac{1}{2} \cos(2x) + \cos(y).$$

Remark 4.1.1. Since this is the periodic setting, all dependent variables are supposed to have zero mean value, that is, their zero Fourier modes are zero.

Recall that the initial velocity can be recovered from ω_0 by solving Poisson's equation:

$$-\Delta u_0 = \nabla \times \omega_0.$$

For the two-fluid system, the initial vorticity and vector potential should be the same as the initial data for MHD. However, the two-fluid system needs more initial data: j_0, E_0, B_0, Ω_0, noting that B_0, Ω_0 should have two components. To ensure two systems are comparable, these initial data should be computed via MHD system:

$$B_0 = \nabla \times A_0, \quad j_0 = \nabla \times B_0, \quad \Omega_0 = \nabla \times j_0, \quad E_0 = \frac{1}{\sigma} j_0 - u_0 \times B_0.$$

4.2 Numerical Scheme

Since we are on the periodic setting, it is easy to use pseudo spectral method. Observing that the diffusion terms are in the equations, an implicit method should be used to ensure the stability of the scheme. Overall, Crank-Nicolson plus pseudo spectral method will both ensure the accuracy and calculation speed. For the following partial differential equation in periodic boundary condition,

$$\partial_t u = L(u) + N(u),$$

where L and N are the linear and nonlinear part respectively, the scheme is

$$\frac{\hat{u}_k^{n+1} - \hat{u}_k^n}{\delta t} = L(\frac{\hat{u}_k^{n+1} + \hat{u}_k^n}{2}) + \widehat{N(u^n)}_k. \tag{4.6}$$

$\hat{u}_k^n$ means the k-th mode of u at the n-th time step. In other words, we are looking for the solution in Fourier side instead of in the physical side. To recover the original function u, we just apply the Fourier inverse transform to $\hat{u}_k^n$. For the calculation of nonlinear term, for example if $N(u) = u^2$, in Fourier side it becomes $(\hat{u}^n * \hat{u}^n)_k$ at n-th time step. Knowing $\hat{u}_k^n$, the cost is still expensive if the convolution is directly calculated. A much faster way is taking advantages of fast Fourier transform algorithm(FFT), finding u^n by taking inverse FFT and calculating $(u^n)^2$ then applying FFT again to get the desired result in Fourier side.

4.2.1 Numerical Schemes of the two fluid model and MHD

For simplicity, we use matrix form to present the numerical schemes of the two systems.

Recall the two fluid model (4.3) is

$$\begin{cases} \partial_t \omega - \nu \Delta \omega = B \cdot \nabla j - u \cdot \nabla \omega, \\ \varepsilon^2 \partial_t \Omega + \dfrac{1}{\sigma}\Omega - \varepsilon^2 \nu \Delta \Omega - \nabla \times E = \nabla \times (u \times B - \varepsilon^2 u \cdot \nabla j), \\ \dfrac{1}{c^2}\partial_t E + \Delta A - \Delta^{-1}(\nabla \times \Omega) = 0, \\ \partial_t A + E = 0, \end{cases} \tag{4.3}$$

Applying Fourier transform in space yields at the k-th mode

$$\begin{cases} \partial_t \hat{\omega}_k + \nu|k|^2 \hat{\omega}_k = \hat{f}_{1k}, \\ \varepsilon^2 \partial_t \hat{\Omega}_{1k} + (\dfrac{1}{\sigma} + \varepsilon^2 \nu |k|^2) \hat{\Omega}_{1k} - ik_2 \hat{E}_k = \hat{f}_{21k} - \varepsilon^2 \hat{f}_{31k}, \\ \varepsilon^2 \partial_t \hat{\Omega}_{2k} + (\dfrac{1}{\sigma} + \varepsilon^2 \nu |k|^2) \hat{\Omega}_{2k} + ik_1 \hat{E}_k = \hat{f}_{22k} - \varepsilon^2 \hat{f}_{32k}, \\ \dfrac{1}{c^2} \partial_t \hat{E}_k - |k|^2 \hat{A}_k + \dfrac{i}{|k|^2}(k_1 \hat{\Omega}_{2k} - k_2 \hat{\Omega}_{1k}) = 0, \\ \partial_t \hat{A}_k + \hat{E}_k = 0, \end{cases} \tag{4.7}$$

where

$$f_1 = B \cdot \nabla j - u \cdot \nabla \omega,$$
$$f_{21} = (\nabla \times (u \times B))_1, \quad f_{22} = (\nabla \times (u \times B))_2,$$
$$f_{31} = (\nabla \times (u \cdot \nabla j))_1, \quad f_{32} = (\nabla \times (u \cdot \nabla j))_2.$$

Let $\hat{X}_k^n = (\hat{\omega}_k^n, \hat{\Omega}_{1k}^n, \hat{\Omega}_{2k}^n, \hat{E}_k^n, \hat{A}_k^n)^T$, $\hat{F}_{1k}^n = (\hat{f}_{1k}^n, \hat{f}_{21k}^n, \hat{f}_{22k}^n, 0, 0)^T$ and $\hat{F}_{2k}^n = (0, \hat{f}_{31k}^n, \hat{f}_{32k}^n, 0, 0)^T$, and define the coefficient matrix by

$$C_1^{\varepsilon,c} = \frac{1}{\delta t} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon^2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c^2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C_2^{\varepsilon} = \frac{1}{2} \begin{pmatrix} \nu|k|^2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma} + \varepsilon^2 \nu |k|^2 & 0 & -ik_2 & 0 \\ 0 & 0 & \frac{1}{\sigma} + \varepsilon^2 \nu |k|^2 & ik_1 & 0 \\ 0 & -\frac{ik_2}{|k|^2} & \frac{ik_1}{|k|^2} & 0 & -|k|^2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then applying Crank-Nicolson method gives us

$$C_1^{\varepsilon,c}(\hat{X}_k^{n+1} - \hat{X}_k^n) + C_2^{\varepsilon}(\hat{X}_k^{n+1} + \hat{X}_k^n) = \hat{F}_{1k}^n - \varepsilon^2 \hat{F}_{2k}^n. \tag{4.8}$$

Hence the procedure is that knowing the n-th step data $\hat{X}_k^n$, updating the intermediate variables:

$$\hat{u}_k^n = (\frac{ik_2}{|k|^2}\hat{\omega}_k^n, -\frac{ik_1}{|k|^2}\hat{\omega}_k^n), \quad \hat{j}_k^n = \frac{ik_1\hat{\Omega}_{2k}^n - ik_2\hat{\Omega}_{1k}^n}{|k|^2}, \quad \hat{B}_k^n = (ik_2\hat{A}_k^n, -ik_1\hat{A}_k^n),$$

then calculating the nonlinear terms $\hat{F}_{1k}^n, \hat{F}_{2k}^n$. Finally the next step $\hat{X}_k^{n+1}$ can be calculated by

$$\hat{X}_k^{n+1} = (C_1^{\varepsilon,c} + C_2^{\varepsilon})^{-1}\left[(C_1^{\varepsilon,c} - C_2^{\varepsilon})\hat{X}_k^n + \hat{F}_{1k}^n - \varepsilon^2\hat{F}_{2k}^n\right]. \tag{4.9}$$

For MHD (4.5):

$$\begin{cases} \partial_t\omega - \nu\Delta\omega = B\cdot\nabla j - u\cdot\nabla\omega, \\ \partial_t A - \dfrac{1}{\sigma}\Delta A = -u\cdot\nabla A, \quad j = -\Delta A. \end{cases} \tag{4.5}$$

Similar to the two-fluid model, applying the Fourier transform in space yields

$$\begin{cases} \partial_t\hat{\omega}_k + \nu|k|^2\hat{\omega}_k = \hat{g}_{1k}, \\ \partial_t\hat{A}_k + \dfrac{1}{\sigma}|k|^2\hat{A}_k = \hat{g}_{2k}, \end{cases} \tag{4.10}$$

where $g_1 = B\cdot\nabla j - u\cdot\nabla\omega, g_2 = -u\cdot\nabla A$.
let $\hat{Y}_k^n = (\hat{\omega}_k^n, \hat{A}_k^n)^T$, $\hat{G}_k^n = (\hat{g}_{1k}^n, \hat{g}_{2k}^n)^T$ and the coefficient matrix

$$C_1 = \frac{1}{\delta t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 = \frac{1}{2}\begin{pmatrix} \nu|k|^2 & 0 \\ 0 & \frac{1}{\sigma}|k|^2 \end{pmatrix}.$$

Then the scheme for MHD is

$$\hat{Y}_k^{n+1} = (C_1 + C_2)^{-1}\left[(C_1 - C_2)\hat{Y}_k^n + \hat{G}_k^n\right]. \tag{4.11}$$

Like two-fluid model's procedure, the intermediate variables for MHD simulation should be updated by the following formula

$$\hat{u}_k^n = (\frac{ik_2}{|k|^2}\hat{\omega}_k^n, -\frac{ik_1}{|k|^2}\hat{\omega}_k^n), \quad \hat{j}_k^n = \frac{1}{|k|^2}\hat{A}_k^n, \quad \hat{B}_k^n = (ik_2\hat{A}_k^n, -ik_1\hat{A}_k^n).$$

4.3 Numerical Result

Recall that the periodic domain is $[0, 2\pi) \times [0, 2\pi)$ and the initial data is

$$\omega_0 = \cos(x) + \cos(y), \quad A_0 = \frac{1}{2}\cos(2x) + \cos(y).$$

Remark 4.3.1. Although Crank-Nicolson is unconditional stable when the system is linear, in our nonlinear system, it seems the big initial data causes unstable of the scheme. So in our test, we choose the initial data size around 1. A very tiny step is needed if we increase the initial data size.

We set the fluid and magnetic viscosities are the same, i.e. $\nu = \frac{1}{\sigma} = 0.005$. Grid sizes in both x, y directions are set to $N = 128$ and time step is $\Delta t = 0.00125$. Let us first fix the value of c, ε and see the solutions of two system. Figure 4.1 shows the solution comparison when $c = 10^8, \varepsilon = 10^{-8}$ and at time $t = 1$. One can see that two systems' solutions are extremely similar the differences between them are about 10^{-13}. Figure 4.2 shows the solutions at a longer time $t = 30$. The value of solutions decreases and the differences between two systems does not increase too much and still stay very small. Eventually, everything will go to zero because of the viscosity. If one increases the value of $\nu, \frac{1}{\sigma}$, the tendency of solutions to zero will appears at first few seconds however doing so, the changes of dynamics of the solutions will be hard to capture since it will happen at the very start then quickly go to zero.

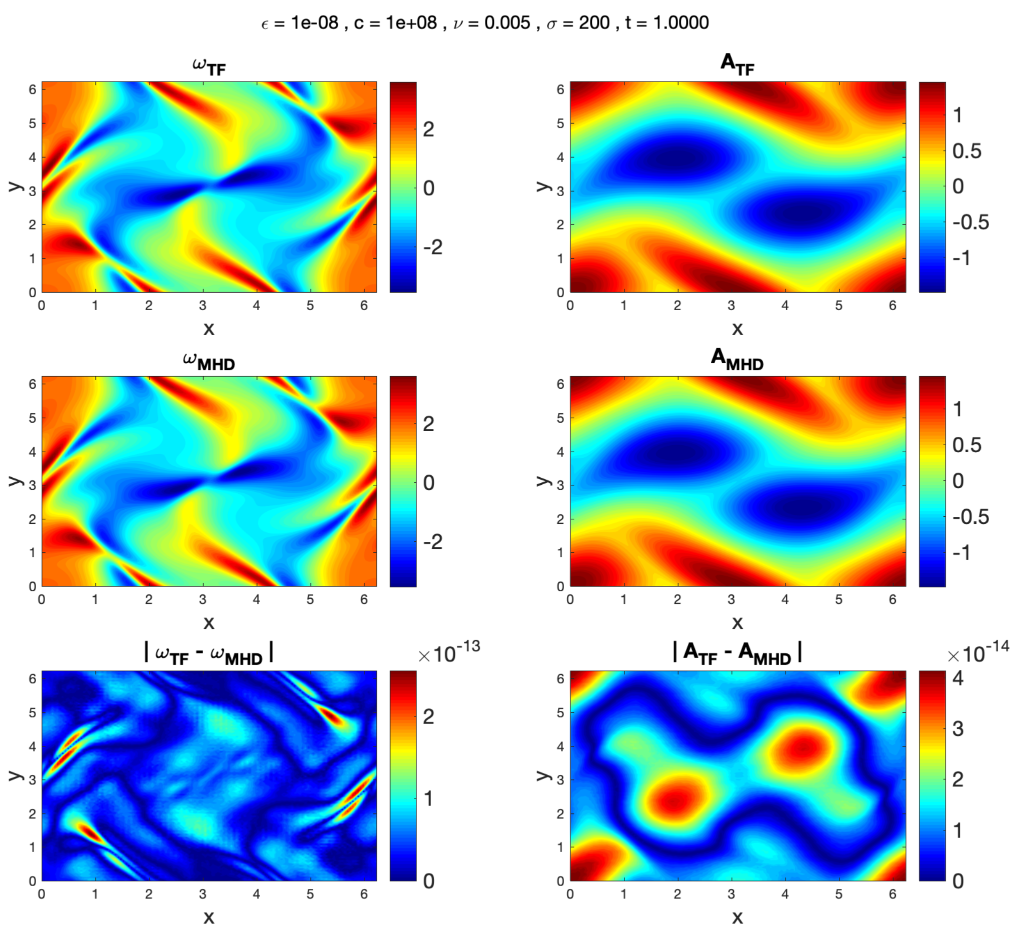

Figure 4.1: Solutions comparison at $t = 1$.

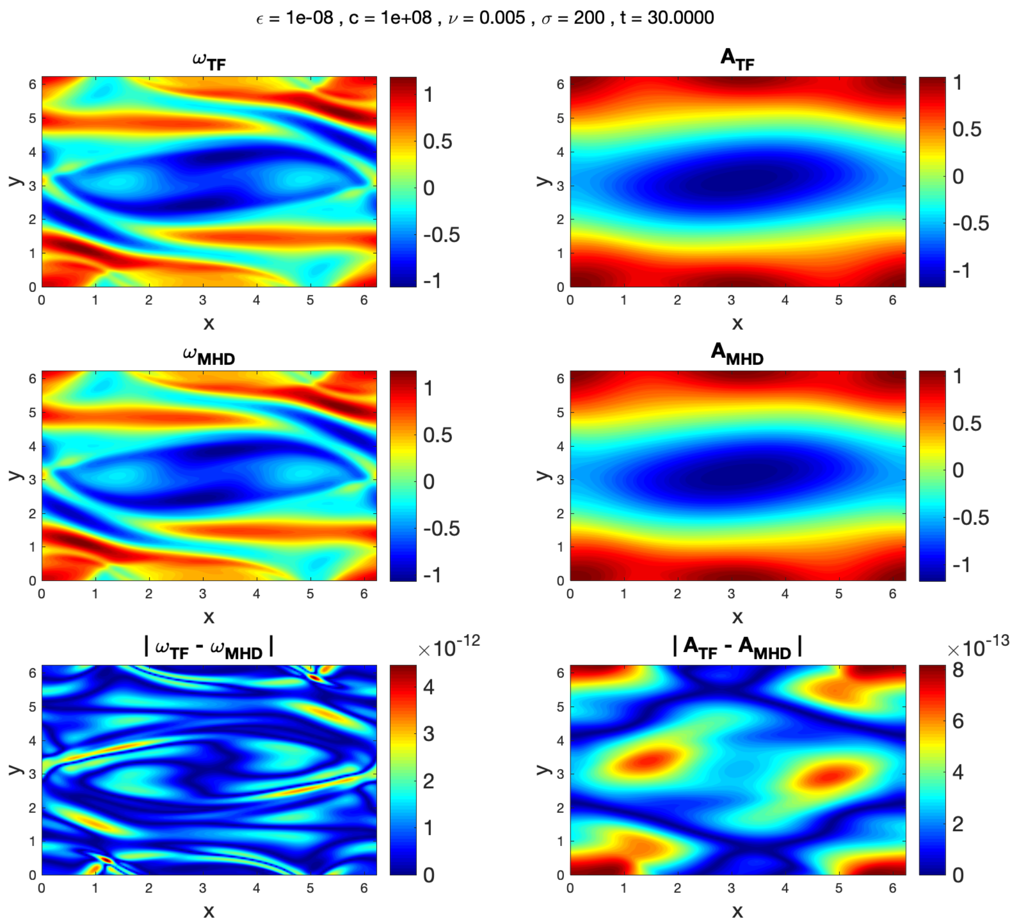

Figure 4.2: Solutions comparison at $t = 30$.

4.3.1 Numerical convergence rate

Now we focus on how the differences of two system changes with respect to c, ε. In the next numerical test, we set $\log_{10}(c)$ changes from $1 \sim 8$ and $\log_{10}(\varepsilon)$ changes from $-1 \sim -8$. For each pair of (c, ε), calculating $\log_{10}\left(\|\omega_{TF} - \omega_{MHD}\|_{L^2} + \|A_{TF} - A_{MHD}\|_{L^2}\right)$.

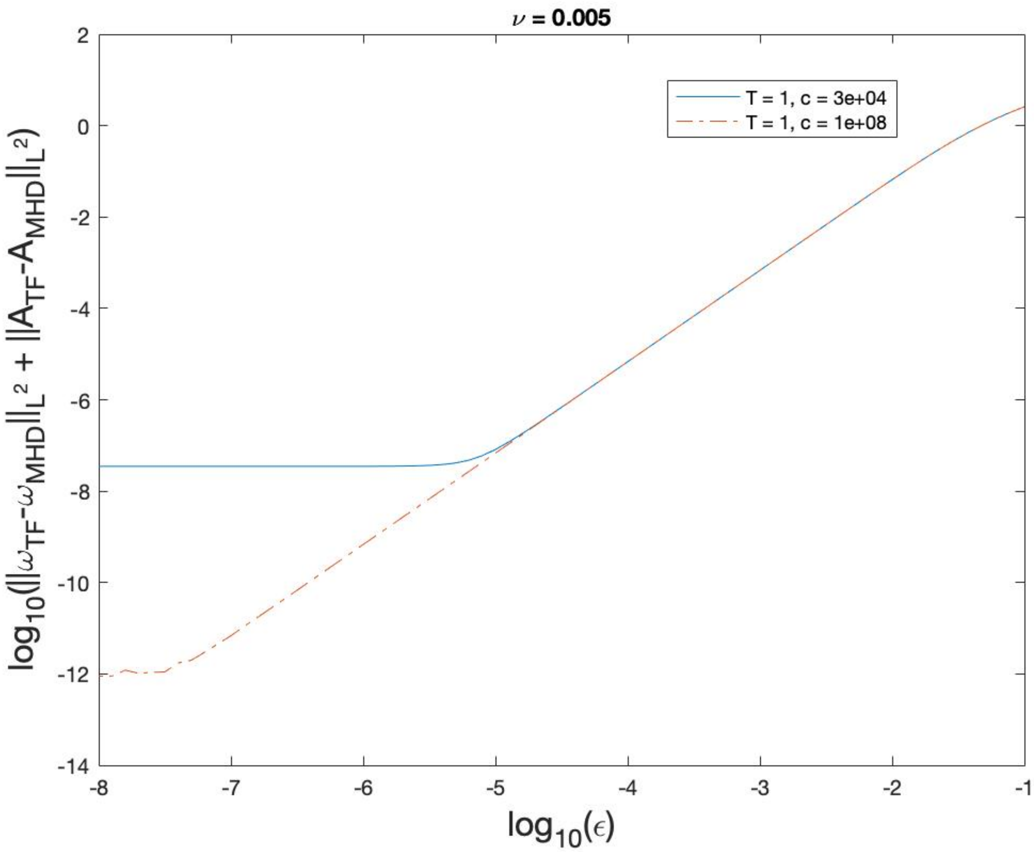

Figure 4.3: Fix c, the difference of the solution vs. $\log_{10}(\varepsilon)$ at time $t = 1$.

Figure 4.3 and Figure 4.4 shows the results at time $t = 1$ when fixing c and fixing ε respectively. One can observe clearly that, for example, when fixing c, $\log_{10}\left(\|\omega_{TF} - \omega_{MHD}\|_{L^2} + \|A_{TF} - A_{MHD}\|_{L^2}\right)$ changes linearly with respect to $\log_{10}(\varepsilon)$ then becomes a constant. The oscillation for big c case mean the calculation reaches the computer's limit, because in the scheme, $1/c^2, \varepsilon^2$ will be calculated and for big c and small ε, and the quantities $1/c^2, \varepsilon^2$ can easily reach the floating-point relative accuracy $2^{-52} \sim 10^{-15.6}$. Similar behaviour happens for the case of fixing ε (see Figure 4.4).

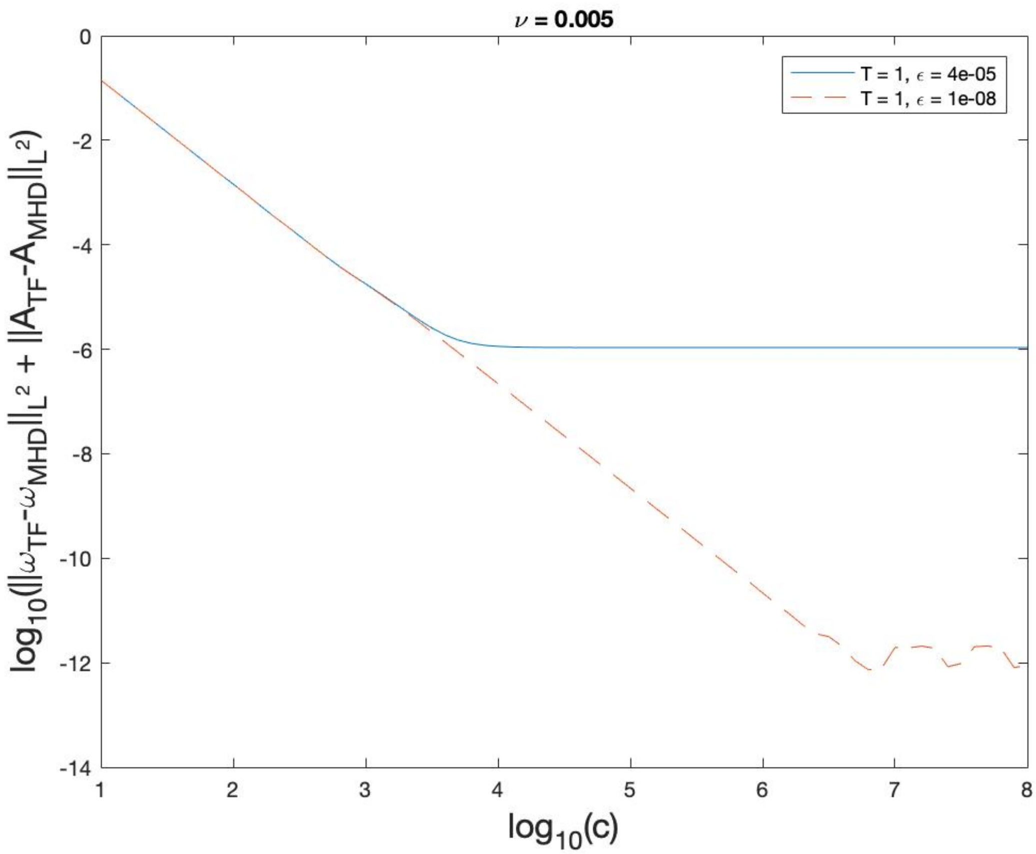

Figure 4.4: Fix ε, the difference of the solution vs. $\log_{10}(c)$ at time $t = 1$.

We can fit the slope of the linear changing part which gives us the slope is about 2 for $\log_{10}(\varepsilon)$ and -2 for $\log_{10}(c)$. It is also clear that the turning point from linear relation to constant depends on c, ε. To figure out how these turning points related on c or ε, we show the contour plot of the solution difference vs. $\log_{10}(c), \log_{10}(\varepsilon)$ (see Figure 4.5). The $*$ points are where the linear relation changes to a constant. For example, fixing $\log_{10}(c)$ the difference between two system at first is nearly a constant with respect to $\log_{10}(\varepsilon)$ then when passing the corresponding $*$ point, the difference changes linearly (according to Figure 4.3) with respect to $\log_{10}(\varepsilon)$.
Next, we need write down the equation of the curve formed by $*$ points. According to Figure 4.4, for different $\log_{10}(\varepsilon)$, the linear change part share the same equation. So, we can assume that in the domain below $*$ points,

$$D := \log_{10}\left(\|\omega_{TF} - \omega_{MHD}\|_{L^2} + \|A_{TF} - A_{MHD}\|_{L^2}\right) = -2\log_{10}(c) + C_c,$$

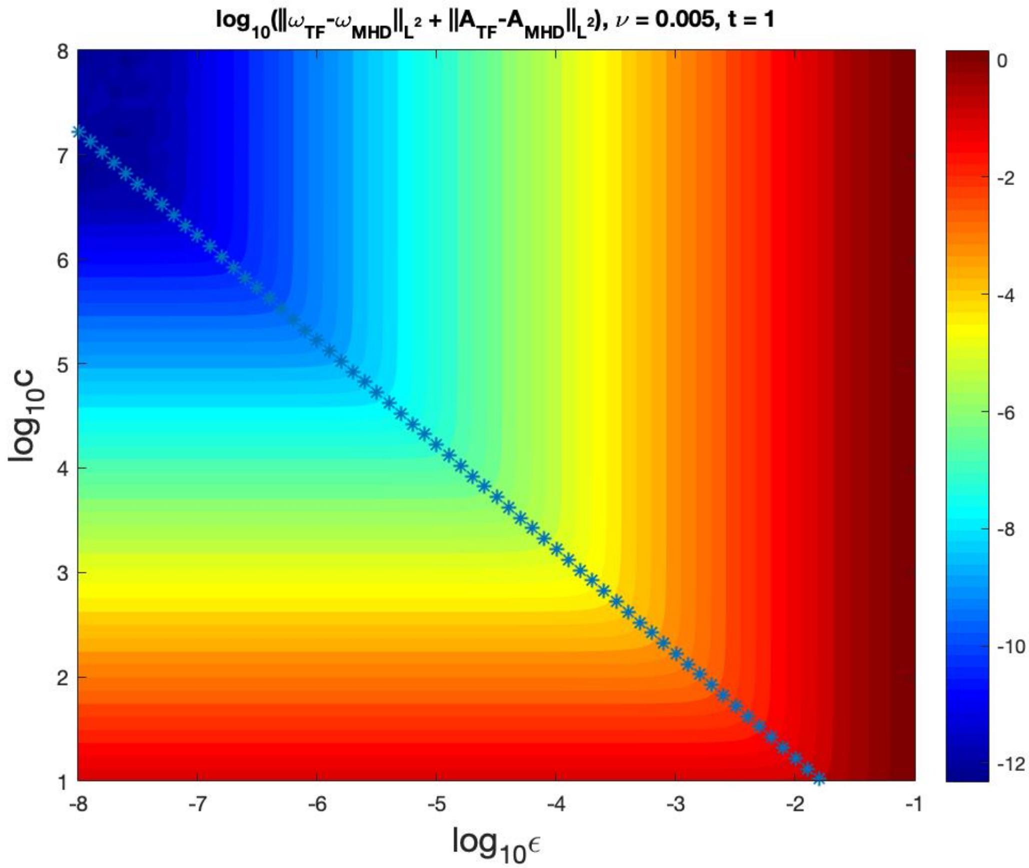

Figure 4.5: Contour plot of the solution difference vs. $\log_{10}(c), \log_{10}(\varepsilon)$ at time $t = 1$. The point $*$ means the turning point.

Similarly, in the domain above the $*$ points,

$$D = 2\log_{10}(\varepsilon) + C_\varepsilon.$$

At the $*$ point, we have $2\log_{10}(\varepsilon) + C_\varepsilon = -2\log_{10}(c) + C_c$ which gives the equation of $*$ lines:

$$\log_{10}(c) \approx -\log_{10}(\varepsilon) + \frac{C_c - C_\varepsilon}{2}.$$

In this case, the data gives us $C_c \approx 1.2089$ and $C_\varepsilon \approx 2.7640$.

After all the above work, the converge rate with respect to c, ε is

$$D \approx \begin{cases} 2\log_{10}(\varepsilon) + C_\varepsilon, & \text{if } \log_{10}(c) \geq -\log_{10}(\varepsilon) + \frac{C_c - C_\varepsilon}{2} \\ -2\log_{10}(c) + C_c, & \text{if } \log_{10}(c) < -\log_{10}(\varepsilon) + \frac{C_c - C_\varepsilon}{2}. \end{cases}$$

In other words, we have

$$\|\omega_{TF} - \omega_{MHD}\|_{L^2} + \|A_{TF} - A_{MHD}\|_{L^2} \approx \begin{cases} 10^{C_\varepsilon}\varepsilon^2, & \text{if } c\varepsilon \geq 10^{\frac{C_c - C_\varepsilon}{2}}, \\ 10^{C_c}c^{-2}, & \text{if } c\varepsilon < 10^{\frac{C_c - C_\varepsilon}{2}}. \end{cases} \tag{4.12}$$

Simplify a little bit, the above estimate (4.12) becomes

$$\|\omega_{TF} - \omega_{MHD}\|_{L^2} + \|A_{TF} - A_{MHD}\|_{L^2} \approx \max\left\{10^{C_\varepsilon}\varepsilon^2, 10^{C_c}c^{-2}\right\}, \tag{4.13}$$

where two constant C_c, C_ε may depend on $t, \nu, \frac{1}{\sigma}$. We also test for different time, different $\mu, \frac{1}{\sigma}$ to see the corresponding dependence.

Figure 4.6 and Figure 4.7 plot the relations between D and $\log_{10}(\varepsilon), \log_{10}(c)$ respec-

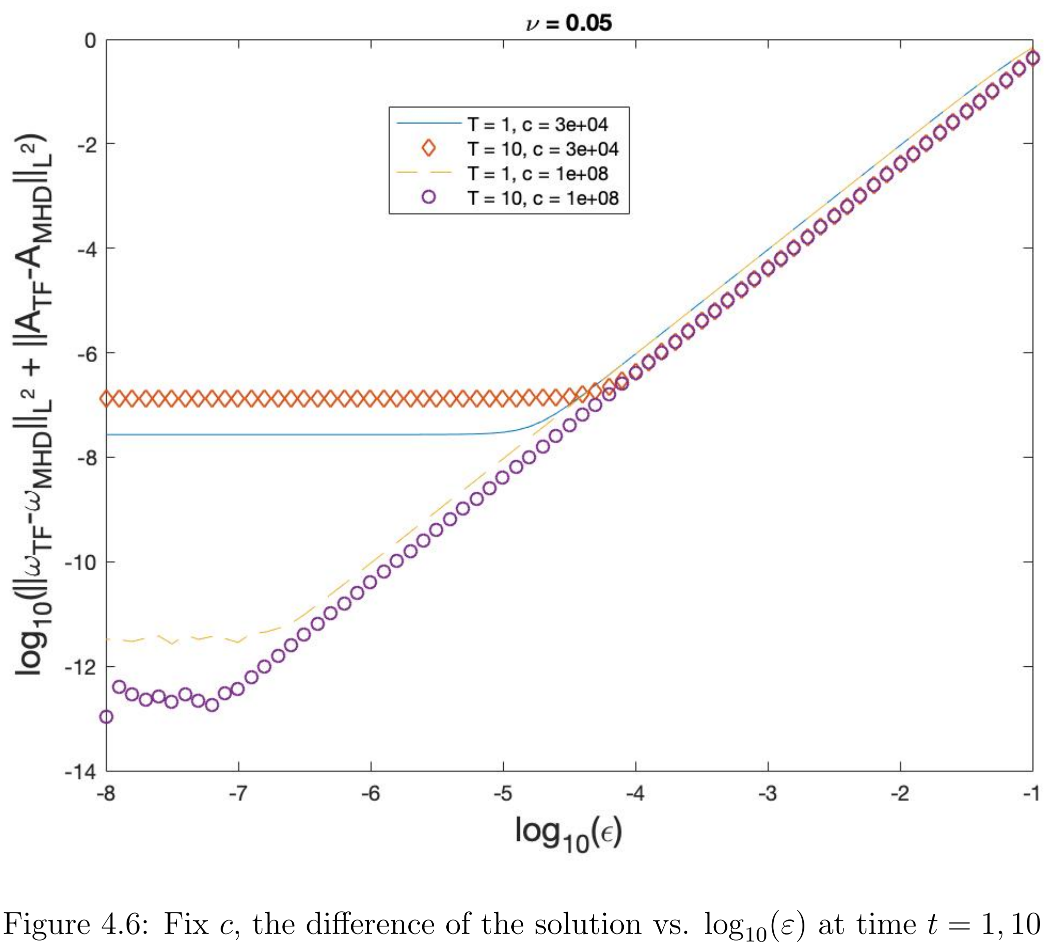

Figure 4.6: Fix c, the difference of the solution vs. $\log_{10}(\varepsilon)$ at time $t = 1, 10$.

tively at two different time $t = 1, 10$. Through these figure, we can conclude that

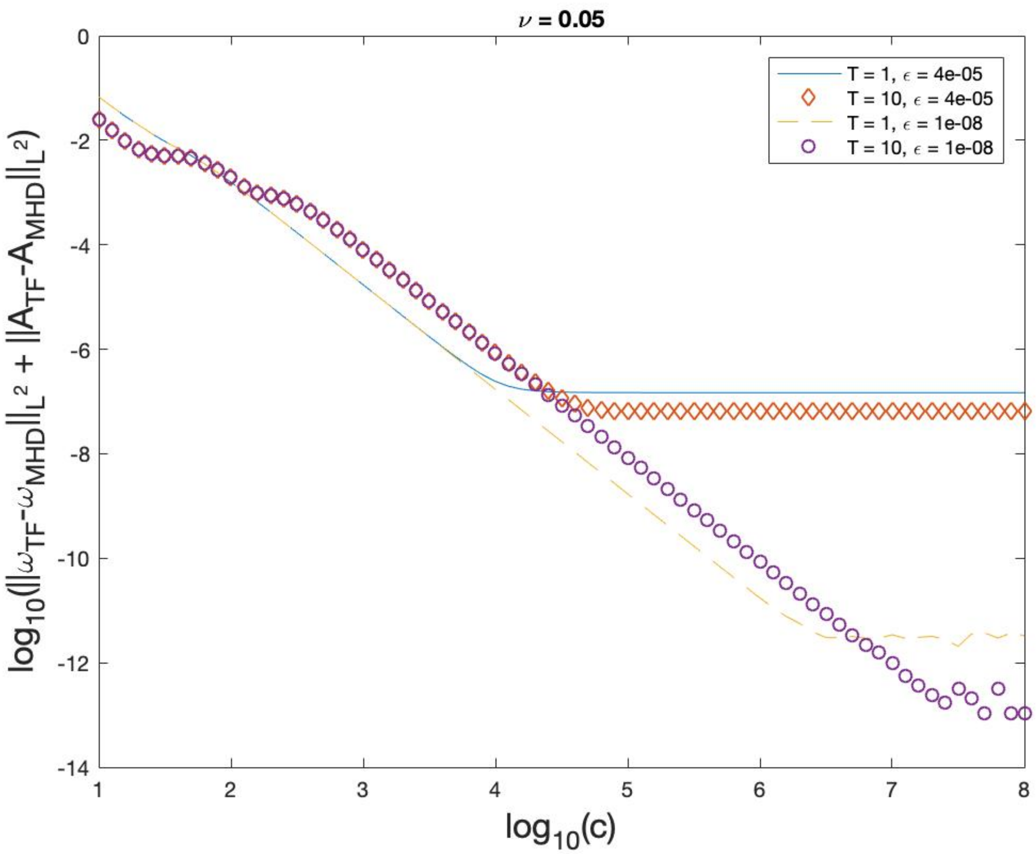

Figure 4.7: Fix ε, the difference of the solution vs. $\log_{10}(c)$ at time $t = 1, 10$.

even at different time, the linear part can still take the form:

$$D = -2\log_{10}(c) + C_c, \quad D = 2\log_{10}(\varepsilon) + C_\varepsilon,$$

with $C_c \approx 1.2308$ and $C_\varepsilon \approx 1.7826$. Comparing to the previous $\mu = 0.005$ case, we can conclude that C_c, C_ε only depend on $\mu, \frac{1}{\sigma}$ and are independent of time.

Remark 4.3.2. In [2], the convergence rate in 2D case is given by $\frac{1}{e^{c^2}\varepsilon}$. And if we want this quantity goes to zero when $\varepsilon \to 0, c \to \infty$, ε should be a function of c and $\lim_{c\to\infty} \varepsilon = 0$. However, the numerical result gives that the convergence rate is $\max(10^{C_c}c^{-2}, 10^{C_\varepsilon}\varepsilon^2)$ which shows no dependence required between c and ε.

Appendix A

A.1 Sobolev spaces

Sobolev spaces are very useful to find the solutions to partial differential equations. We denote the Fourier transform:

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(x)dx,$$

where $\xi, x \in \mathbb{R}^d$. For $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ (sometimes denoted by $\dot{H}^s$) consists of functions with the following norm(see [3])

$$\|u\|_{\dot{H}^s}^2 := \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty.$$

The inhomogeneous Sobolev space H^s equips the norm:

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty.$$

Remark A.1.1. In the definition of Sobolev spaces, the index s can be any real number. If s is positive integer, the interpretation of Sobolev spaces is much easier to understand: H^s consists of functions with all of the 1st up to s-th derivatives and function itself in L^2 while $\dot{H}^s$ only consists of functions with their s-th derivatives in L^2.

One important property of Sobolev spaces is the Sobolev embedding which is frequently used in the whole thesis.

Theorem A.1.1. Let $s \in [0, d/2)$, then $\dot{H}^s$ is continuously embedded in $L^{\frac{2d}{d-2s}}$. On the other hand, if $p \in (1, 2]$, then L^p is continuously embedded in $\dot{H}^{d/2-d/p}$.

Theorem A.1.2. The space H^s is embedded continuously in L^p, if $0 \le s < d/2$ and $2 \le p \le 2d/(d-2s)$.

The details of the proofs can be found in [3]. We point out here that the relation between homogeneous Sobolev space index s and Lebesgue space index p is

$$p = \frac{2d}{d-2s},$$

which could be easily calculated through a scaling argument. We could define the scaling function of u by $u_\lambda = u(\lambda x)$, where u is a function on $\mathbb{R}^d$. Then

$$\begin{aligned} \|u_\lambda\|_{\dot{H}^s}^2 &= \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}_\lambda(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} |\xi|^{2s} \lambda^{-2d} |\hat{u}(\lambda^{-1}\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} |\lambda\eta|^{2s} \lambda^{-d} |\hat{u}(\eta)|^2 d\eta \\ &= \lambda^{2s-d} \|u\|_{\dot{H}^s}^2, \end{aligned}$$

where the second step we use one property of Fourier transform: $\hat{u}_\lambda(\xi) = \lambda^{-d}\hat{u}(\lambda^{-1}\xi)$. Besides,

$$\begin{aligned} \|u_\lambda\|_{L^p}^p &= \int_{\mathbb{R}^d} |u_\lambda(x)|^p dx \\ &= \int_{\mathbb{R}^d} \lambda^{-d} |u(y)|^p dy \\ &= \lambda^{-d} \|u\|_{L^p}^p. \end{aligned}$$

Therefore

$$\|u_\lambda\|_{\dot{H}^s} = \lambda^{s-d/2} \|u\|_{\dot{H}^s}, \quad \|u_\lambda\|_{L^p} = \lambda^{-d/p} \|u\|_{L^p}.$$

If the embedding $\dot{H}^s \subset L^p$ is true, then $\|u\|_{L^p} \le C\|u\|_{\dot{H}^s}$ and $\|u_\lambda\|_{L^p} \le C\|u_\lambda\|_{\dot{H}^s}$ must be true for the same universal constant C and for any scaling λ. Then it must have $s - d/2 = -d/p$. If the embedding is $L^p \subset \dot{H}^s$, the argument is similar and leads to the same result.

The next property used in thesis is the interpolation.

Proposition A.1.1. Assume $s_0 \le s \le s_1$, $0 \le \theta \le 1$. Then we have

$$\|u\|_{\dot{H}^s} \le \|u\|_{\dot{H}^{s_0}}^{1-\theta} \|u\|_{\dot{H}^{s_1}}^{\theta},$$

where $s = (1-\theta)s_0 + \theta s_1$.
The same result holds for inhomogeneous Sobolev spaces:

$$\|u\|_{H^s} \leq \|u\|_{H^{s_0}}^{1-\theta} \|u\|_{H^{s_1}}^{\theta}.$$

Proof. We only prove the inhomogeneous version. The homogeneous version is similar. Rewrite $\|u\|_{\dot{H}^s}^2$ by

$$\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^2} |\xi|^{2(1-\theta)s_0} |\hat{u}|^{2(1-\theta)} |\xi|^{2\theta s_1} |\hat{u}|^{2\theta} d\xi.$$

Then applying Hölder's inequality for $p = 1/(1-\theta)$ and $q = 1/\theta$ proves the proposition. □

A.2 Littlewood-Paley Decomposition

The standard Littlewood-Paley decomposition (see [3]) splits a tempered distribution into dyadic blocks. Basically, it is a frequency based decomposition. Let us first introduce two smooth positive, compact supported functions: $\psi(\xi), \phi(\xi) \in C_c^\infty(\mathbb{R}^d)$

$$\psi, \phi > 0, \quad \text{supp } \psi \subset \{|\xi| \leq 1\}, \quad \text{supp } \phi \subset \left\{\frac{1}{2} \leq |\xi| \leq 2\right\},$$

and

$$\psi(\xi) + \sum_{k=0}^{\infty} \phi(2^{-k}\xi) = 1, \text{ for all } \xi \in \mathbb{R}^d.$$

The functions satisfying the property above exist (see [3]). Let $\phi_k(\xi) = \phi(2^{-k}\xi)$, then for $\xi \neq 0$, it holds that

$$\sum_{k=-\infty}^{\infty} \phi_k(\xi) = 1.$$

The k-th dyadic block of a tempered distribution u is defined by

$$\Delta_k u := \mathcal{F}^{-1}\left(\phi_k(\xi)\hat{u}(\xi)\right),$$

where $\mathcal{F}^{-1}$ is inverse Fourier transform. Thus u could be decomposed into many dyadic blocks: $u = \sum_{k=-\infty}^{\infty} \Delta_k u$.
The Littlewood-Paley decomposition plays key role in parabolic regularization. The

classic estimate on each dyadic block shows the smoothing effect of heat operator: $e^{t\Delta}$.

Proposition A.2.1. For any dyadic block Δ_k, it holds that

$$\|e^{t\Delta}\Delta_k u\|_{L^2} \leq e^{-ct2^k}\|\Delta_k u\|_{L^2},$$

where c is a universal constant independent of k.

Proof. By FourierPlancherel formula,

$$\|e^{t\Delta}\Delta_k u\|_{L^2}^2 = \int_{\mathbb{R}^d} e^{-t|\xi|^2}\phi_k^2(\xi)\hat{u}^2(\xi)d\xi.$$

Noticing that supp $\phi_k \subset \left\{2^{k-1} \leq |\xi| \leq 2^{k+1}\right\}$, then it is clear that

$$\|e^{t\Delta}\Delta_k u\|_{L^2}^2 \leq e^{-ct2^k}\|\Delta_k u\|_{L^2}^2,$$

and the proposition is proved. □

Remark A.2.1. The above proposition can be extended to any L^p with $p \in [1, \infty]$:

$$\|e^{t\Delta}\Delta_k u\|_{L^p} \leq Ce^{-ct2^k}\|\Delta_k u\|_{L^p},$$

where C, c are universal constants independent of k. We refer the proof to [3].

A.3 Chemin−Lerner spaces

The usual Chemin−Lerner spaces were first introduced by Chemin and Lerner [7]. This kind of space appears naturally if one needs to establish a parabolic regularization. Before we introduce this kind of spaces, let us see the definition of classic time dependent Sobolev spaces $L^p(0,T;H^s)$ (denoted by $L^p_T H^s$). The space $L^p_T H^s$ is a set of tempered distributions with the following norm being finite.

$$\|u\|_{L^p_T H^s}^p = \int_0^T \|u(t)\|_{H^s}^p dt = \int_0^T \|(1+2^k)^s \Delta_k u(t)\|_{l^2(\mathbb{Z})}^p dt < \infty.$$

One could see the norms of classic time dependent Sobolev spaces are taking H^s norm in space first then taking L^p norm in time. While for the Chemin−Lerner spaces $\tilde{L}^p H^s$, the norms are taking $L^p L^2$ first for each dyadic block then taking $l^2(\mathbb{Z})$ to sum all the dyadic blocks. The more precise definition is

$$\|u\|_{\tilde{L}^p_T H^s} := \|(1+2^k)^s \|\Delta_k u\|_{L^p_T L^2}\|_{l^2(\mathbb{Z})}.$$

And the homogeneous version is

$$\|u\|_{\tilde{L}^p_T \dot{H}^s} := \|2^{ks} \|\Delta_k u\|_{L^p_T L^2}\|_{l^2(\mathbb{Z})}.$$

The classic parabolic regularization usually applies Proposition A.2(or similar estimate) to each dyadic block and then integrating in time leads to the $L^p L^2$ estimate for each block. Lastly, applying some weighted $l^2(\mathbb{Z})$ norm, we obtain the estimate in a space of type Chemin−Lerner.
The Chemin−Lerner spaces could be linked with classic time dependent Sobolev spaces through Minkowski's integral inequality: for $p \geq 1$, it holds that

$$\left(\int \left|\int f(x,y)dx\right|^p dy\right)^{\frac{1}{p}} \leq \int \left(\int |f(x,y)|^p dy\right)^{\frac{1}{p}} dx,$$

where the inequality is still true if the integral changes to infinite summation. Thus applying the above inequality, we have

Proposition A.3.1.

- If $1 \leq p \leq 2$, then

$$\|u\|_{\tilde{L}^p H^s} \lesssim \|u\|_{L^p H^s}, \quad \|u\|_{\tilde{L}^p \dot{H}^s} \lesssim \|u\|_{L^p \dot{H}^s}$$

- If $p \geq 2$, then

$$\|u\|_{\tilde{L}^p H^s} \gtrsim \|u\|_{L^p H^s}, \quad \|u\|_{\tilde{L}^p \dot{H}^s} \gtrsim \|u\|_{L^p \dot{H}^s}$$

- If $p \leq 2, s < s'$, then

$$\|u\|_{L^p H^s} \lesssim \|u\|_{\tilde{L}^p H^{s'}}.$$

For more details of Chemin−Lerner spaces, we refer to [3].

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