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A multiplicative ergodic theoretic characterization of relative equilibrium states

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Abstract. In this article, we continue the structural study of factor maps between symbolic dynamical systems and the relative thermodynamic formalism. Here, one is studying a factor map from a shift of finite type X (equipped with a potential function) to a sofic shift Z, equipped with a shift-invariant measure ν. We study relative equilibrium states, that is, shift-invariant measures on X that push forward under the factor map to ν which maximize the relative pressure: the relative entropy plus the integral of ϕ. In this paper, we establish a new connection to multiplicative ergodic theory by relating these factor triples to a cocycle of Ruelle–Perron–Frobenius operators, and showing that the principal Lyapunov exponent of this cocycle is the relative pressure; and the dimension of the leading Oseledets space is equal to the number of measures of relative maximal entropy, counted with a previously identified concept of multiplicity.

Key words: relative thermodynamic formalism, multiplicative ergodic theory, transfer operators
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1. *Introduction*

Let A and B be finite non-empty sets, let $X \subset A^{\mathbb{Z}}$ be an irreducible shift of finite type, and let $\pi : X \to B^{\mathbb{Z}}$ be a shift-commuting map so that $Z = \pi(X)$ is a sofic factor of X. Given a shift-invariant measure ν on Z, we are interested in $\pi_*^{-1}\{\nu\}$, the set of shift-invariant measures on X whose push-forward is ν. *Relative thermodynamic formalism* gives a means to identify distinguished elements of $\pi_*^{-1}\{\nu\}$ similar to standard thermodynamic formalism. In fact, standard thermodynamic formalism is the special case of relative thermodynamic formalism, where Z is the one-point space.

We make a standing assumption that the factor map has the property that $\pi(x)_0$ only depends on $x_0, x_1, \ldots$ (and in fact in this case, by the Curtis–Hedlund–Lyndon theorem, $\pi(x)_0$ only depends on $x_0, \ldots, x_{k-1}$ for some $k \in \mathbb{N}_0$). We call this a *forward-looking factor map*. The simplest case of such factor maps is the case where $\pi(x)_0$ depends only on x_0, that is, π is a one-block factor map. It is well-known [8, Proposition 1.5.12] that up to conjugacy, this is the general case. Let $X^+ \subset A^{\mathbb{N}_0}$ and $Z^+ \subset B^{\mathbb{N}_0}$ denote the one-sided versions of X and Z. By the assumption on the factor map, π induces a map from X^+ to Z^+, which we also call π.

Given an invariant measure ν on Z, and a Hölder continuous function ϕ on X^+ (which we call a *potential* and which we also view as a function on X), recall that a *relative equilibrium state* of ϕ over ν is an invariant measure μ on X such that $\pi_*(\mu) = \nu$ and $h_\mu + \int \phi \, d\mu = r_\pi(\nu)$, where $r_\pi(\nu) := \max_{\{\lambda :\, \pi_*(\lambda)=\nu\}}(h_\lambda + \int \phi \, d\lambda)$. Since ϕ is continuous, the existence of a relative equilibrium state follows from the compactness of the space of measures and upper semi-continuity of entropy. A number of papers [1, 3, 10] have given bounds on the number of ergodic relative equilibrium states and measures of relative maximal entropy (relative equilibrium states in the case where $\phi = 0$) in this setting.

We are seeking to relate the number of relative equilibrium states of ϕ over ν to the Lyapunov exponents and Oseledets spaces of a cocycle of Perron–Frobenius operators that we describe below. For $0 < \beta < 1$, we introduce a metric d_β on X^+ given by $d_\beta(x, x') = \beta^{\min\{n :\, x_n \neq x'_n\}}$ and write $C^\beta(X^+)$ for the Lipschitz functions with respect to this metric. For $\beta \in [\frac{1}{2}, 1)$, $C^\beta(X^+)$ is precisely the collection of $(-\log_2 \beta)$-Hölder continuous functions with respect to the standard metric $d_{1/2}$. More generally, as is common in symbolic dynamics, we refer to the spaces $C^\beta(X^+)$ as the Hölder continuous functions on X^+ (even when $0 < \beta < \frac{1}{2}$). Once the potential ϕ is fixed, we choose β so that $\phi \in C^\beta(X^+)$.

For $j \in B$, define an operator $\mathcal{L}_j$ on $C^\beta(X^+)$ by

$$\mathcal{L}_j f(x) = \sum_{\{i :\, ix \in X^+; (\pi(ix))_0 = j\}} e^{\phi(ix)} f(ix),$$

where ix denotes the point in $A^{\mathbb{N}_0}$ defined by $(ix)_0 = i$, $(ix)_n = x_{n-1}$ for $n \geq 1$. Similarly for a word $w = w_0 \ldots w_{k-1}$, $(wx)_n = w_n$ if $n < k$ and $(wx)_n = x_{n-k}$ if $n \geq k$. If x is an element of X or X^+, we use the notation x_0^{n-1} to denote the word $x_0 \ldots x_{n-1}$. We shall study the cocycle over the dynamical system $\sigma : Z \to Z$ where the map corresponding to z is $\mathcal{L}_z := \mathcal{L}_{z_0}$. As usual, we define $\mathcal{L}_z^{(n)} = \mathcal{L}_{\sigma^{n-1}z} \circ \cdots \circ \mathcal{L}_z$.

An inductive calculation shows that

$$\mathcal{L}_z^{(n)} f(x) = \sum_{\{w :\, (\pi(wx))_0^{n-1} = z_0^{n-1}, w_0 \ldots w_{k-1} x \in X^+\}} e^{S_n\phi(wx)} f(wx),$$

where, as usual, $S_n\phi(wx)$ denotes the sum $\phi(wx) + \cdots + \phi(T^{n-1}(wx))$.

Our main theorem states that for an ergodic invariant measure ν on Z and a Hölder continuous potential, the number of ergodic relative equilibrium states on X is the multiplicity of the top Lyapunov exponent of the above cocycle. While we defer detailed definitions and statements which the theorem relies on, we mention Theorems 5 and 6 of Jisang Yoo which establish that a factor map $\pi : X \to Z$ of the type that we consider

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may be expressed as a composition of factor maps $\pi_1\colon X \to Y$ and $\pi_2\colon Y \to Z$ with good properties defined in detail below: π_1 is of class degree 1, and π_2 is finite-to-one of degree c_π, which is the class degree of the original map π. Recall that for a finite-to-one factor map $\pi_2\colon Y \to Z$ from one irreducible two-sided sofic system to another, the *degree* of π_2 is the minimal cardinality of $\pi_2^{-1}(z)$ as z runs over Z. The minimum is attained for all doubly transitive (that is, right and left transitive) points (see [8, Theorem 9.1.11]).

If ν is a fully supported ergodic shift-invariant measure on Z, then since $\pi_2\colon Y \to Z$ is of degree c_π, ν-almost every (a.e.) $\omega \in Z$ has c_π preimages. It may be shown that there are only finitely many ergodic invariant measures on Y that factor onto ν, $\nu_1, \ldots, \nu_k$ say. Yoo defines *multiplicities* $m_1, \ldots, m_k$ of these measures with $m_1 + \cdots + m_k = c_\pi$ and shows that for ν-a.e. $z \in Z$, of the c_π elements of $\pi_2^{-1}(z)$, m_i are generic for ν_i for each i.

Since the multiplicative ergodic theory of infinite-dimensional operators is less well-known than in the finite-dimensional case, we include a quick summary. While there are multiple versions of operator-valued multiplicative ergodic theorems, we focus on the context in this article. Assume that there is a 'base' dynamical system $\sigma\colon Z \to Z$ which is a continuous homeomorphism from a compact metric space to itself. The space Z is assumed to be equipped with a σ-invariant ergodic Borel probability measure ν. A function on Z is said to be ν-*continuous* if for any $\epsilon > 0$, there exists a subset $Z' \subset Z$ of measure at least $1 - \epsilon$ on which the restriction of the function is continuous. Recall that by Lusin's theorem, any Borel-measurable real-valued function on Z is ν-continuous, but this is not necessarily true for functions with non-separable ranges. There is also a Banach space B and a collection $\{\mathcal{L}_z\colon z \in Z\}$ of linear maps from B to itself. One then studies the operators $\mathcal{L}_z^{(n)}$, defined by $\mathcal{L}_z^{(n)} = \mathcal{L}_{\sigma^{n-1}z} \circ \cdots \circ \mathcal{L}_z$. Under conditions of quasi-compactness (which are satisfied in our context), there exists a leading Lyapunov exponent λ_1, an exponent $\lambda_2 < \lambda_1$, a *multiplicity* M, a ν-continuous map E from Z into $\mathcal{G}_M(B)$, the collection of M-dimensional subspaces of B, and ν-continuous maps $\eta_1, \ldots, \eta_M$ from Z into B^* satisfying:

- (equivariance) $\mathcal{L}_z(E(z)) = E(\sigma(z))$, ν-a.e.; and $\mathcal{L}_z(F(z)) \subset F(\sigma(z))$, where $F(z) = \bigcap_{i=1}^M \ker \eta_i(z)$, ν-a.e.;
- (growth) $\lim_{n\to\infty}(1/n) \log \|\mathcal{L}_z^{(n)} f\| = \lambda_1$ for all $f \in E(z) \setminus \{0\}$; and $\lim_{n\to\infty}(1/n) \log \|\mathcal{L}_z^{(n)}|_{F(z)}\| = \lambda_2$.

A one sentence summary of this is that over a.e. z, B decomposes into two equivariant spaces of dimension M and co-dimension M on which the growth rates of the operator cocycle are λ_1 and λ_2 respectively.

Some of the terms appearing in the statement below of the main theorem will be defined in §3.

THEOREM 1. *Let π be a forward-looking factor map from an irreducible (two-sided) shift of finite type X to a sofic shift Z and let $\phi \in C^\beta(X^+)$. Let $\pi = \pi_2 \circ \pi_1$ be the factorization described above where $\pi_1\colon X \to Y$ is of class degree one and map $\pi_2\colon Y \to Z$ has degree equal to the class degree of π.*

Let ν be a fully supported ergodic invariant measure on Z and let $\nu_1, \ldots, \nu_k$ be the lifts of ν under π_2 with multiplicities $m_1, \ldots, m_k$ respectively. Let $r_{\pi_1}(\nu_i)$ be the relative pressure of ϕ over ν_i and $r_\pi(\nu)$ be the relative pressure of ϕ over ν.

Let $(\mathcal{L}_z^{(n)})$ be the cocycle of linear operators over Z acting on $C^\beta(X^+)$ described above. Then the largest Lyapunov exponent of the cocycle is $r_\pi(\nu)$ and the multiplicity of this exponent is

$$\sum_{r_{\pi_1}(\nu_i)=r_\pi(\nu)} m_i.$$

In the case where ϕ is locally constant, we can give a more tangible description of this multiplicity as the multiplicity of the leading exponent of an explicit matrix cocycle.

COROLLARY 2. *Let π, X, Z and ν be as in the statement of Theorem 1 and suppose additionally that ϕ is locally constant. Then the action of $\mathcal{L}_z^{(n)}$ restricts to the space of locally constant functions (constant on cylinders of the same length). The multiplicity of the leading exponent of this cocycle is the same as that of the cocycle in Theorem 1.*

In the proof of this corollary, we assume (without loss of generality) that X is a 1-step shift of finite type and $\phi(x)$ depends only on x_0. In this case, the locally constant functions in the proof also depend only on the zeroth coordinate. It is then straightforward to write down the matrix cocycle representing the action of $\mathcal{L}_z^{(n)}$ on these functions.

2. *An example*

In this section, we give an example illustrating the objects appearing in the theorem and the corollary. The example is mostly self-contained, but we rely on one fact from the proof of the main theorem while presenting the example. In fact, where the corollary would give a cocycle of 3×3 matrices, we are able to exploit some symmetries to build instead a cocycle of 2×2 matrices. Let $X = \{0, 1, 2\}^{\mathbb{Z}}$ and $Z = \{F, N\}^{\mathbb{Z}}$. The factor map is defined by $\pi(x)_j = F$ (for flip) if x_j and x_{j+1} have opposite parities and $\pi(x)_j = N$ if x_j and x_{j+1} have the same parity (modulo 2). We define the potential ϕ by $\phi(x) = x_0$. For any $z \in Z$, the preimage set $\pi^{-1}\{z\}$ consists of two classes, one consisting of points where x_0 is 0 or 2 and the other consisting of points with $x_0 = 1$. These two classes are *mutually separated*: at each $j \in \mathbb{Z}$, points in one class have even symbols while points in the other class have odd symbols, so that π has class degree 2. A suitable factorization of π into the composition of a map of class degree 1 and a map of degree 2 is given by $\pi = \pi_2 \circ \pi_1$, where $Y = \{0, 1\}^{\mathbb{Z}}$, $\pi_1(x)_i = x_i \bmod 2$ and π_2 is $\pi|_Y$. To see that π_1 is of class degree 1, notice that if $x, x' \in \pi_1^{-1}(y)$, then the hybrid point $\bar{x}$ agreeing with x on symbols up to the $(k-1)$st and agreeing with x' thereafter belongs to X and is a preimage of y, so that x transitions to x' for any two elements of $\pi_1^{-1}(y)$ (as defined in §3). To see that π_2 has degree 2, notice that each $z \in Z$ has exactly two preimages, one the image of the other under the map $r: Y \to Y$ given by $r(y)_j = 1 - y_j$.

If μ_p is the Bernoulli measure on Y with 0's with probability p and 1's with probability $1-p$, then $\mu_p \circ \pi_2^{-1} = \mu_{1-p} \circ \pi_2^{-1}$ (this follows from the facts that $\pi_2 \circ r = \pi_2$ and $\mu_p \circ r^{-1} = \mu_{1-p}$). We let $\nu_p = \mu_p \circ \pi_2^{-1}$. It is not hard to see that if $p = \frac{1}{2}$, then ν_p is the uniformly distributed Bernoulli measure on Z. However, for $p \neq \frac{1}{2}$, the measure ν_p is a Hidden Markov Measure where there is long range dependence between the entries (see for example [5]).

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We then look at the equilibrium states on X for ϕ relative to the factor ν_p on Z. We find these by first understanding the lifts of ν_p to Y under π_2. The ergodic lifts of ν_p to Y are μ_p and μ_{1-p}, each with multiplicity 1 in the case $p \neq \frac{1}{2}$; and $\mu_{1/2}$ with multiplicity 2 in the case $p = \frac{1}{2}$. To understand this, notice that a typical point of $\nu_{1/2}$ has two preimages, both generic for the Bernoulli measure $\mu_{1/2}$ on Y where each is the image of the other under r.

To find the relative equilibrium states for ϕ over (Z, ν_p) with the factor map π, we then look for the relative equilibrium states of ϕ over (Y, μ_p) and (Y, μ_{1-p}) with the factor map π_1. By [1], since π_1 has class degree 1, there is a unique relative equilibrium state of ϕ over (Y, μ_p) for each p. The relative pressures with the factor map π_1 over (Y, μ_p) and (Y, μ_{1-p}) are $p \log(1 + e^2) + (1 - p)$ and $(1 - p) \log(1 + e^2) + p$. To see this, notice that to lift (Y, μ_p), the symbol 0 in Y with probability p is to be split into two states 0 and 2. Choosing 0 with probability $p/(1 + e^2)$ and 2 with probability $pe^2/(1 + e^2)$ maximizes the relative pressure. For $p = \frac{1}{2}$, the relative equilibrium state is a lift of $\mu_{1/2}$ to X under π_1. The relative equilibrium state is the Bernoulli measure with 0, 1 and 2 having weights $\frac{1}{2}/(1 + e^2)$, $\frac{1}{2}$ and $\frac{1}{2}e^2/(1 + e^2)$ respectively. The relative pressure is $\frac{1}{2}(1 + \log(1 + e^2))$.

Since it is not easy to directly compute exponents of Perron–Frobenius operator cocycles, we identify a finite-dimensional space V of piecewise constant functions, invariant under the cocycle, and do computations there. That this is possible is because the function ϕ is piecewise constant.

We also need to see why the growth rates appearing in the subspace V are the maximal growth rates in the full Banach space. This follows since V intersects each of the cones $C_a \cap S_P$ appearing in Lemma 12 for P satisfying the conditions appearing in Lemma 13.

Let the two-dimensional space V be the collection of functions on X, constant on cylinders of length 1, with the property that the value on the [0] and [2] cylinders are equal. We claim that both $\mathcal{L}_N$ and $\mathcal{L}_F$ map V onto itself. We represent an element f of V by a vector consisting of its values on the cylinder sets $[0] \cup [2]$ and [1] respectively. We then compute the action of $\mathcal{L}_N$ and $\mathcal{L}_F$ on V as follows.

We have

$$\mathcal{L}_N f(x) = \begin{cases} e^{\phi(1x)} f(1x) & \text{if } x_0 = 1, \\ e^{\phi(0x)} f(0x) + e^{\phi(2x)} f(2x) & \text{if } x_0 \text{ is 0 or 2;} \end{cases}$$

$$\mathcal{L}_F f(x) = \begin{cases} e^{\phi(0x)} f(0x) + e^{\phi(2x)} f(2x) & \text{if } x_0 = 1, \\ e^{\phi(1x)} f(1x) & \text{if } x_0 \text{ is 0 or 2.} \end{cases}$$

Recalling that $\phi(x) = x_0$ and representing both f and its image by vectors in the order described above, we have $\mathcal{L}_N$ and $\mathcal{L}_F$ are represented on V by the matrices

$$A_N = \begin{pmatrix} 1 + e^2 & 0 \\ 0 & e \end{pmatrix} \quad \text{and} \quad A_F = \begin{pmatrix} 0 & e \\ 1 + e^2 & 0 \end{pmatrix}.$$

Let $A_z^{(n)}$ denote the cocycle over z generated by these matrices. If y is a μ_p-generic point, then $r(y)$ is μ_{1-p}-generic and $z = \pi_2(y) = \pi_2(r(y))$ is ν_p-generic. We can verify that if $y_0 \ldots y_n$ starts and ends with even symbols (which implies that $z_0 \ldots z_{n-1}$ has an even

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number of F's), then

$$A_z^{(n)} = \begin{pmatrix} (1+e^2)^E e^O & 0 \\ 0 & (1+e^2)^O e^E \end{pmatrix},$$

where E denotes the number of even symbols in $y_0 \ldots y_n$ and O is the number of odd symbols. Similarly, if $y_0 \ldots y_n$ begins and ends with odd symbols,

$$A_z^{(n)} = \begin{pmatrix} (1+e^2)^O e^E & 0 \\ 0 & (1+e^2)^E e^O \end{pmatrix}.$$

Finally, if $y_0 \ldots y_n$ begins with an even symbol and ends with an odd symbol, or begins with an odd symbol and ends with an even symbol, then $A_z^{(n)}$ is respectively

$$\begin{pmatrix} 0 & (1+e^2)^E e^O \\ (1+e^2)^O e^E & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & (1+e^2)^O e^E \\ (1+e^2)^E e^O & 0 \end{pmatrix}.$$

In all of these cases, we see that the exponential growth rates of the matrix product along the orbit (and hence of the restriction of the Perron–Frobenius cocycle to V) are $\log((1+e^2)^p e^{1-p})$ and $\log((1+e^2)^{1-p} e^p)$ as computed above. In the case where $p = \frac{1}{2}$, the two exponents are both equal to $\log((1+e^2)^{1/2} e^{1/2})$ as expected.

3. *Background*

In this section, we collect a number of theorems and definitions that we will need for the proof, as well as setting out a number of related articles in the literature.

If (Z, S) is a subshift, $\mathsf{A}(Z)$ is its *alphabet* (so that $Z \subset \mathsf{A}(Z)^{\mathbb{Z}}$) and $\mathsf{L}(Z)$ denotes its *language*, that is, the set of all finite strings that appear in points of Z. A point $z \in Z$ is said to be *right transitive* if $\{S^n(z) : n \geq 0\}$ is dense in Z.

If $\pi : X \to Z$ is a factor map from a shift of finite type to a sofic shift and ν is an ergodic invariant measure on Z, Petersen, Quas and Shin [10] established that the collection of ergodic invariant measures of relative maximal entropy is finite. These measures are the relative equilibrium states in the case where the potential function ϕ is taken to be 0. In the case where the factor map π is a one-block map (that is $\pi(x)_0$ depends only on x_0), they established that the number of ergodic measures of relative maximal entropy over any ergodic invariant measure ν on Z is bounded above by $\min_{j \in \mathsf{A}(Z)} |\rho^{-1}(j)|$, where ρ is the symbol map giving rise to π. This result shows that, in particular, the number of these measures is finite. The bound suffers from a failure to be invariant under conjugacies. This deficiency was remedied and the bound improved in the paper [3] of Allahbakshi and Quas, some ideas from which will play an important role here.

For $z \in Z$, if $x, x' \in \pi^{-1}z$, we say that x *transitions to* x', and write $x \to x'$, if for all n, there exists $\bar{x} \in \pi^{-1}z$ such that $\bar{x}_{-\infty}^n = x_{-\infty}^n$ and $\bar{x}_m = x'_m$ for all sufficiently large m. We then define an equivalence relation on $\pi^{-1}z$ by $x \leftrightarrow x'$ if $x \to x'$ and $x' \to x$. The equivalence classes are called *transition classes* (a pigeonhole argument using the finite type property shows there are finitely many transition classes). Let $\mathcal{T}(z)$ denote the collection of transition classes over z. The paper [3] establishes that the number of transition classes over any right transitive point $z \in Z$ is a constant c_π independent of z. This constant is called the *class degree* of π.

THEOREM 3. (Allahbakhshi and Quas [3]) *Let X be a shift of finite type and Z be a sofic shift. Let $\pi : X \to Z$ be a one-block factor map. There exists a word $W = w_0^{n-1}$ in $\mathsf{L}(Z)$, a position $0 \le l < n$, and a subset $B \subset \mathsf{A}(X)$ whose cardinality is the class degree c_π, so that for each element u_0^{n-1} of $\pi^{-1}(W)$, there is a word $v_0^{n-1} \in \pi^{-1}(W)$ such that $u_0 = v_0$, $u_{n-1} = v_{n-1}$ and $v_l \in B$.*

The number of measures of relative maximal entropy over ν is bounded above by c_π.

The bound on the number of measures of relative maximal entropy was extended by Allahbakhshi, Antonioli and Yoo [1] to the number of relative equilibrium states of a Hölder continuous (or Bowen) potential function.

In the situation described in the above theorem, W is called a *minimal transition block*; B is a set of *representatives* and the word u is said to be *routed through* v_l. (The minimality in the name refers to the fact that the set of representatives is as small as possible.) A pair of elements x, x' of X is said to be *mutually separated* if $x_n \neq x'_n$ for each n. Two subsets S_1 and S_2 of X are mutually separated if for each $x \in S_1$ and $x' \in S_2$, x and x' are mutually separated.

THEOREM 4. (Allahbakhshi, Hong and Jung [2]) *Let $\pi : X \to Z$ be a one-block factor map from an irreducible two-sided shift of finite type X to a two-sided sofic shift Z. If $z \in Z$ is right transitive, then the elements of $\mathcal{T}(z)$ are mutually separated. In particular, for each copy of W in z, there exists a bijection between $\mathcal{T}(z)$ and B so that for each $C \in \mathcal{T}(z)$, there exists a representative $s \in B$ such that each $x \in C$ may be routed through s over that copy of W and through no other element of B.*

THEOREM 5. (Yoo [12]) *Let X be an irreducible two-sided shift of finite type, Z a two-sided sofic shift, and $\pi : X \to Z$ be a continuous factor map. Then there is a sofic shift Y and factorization of $\pi : X \to Z$ as a composition of factor maps, $\pi_2 \circ \pi_1$ where $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$ with the properties that π_2 is finite-to-one of degree c_π, the class degree of π and π_1 is of class degree 1.*

THEOREM 6. (Yoo [13]) *Let π be a finite-to-one continuous factor map from a homeomorphism S of a compact metric space Y to a homeomorphism T of a compact metric space Z. Suppose that ν is an ergodic T-invariant measure. Then:*

(1) *there exists $d \in \mathbb{N}$ such that for ν-a.e. $z \in Z$, $|\pi^{-1} z| = d$;*

(2) *there are only finitely many ergodic measures $\mu_1, \ldots, \mu_k$ on Y such that $\pi_* \mu_i = \nu$ and $k \le d$; if ν is fully supported, then so are the $\mu_1, \ldots, \mu_k$;*

(3) *there exist multiplicities $m_1, \ldots, m_k$ whose sum is d.*

In the case where Y is a shift space and π is a one-block map, there exists a joining $\bar{\mu}$ on Y^d such that for $\bar{\mu}$-a.e. $(y^1, \ldots, y^d)$, $\pi(y^1) = \cdots = \pi(y^d)$; the y^i are mutually separated; and $y^{M_i+1}, \ldots, y^{M_i+m_i}$ are generic for μ_i, where $M_i = m_1 + \cdots + m_{i-1}$.

The joining $\bar{\mu}$ constructed in the above theorem is called an *ergodic degree joining*.

The following theorem gives a criterion for simplicity of the top Lyapunov exponent of an operator cocycle based on contraction of cones and Birkhoff's theorem on contraction

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of the Hilbert metric. Recall that a *cone* is a closed subset C of a real Banach space B that is closed under addition and scalar multiplication by a non-negative real number.

For $f, g \in C$, let $m(f, g) = \sup\{t \geq 0\colon f - tg \in C\}$ and let $M(f, g) = \inf\{s \geq 0\colon sg - f \in C\}$. The projective distance between two points in the cone is defined as $\Theta_C(f, g) = \log(M(f, g)/m(f, g))$. (Note that this is not a metric as it may be infinite; also $\Theta_C(\beta f, \gamma g) = \Theta_C(f, g)$ for all $\beta, \gamma > 0$.) The *diameter* of a subset S of C is $\sup_{f,g\in S\setminus\{0\}} \Theta_C(f, g)$. A cone is said to be *D-adapted* if whenever $f \in B$ and $g \in C$, then $g \pm f \in C$ implies $\|f\| \leq D\|g\|$.

THEOREM 7. (Horan [6, Theorem 2.14]) *Let Y be a compact metric space and $S\colon Y \to Y$ be a continuous invertible transformation. Let ν be an ergodic S-invariant Borel probability measure on Y. Let B be a Banach space and let C be a D-adapted cone in B such that $C - C = B$, $C \cap (-C) = \{0\}$.*

Suppose that for each $y \in Y$, $\mathcal{L}_y$ is a linear operator from B to B such that $y \mapsto \mathcal{L}_y$ is continuous (where the linear operators on B are equipped with the norm topology), that $\mathcal{L}_y(C) \subset C$ for each y and that there is a measurable subset $A \subset Y$ with $\nu(A) > 0$ and an $n > 0$ such that $\operatorname{diam}(\mathcal{L}_y^{(n)}C) < \infty$ for all $y \in A$.

Then the leading Lyapunov exponent of the cocycle $(\mathcal{L}_y^{(n)})_{y\in Y}$ is simple. That is there exist $\alpha > \beta$, a measurable function $v\colon Y \to B$ and a measurable function $\psi\colon Y \to B^*$ such that $\mathcal{L}_y(v(y))$ is a multiple of $v(S(y))$; $(1/n) \log \|\mathcal{L}_y^{(n)} v(y)\| \to \alpha$ a.e.; and $\limsup_{n\to\infty}(1/n) \log \|\mathcal{L}_y^{(n)} w\| \leq \beta$ whenever $w \in \ker \psi(y)$.

This theorem should be thought of as a skew product version of the Perron–Frobenius theorem.

We will use the relative variational principle of Ledrappier and Walters [7]. Recall the Bowen definition of pressure:

$$P(\phi) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} e^{S_n\phi(x)},$$

where the supremum is taken over (n, ϵ)-separated sets, that is, sets E such that for any distinct elements x, x' of E, there is $0 \leq j < n$ such that $d_\beta(T^j x, T^j x') \geq \epsilon$. In the case of shift spaces, this may be simplified, fixing ϵ to be 1 and taking E to be any set consisting of exactly one point in each cylinder set of length n (so that E has the same cardinality as $\mathcal{L}_n(X)$). For symbolic systems,

$$P(\phi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in E} e^{S_n\phi(x)},$$

where E is any set with one representative of each cylinder set of length n. This definition is further refined by restricting the elements of E to lie in a fixed subset K:

$$P(\phi, K) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in E\subset K} e^{S_n\phi(\omega)},$$

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where E is any maximal $(n, 1)$-separated collection of points of K. We define $p_n(\phi, K) = \sup_{E\subset K;\ (n,1)\text{-separated}} \sum_{\omega\in E} e^{S_n\phi(\omega)}$ so that

$$P(\phi, K) = \limsup_{n\to\infty} \frac{1}{n} \log p_n(\phi, K).$$

THEOREM 8. (Relative variational principle) *Let $T: X \to X$ and $S: Y \to Y$ be continuous dynamical systems on compact spaces; let ν be an ergodic invariant measure for S and let $\pi: X \to Y$ be a continuous factor map from (X, T) to (Y, S). Then for ν-a.e. y, $P(\phi, \pi^{-1}y) = r_\pi(\nu)$.*

4. *Proofs*

In this section, we start with some preliminary lemmas and then establish Theorem 14 (which is the special case of the main theorem in the case where π has class degree 1), before using it to prove the main theorem.

The proof structure is as follows. We start with a factor map $\pi: X \to Z$ and an ergodic invariant measure ν on Z. Given a ν-typical point $z \in Z$, its preimages in X can be separated into a number of transition classes as described in the previous section. Those results show that one can associate pressure-maximizing measures on X to these classes, and that ν-a.e. z gives rise to the same collection of measures on X. Theorem 14 deals with the case where the class degree is 1 (so there is a single transition class). Some preparatory lemmas show that the cocycle of operators maps a family of cones inside itself, and from time to time maps a cone in the family into a finite diameter sub-cone of the cone in the family. This allows us to apply Theorem 7 showing that there is a simple leading Lyapunov exponent. A calculation shows that this exponent is the quantity appearing on the left side of the equality in the relative variational principle (while the conclusion of Theorem 14 is that the exponent is the right side of the equality). To deal with the case of class degree greater than 1, we express π as $\pi_2 \circ \pi_1$ as in Theorem 5, and express the Perron–Frobenius cocycle as a sum of non-interacting cocycles, each of which satisfies the hypotheses of Theorem 14, with one summand per transition class.

Finally, in the case where ϕ is locally constant, there is a corresponding family of locally constant functions that is mapped into itself by the Perron–Frobenius cocycle. We show that this family intersects each of the cones described above, so that the multiplicity of the top Lyapunov exponent is captured by the action on this finite-dimensional subspace.

For this section, let ϕ be a fixed Hölder continuous function. Given $\beta < 1$, we define a semi-norm on $C^\beta(X^+)$ by $|f|_\beta = \sup_{x\neq x'} |f(x) - f(x')|/d_\beta(x, x')$ (that is, the Lipschitz constant of f with respect to d_β) and a norm by $\|f\|_\beta = \max(\|f\|_\infty, |f|_\beta)$. Let β be such that $\|\phi\|_\beta < \infty$. This quantity will be fixed from here on. We also assume throughout this section that the factor map π is a one-block map as this is the context in the proof of the main theorem.

We define a family of cones, one for each real $a > 0$, by

$$C_a = \{f \in C^\beta(X^+):\ f \geq 0;\ f(x') \leq e^{ad_\beta(x,x')} f(x) \text{ whenever } x_0 = x'_0\}.$$

These cones are widely used in symbolic dynamics and appear, for instance, in the work of Parry and Pollicott [9], although our usage differs slightly as we do not impose any

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condition on $f(x)/f(x')$ when $x_0 \neq x_0'$. This is important for us, since some operators that we consider yield functions that are 0 on part of X^+.

LEMMA 9. *Let a be large enough that* $b := \beta(a + |\phi|_\beta) < a$. *Then* $\mathcal{L}_j C_a \subset C_b$ *for each* $j \in \mathsf{A}(Y)$.

Proof. Let $f \in C_a$. For each symbol $i \in \mathsf{A}(X)$, set $\tilde{\mathcal{L}}_i f(x) = e^{\phi(ix)} f(ix)$. Suppose x and x' agree for n symbols for some $n \geq 1$ and suppose $f(ix') > 0$ (so that $f(ix) > 0$ also). Then

$$\frac{\tilde{\mathcal{L}}_i f(x')}{\tilde{\mathcal{L}}_i f(x)} = \frac{e^{\phi(ix')} f(ix')}{e^{\phi(ix)} f(ix)}$$
$$\leq e^{|\phi|_\beta \beta^{n+1}} e^{a\beta^{n+1}}$$
$$\leq e^{\beta(|\phi|_\beta + a) d_\beta(x,x')} = e^{b d_\beta(x,x')}.$$

Since for $j \in \mathsf{A}(Y)$, $\mathcal{L}_j = \sum_{i \in \pi^{-1} j} \tilde{\mathcal{L}}_i$ (where $\pi^{-1} j$ denotes the symbols in $\mathsf{A}(X)$ that map to j under the alphabet map defining π), the result follows. □

LEMMA 10. *For* $f \in C_a$, $\|f\|_\beta \leq \max(3, 1 + ae^a)\|f\|_\infty$. *It follows that* C_a *is D-adapted with* $D = \max(6, 2 + 2ae^a)$.

Proof. Let $f \in C_a$. If $x, x' \in X^+$ have different initial symbols, then $|f(x) - f(x')| \leq 2\|f\|_\infty \leq \max(2, ae^a)\|f\|_\infty d_\beta(x, x')$. If they have the same initial symbol, then $|f(x) - f(x')| \leq |f(x)|(e^{ad_\beta(x,x')} - 1) \leq \|f\|_\infty ae^a d_\beta(x, x') \leq \max(2, ae^a)\|f\|_\infty d_\beta(x, x')$, where we used the mean value theorem for the second inequality. Hence $|f|_\beta \leq \max(2, ae^a)\|f\|_\infty$ so that $\|f\|_\beta \leq \max(3, 1 + ae^a)\|f\|_\infty$.

For the second statement in the lemma, $g \pm f \in C_a$ implies $\|f\|_\infty \leq \|g\|_\infty$, so that $\|g \pm f\|_\infty \leq 2\|g\|_\infty$ and $\|g \pm f\|_\beta \leq \max(6, 2 + 2ae^a)\|g\|_\infty \leq \max(6, 2 + 2ae^a)\|g\|_\beta$. Subtracting $g - f$ from $g + f$, we obtain the desired bound. □

For these cones, we have the following lemma (which can be seen as a special case of a result of Andô [4]). Expressing arbitrary Hölder continuous functions as a difference of elements of the cone will allow us to prove the simplicity of the top Lyapunov exponent.

LEMMA 11. *For all* $f \in C^\beta(X^+)$, *there exist* $g, h \in C_a$ *with* $\|g\|_\beta, \|h\|_\beta \leq (2 + 1/a)\|f\|_\beta$ *such that* $f = g - h$.

Proof. Let $f \in C^\beta(X^+)$, let $g = f + (1 + 1/a)\|f\|_\beta$ and $h = (1 + 1/a)\|f\|_\beta$. Clearly $h \in C_a$. Notice that $\min g \geq (1/a)\|f\|_\beta$, so that

$$\frac{g(x)}{g(x')} = 1 + \frac{g(x) - g(x')}{g(x')} \leq 1 + \|f\|_\beta d_\beta(x, x')/(\|f\|_\beta/a)$$
$$= 1 + ad_\beta(x, x') \leq e^{ad_\beta(x,x')}.$$

In particular, $g \in C_a$ and $\|g\|_\beta, \|h\|_\beta$ are bounded above by $(2 + 1/a)\|f\|_\beta$. □

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LEMMA 12. *Let $0 < b < a$ and $A \geq 1$. Let P be a non-empty subset of the alphabet of X. Write $[P] = \bigcup_{j \in P}[j]$ and*

$$S_P = \{f : f(x) > 0 \text{ iff } x \in [P];\ f(x) \leq Af(x') \text{ for all } x, x' \in [P]\}.$$

Then there exists $K > 0$ such that $\Theta_{C_a}(f, g) \leq K$ for all $f, g \in S_P \cap C_b$.

The conclusion here states that the diameter of the set is finite. This is a key hypothesis in Birkhoff's cone contraction argument.

Proof. Let $t > 0$ be chosen sufficiently small to ensure that $2btA/(1 - tA) \leq a - b$. Let $f, g \in S_P \cap C_b$. Using the scale-homogeneity of Θ_{C_a}, we may scale f and g so that $\min_{[P]} f = \min_{[P]} g = 1$, and hence $\max f, \max g \leq A$.

We claim that $f - tg \in C_a$. Let $x, x' \in X^+$ have a common first symbol belonging to P (if x, x' have a common first symbol outside P, then $(f - tg)(x)$ is trivially bounded above by $e^{ad_\beta(x,x')}(f - tg)(x')$ since both of these quantities are zero). We have

$$\begin{aligned}
\frac{f(x') - tg(x')}{f(x) - tg(x)} &\leq \frac{e^{bd_\beta(x,x')} f(x) - te^{-bd_\beta(x,x')} g(x)}{f(x) - tg(x)} \\
&= e^{bd_\beta(x,x')} + \frac{t(e^{bd_\beta(x,x')} - e^{-bd_\beta(x,x')})g(x)}{f(x) - tg(x)} \\
&\leq e^{bd_\beta(x,x')} \left(1 + \frac{At}{1 - At}(1 - e^{-2bd_\beta(x,x')})\right) \\
&\leq e^{bd_\beta(x,x')} \left(1 + \frac{2Atb}{1 - At} d_\beta(x, x')\right) \\
&\leq e^{bd_\beta(x,x')} e^{(a-b)d_\beta(x,x')} = e^{ad_\beta(x,x')},
\end{aligned}$$

so that $f - tg \in C_a$. By symmetry, $g - tf \in C_a$, or equivalently $(1/t)g - f \in C_a$. Hence $\Theta_{C_a}(f, g) \leq \log(1/t^2)$ for all $f, g \in S_P \cap C_b$. □

LEMMA 13. *Let $\pi : X \to Y$ be a factor map of class degree 1. Let $W = w_0^{n-1}$ be a minimal transition block in Y. Then there exists an $A \geq 1$ such that for any $y \in [W]$, and any $f \in C_a$, $\mathcal{L}_y^{(n)} f \in S_P \cup \{0\}$, where S_P is the set in Lemma 12 (whose definition involves the constant A) and P is $\{j \in \mathsf{A}(X) :$ there exists $U \in \pi_b^{-1}(W) : Uj \in \mathsf{L}(X)\}$.*

Proof. Since $S_P \cup \{0\}$ is closed under addition and $\mathcal{L}_y^{(n)}$ is linear, it suffices to prove the statement for a function f supported on a single cylinder set. Suppose that f is supported on $[k]$. If there is no preimage of W whose initial symbol is k, we see that $\mathcal{L}_y^{(n)} f = 0$ since there are no positive summands. Suppose however that U is a preimage of W under π starting with a k. Let $j \in P$ and let V be a preimage of W under π such that $Vj \in \mathsf{L}(X)$. Since W is a minimal transition block with a single representative, there exists a preimage U' of W starting with the first symbol of U and ending with the last symbol of V. If $x \in [j]$, we now calculate

$$\mathcal{L}_y^{(n)} f(x) \geq e^{S_n\phi(U'x)} f(U'x) \geq e^{n \min \phi} \|f\|_\infty / e^a.$$

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However, it is clear that $\mathcal{L}_y^{(n)} f(x) \le |\mathsf{A}(X)|^n e^{n \max \phi} \|f\|_\infty$ for any $x \in X$. Hence we have demonstrated the hypothesis of Lemma 12 is satisfied with $A = e^{a+n(\max \phi - \min \phi)} |\mathsf{A}(X)|^n$. □

We point out that the idea of studying the Ruelle–Perron–Frobenius cocycle over a factor Y and expressing the operators corresponding to symbols in $\mathsf{A}(Y)$ as sums of operators indexed by symbols in $\mathsf{A}(X)$, as well as some of the cones that we study here and the description of S_P above, appear in the work of Piraino [11].

THEOREM 14. (Main theorem, class degree 1 case) *Let X be an irreducible shift of finite type, let $\pi : X \to Y$ be a forward-looking factor map of class degree 1 and let ϕ be a Hölder continuous function on X^+. Suppose ν is a fully supported invariant measure on Y. Then the cocycle $(\mathcal{L}_y^{(n)})$ has a simple top Lyapunov exponent, whose value is $r_\pi(\nu)$, the relative pressure of ϕ over ν.*

Further, for ν-a.e. y,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_y^{(n)} \mathbf{1}\| = \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_y^{(n)} \mathbf{1}_{\pi^{-1}[y_0]}\| = r_\pi(\nu).$$

Proof. By conjugating X and Y if necessary, we may assume that π is a one-block map. This does not affect any of the hypotheses or conclusions of the theorem (see [8, Proposition 1.5.12] and [3] for more details). Let β be such that $\|\phi\|_\beta < \infty$ and let a satisfy the hypothesis in Lemma 9. Let W be a minimal transition block for the factor map $\pi : X \to Y$. By Lemma 11, $C_a - C_a = C^\beta(X^+)$. By Lemmas 9, 12 and 13, we see that $\mathcal{L}_y^{(|W|)} C_a$ is a finite diameter subset of C_a for any $y \in [W]$. Since the hypotheses of Theorem 7 are satisfied (the continuity of $y \mapsto \mathcal{L}_y$ is because the map is piecewise constant and the D-adaptedness condition on C_a is satisfied by the second statement of Lemma 10), the top Lyapunov exponent of the cocycle $(\mathcal{L}_y^{(n)})$ acting on $C^\beta(X^+)$ is simple.

Notice that for $y \in Y$ and $g \in C_a$,

$$\mathcal{L}_y^{(n)} g(x) = \sum_{W \in \pi^{-1}(y_0^{n-1}) :\ W x_0 \in \mathsf{L}(X)} e^{S_n \phi(Wx)} g(Wx),$$

so that $\|\mathcal{L}_y^{(n)} g\|_\infty$ is bounded above by $p_n(\phi, \pi^{-1} y)\|g\|_\infty$. By Lemmas 9 and 10, $\|\mathcal{L}_y^{(n)} g\|_\beta$ is bounded above by $\max(3, 1 + e^a) p_n(\phi, \pi^{-1} y)\|g\|_\infty$. If $f \in C^\beta(X^+)$, using Lemma 11, we may write f as the difference $g - h$ with $g, h \in C_a$, each of $\|\cdot\|_\beta$ norm at most $(2 + 1/a)\|f\|_\beta$. Hence $\|\mathcal{L}_y^{(n)} f\|_\beta \le 2(2 + 1/a) \max(3, 1 + e^a) p_n(\phi, \pi^{-1} y)\|f\|_\beta$. As noted above, we have $\limsup_{n\to\infty} (1/n) \log p_n(\phi, \pi^{-1} y) = P(\phi, \pi^{-1} y)$, which is $r_\pi(\nu)$ by Theorem 8, so that the top Lyapunov exponent is bounded above by $r_\pi(\nu)$.

For the converse inequality, let $(x^i)_{i \in \mathsf{A}(X)}$ be a collection of points in X^+, where x^i starts with the symbol i. Now

$$\sum_{i \in \mathsf{A}(X)} \mathcal{L}_y^{(n)} \mathbf{1}(x^i) \ge e^{-c} p_n(\phi, \pi^{-1} y),$$

where c is a constant independent of n, where $|S_n\phi(x) - S_n\phi(x')| \leq c$ whenever $x_0^{n-1} = x'{}_0^{n-1}$ (such a c exists since ϕ is Hölder). Then

$$\|\mathcal{L}_y^{(n)}\mathbf{1}\|_\beta \geq \|\mathcal{L}_y^{(n)}\mathbf{1}\|_\infty \geq \frac{e^{-c}}{|\mathsf{A}(X)|} p_n(\phi, 1, \pi^{-1}y).$$

In particular, for ν-a.e. y, the limit superior growth rate of $\|\mathcal{L}_y^{(n)}\mathbf{1}\|_\beta$ is at least $r_\pi(\nu)$, as required. □

Proof of Theorem 1. We assume without loss of generality as above that π is a one-block map. Using Theorem 5, $\pi : X \to Z$ may be factorized as $\pi_2 \circ \pi_1$, where π_1 is of class degree 1 from X to a sofic shift Y; and π_2 is finite-to-one, and for ν-a.e. point, $\pi_2^{-1}(z)$ consists of c_π preimages.

We need a more precise description of the construction of the sofic shift Y and the factor code π_1, for which we will follow [12]. The space Y is built from a minimal transition block W in $\mathsf{L}(Z)$. Recall the representatives of the transition block are a subset B of $\mathsf{A}(X)$ of cardinality c_π such that if $\pi(x) \in [W]$, then x may be locally modified on the coordinates $(0, n-1)$ to give a point $x' \in X$ with $x'_l \in B$.

The alphabet of Y is then $\mathsf{A}(Z) \times (B \cup \{\star\})$. The factor map π_1 is defined as follows:

$$\pi_1(x)_m = \begin{cases} (\pi(x)_m, s), & \pi(x)_{m-l}^{m-l+n-1} = W, x_{m-l}^{m-l+n-1} \text{ routable through } s; \\ (\pi(x)_m, \star), & \pi(x)_{m-l}^{m-l+n-1} \neq W. \end{cases}$$

That is, $\pi_1(x)$ records the image in Z, together with the representatives in B through which the orbit of x may be routed each time that the orbit passes through a transition block. The factor map π_2 is the one-block factor map from Y to Z defined by the symbol map sending (a, b) to a for any $(a, b) \in \mathsf{A}(Z) \times (B \cup \{\star\})$.

We then define an operator cocycle over Y. For each $s \in B$, let $R_s \subset \mathsf{A}(X)$ be the collection of symbols in X that a preimage of W may pass through if it is routable through s. By Theorem 4, these sets are disjoint. Write $[R_s]$ for $\bigcup_{i \in R_s}[i]$ and let q be the lth symbol of W.

The generator of the cocycle is then defined by

$$\bar{\mathcal{L}}_{(j,\star)} f(x) = \mathcal{L}_j f(x) \quad \text{for } j \in \mathsf{A}(Z);$$
$$\bar{\mathcal{L}}_{(q,s)} f(x) = \mathcal{L}_q(\mathbf{1}_{[R_s]} f)(x) \quad \text{for } s \in B.$$

That is, each time $\pi(x)$ passes through a transition block, the operator projects to the part of the function routable through the specified representative. Note that $(q, \star) \in \mathsf{A}(Y)$, and this appears in the image of points under π_1 for points in X whose symbol maps to q, but where the word $\pi(x)_{m-l}^{m-l+n-1}$ is not equal to W.

By Theorem 6, there are finitely many ergodic invariant measures on Y projecting to ν, say $\mu_1, \ldots, \mu_k$, each fully supported; as well as multiplicities $m_1, \ldots, m_k$ summing to c_π such that a ν-generic $z \in Z$ has m_i μ_i-generic preimages under π_2 for each i, with the whole collection of c_π preimages mutually separated. Further, there exists an ergodic measure $\bar{\mu}$ on Y^{c_π} where $\bar{\mu}$-a.e. point is supported on the c_π preimages of some point $z \in Z$; the first m_1 being generic points for the ergodic measure μ_1 on Y; the next

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m_2 being generic for the measure μ_2 etc. We assume without loss of generality that $r_{\pi_1}(\mu_1) \geq r_{\pi_1}(\mu_2) \geq \cdots$, where $r_{\pi_1}(\mu_i)$ is the π_1-relative pressure of ϕ over μ_i; and that the maximal value of $r_{\pi_1}(\mu_i)$ is attained for $i = 1, \ldots, p$ (but not for $i = p+1, \ldots, k$). Notice that $r_{\pi_1}(\mu_i) = r_\pi(\nu)$ for $i = 1, \ldots, p$ since any ergodic measure on X in $\pi_*^{-1}\{\nu\}$ lies in one of the ${\pi_1}_*^{-1}\{\mu_i\}$ for some μ_i; and π_2 is finite-to-one, so does not decrease entropy.

Since $\pi_1 \colon X \to Y$ and each μ_i satisfies the conditions of Theorem 14, we see that there is a simple top exponent λ_i for the cocycle $(\bar{\mathcal{L}}_y^{(n)})_{y\in Y}$ for each of the measures μ_i. The set of $y \in Y$ for which the exponent λ_i is achieved and for which the second Lyapunov exponent is strictly smaller is a collection of full μ_i-measure.

For $\bar{\mu}$-a.e. $(y^1, \ldots, y^{c_\pi}) \in Y^{c_\pi}$, the simple top exponent of the cocycle $(\bar{\mathcal{L}}_y)^{(n)}$ is λ_i for each $y = y^{M_i+k}$ with $k = 1, \ldots, m_i$ (where $M_i = m_1 + \cdots + m_{i-1}$ and $M_1 = 0$). In particular, the top exponent of the cocycle is almost surely simple with exponent $\lambda_1 = r_{\pi_1}(\mu_1)$ over each of $y^1, \ldots, y^{m_1+\cdots+m_p}$ and strictly smaller for the other y's.

We now derive a relationship between the cocycle $(\mathcal{L}_z^{(n)})$ over Z and the cocycle $(\bar{\mathcal{L}}_y^{(n)})$ over Y. Recall that for $\bar{\mu}$-a.e. $\bar{y} = (y^1, \ldots, y^{c_\pi})$, one has the equality $\pi(y^1) = \cdots = \pi(y^{c_\pi})$. Write $\bar{\pi}(\bar{y})$ for this common value. Next, we claim that for $\bar{\mu}$-a.e. $\bar{y}$,

$$\mathcal{L}_{\bar{\pi}(\bar{y})}^{(n)} f(x) = \sum_{i=1}^{c_\pi} \bar{\mathcal{L}}_{y^i}^{(n)} f(x) \tag{1}$$

for all n such that $\bar{\pi}(\bar{y})_0^{n-1}$ contains a copy of W, the minimal transition block used in the definition of π_1.

To see this, notice that if $q = w_l$ is the symbol in W over which the representatives lie, then the following identities hold:

$$\mathcal{L}_q = \sum_{i\in S} \bar{\mathcal{L}}_{q,i};$$
$$\mathcal{L}_j = \bar{\mathcal{L}}_{j,\star} \quad \text{for each } j \in \mathsf{A}(Z) \text{ (including } q).$$

So $\mathcal{L}_z^{(n)}$ is a composition in which a number of the terms (those occurring when z contains a copy of W) are replaced by a sum of $\bar{\mathcal{L}}_{q,i}$. Since the $\bar{\mathcal{L}}$ are linear, we may distribute the composition over the sum. Since when z is right transitive, the transition classes are mutually separated (Theorem 4), almost all of the terms in the summation vanish; the only ones that survive are those in which the choices of representative are consistent: the representative over one instance of W together with the point z determines the representative over all of the other instances of W by virtue of the mutual separation of the classes in $\pi^{-1}(z)$.

For $\bar{\mu}$-a.e. $\bar{y} = (y^1, \ldots, y^{c_\pi})$, each of the y^i terms is right transitive; and the map $\pi_1 \colon X \to Y$ is of class degree 1. The preimages $\pi_1^{-1}(y^i)$ for $i = 1, \ldots, c_\pi$ form the transition classes, $\mathcal{T}(\bar{\pi}(\bar{y}))$ in X over $\bar{\pi}(\bar{y}) \in Z$. By Theorem 14, applied to $\pi_2 \colon X \to Y$, for ν-a.e. $\bar{\pi}(\bar{y})$ and each y^i with $1 \leq i \leq m_1 + \cdots + m_p$, the cocycle $(\bar{\mathcal{L}}_{y^i}^{(n)})$ has an equivariant one-dimensional space of functions growing at rate $r_{\pi_1}(\mu_1)$. In particular, the functions $\mathbf{1}_{\pi_1^{-1}[y_0^i]}$ for $i = 1, \ldots, M_p$ grow at rate $r_{\pi_1}(\mu_1)$ under the respective cocycles

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$\bar{\mathcal{L}}^{(n)}_{y^i}$ (and are eventually annihilated by the other cocycles). Since the $\mathcal{T}(\bar{\pi}(\bar{y}))$ are mutually separated, for each n, the $\bar{\mathcal{L}}^{(n)}_{y^i}\mathbf{1}_{\pi_1^{-1}[y_0^i]}$ are disjointly supported. By (1), the span of $\{\mathbf{1}_{\pi_1^{-1}[y_0^i]}\colon 1 \le i \le M_p\}$ is an M_p-dimensional space of functions, where the entire space grows under the cocycle $\mathcal{L}^{(n)}_{\bar{y}}$ at rate $r_{\pi_1}(\mu_1) = r_\pi(\nu)$.

Theorem 14 implies that in any two-dimensional space of functions supported on $\pi_1^{-1}(y^i)$, there is a function whose growth rate is strictly smaller than $r_\pi(\nu)$.

However, by Theorem 14 for $i > M_p$, the growth rate on $\pi_1^{-1}(y^i)$ is at most $r_{\pi_1}(\mu_{p+1})$, which is strictly smaller. Combining these facts, it follows that the dimension of the fastest growing space is precisely $m_1 + \cdots + m_p$ as required. □

Proof of Corollary 2. For this proof, we assume that π is a one-block factor map, given by the map $\rho\colon \mathsf{A}(X) \to \mathsf{A}(Z)$, (as in the previous theorem) and the locally constant function $\phi(x)$ depends only on x_0. The key observation in this case is that $\mathcal{L}_z$ maps LC, the finite-dimensional subspace of functions depending only on the zeroth coordinate, onto itself.

Specifically, if f is the function taking the value f_i on the cylinder $[i]$, then

$$\mathcal{L}_z f(x) = \sum_{i:ix_0\in\mathsf{L}(X),\rho(i)=z_0} e^{\phi_i} f_i,$$

another function whose value is determined by x_0. That is, $\mathcal{L}_z$ is represented by the matrix with entries

$$(A_z)_{ij} = \mathbf{1}_{ij\in\mathsf{L}(X)}\mathbf{1}_{\rho(i)=z_0}e^{\phi_i}.$$

Let $\pi = \pi_1 \circ \pi_2$ as in the proof of Theorem 1, so that the symbol map ρ is the composition of maps ρ_1 and ρ_2. We use the notation of the proof of Theorem 1. Let $\bar{y} = (y^1, \ldots, y^{c_\pi})$ be a generic element of the degree joining, where we assume that $y^1, \ldots, y^{M_p}$ are generic for measures $\mu_1, \ldots, \mu_p$ with $r_{\pi_1}(\mu_i) = r_\pi(\nu)$ for $i = 1, \ldots, p$. Then we showed above that $\bar{\mathcal{L}}^{(n)}_{y_i}\mathbf{1}_{\pi_1^{-1}[y_0^i]}$ grows at rate $r_\pi(\nu)$ for $i = 1, \ldots, M_p$ and for each n, these functions are disjointly supported. Further, $\bar{\mathcal{L}}^{(n)}_{y_i}\mathbf{1}_{\pi_1^{-1}[y_0^i]} = \mathcal{L}^{(n)}_{\bar{\pi}(\bar{y})}\mathbf{1}_{\pi_1^{-1}[y_0^i]}$.

Since $\mathbf{1}_{\pi_1^{-1}[y_0^i]} \in \mathsf{LC}$, we see that the multiplicity of the exponent $r_\pi(\nu)$ for the matrix cocycle $A_z^{(n)}$ is at least M_p. However, multiplicity of the exponent $r_\pi(\nu)$ for the action of the cocycle on $C^\alpha(X^+)$ is an upper bound for the multiplicity on the subspace LC. Hence the multiplicity of the leading exponent for the matrix cocycle is exactly M_p as required. □

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REFERENCES

[1] M. Allahbakhshi, J. Antonioli and J. Yoo. Relative equilibrium states and class degree. *Ergod. Th. & Dynam. Sys.* **39** (2019), 865–888.

[2] M. Allahbakhshi, S. Hong and U. Jung. Structure of transition classes for factor codes on shifts of finite type. *Ergod. Th. & Dynam. Sys.* **35** (2015), 2353–2370.

[3] M. Allahbakhshi and A. Quas. Class degree and relative maximal entropy. *Trans. Amer. Math. Soc.* **365** (2013), 1347–1368.

[4] T. Andô. On fundamental properties of a Banach space with a cone. *Pacific J. Math.* **12** (1962), 1163–1169.

[5] D. Blackwell. The entropy of functions of finite-state Markov chains. *Transactions of the First Prague Conference on Information Theory, Statistical Decision Functions, Random Processes held at Liblice near Prague from November 28–30, 1956*. Ed. A. Spacek. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957, pp. 13–20.

[6] J. Horan. Asymptotics for the second-largest Lyapunov exponent for some Perron–Frobenius operator cocycles. *Nonlinearity* **34** (2021), 2563–2610.

[7] F. Ledrappier and P. Walters. A relativised variational principle for continuous transformations. *J. Lond. Math. Soc. (2)* **16** (1977), 568–576.

[8] D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.

[9] W. Parry and M. Pollicott. *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics (Astérisque, 187–188)*. Société Mathématique de France, Paris, 1990.

[10] K. Petersen, A. Quas and S. Shin. Measures of relative maximal entropy. *Ergod. Th. & Dynam. Sys.* **23** (2003), 207–223.

[11] M. Piraino. Projections of Gibbs states for Hölder potentials. *J. Stat. Phys.* **170** (2018), 952–961.

[12] J. Yoo. Decomposition of infinite-to-one factor codes and uniqueness of relative equilibrium states. *J. Mod. Dyn.* **13** (2018), 271–284.

[13] J. Yoo. Multiplicity structure of preimages of invariant measures under finite-to-one factor maps. *Trans. Amer. Math. Soc.* **370** (2018), 8111–8133.