FACTORIZATION OF MATRICES INTO MATRICES WITH PRESCRIBED EIGENVALUES

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ABSTRACT

Necessary and sufficient conditions that a singular square complex matrix can be written as a product of two square matrices with prescribed eigenvalues are given We will also use our results together with some old results to give new proofs or shorter proofs of some known theorems such as Radjavi's theorem about characterization of products of four Hermitian matrices and Wu's theorem about products of four positive semidefinite matrices

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DEDICATION

With grateful love

to

My Parents

and

Brother, Kuniyo

CHAPTER I

INTRODUCTION

1.1 The General Factorization Problems

It has always been of great interest to represent an arbitrary square matrix, or a matrix from a wide class of matrices, as a product of matrices of special type. In addition to its intrinsic interest, this factorization problem has connections with group theory and geometry. Some of the classical factorization results are the following:

(a) Every real orthogonal matrix is a product of reflections [4, Prop. 5, Ch. IX, §6, Sect. 4]

(b) Every invertible matrix with determinant 1 is a product of transvections [1, p. 163]

(c) The polar decomposition: Every complex matrix is a product of a unitary matrix and a positive semidefinite matrix [9, p. 343]

(d) The singular value decomposition: Every complex matrix is a product UDV where U and V are unitary and D is diagonal [10, p. 157, 414ff].

When the factors in the decomposition belong to a special class $\mathscr{C}$ of matrices, it is also instructive to determine the smallest number of factors in such a factorization. More precisely, if we denote the set of all products of a finite, but arbitrary, number of matrices from $\mathscr{C}$

by $\mathscr{C}^{\infty}$, it is of great interest to determine, for $T \in \mathscr{C}^{\infty}$, the length of T with respect to $\mathscr{C}$, denoted by $\ell(T,\mathscr{C})$ and defined as the smallest integer m such that T can be expressed as a product $T_1T_2 \cdots T_m$ with $T_j \in \mathscr{C}$ for every j. A related problem is to determine, for every positive integer m, the set of matrices, $\mathscr{C}^m$, which may be written as a product of m matrices from $\mathscr{C}$. Another length problem, which is often easier to determine than the aforementioned two, is to determine the length of $\mathscr{C}$, denoted by $\ell(\mathscr{C})$ and defined by $\ell(\mathscr{C}) = \sup\{\ell(T,\mathscr{C}) : T \in \mathscr{C}^{\infty}\}$. One classical result in this vein is that the length of the set of $n{\times}n$ reflections is n [1, p. 131].

In addition to the classical results stated above, quite a few results have been obtained in the last twenty years. Some of these results turned out to be quite elegant. For example, in 1969 [15], Radjavi proved that a complex matrix A is a product of four Hermitian matrices if and only if $\det A$ is real; he further proved that a complex matrix A is a product of two Hermitian matrices if and only if A is similar to A^* [16]. For more recent matrices factorization results, see [7, 12, 13, 18, 22, 23].

A survey of numerous matrix factorizations is given by Wu [24]. Classes of matrices considered in that paper include symmetric matrices, positive semidefinite matrices, involutions, and many others.

Although we only consider complex matrices in this thesis, one might be interested in factoring matrices over an arbitrary field (see [13, 18] for examples). Factorization of matrices over rings has also been considered by many authors (see [5, 6, 21] for examples).

Moreover, some are interested in factoring an integer matrix. For example, in [14], Laffey proved that if $n \geq 3$, then every $n \times n$ integer matrix with determinant zero is the product of $36n + 217$ idempotent integer matrices and $72n + 434$ nilpotent integer matrices.

1.2 Our Main Problem

In this thesis we restrict ourselves to complex square matrices even though some of the results are valid for matrices over arbitrary fields. We denote the set of all $n \times n$ complex matrices by M_n.

The main problem discussed below is the following: Given $A \in M_n$ and given complex numbers $b_1, \cdots, b_n$ and $c_1, \cdots, c_n$, can A be expressed as a product of two matrices B and C in M_n, such that B has eigenvalues $b_1, b_2, \cdots, b_n$ and C has eigenvalues $c_1, c_2, \cdots, c_n$? Here, the eigenvalues of matrices are always repeated according to their algebraic multiplicity, that is, their multiplicity as a root of the characteristic polynomial of the matrix. In the case where A is invertible and nonscalar, this problem has been solved by Sourour [18], who proved that as long as $b_1 b_2 \cdots b_n c_1 c_2 \cdots c_n = \det A$, such B and C exist. A more precise statement of this theorem is given in Chapter II. Now the natural question is 'what about when A is singular?' This will be our main concern in this thesis. It has been observed that the proof in Sourour's paper [18] establishes one result about factorization of singular matrices, namely that a sufficient condition for the existence of B and C as described above is that the number of zeros among the b_i's and c_i's be exactly equal to the nullity of A. This is given in the recent book by Horn and Johnson [11]. Another key result in this circle of ideas is due to Wu [22] who states that except for nonzero 2×2 nilpotent matrices, every complex singular matrix is a product of two nilpotent matrices. The above two results will serve as a tool to prove our main theorem. They will be fully stated in Chapter II, we will also outline the proofs of some of these theorems.

Chapter III contains our main theorem which gives a necessary and sufficient condition for factoring a singular matrix A as a product of two matrices with prescribed eigenvalues. The theorem basically says that as long as the number of zeros among the b_i's and c_i's is greater than or equal to the nullity of A, such B and C can be found. Since we are mainly using induction to prove our theorems, and also since 2×2 matrices are exceptional in as much as the general statement does not hold for them, we will spend one section to consider the 2×2 cases. In order to prove our main theorem, we are going to prove three propositions which can also be viewed as special cases to our main problem. The first proposition says that for a singular matrix A which is not a 2×2 nilpotent matrix, if those b_i's and c_i's contain an equal number of zeros, and the total number of zeros among the b_i's and c_i's is greater than or equal to the nullity of A, then we can find such B and C. The second proposition says that for a singular matrix A, if all the b_i's are zero and all the c_i's are nonzero, such B and C exist. The third proposition says that if all the c_i's are nonzero and if the number of zeros among the b_i's is greater than or equal to the nullity of A, then B and C can be found. These propositions will be the "bridges" to our final result, and the complete statements and proofs are given in Chapter III.

In Chapter IV, we will show how the factorization theorem described above can be applied to give new proofs or to shorten proofs of some previously known results. In Section 4.1 we give a new proof to Radjavi's theorem [15] about factoring a matrix into four Hermitian matrices. In Section 4.2, we apply the factorization theorems to shorten the proof given by Wu [23] about products of positive semidefinite matrices. In fact, more applications can be found in [18], we will discuss them in Sections 4.3 and 4.4.

1.3 Notation and Definitions

M_n	the set of all $n{\times}n$ complex matrices.
$GL(n,\mathbb{C})$	the set of all invertible matrices in M_n.
$SL(n,\mathbb{C})$	the set of all matrices in $GL(n,\mathbb{C})$ with determinant 1.

For the following, we assume A and B are in M_n.

$\sigma(A)$	the set of all eigenvalues of A repeated according to their algebraic multiplicity.
$\det A$	the determinant of A.
$\mathcal{N}(A)$	the null space of A, *i.e.* the set of vectors $x \in \mathbb{C}^n$ such that $Ax = 0$.
$\nu(A)$	the nullity of A, *i.e.* the dimension of $\mathcal{N}(A)$.

We say A is similar to B, denoted by $A \sim B$, if there exists an invertible matrix P such that $A = P^{-1}BP$. A is called a scalar matrix if it is of the form αI where α is a scalar. A is called a nilpotent matrix if $A^m = 0$ for some positive integer m. Moreover, it can be easily shown that A is a nilpotent matrix if and only if $\sigma(A)$ contains zero only.

If $A = (a_{ij})$, we use $A^t = (a_{ji})$ to denote the transpose of A. If $A = (a_{ij})$, we use $A^* = (\overline{a_{ji}})$ to denote the adjoint of A. Moreover, we call A a Hermitian matrix if $A = A^*$. A is positive semidefinite if $x^*Ax \geq 0$ for every $x \in \mathbb{C}^n$. Similarly, A is positive definite if $x^*Ax > 0$ for every nonzero $x \in \mathbb{C}^n$. A is a direct sum of matrices $A_1, A_2, \ldots, A_r$, written $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$ if

$$A = \begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_r \end{bmatrix}$$

CHAPTER II

LITERATURE REVIEW

In [18] Sourour gave the following result.

THEOREM 2.1. *Let* $A \in M_n$ *be invertible and nonscalar and let* $b_1, b_2, \cdots, b_n$ *and* $c_1, c_2, \cdots, c_n$ *be given complex numbers such that* $\prod_{i=1}^{n} b_i c_i = \det A$. *Then there exists a matrix* $B \in M_n$ *with eigenvalues* $b_1, b_2, \cdots, b_n$ *and a matrix* $C \in M_n$ *with eigenvalues* $c_1, c_2, \cdots, c_n$ *such that* $A = BC$.

Later, C R. Johnson and others have observed that by using virtually the same method one can prove the following theorem which appeared in [11].

THEOREM 2.2. *Let* $A \in M_n$ *be given with* $\operatorname{rank} A = k < n$, *let* $b_1, b_2, \cdots, b_n$ *and* $c_1, c_2, \cdots, c_n$ *be given complex numbers, exactly* $n - k$ *of which are zero. Then there exist* B *and* $C \in M_n$ *with eigenvalues* $b_1, b_2, \cdots, b_n$ *and* $c_1, c_2, \cdots, c_n$ *respectively such that* $A = BC$.

Notice that to prove Theorem 2.2 (and all the theorems that follow), it suffices to prove it for some matrix similar to A because similarity preserves eigenvalues, *i.e.* $S^{-1}BCS = S^{-1}BSS^{-1}CS$ and $\sigma(B) = \sigma(S^{-1}BS)$ while $\sigma(C) = \sigma(S^{-1}CS)$. We will outline the proof of Theorem 2.2, but before that, we need three lemmas.

Lemma 2.3. *If* $A \in M_n$ *and if every vector* $v \in \mathbb{C}^n$ *is an eigenvector of* A, *then* A *is a scalar.*

Proof. We show that A has only one eigenvalue. To prove this assume, to the contrary, that $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$ where $\lambda_1 \neq \lambda_2$ and v_1 and v_2 are nonzero. Let $v_3 = v_1 + v_2$. By assumption $Av_3 = \lambda_3 v_3$ for some scalar λ_3. Therefore $\lambda_1 v_1 + \lambda_2 v_2 = \lambda_3(v_1+v_2)$. Thus, v_1 and v_2 are linearly dependent. But this implies that $\lambda_1 = \lambda_2$, a contradiction. This shows that A has only one eigenvalue λ. Since every vector is an eigenvector, we have $Av = \lambda v$ for every v, and so $A = \lambda I$. ∎

Lemma 2.4. *Suppose that* $A \in M_n$ *is not a scalar matrix, and let* a *be any complex number. Then* A *is similar to a matrix which has* a *in its* $(1,1)$ *position.*

Proof. Since A is not a scalar matrix, by Lemma 2.3, we can choose a vector e_1 such that e_1 is not an eigenvector of $A - aI$. Now let $e_2 = (A-aI)e_1$ and choose an ordered basis $\mathcal{B}$ with e_1 and e_2 as its first and second vectors. The matrix of A with respect to $\mathcal{B}$ has first column $(a,1,0,\cdots,0)^t$. ∎

Lemma 2.5. *Let* $x,y \in \mathbb{C}^n$ and $B \in M_n$. Set $A \equiv \begin{bmatrix} \alpha & y^t \\ x & B \end{bmatrix} \in M_{n+1}$ where α is some nonzero complex number. Then

$$\operatorname{rank} A = \operatorname{rank}\left[B - \frac{1}{\alpha}\, xy^t\right] + 1.$$

Proof. $A = \begin{bmatrix} \alpha & y^t \\ x & B \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ x & I \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{\alpha}\, y^t \\ 0 & B - \frac{1}{\alpha}\, xy^t \end{bmatrix}$. Notice that $\begin{bmatrix} \alpha & 0 \\ x & I \end{bmatrix}$ is invertible. Thus,

$$\operatorname{rank} A = \operatorname{rank}\begin{bmatrix} 1 & \frac{1}{\alpha}\, y^t \\ 0 & B - \frac{1}{\alpha}\, xy^t \end{bmatrix}$$

$$= \operatorname{rank}\left[B - \frac{1}{\alpha}\, xy^t\right] + 1. \quad \blacksquare$$

Proof of Theorem 2.2. First we consider the case $A = 0$. Since there are exactly n zeros among $b_1, b_2, \cdots, b_n$, $c_1, c_2, \cdots, c_n$, we may label them so that $b_j c_j = 0$ for every j. The matrix A may now be factored as follows:

$$A = 0 = \begin{bmatrix} b_1 & & O \\ & \ddots & \\ O & & b_n \end{bmatrix}\begin{bmatrix} c_1 & & O \\ & \ddots & \\ O & & c_n \end{bmatrix}.$$

Next we consider the case $A \neq 0$ and proceed by induction on n. If $n = 2$, then $\operatorname{rank} A = 1$ and so exactly one of b_1, b_2, c_1, c_2 is zero, so we may assume that $b_1 \neq 0$ and $c_1 \neq 0$. By Lemma 2.4, A is similar to $\begin{bmatrix} b_1c_1 & y \\ x & z \end{bmatrix}$. Since A is singular, we have $z = \frac{xy}{b_1c_1}$. So one factorization is

$$\begin{bmatrix} b_1c_1 & y \\ x & z \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ \frac{1}{c_1}x & 0 \end{bmatrix} \begin{bmatrix} c_1 & \frac{1}{b_1}y \\ 0 & c_2 \end{bmatrix}$$

Now we assume that $n \geq 3$. Again, the number of zeros among the b_i's and c_i's is less than n and so we may assume that b_1 and c_1 are nonzero. By Lemma 2.4, A is similar to $\begin{bmatrix} b_1c_1 & y^t \\ x & S \end{bmatrix}$ where x is a column vector, y^t is a row vector and $S \in M_{n-1}$. Now Lemma 2.5 says that

$$\text{rank } A = k = \text{rank}\left[S - \frac{1}{b_1c_1} xy^t\right] + 1.$$

Therefore, we have

$$(n-1)-\text{rank}\left[S - \frac{1}{b_1c_1} xy^t\right] = n - k$$

which is the total number of zeros among the b_i's and c_i's. Hence by the induction hypothesis, $S - \frac{1}{b_1c_1} xy^t = B'C'$ with

$$\sigma(B') = \{b_2, \cdots, b_n\} \text{ and } \sigma(C') = \{c_2, \cdots, c_n\}.$$

We can factor A in the following way:

$$A = \begin{bmatrix} b_1c_1 & y^t \\ x & S \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ \frac{1}{c_1}x & B' \end{bmatrix} \begin{bmatrix} c_1 & \frac{1}{b_1}y^t \\ 0 & C' \end{bmatrix} \qquad \blacksquare$$

The following key result was given by Wu [22] about factorization of a singular matrix into two nilpotent matrices.

THEOREM 2.6. *Suppose that* $A \in M_n$ *is singular but not a* 2×2 *nilpotent matrix. Then* A *is a product of two nilpotent matrices both of which have rank equal to* rank A.

Since this is one of the key results used in this thesis and for the sake of completeness, we will include the proof of the theorem, but first we need two lemmas. In the following we denote the $m\times m$ nilpotent Jordan block by J_m, that is

$$J_m = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & & & \vdots \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

In particular, J_1 is the 1×1 matrix 0.

LEMMA 2.7. *For* $m \neq 2$, J_m *is the product of two nilpotent matrices both of which have rank equal to* rank J_m.

Proof. The proof is obvious for $m = 1$. For odd $m \geq 3$, we have

$$J_m = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & 1 & 0 \\ 0 & 0 & & \cdots & 0 & 0 \\ 1 & 0 & & \cdots & 0 & 0 \end{bmatrix}$$

and for m even, $m \neq 2$, we have

$$J_m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & & & \vdots & \vdots \\ \vdots & & & & & \vdots & \vdots \\ \vdots & & & & & \vdots & \vdots \\ 0 & & \cdots & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & & 1 & 0 \\ 0 & 0 & & \cdots & & 0 & 0 \\ 1 & 0 & & \cdots & & 0 & 0 \end{bmatrix} . \qquad \blacksquare$$

LEMMA 2.8. *For* $n \neq 2$, *every* $n \times n$ *nilpotent matrix* A *is a product of two nilpotent matrices both of which have rank equal to* rank A.

Proof. The result is obvious for $n = 1$. For $n \geq 3$, A is similar, *via* the Jordan canonical form, to a direct sum of nilpotent Jordan blocks, *i.e.* a block diagonal matrix

$$\begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_r \end{bmatrix}$$

where every A_j is an $m_j \times m_j$ nilpotent Jordan block J_{m_j} for some positive integer m_j. The existence of 2×2 Jordan blocks complicates the proof since they cannot be written as a product of two nilpotents. If there is more than one 2×2 Jordan block we group them in groups of twos and threes. If there is only one of them, we group it with another Jordan block of any size (such a block exists because $n > 2$). Therefore A is similar to a direct sum of matrices

$$R_1 \oplus R_2 \oplus \cdots \oplus R_\ell$$

where each R_ℓ is similar to one of the matrices: J_m with $m \neq 2$, $J_2 \oplus J_1$, $J_2 \oplus J_m$ with $m \geq 2$ and $J_2 \oplus J_2 \oplus J_2$. It suffices to prove the theorem for these matrices, since if $R_j = N_j M_j$ with N_j and M_j nilpotent and $\operatorname{rank} N_j = \operatorname{rank} M_j = \operatorname{rank} R_j$, then

$$R_1 \oplus \cdots \oplus R_\ell = NM$$

where $N = N_1 \oplus \cdots \oplus N_\ell$ and $M = M_1 \oplus \cdots \oplus M_\ell$. Furthermore, the direct sum of nilpotent matrices is always nilpotent and the rank of a direct sum equals that sum of ranks of the individual direct summands.

The matrices J_m with $m \neq 2$ have been dealt with in Lemma 2.7. For the matrix $J_2 \oplus J_1$ we have the following factorization

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

For $J_m \oplus J_2$ with $m \geq 2$, we have

$$\begin{bmatrix} J_m & 0 \\ 0 & J_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & & & \\ \vdots & \vdots & & J_m & \\ 0 & 0 & & & \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & & J_2 \\ 0 & \cdots & 0 & & \\ 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ & & 1 & & \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \quad \text{if } m \text{ is even,}$$

and

$$\begin{bmatrix} J_m & 0 \\ 0 & J_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 1 \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & J_2 \\ J_m & 0 \end{bmatrix} \quad \text{if } m \text{ is odd.}$$

Finally,

$$\begin{bmatrix} J_2 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & P \\ J_2 & 0 & 0 \\ 0 & J_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & Q & 0 \\ 0 & 0 & Q \\ J_2 & 0 & 0 \end{bmatrix}$$

where

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad \blacksquare$$

Proof of Theorem 2.6. By the Jordan canonical form, A is similar to a matrix $L = L_1 \oplus \cdots \oplus L_m$ where

$$L_1 = \begin{bmatrix} a_1 & & \cdots & 0 \\ 1 & a_1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & a_1 \end{bmatrix}.$$

Since the nilpotent case has been dealt with in Lemma 2.8, we need only consider the nonnilpotent case. So at least one eigenvalue is nonzero and we may assume without loss of generality that $a_m \neq 0$. Notice that it suffices to prove that $L = YZ$ for some nilpotent matrices Y and Z such that

(i) The ith row of Y is zero if and only if the ith row of L is zero;

(ii) the jth column of Z is zero if and only if the jth column of L is zero;

(iii) the nonzero rows of Y are independent;

(iv) the nonzero columns of Z are independent.

We use induction on n. For $n = 2$, A is either 0, in which case we take each factor to be 0, or A is similar to $\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ with $a \neq 0$. We have

$$\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Next we consider $n = 3$. It suffices again to consider nonnilpotent L. There are three possibilities for L which are factored as follows.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1/\alpha \\ 1 & -\alpha & -1 \end{bmatrix} \begin{bmatrix} \alpha & 0 & \alpha \\ 2 & 0 & 1 \\ -\alpha & 0 & -\alpha \end{bmatrix} \quad \text{where} \quad \alpha \neq 0.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix} \quad \text{where} \quad \alpha \neq 0.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \gamma & \beta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{where} \quad \alpha, \beta \neq 0.$$

Now assume $n \geq 4$ (and recall that $\alpha_m \neq 0$). Let L' be the singular $(n-1)\times(n-1)$ matrix obtained by deleting the last row and the last column of L. So

$$L = \left[\begin{array}{c|c} L' & 0 \\ \hline \mu^t & \alpha_m \end{array}\right] \quad \text{where} \quad \mu^t = [0,\cdots,0,d]$$

with $d = 0$ or 1. By the induction hypothesis $L' = Y'Z'$ where Y' and Z' are $(n-1)\times(n-1)$ nilpotent matrices with properties (i), (ii), (iii), and (iv). Let y_i^t be the ith row of Y' and z_j be the jth column of Z', we want to show that there exist y_n^t and z_n such that

$$L = \left[\begin{array}{c|c} Y' & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline y_n^t & 0 \end{array}\right] \left[\begin{array}{c|c} Z' & z_n \\ \hline 0 \cdots 0 & 0 \end{array}\right]$$

with properties (i), (ii), (iii), and (iv) still holding. In other words, we want to find y_n^t and z_n such that

$$(1) \qquad y_n^t Z' = \mu^t, \qquad y_n^t z_n = a_m,$$

$$(2) \qquad Y' z_n = 0,$$

and y_n^t is independent of $y_1^t, \ldots, y_{n-1}^t$ while z_n is independent of $z_1, \ldots, z_{n-1}$.

Since rank Y' = rank L' and L' is singular, we have that Y' is also singular. Therefore there exists a nonzero z_n such that (2) holds. Next we show that z_n is independent of the other z_j's. Suppose not — that is, suppose that

$$z_n = \sum_{j=1}^{n-1} w_j z_j \quad \text{for some scalars } w_j\text{'s.}$$

We may assume that $w_j = 0$ if z_j is a zero column. In other words $z_n = Z'w$ where $w = [w_1, \cdots, w_{n-1}]^t$. Now

$$L'w = Y'Z'w = Y'z_n = 0.$$

Notice that the nonzero columns of L' are linearly independent and $w_j = 0$ if the jth column of L' is zero. Now let $L_{i_1}, \cdots, L_{i_r}$ be all the nonzero columns of L', then the $(n-1)\times r$ matrix

$$L' = [L_{i_1}, \cdots, L_{i_r}]$$

formed by these nonzero columns has linearly independent columns, and so it has rank r. This implies that its null space is $\{0\}$, and the equation $L'w = 0$ is satisfied only when $w = 0$. Therefore every $w_j = 0$ and so $z_n = 0$ which is a contradiction.

Next we prove the existence of a vector y_n satisfying equation (1). First we assume that $d = 0$. Let $z_{i_1}, \cdots, z_{i_r}$ be the nonzero columns of Z', then the $(n-1)\times(r+1)$ matrix

$$Z' = [z_{i_1}, \cdots, z_{i_r}, z_n]$$

formed by the nonzero columns of Z' and z_n has linearly independent columns and therefore the system

$$[x_1, \cdots, x_{n-1}]Z' = [0, \cdots, 0, a_m]$$

is consistent. Thus there exists a vector y_n^t satisfying equation (1). On the other hand, if $d = 1$, then the size of L_m (the last Jordan block of L) is greater than or equal to two, and hence the $(n-1)$st column of L' is not zero. By our assumption, we must have that z_{n-1} is not a zero column. As before the system of equations

$$[x_1, \cdots, x_{n-1}]Z' = [0, \cdots, 0, 1, a_m]$$

is consistent. It remains only to show that y_n^t is not linearly dependent on the rows of Y'. To prove this we notice that

$$L = YZ \quad \text{with} \quad Y = \left[\begin{array}{c|c} & 0 \\ Y' & \vdots \\ & 0 \\ \hline y_n^t & 0 \end{array}\right] \quad \text{and} \quad Z = \left[\begin{array}{c|c} Z' & z_n \\ & \\ \hline 0 \cdots 0 & 0 \end{array}\right]$$

and so

$$\text{rank } Y \geq \text{rank } L = 1 + \text{rank } L' = 1 + \text{rank } Y'$$

which implies the required independence. ∎

CHAPTER III

MAIN THEOREM

THEOREM 3.1 (the main theorem). *Suppose that* $A \in M_n$ *is not a nonzero* 2×2 *nilpotent matrix and* rank $A = k < n$. *Let* $b_1, b_2, \cdots, b_n$ *and* $c_1, c_2, \cdots, c_n$ *be given complex numbers, exactly* t *of which are zero. Then there exist matrices* B *and* C *in* M_n *with eigenvalues* $b_1, b_2, \cdots, b_n$ *and* $c_1, c_2, \cdots, c_n$ *respectively such that* $A = BC$ *if and only if* $t \geq n - k$.

REMARK 3.2. To prove the theorem, we may assume, without loss of generality, that the number of zeros among the b_i's is greater than or equal to the number of zeros among the c_i's. Indeed, in the opposite case we may factor A^t as a product RS with $\sigma(R) = \{c_1, c_2, \cdots, c_n\}$ and $\sigma(S) = \{b_1, b_2, \cdots, b_n\}$. Since rank A^t = rank A, $\sigma(R^t) = \sigma(R)$ and $\sigma(S^t) = \sigma(S)$, and since $A = (RS)^t = S^t R^t$, we get the desired factorization.

3.1 The 2×2 matrices

In this section we prove Theorem 3.1 for 2×2 matrices and we also establish a necessary and sufficient condition for factoring the nonzero 2×2 nilpotent matrices — the exceptional case. The proof is given in a sequence of steps.

(i) If A is a nonzero 2×2 nilpotent matrix, then A cannot be written as a product of two nilpotent matrices.

This has first been observed in the paper by Fong and Sourour [7]. We give a proof.

Proof. Every nonzero 2×2 nilpotent is similar to

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Suppose that $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = N_1N_2$ where N_1, N_2 are nilpotent matrices. Take a nonzero $x = (x_1,x_2) \in \mathcal{N}(N_2)$; we know that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N_1N_2x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies $x_1 = 0$ and $x_2 \neq 0$. Hence, N_2 is of the form $\begin{bmatrix} b_2 & 0 \\ a_2 & 0 \end{bmatrix}$. But N_2 is a nilpotent matrix, and so $b_2 = 0$, this means N_2 is of the form $\begin{bmatrix} 0 & 0 \\ a_2 & 0 \end{bmatrix}$. On the other hand $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = N_2^t\, N_1^t$, the same argument shows that N_1 is of the form $\begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix}$. However,

$$\begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ a_2 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

for any a_1 and a_2. ■

(ii) Let $E_1 = \{0,0\}$, $E_2 = \{c_1,0\}$, $c_1 \neq 0$. For every 2×2 singular matrix A, there exist matrices B and C with $\sigma(B) = E_1$ and $\sigma(C) = E_2$ such that $A = BC$.

Proof. Notice that A is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ where $a \neq 0$ and so it suffices to establish the desired factorization for these three matrices. We have

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/c_1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 1 & 0 \end{bmatrix}. \quad \blacksquare$$

(iii) Let $E_1 = \{0,0\}$, $E_2 = \{c_1,c_2\}$, $c_1 \neq 0$, $c_2 \neq 0$. For every 2×2 singular matrix A, there exist matrices B and C with $\sigma(B) = E_1$ and $\sigma(C) = E_2$ such that $A = BC$.

Proof. Again it suffices to consider the three canonical forms

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/c_1 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}.$$

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1+c_2 & -c_1c_2 \\ 1 & 0 \end{bmatrix}. \quad \blacksquare$$

(iv) Let $E_1 = \{b_1,0\}$, $E_2 = \{c_1,0\}$, $b_1 \neq 0$, $c_1 \neq 0$. For every 2×2 singular matrix A, there exist matrices B and C with $\sigma(B) = E_1$ and $\sigma(C) = E_2$ such that $A = BC$.

Proof. If $A \neq 0$, then by Lemma 2.4, A is similar to $\begin{bmatrix} b_1c_1 & y \\ x & z \end{bmatrix}$. Since A is singular, we have $z = \frac{xy}{b_1c_1}$. Hence we can factor A in the following way:

$$\begin{bmatrix} b_1c_1 & y \\ \\ x & z \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ \\ \frac{1}{c_1}x & 0 \end{bmatrix} \begin{bmatrix} c_1 & \frac{1}{b_1}y \\ \\ 0 & 0 \end{bmatrix}.$$

If $A = 0$ then

$$A = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & c_1 \end{bmatrix}. \quad \blacksquare$$

(v) Let A be a 2×2 matrix, $E_1 = \{b_1, 0\}$ and $E_2 = \{c_1, c_2\}$ where b_1, c_1, c_2 are nonzero. Then A is a product of some matrices B and C with $\sigma(B) = E_1$ and $\sigma(C) = E_2$ if and only if rank $A = 1$.

Proof. If $A = BC$ with $\sigma(B) = E_1$ and $\sigma(C) = E_2$, then C is invertible and so rank A = rank $B = 1$. The converse is contained in Theorem 2.2. $\blacksquare$

(vi) If A is a singular 2×2 matrix which is not a nonzero nilpotent matrix, then $A = BC$ for some matrices B and C such that $\sigma(B) = \sigma(C) = \{0,0\}$.

Proof. If $A = 0$, we may take $B = C = 0$. If $A \neq 0$, then, by the hypothesis, A is similar to $\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$ for some $\alpha \neq 0$. The factorization

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

establishes the conclusion. ■

The preceding steps establish a proof of Theorem 3.1 for 2×2 matrices. They also establish the following proposition regarding 2×2 nonzero nilpotent matrices.

PROPOSITION 3.3. *Let* A *be a* 2×2 *nonzero nilpotent, and let* b_1, b_2, c_1, c_2 *be complex numbers. Then there exist* 2×2 *matrices* B *and* C *with eigenvalues* $\{b_1,b_2\}$ *and* $\{c_1,c_2\}$ *respectively such that* $A = BC$ *if and only if the total number of zeros* t *among* b_1, b_2, c_1, c_2 *satisfies the inequalities* $1 \leq t \leq 3$.

3.2 Proof of the Main Theorem

We start by proving the necessity of the condition $t \geq n - k$ for the factorization in Theorem 3.1.

Proof of Necessity: Suppose $A = BC$ with $\sigma(B) = \{b_1, \cdots, b_n\}$ and $\sigma(C) = \{c_1, \cdots, c_n\}$. Two standard facts from linear algebra are that the nullity is submultiplicative, *i.e.* $\nu(DE) \leq \nu(D) + \nu(E)$ for all $n\times n$ matrices D, E and that the nullity of a matrix is less than or equal to the number of zeros among its eigenvalues. Using these, we have

$$\begin{aligned} n - k &= \nu(A) \\ &= \nu(BC) \\ &\leq \nu(B) + \nu(C) \\ &\leq \text{the number of zero eigenvalues of } B \\ &\quad + \text{the number of zero eigenvalues of } C \\ &= t. \quad \blacksquare \end{aligned}$$

Before proceeding with the proof of the sufficiency of the condition $t \geq n - k$ we need one lemma.

Lemma 3.4. *Let S be an invertible matrix in M_n and set*

$$A = \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} \in M_{n+1}.$$

Then A is similar to $\begin{bmatrix} 0 & 0 \\ x & S \end{bmatrix}$ where x is any $n \times 1$ vector.

Proof. Let $y = S^{-1}x$.

$$\begin{aligned} A &\sim \begin{bmatrix} 1 & 0 \\ -y & I \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ y & I \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ Sy & S \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x & S \end{bmatrix}. \quad \blacksquare \end{aligned}$$

Now the proof of the sufficiency will be established first for several special cases pertaining to the distribution of zeros among the two sets of eigenvalues $\{b_1,\cdots,b_n\}$ and $\{c_1,\cdots,c_n\}$. This is established in the following three propositions.

PROPOSITION 3.5. *Suppose that* $A \in M_n$ *is not a* 2×2 *nilpotent and* $\operatorname{rank} A = k < n$. *Let* $b_1,\cdots,b_m,0,\cdots,0$, *and* $c_1,\cdots,c_m,0,\cdots,0$ *be* $2n$ *complex numbers such that all the* b_i*'s and* c_i*'s are nonzero and* $2(n-m) \geq n-k$. *Then there exist matrices* B *and* C *with eigenvalues* $b_1,\cdots,b_m,0,\cdots,0$ *and* $c_1,\cdots,c_m,0,\cdots,0$ *respectively such that* $A = BC$.

Proof. We are given that $n - k \leq 2(n-m)$ which is equivalent to $k \geq 2m - n$. Our proof will proceed by using induction on m. If $m = 0$, the proposition is true by Theorem 2.6. Now assume $m = 1$. If $n = 2$, then by Section 3.1(iv), we know that the proposition is true. If $n = 3$, then A is similar to

$$\begin{bmatrix} \alpha & 0 & 0 \\ a & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$

where α, β are nonzero, and a, b are either 0 or 1. These matrices can be factored as follows:

$$\begin{bmatrix} \alpha & 0 & 0 \\ a & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & b_1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ \alpha & 0 & 0 \\ -b_1\alpha+a & \beta & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{c_1} a & 0 & 0 \\ 0 & a & b_1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and
$$\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{c_1} a & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{b_1} b & 0 \end{bmatrix}$$

Hence, when $m = 1$, $n = 3$, the proposition is true. Suppose now that we have $m = 1$ and $n \geq 4$. If $A \neq 0$, then by Lemma 2.4, A is similar to

$$\left[\begin{array}{c|c} b_1c_1 & y^t \\ \hline x & S \end{array}\right],$$

where x is a $(n-1)\times 1$ column vector, y^t is a $1\times(n-1)$ row vector and $S \in M_{n-1}$. By Lemma 2.5, we know that

$$\operatorname{rank} A = k = \operatorname{rank}\left[S - \frac{1}{b_1c_1} x\, y^t\right] + 1.$$

So $\operatorname{rank}\left[S - \frac{1}{b_1c_1} x\, y^t\right] = k - 1 < n - 1$, and hence $S - \frac{1}{b_1c_1} x\, y^t$ is singular. Therefore, by Theorem 2.6, there exist nilpotent matrices B' and C' such that $S - \frac{1}{b_1c_1} x\, y^t = B'C'$. Now the desired factorization of A follows from the equations:

$$A \sim \left[\begin{array}{c|c} b_1c_1 & y^t \\ \hline x & S \end{array}\right] = \left[\begin{array}{c|c} b_1 & 0 \\ \hline \frac{1}{c_1} x & B' \end{array}\right] \left[\begin{array}{c|c} c_1 & \frac{1}{b_1} y^t \\ \hline 0 & C' \end{array}\right].$$

So the proposition is true when $m = 1$.

Now we complete the proof by using induction on m. Assume $m \geq 2$.

Case (1). $A = 0$. Since $\operatorname{rank} A = 0 \geq 2m - n$, we have $n \geq 2m$. Therefore

$$A = 0 = \begin{bmatrix} b_1 & & & & 0 \\ & \ddots & & & \\ & & b_m & & \\ & & & 0 & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & & c_1 & \\ & 0 & & & \ddots & \\ & & & & & c_m \end{bmatrix}$$

Case (2). $A \neq 0$. By Lemma 2.4,

$$A \sim \begin{bmatrix} b_m c_m & y^t \\ x & S \end{bmatrix}$$

where x is a $(n-1)\times 1$ column vector, y^t is a $1\times(n-1)$ row vector and $S \in M_{n-1}$. Now $\operatorname{rank} A = k = \operatorname{rank}\left[S - \frac{1}{b_m c_m} x\, y^t\right] + 1$ by Lemma 2.5. Hence we have

$$\begin{aligned} \operatorname{rank}\left[S - \frac{1}{b_m c_m} x\, y^t\right] &= k - 1 \geq 2m - n - 1 \\ &= 2(m-1) - (n-1). \end{aligned}$$

Therefore, by the induction hypothesis, there exist $(n-1)\times(n-1)$ matrices B' and C' with

$$\sigma(B) = \{b_1, \cdots, b_{m-1}, 0, \cdots, 0\}$$

and

$$\sigma(C) = \{c_1, \cdots, c_{m-1}, 0, \cdots, 0\}$$

such that $\left[S - \frac{1}{b_m c_m} x\, y^t\right] = B'C'$. Consequently, we have

$$\left[\begin{array}{c|c} b_m c_m & y^t \\ \hline x & S \end{array}\right] = \left[\begin{array}{c|c} b_m & 0 \\ \hline \frac{1}{c_m} x & B' \end{array}\right] \left[\begin{array}{c|c} c_m & \frac{1}{b_m} y^t \\ \hline 0 & C' \end{array}\right],$$

and this completes the proof of Proposition 3.5. ■

PROPOSITION 3.6. *Suppose that* $A \in M_n$ *is singular, and let* $c_1, c_2, \cdots, c_n$ *be nonzero complex numbers. Then there exist matrices* B *and* C *in* M_n *with eigenvalues* $0, 0, \cdots, 0$, *and* $c_1, c_2, \cdots, c_n$ *respectively such that* $A = BC$.

Proof. If $n = 1$, the proof is obvious. If $n = 2$, then by Section 3.1(iii), the proposition is true. Now we proceed by induction on n. Assume $n \geq 3$, we consider two cases.

Case (1): A has more than one zero eigenvalue. Then A is similar to

$$\left[\begin{array}{c|c} 0 & y' \\ \hline 0 & S \end{array}\right]$$

for some $(n-1)\times(n-1)$ singular matrix S. So by the induction hypothesis, there exist B', C' in M_{n-1} with $\sigma(B) = \{0, \cdots, 0\}$ and $\sigma(C) = \{c_1, \cdots, c_{n-1}\}$ such that $S = B'C'$. Hence we have

$$\left[\begin{array}{c|c} 0 & y^t \\ \hline 0 & S \end{array}\right] = \left[\begin{array}{c|c} 0 & y^tC'^{-1} \\ \hline 0 & B' \end{array}\right]\left[\begin{array}{c|c} c_n & 0 \\ \hline 0 & C' \end{array}\right]$$

Case (2): A has exactly one zero eigenvalue. Then A is similar to

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & S \end{array}\right]$$

for some invertible $(n-1)\times(n-1)$ matrix S. By Lemma 3.4, we know that

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & S \end{array}\right] \text{ is similar to } \left[\begin{array}{c|c} 0 & 0 \\ \hline x & S \end{array}\right]$$

for some nonzero $(n-1)\times1$ column vector x. Now let $w = S^{-1}\frac{1}{c_n}x$, then choose a $1\times(n-1)$ vector z^t such that $z^tw = 1$. So we have

$$\begin{aligned}\left[S - \frac{1}{c_n}xz^t\right]w &= Sw - \frac{1}{c_n}xz^tw \\ &= Sw - \frac{1}{c_n}x \\ &= 0.\end{aligned}$$

Hence, $S - \frac{1}{c_n} xz^t$ is singular. So, by the induction hypothesis, there exist B', C' in M_{n-1} with $\sigma(B') = \{0,\cdots,0\}$ and

$$\sigma(C') = \{c_1,\cdots,c_{n-1}\}$$

such that $S - \frac{1}{c_n} xz^t = B'C'$. Consequently, we have the following factorization:

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline x & S \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline \frac{1}{c_n} x & B' \end{array}\right]\left[\begin{array}{c|c} c_n & z^t \\ \hline 0 & C' \end{array}\right] \qquad \blacksquare$$

PROPOSITION 3.7. *Let* $A \in M_n$ *be given with* rank $A = k < n$. *Let* $b_1,\cdots,b_m,0,\cdots,0$, *and* $c_1,\cdots,c_m,c_{m+1},\cdots,c_n$ *be* $2n$ *complex numbers such that all of the* b_i's *and* c_j's *are nonzero and* $n - m \geq n - k$. *Then there exist matrices* B *and* C *in* M_n *with eigenvalues* $b_1,\cdots,b_m,0,\cdots,0$ *and* $c_1,\cdots,c_n$ *respectively such that* $A = BC$.

Proof. If $m = 0$, then the proposition is true by Proposition 3.6. Now assume $m \geq 1$. Since

$$\text{rank } A = k \geq m \geq 1,$$

we know that $A \neq 0$. Hence by Lemma 2.4, A is similar to

$$\left[\begin{array}{c|c} b_mc_m & y^t \\ \hline x & S \end{array}\right]$$

where x is a $(n-1)\times 1$ column vector, y^t is a $1\times(n-1)$ row vector and $S \in M_{n-1}$. So by Lemma 2.5, $\operatorname{rank}\left[S - \frac{1}{b_mc_m}xy^t\right] = k - 1 \geq m - 1$. Therefore, by the induction hypothesis, there exist $(n-1)\times(n-1)$ matrices B' and C' with

$$\sigma(B') = \{b_1,\cdots,b_{m-1},0,\cdots,0\} \quad \text{and} \quad \sigma(C') = \{c_1,\cdots,c_{m-1},c_{m+1},\cdots,c_n\}$$

such that $S - \frac{1}{b_mc_m}xy^t = B'C'$. Consequently, we can factor A in the following way:

$$\left[\begin{array}{c|c} b_mc_m & y^t \\ \hline x & S \end{array}\right] = \left[\begin{array}{c|c} b_m & 0 \\ \hline \frac{1}{c_m}x & B' \end{array}\right]\left[\begin{array}{c|c} c_m & \frac{1}{b_m}y^t \\ \hline 0 & C' \end{array}\right],$$

and this completes the proof of Proposition 3.7. ■

Proof of Theorem 3.1. Without loss of generality, we may assume that the number of zeros among the b_i's is greater than or equal to the number of zeros among the c_i's since by factoring A^t instead of A we may reverse the roles of the b_i's and c_i's. Suppose that exactly m of the b_i's are nonzero; we may assume that $b_1,\cdots,b_m$ are nonzero and $b_{m+1},\cdots,b_n$ are zero. Now suppose that exactly ℓ of the c_i's are zero. By our assumption $\ell \le n - m$. We may assume that $c_1, \cdots, c_m, c_{m+\ell+1}, \cdots, c_{m+\ell+j} = c_n$ are all nonzero and $c_{m+1}, \cdots, c_{m+\ell}$ are zero. Notice that it is possible that ℓ or j is zero. So now, we have n-tuples of complex numbers: $\{b_1, \cdots, b_m, 0, \cdots, 0\}$ and $\{c_1, \cdots, c_m, 0, \cdots, 0, c_{m+\ell+1}, \cdots, c_{m+\ell+j}\}$ where every b_i and c_i is nonzero. Notice that if t is the total number of zeros among these $2n$ complex numbers, then $t = (n-m) + \ell$. So the condition $t \ge n - k$ in the statement of Theorem 3.1 is equivalent to $k \ge m - \ell$.

Now we break down the theorem into the following cases.

Case (1): $\ell = 0$. Then the theorem is true by Proposition 3.7.

Case (2): $\ell \ge 1$, $j = 0$. Then the theorem is true by Proposition 3.5.

Case (3): $\ell \ge 1$, $j \ge 1$. The condition $t \ge n - k$ of Theorem 3.1 gives $n - m + \ell \ge n - k$ and so $k \ge m - \ell$. The case $k = m - \ell$, equivalently $t = n - k$, is dealt with in Theorem 2.2, thus we may assume that $k \ge m - \ell + 1$ and we will use induction on j.

Subcase (3a): A has more than one zero eigenvalue. In this case $A \sim \begin{bmatrix} 0 & 0 \\ x & S \end{bmatrix}$ for some $(n-1)\times(n-1)$ singular matrix S. Now rank $S \ge$ rank $A - 1 \ge m - \ell$ and so if S is not a 2×2 nonzero nilpotent matrix, we get, by the induction hypothesis, that S can be factored into matrices B' and C' with $\sigma(B') = \{b_1, \cdots, b_m, 0, \cdots, 0\}$ and $\sigma(C') = \{c_1, \cdots, c_m, 0, \cdots, 0, c_{m+\ell+1}, \cdots, c_{m+\ell+j-1}\}$. Hence, we may factor A as follows:

$$\begin{bmatrix} 0 & 0 \\ x & S \end{bmatrix} = \begin{bmatrix} 0 & 0 \cdots 0 \\ c_{m+\ell+j}^{-1}\, x & B' \end{bmatrix} \begin{bmatrix} c_{m+\ell+j} & 0 \cdots 0 \\ 0 & \\ \vdots & C' \\ 0 & \end{bmatrix}.$$

If S is a 2×2 nonzero nilpotent matrix, then by Proposition 3.3, S may be factored into two matrices with eigenvalues as above except for the case where all the eigenvalues of the factors of S are required to be zero. It remains only to establish this case. This means that A is a 3×3 matrix and there are at least two zeros among the b_i's and at least two zeros among the c_i's. The condition that $j \geq 1$ implies that exactly two of the c_i's are zero and that all of the b_i's are zero. In other words we wish to factor A as product BC such that $\sigma(B) = \{0,0,0\}$ and $\sigma(C) = \{0,0,c_3\}$, $c_3 \neq 0$. We are still assuming that the multiplicity of zero as an eigenvalue of A is at least 2, so A is similar to one of the matrices:

$$\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 \end{bmatrix}$$

where $\alpha \neq 0$ and $a = 0$ or 1. These matrices can be factored as follows:

$$\begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{a}{c_3} & 0 & 0 \\ 0 & \alpha & 0 \end{bmatrix} \begin{bmatrix} c_3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & c_3 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{c_3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \\ O \\ \\ \end{bmatrix} = \begin{bmatrix} \\ O \\ \\ \end{bmatrix} \begin{bmatrix} c_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Subcase (3b) A has exactly one zero eigenvalue. In this case A is similar to

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_r \end{matrix} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & S \end{array}\right]$$

where

$$B_1 = \begin{bmatrix} \alpha_1 & & 0 \\ 1 & \ddots & \\ 0 & 1 & \alpha_1 \end{bmatrix}$$

with $\alpha_1 \neq 0$ This is merely the Jordan form of A. By Lemma 3.4, we know that

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & S \end{array}\right] \text{ is similar to } \left[\begin{array}{c|c} 0 & 0 \\ \hline x & S \end{array}\right]$$

where $x = (1,0,\cdots,0)^t$ Define a $1 \times (n-1)$ vector z^t as $(\alpha_1 c_n, 0, \cdots, 0)$ Now we have

$$S - c_n^{-1} x z^t = S - c_n^{-1}(1,0,\cdots,0)^t(\alpha_1 c_n, 0, \cdots, 0)$$

$$= S - \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ \vdots & & & \ddots \\ 0 & \cdots & & 0 \end{bmatrix}$$

which is singular and has rank $n - 2$. So we have

$$\operatorname{rank}(S - c_n^{-1} x z^t) = n - 2 = \operatorname{rank} A - 1 \geq m - \ell$$

Hence, by the induction hypothesis, $S - c_n^{-1} x\, z^t$ can be factored into some $(n-1)\times(n-1)$ matrices B' and C' with

$$\sigma(B') = \{b_1, \cdots, b_m, 0, \cdots, 0\} \text{ and}$$

$$\sigma(C') = \{c_1, \cdots, c_m, 0, \cdots, 0, c_{m+\ell+1}, \cdots, c_{m+\ell+j-1}\}$$

Now

$$A \sim \begin{bmatrix} 0 & 0 \\ x & S \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c_n^{-1}x & B' \end{bmatrix} \begin{bmatrix} c_n & z^t \\ 0 & C' \end{bmatrix},$$

which is the desired factorization. This completes the proof of our main theorem. ∎

CHAPTER IV

APPLICATIONS

4.1 Products of Hermitian Matrices

In [15], Radjavi proved that if the determinant of a matrix A is real, then A is a product of four Hermitian matrices. In the case when A is not a scalar matrix, we can apply Theorem 2.1 and 2.6 to give a different proof.

THEOREM 4.1. *An* $n\times n$ *complex matrix* A *is the product of four Hermitian matrices if and only if its determinant is real.*

Proof. First we prove that the property of a matrix S being the product of two (or four) Hermitian matrices is invariant under similarity. That is, if a matrix S has the property of being the product of two (or four) Hermitian matrices, then every matrix similar to S also has that property. Now suppose $S = H_1H_2$ where H_1 is Hermitian, and $P^{-1}SP$ is a matrix similar to S. Then

$$P^{-1}SP = (P^{-1}H_1P^{*-1})(P^*H_2P),$$

and both matrices $P^{-1}H_1P^{*-1}$ and P^*H_2P are Hermitian matrices, *i.e.* $P^{-1}SP$ is the product of two Hermitian matrices.

Similarly, suppose that $S = H_1H_2H_3H_4$ where H_i is Hermitian. Then

$$P^{-1}SP = (P^{-1}H_1P^{*-1})(P^*H_2P)(P^{-1}H_3P^{*-1})(P^*H_4P)$$

and $P^{-1}H_1P^{*-1}$, P^*H_2P, $P^{-1}H_3P^{*-1}$, and P^*H_4P are all Hermitian matrices.

This invariance will be used repeatedly in the remainder of this section.

Now we prove Theorem 4.1 by considering the following cases.

Case (1): $A = \lambda I$, and this is a special case of Radjavi's original proof. If $\lambda = 0$, the result is obvious. So we assume that $\lambda \neq 0$. We have

$$A = \begin{bmatrix} \lambda & & O \\ & \ddots & \\ O & & \lambda \end{bmatrix} = \begin{bmatrix} 0 & \lambda & & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda \\ \lambda & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}. \tag{4.1.1}$$

Now

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & 1 \\ \vdots & & \iddots & \\ 0 & 1 & & 0 \end{bmatrix} \begin{bmatrix} O & & 1 \\ & \iddots & \\ 1 & & O \end{bmatrix} \tag{4.1.2}$$

which are both Hermitian. On the other hand, let

$$B = \begin{bmatrix} 0 & \lambda & & 0 \\ \vdots & & \ddots & \\ & & & \\ 0 & & & \lambda \\ \lambda & 0 & \cdots & 0 \end{bmatrix}.$$

Then B is similar to C where

$$C = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \lambda^n & 0 & \cdots & 0 \end{bmatrix}$$

since

$$C = \begin{bmatrix} 1 & & & & O \\ & \lambda & & & \\ & & \lambda^2 & & \\ & O & & \ddots & \\ & & & & \lambda^{n-1} \end{bmatrix} B \begin{bmatrix} 1 & & & & O \\ & \lambda^{-1} & & & \\ & & \lambda^{-2} & & \\ & O & & \ddots & \\ & & & & \lambda^{1-n} \end{bmatrix}.$$

However,

$$C = \begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{bmatrix} \begin{bmatrix} \lambda^n & 0 & \cdots & 0 \\ 0 & & & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & 0 \end{bmatrix}$$

which are both Hermitian since $\lambda^n = \det A$ is real. It follows that B is the product of two Hermitian matrices, and this, together with equation (4.1.1) and (4.1.2) implies that A is the product of four

Hermitian matrices.

Case (2): A is invertible but $A \neq \lambda I$. Suppose that $\det A = c \neq 0$. If n is even, choose $E_1 = \left\{1, c, \beta_1, \bar{\beta}_1, \cdots, \beta_{\frac{n}{2}-1}, \bar{\beta}_{\frac{n}{2}-1}\right\}$ if $c \neq 1$ (or choose $E_1 = \left\{\frac{1}{2}, 2, \beta_1, \bar{\beta}_1, \cdots, \beta_{\frac{n}{2}-1}, \bar{\beta}_{\frac{n}{2}-1}\right\}$ if $c = 1$) and $E_2 = \left\{\gamma_1, \bar{\gamma}_1, \cdots, \gamma_{\frac{n}{2}}, \bar{\gamma}_{\frac{n}{2}}\right\}$ such that the β_i's (respectively the γ_j's) are distinct non-real complex numbers with $|\beta_i| = |\gamma_j| = 1$ for every i and j. If n is odd, choose

$$E_1 = \left\{c, \beta_1, \bar{\beta}_2, \cdots, \beta_{\frac{n-1}{2}}, \bar{\beta}_{\frac{n-1}{2}}\right\} \text{ and } E_2 = \left\{1, \gamma_1, \bar{\gamma}_1, \cdots, \gamma_{\frac{n-1}{2}}, \bar{\gamma}_{\frac{n-1}{2}}\right\}$$

such that the β_i's (respectively the γ_j's) are distinct non-real complex numbers with $|\beta_i| = |\gamma_j| = 1$ for every i and j. By Theorem 2.1, we know that there exist $n \times n$ matrices B and C with $\sigma(B) = E_1$ and $\sigma(C) = E_2$ such that $A = BC$. Since every $n \times n$ matrix with n distinct eigenvalues is similar to a diagonal matrix, it follows that both B and C are diagonalizable, so each of B and C is similar to a direct sum of 2×2 matrices of the form

$$\begin{bmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{bmatrix}$$

together with possible matrices of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} c \end{bmatrix}, \text{ or } \begin{bmatrix} 1 \end{bmatrix}$$

It suffices to show that each of these matrices is a product of two Hermitian matrices. This is quite obvious for the Hermitian ones by taking one factor to be I, the identity matrix. On the other hand,

$$\begin{bmatrix} \beta & 0 \\ 0 & \overline{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \overline{\beta} \\ \beta & 0 \end{bmatrix}$$

which are both Hermitian.

Case (3): $\det A = 0$. If $A = 0$, then the result is obvious. Now if A is a nonzero 2×2 nilpotent matrix, then

$$A \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore A is a product of two Hermitian matrices and hence is also a product of four Hermitian matrices, as we may take two factors to be the identity. It remains to consider the case when A is not a 2×2 nilpotent matrix. Then, by Theorem 2.6, $A = N_1N_2$ where N_1, N_2 are nilpotent matrices. Since each of N_1 and N_2 is similar to a direct sum of nilpotent Jordan blocks, it suffices to show that every nilpotent Jordan block is a product of two Hermitian matrices. In fact,

$$\begin{bmatrix} 0 & & & 0 \\ 1 & & & \\ & \ddots & & \\ 0 & & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & 0 \\ & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & 0 \end{bmatrix} \begin{bmatrix} \text{O} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \text{O} \end{bmatrix}. \blacksquare$$

4.2 Products of Positive Semidefinite Matrices

In [23], Wu proved that every singular matrix is the product of four positive semidefinite matrices. We will give a shorter proof of this result using Theorem 3.1. But first we need the following lemma from Wu's paper, the proof of which will not be given here.

LEMMA 4.2. *Every nilpotent matrix is the product of three positive semidefinite matrices.*

The proof can be found in [23].

THEOREM 4.3. *Every singular matrix is a product of four positive semidefinite matrices.*

Proof. Suppose that A is an $n \times n$ singular matrix. By Theorem 3.1, we know that $A = BC$ where B has eigenvalues $1, 2, \cdots, n$ and C is nilpotent. So B is similar to a positive definite matrix P_1, *i.e.* $B = S^{-1}P_1S$ for some invertible matrix S. On the other hand, by Lemma 4.2, SCS^{-1} is a product of three positive semidefinite matrices, say, P_2, P_3, and P_4. Now we have

$$
\begin{aligned}
A &= BC \\
&= (S^{-1}P_1S)(S^{-1}P_2P_3P_4S) \\
&= S^{-1}P_1P_2P_3P_4S \\
&= (S^{-1}P_1S^{-1*})(S^*P_2S)(S^{-1}P_3S^{-1*})(S^*P_4S)
\end{aligned}
$$

and each factor is positive semidefinite. ∎

4.3 Products of Involutions

A few more applications of Theorem 2.1 can be found in Sourour's paper [18]. One of them is about factorization of involutions which was first proved in [8] by Gustafson, Halmos, and Radjavi. Recall that a square matrix is called an involution if its square is the identity matrix. In [18], a short proof of this result is given using Theorem 2.1. Here we present the theorem and its short proof.

THEOREM 4.4. *Suppose that* $A \in M_n$. *If* $\det A = \pm 1$, *then* A *is the product of at most four involutions.*

Proof. First we observe that the set of involutions is invariant under similarity. Indeed, if B is an involution, then $P^{-1}BP$ is an involution for any invertible matrix P. It follows that the property of a matrix being the product of k involutions is invariant under similarity for any positive integer k.

To prove Theorem 4.4, first we assume that $A = \lambda I$. From the proof of Theorem 4.1 in Section 4.1, we know that, for $\lambda \neq 0$, $A = \lambda I$ can be factored into four matrices, each of which is similar to

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{bmatrix}, \text{ or } \begin{bmatrix} \lambda^n & 0 & \cdots & 0 \\ 0 & & & 1 \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & 0 \end{bmatrix}.$$

Since in this case $\lambda^n = \det A = \pm 1$, we observe that all of these three matrices are involutions, and so the result follows.

Now we assume that A is non-scalar and that $\det A = 1$. By Theorem 2.1, we may write A as a product BC where B has distinct eigenvalues of the form

$$\left\{\beta_1, \beta_1^{-1}, \cdots, \beta_{\frac{n}{2}}, \beta_{\frac{n}{2}}^{-1}\right\} \text{ or } \left\{1, \beta_1, \beta_1^{-1}, \cdots, \beta_{\frac{n-1}{2}}, \beta_{\frac{n-1}{2}}^{-1}\right\}$$

(depending on whether n is even or odd), and C has distinct eigenvalues of the form

$$\left\{\gamma_1, \gamma_1^{-1}, \cdots, \gamma_{\frac{n}{2}}, \gamma_{\frac{n}{2}}^{-1}\right\} \text{ or } \left\{1, \gamma_1, \gamma_1^{-1}, \cdots, \gamma_{\frac{n-1}{2}}, \gamma_{\frac{n-1}{2}}^{-1}\right\}.$$

Notice that since B and C both have n distinct eigenvalues, they are diagonalizable. On the other hand, we know that

$$\begin{bmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \delta^{-1} \\ \delta & 0 \end{bmatrix}$$

for any nonzero $\delta \in \mathbb{C}$. So let

$$B_1 = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_1^{-1} \end{bmatrix}, \quad B_1' = \begin{bmatrix} 0 & \beta_1^{-1} \\ \beta_1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then we have

$$B \sim \begin{bmatrix} B_1 & & & O \\ & B_2 & & \\ & & \ddots & \\ O & & & B_{\frac{n}{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & & & O \\ & \mathbf{Q} & & \\ & & \ddots & \\ O & & & \mathbf{Q} \end{bmatrix} \begin{bmatrix} B_1' & & & O \\ & B_2' & & \\ & & \ddots & \\ O & & & B_{\frac{n}{2}}' \end{bmatrix}$$

if n is even, or

$$B \sim \begin{bmatrix} 1 & & & O \\ & B_1 & & \\ & & \ddots & \\ O & & & B_{\frac{n-1}{2}} \end{bmatrix} = \begin{bmatrix} 1 & & & O \\ & \mathbf{Q} & & \\ & & \ddots & \\ O & & & \mathbf{Q} \end{bmatrix} \begin{bmatrix} 1 & & & O \\ & B_1' & & \\ & & \ddots & \\ O & & & B_{\frac{n-1}{2}}' \end{bmatrix}$$

if n is odd. All four factors above are easily seen to be involutions. Similar factorization can be obtained for C, and it follows that A is a product of four involutions.

Next assume that $\det A = -1$. If n is odd, then $\det(-A) = 1$ and from above we know that $-A = B_1B_2B_3B_4$ where B_i is an involution.

It follows that $A = (-B_1)B_2B_3B_4$ and we have the desired factorization. If n is even, then we may write A as a product BC where B has distinct eigenvalues $\left\{1,-1,\beta_1,\beta_1^{-1},\cdots,\beta_{\frac{n-2}{2}},\beta_{\frac{n-2}{2}}^{-1}\right\}$ and C has distinct eigenvalues $\left\{1,-1,\gamma_1,\gamma_1^{-1},\cdots,\gamma_{\frac{n-2}{2}},\gamma_{\frac{n-2}{2}}^{-1}\right\}$. Similarly, we have

$$B \sim \begin{bmatrix} R & & & O \\ & B_1 & & \\ & & \ddots & \\ O & & & B_{\frac{n-2}{2}} \end{bmatrix} = \begin{bmatrix} R & & & O \\ & Q & & \\ & & \ddots & \\ O & & & Q \end{bmatrix} \begin{bmatrix} I_2 & & & O \\ & B_1' & & \\ & & \ddots & \\ O & & & B_{\frac{n-2}{2}}' \end{bmatrix},$$

where

$$B_1 = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_1^{-1} \end{bmatrix}, \quad B_1' = \begin{bmatrix} 0 & \beta_1^{-1} \\ \beta_1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the two matrix factors are involutions. Similar factorization can be obtained for C, and hence A is a product of four involutions. ■

4.4 Products of Positive Definite Matrices and Commutators

In [18], Sourour also gives short proofs to Ballantine's results [2, 3] about products of positive definite matrices, and to Shoda's and Thompson's results [17, 19, 20] about commutators of matrices. Here, we present these two theorems, details of the short proofs can be found in [18].

Theorem 4.5. *Suppose that* $A \in M_n$ *Then*

(*a*) A *is a product of four positive definite matrices if and only if* $\det A > 0$ *and* A *is not a scalar matrix* λI *where* λ *is not positive*

(*b*) A *is a product of five positive definite matrices if and only if* $\det A > 0$

For the following theorem, given that $S \subsetneq GL(n,\mathbb{C})$, we say that a matrix $A \in M_n$ is a commutator of matrices in S if $A = BCB^{-1}C^{-1}$ for some B and C in S

Theorem 4.6. *Let* $A \in SL(n,\mathbb{C})$.

(*a*) *Then* A *is a commutator of matrices in* $GL(n,\mathbb{C})$.

(*b*) *If* A *is nonscalar, then* A *is a commutator of matrices in* $SL(n,\mathbb{C})$.

(*c*) *If* A *is nonscalar, then* A *is a commutator of matrices with arbitrarily prescribed nonzero determinants.*

CHAPTER V

CONCLUSION

We have shown that a necessary and sufficient condition for a singular matrix, which is not a nonzero 2×2 nilpotent matrix, to be factored into two matrices with prescribed eigenvalues is that at least $n - k$ of these prescribed eigenvalues are zero, where k is the rank of the matrix. The nonzero 2×2 nilpotent matrices are exceptional cases, for they cannot be written as a product of two nilpotent matrices.

We have also shown that the factorization theorems presented herein can be used to shorten the proofs of some old results, replacing some computational proofs by more transparent conceptional proofs.

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