

A NOTE ON CERTAIN GENERATING
FUNCTIONS FOR THE GENERALIZED
BESSEL POLYNOMIALS

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DMS-572-IR

January 1991

ABSTRACT

The main object of the present paper is to prove a general theorem on bilinear, bilateral, or mixed multilateral generating functions for the generalized Bessel polynomials. It is pointed out that this theorem and some other earlier results on the subject would provide unifications (and generalizations) of numerous generating functions which were obtained recently by using group-theoretic techniques. Some interesting applications of these general results are also indicated.

1. INTRODUCTION AND THE MAIN RESULTS

Over four decades ago, Krall and Frink [5] initiated a systematic study of what are now well-known in the literature as the Bessel polynomials. In their terminology, the *generalized Bessel polynomials* $y_n(x, \alpha, \beta)$ are defined by

$$(1.1) \quad y_n(x, \alpha, \beta) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} \alpha+n+k-2 \\ k \end{bmatrix} k! \left(\frac{x}{\beta}\right)^k$$

and the *simple Bessel polynomials* $y_n(x)$ are defined by

$$(1.2) \quad y_n(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} k! \left(\frac{x}{2}\right)^k = y_n(x, 2, 2).$$

These polynomials are orthogonal and satisfy a second-order differential equation. Thus they may be viewed as an additional member of the family of classical orthogonal polynomials, the other members being the Jacobi, Hermite, and Laguerre polynomials, and indeed also such special cases of the Jacobi polynomials as the Gegenbauer (or ultraspherical) polynomials, Legendre (or spherical) polynomials, and the Tchebycheff polynomials of the first and second kinds (see, for details, Szegö [9]). The importance of the Bessel

polynomials lies in the fact that they arise naturally in a number of seemingly diverse contexts; *e.g.*, in connection with the solution of the wave equation in spherical polar co-ordinates [5], in network synthesis and design [3], in the representation of the energy spectral functions for a family of isotropic turbulence fields [7], and so on. For further information and details about these polynomials and their applications, we refer the reader to the excellent monograph by Grosswald [4].

In terms of the generalized hypergeometric function ${}_uF_v$ (with u numerator and v denominator parameters) defined by

$$(1.3) \quad {}_uF_v(a_1, \dots, a_u; \beta_1, \dots, \beta_v; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_u)_n}{(\beta_1)_n \cdots (\beta_v)_n} \frac{z^n}{n!},$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$, the Bessel polynomials in (1.1) and (1.2) can be expressed as

$$(1.4) \quad y_n(x, \alpha, \beta) = {}_2F_0(-n, \alpha+n-1; -; -x/\beta)$$

and

$$(1.5) \quad y_n(x) = {}_2F_0(-n, n+1; -; -\frac{1}{2}x),$$

respectively. For these hypergeometric polynomials, a considerably large number of linear, bilinear, bilateral, and mixed multilateral generating functions are known (*cf.*, *e.g.*, Srivastava and Manocha [8]). Of our concern here are the following classes of bilateral or mixed multilateral generating functions for the Bessel polynomials, which were first proven by Srivastava [6, pp. 228–228, Corollaries 1 and 2] and which have since been reproduced in the latest treatise on the subject by Srivastava and Manocha [8, p. 421, Corollaries 1 and 2]:

THEOREM 1 (*cf.* Srivastava [6]). *Corresponding to a non-vanishing function $\Omega_{\mu}(y_1, \dots, y_s)$ of s variables ξ_1, \dots, ξ_s ($s \in \mathbb{N} = \{1, 2, 3, \dots\}$) and of (complex) order μ , let*

$$(1.6) \quad \Lambda_{m,p,q}^{(1)}[x; \xi_1, \dots, \xi_s; \eta] = \sum_{n=0}^{\infty} a_n y_{m+qn}(x, \alpha-qn, \beta) \\ \cdot \Omega_{\mu+pn}(\xi_1, \dots, \xi_s) \frac{\eta^n}{(qn)!} \quad (a_n \neq 0; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p, q \in \mathbb{N}),$$

where α and β ($\neq 0$) are complex parameters. Suppose also that

$$(1.7) \quad M_{n,q}^{p,\mu}(\xi_1, \dots, \xi_s; \eta) \\ = \sum_{k=0}^{[n/q]} \begin{bmatrix} n \\ qk \end{bmatrix} a_k \Omega_{\mu+pk}(\xi_1, \dots, \xi_s) \eta^k,$$

where, as usual, $[\lambda]$ represents the greatest integer in λ .

Then

$$(1.8) \quad \sum_{n=0}^{\infty} y_{m+n}(x, \alpha-n, \beta) M_{n,q}^{p,\mu}(\xi_1, \dots, \xi_s; \eta) \frac{t^n}{n!} \\ = \left[1 - \frac{xt}{\beta}\right]^{1-\alpha-m} \exp(t) \\ \cdot \Lambda_{m,p,q}^{(1)}\left[\frac{\beta x}{\beta-xt}; \xi_1, \dots, \xi_s; \eta t^q\right] \quad (|t| < |\beta/x|),$$

provided that each member of (1.8) exists.

THEOREM 2 (cf. Srivastava [6]). Under the hypotheses of Theorem 1, let

$$(1.9) \quad \Lambda_{m,p,q}^{(2)}[x; \xi_1, \dots, \xi_s; \eta] = \sum_{n=0}^{\infty} a_n y_{m+qn}(x) \\ \cdot \Omega_{\mu+pn}(\xi_1, \dots, \xi_s) \frac{\eta^n}{(qn)!} \quad (a_n \neq 0; m \in \mathbb{N}_0; p, q \in \mathbb{N}).$$

Suppose also that the function

$$M_{n,q}^{p,\mu}(\xi_1, \dots, \xi_s; \eta)$$

is defined, as before, by (1.7).

Then

$$(1.10) \quad \sum_{n=0}^{\infty} y_{m+n}(x) M_{n,q}^{p,\mu}(\xi_1, \dots, \xi_s; \eta) \frac{t^n}{n!} = (1-2xt)^{-\frac{1}{2}(m+1)} \\ \cdot \exp(x^{-1}\{1 - (1-2xt)^{\frac{1}{2}}\}) \\ \cdot \Lambda_{m,p,q}^{(2)}[x(1-2xt)^{-\frac{1}{2}}; \xi_1, \dots, \xi_s; \eta t^q(1-2xt)^{-\frac{1}{2}q}],$$

provided that each member of (1.10) exists.

A closer examination of the assertions (1.8) and (1.10) would reveal the remarkable fact that, in spite of the relationship (1.2), Theorem 2 cannot be deduced as a special case of Theorem 1. As a matter of fact, Theorem 2 happens to yield the most general family of bilateral or mixed multilateral generating functions for the *simple* Bessel polynomials, which has been found so far. Numerous obvious special cases of Theorem 2 are being rediscovered in the literature, especially by group-theoretic means. For example, we may cite the main *result* of Chongdar and Mukherjee [2, p. 59, Theorem 1] which actually is a very specialized version of Theorem 2 above when

$$p = q = s = 1.$$

Motivated by some other results of Chongdar [1] on the *generalized* Bessel polynomials, analogous to those asserted by Theorem 1 above, we shall present here a rather elementary proof (*without* using the group-theoretic techniques employed in [1]) of the following interesting generalization of Theorem 1:

THEOREM 3. *Under the hypotheses of Theorem 1, let*

$$(1.11) \quad \Lambda_{m,p,q}^{(3)}[x; \xi_1, \dots, \xi_s; \eta] = \sum_{n=0}^{\infty} a_n y_{m+qn}(x, \alpha + (\rho-1)qn, \beta) \\ \cdot \Omega_{\mu+pn}(\xi_1, \dots, \xi_s) \frac{\eta^n}{(qn)!} \quad (a_n \neq 0; \quad m \in \mathbb{N}_0; \quad p, q \in \mathbb{N}),$$

where ρ is a suitable complex parameter. Suppose also that

$$\Phi_{n,p,q}^{\alpha, \mu, \rho}(x; \xi_1, \dots, \xi_s; \eta)$$

is a polynomial of degree $[n/q]$ in η (with coefficients depending upon ξ_1, \dots, ξ_s) defined by

$$(1.12) \quad \Phi_{n,p,q}^{\alpha, \mu, \rho}(x; \xi_1, \dots, \xi_s; \eta) = \sum_{k=0}^{[n/q]} \begin{bmatrix} n \\ qk \end{bmatrix} a_k y_{m+n}(x, \alpha - n + \rho qk, \beta) \\ \cdot \Omega_{\mu+pk}(\xi_1, \dots, \xi_s) \eta^k.$$

Then

$$(1.13) \quad \sum_{n=0}^{\infty} \Phi_{n,p,q}^{\alpha, \mu, \rho}(x; \xi_1, \dots, \xi_s; \eta) \frac{t^n}{n!} = \left[1 - \frac{xt}{\beta}\right]^{1-\alpha-m} \exp(t) \\ \cdot \Lambda_{m,p,q}^{(3)}\left[\frac{\beta x}{\beta - xt}; \xi_1, \dots, \xi_s; \eta t^q \left[1 - \frac{xt}{\beta}\right]^{-\rho q}\right] \quad (|t| < |\beta/x|),$$

provided that each member of (1.13) exists.

2. PROOF OF THEOREM 3

For the sake of convenience, let \mathcal{S} denote the first member of the assertion (1.13) of Theorem 3. Then, by substituting for the polynomials

$$\Phi_{n,p,q}^{\alpha,\mu,\rho}(x,\xi_1,\dots,\xi_s;\eta)$$

from (1.12) into the left-hand side of (1.13), we obtain

$$\begin{aligned} (2.1) \quad \mathcal{S} &= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{\lfloor n/q \rfloor} a_k y_{m+n}(x, \alpha - n + \rho qk, \beta) \Omega_{\mu+pk}(\xi_1, \dots, \xi_s) \frac{\eta^k}{(n-qk)!(qk)!} \\ &= \sum_{k=0}^{\infty} a_k \Omega_{\mu+pk}(\xi_1, \dots, \xi_s) \frac{(\eta t^q)^k}{(qk)!} \\ &\quad \cdot \sum_{n=0}^{\infty} y_{m+qk+n}(x, \alpha + (\rho-1)qk - n, \beta) \frac{t^n}{n!}, \end{aligned}$$

where we have inverted the order of the double summation involved.

The inner series in (2.1) can be summed by appealing to the familiar generating function (*cf.*, *e.g.*, [8, p. 419, Equation 8.4(8)]):

$$\begin{aligned} (2.2) \quad \sum_{n=0}^{\infty} y_{m+n}(x, \alpha - n, \beta) \frac{t^n}{n!} &= \left[1 - \frac{xt}{\beta}\right]^{1-\alpha-m} \exp(t) \\ &\quad \cdot y_m\left[\frac{\beta x}{\beta - xt}, \alpha, \beta\right] \quad (m \in \mathbb{N}_0; |t| < |\beta/x|) \end{aligned}$$

with, of course, m and α replaced by $m + qk$ and $\alpha + (\rho-1)qk$, respectively ($k \in \mathbb{N}_0; q \in \mathbb{N}$), and we thus find from (2.1) that

$$(2.3) \quad \mathcal{S} = \left[1 - \frac{xt}{\beta}\right]^{1-\alpha-m} \exp(t)$$

$$\cdot \sum_{k=0}^{\infty} \frac{a_k}{(qk)!} y_{m+qk} \left[\frac{\beta x}{\beta - xt}, \alpha + (\rho - 1)qk, \beta \right] \left\{ \eta t^q \left[1 - \frac{xt}{\beta} \right]^{-\rho q} \right\}^k \quad (|t| < |\beta/x|),$$

which, in view of the definition (1.11), is precisely the second member of the assertion (1.13) of Theorem 3.

This evidently completes the proof of Theorem 3 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 3 holds true for those values of the various parameters and variables involved for which each member of the assertion (1.13) exists.

3. APPLICATIONS

Just as we remarked above about Theorem 2, Theorem 3 provides a considerably deep generalization of all those classes of bilateral and mixed multilateral generating functions for the generalized Bessel polynomials which can be deduced as (known or new) consequences of Theorem 1.

In its special case when $\rho = 0$, Theorem 3 reduced immediately to Theorem 1. Moreover, Theorem 3 with

$$m = 0, \quad \rho = q = 1, \quad \text{and} \quad \Omega_{\mu}(\xi_1, \dots, \xi_s) \equiv 1$$

would yield one of the two main results of Chongdar [1, p. 151, Theorem 1]. His other main result [1, p. 151, Theorem 2] corresponds to the special case of Theorem 3 above when

$$m = 0 \quad \text{and} \quad \rho = p = q = s = 1.$$

It should be noted that our proof of Theorem 3, based upon the familiar result (2.2), is markedly different from the group-theoretic derivations of the aforementioned special cases of Theorem 3 by Chongdar [1].

For each suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable function

$$\Omega_{\mu}(\xi_1, \dots, \xi_s) \quad (s > 1)$$

is expressed as an appropriate product of several simpler functions, each of the results considered here (including, for example, Theorem 3 above) can be applied to derive various families of mixed multilateral generating functions for the generalized Bessel polynomials. We choose to leave the details involved in these applications of Theorem 3 as an exercise for the interested reader.

Acknowledgements

The present investigation was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

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