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Abstract. The max algebra consists of the set of real numbers, along with negative infinity, equipped with two binary operations, maximization and addition. This algebra is useful in describing certain conventionally nonlinear systems in a linear fashion. Eigenvalues and eigenvectors of matrices over the max algebra are investigated, and proofs are presented for new results as well as for some known results not readily available in the literature. Properties of eigenvalues and eigenvectors that depend solely on the pattern of finite and infinite entries in the matrix are studied. Inequalities for the maximal eigenvalue of a matrix, motivated by those for the Perron root of a nonnegative matrix, are proved.

1. Introduction. The algebraic system called "max algebra" has been used to describe in a linear fashion phenomena which are nonlinear in the conventional algebra. Examples include transportation networks, machine scheduling, and parallel computation. A system in which one component must wait for results from other components (a "discrete event dynamic system") can be modelled in max algebra. See [11, Ch. 1] for a detailed description of such systems. As described there, the question of regularizing a system, that is of initiating a system in such a way that all components begin cycles at the same time, is answered by solving the eigenproblem in max algebra.

An early exposition of max algebra is the monograph of Cuninghame-Green [11]. Related works are Carré [6, Ch. 3], and Gondran and Minoux [16], which discuss more general "path algebras" and describe Gaussian and related solutions of linear systems over path algebras. Currently work on max algebra systems is progressing in many directions, see [1, 5, 9, 15, 22]. Eigenproblems for irreducible matrices over the max algebra were studied in [11]. An extension of this work, in a direction other than that taken in this paper, appears in [21].

Briefly put, the scalars in max algebra are the real numbers together with $-\infty$. The sum of two scalars is their maximum, and the product is their conventional sum. Matrix operations are defined as usual, but using the max algebra scalar operations. The exponential function provides a natural one-to-one map from $\mathbb{R} \cup \{-\infty\}$ onto the nonnegative reals. Under this correspondence, matrices over max algebra correspond to nonnegative matrices over the reals, and much of our work is motivated by the theory of nonnegative matrices. Techniques of proof for max algebra sometimes reflect those for conventional algebra. In particular, the directed graph of a matrix, which provides much information in the study of nonnegative matrices, plays an even more central role in matrices over max algebra. A glance at the definition of $\mu(A)$ in Section 2 and at Theorem 2.8 will confirm this.

In Section 2 we give the basic definitions and results for eigenvalues and eigenvectors of general square matrices over the max algebra. Some of these results can be found scattered in the literature [1, 8, 11, 15, 24, 25]; we have put them together in a more accessible form

In Section 3 we study reducible matrices. We give necessary and sufficient conditions for the eigenvalue of an irreducible component (or $-\infty$) to be an eigenvalue of A (Theorems 3.3, 3.4). We also characterize matrices that admit a finite eigenvector. We remark that in the context of a discrete event dynamic system, the existence of a finite eigenvector implies that the system can be regularized. As our work was nearing completion, we discovered that some of these spectral results had been presented in the thesis of Gaubert [15, Ch. 4] and in [24].

In applications to discrete event dynamic systems such as machine scheduling or parallel computing, it may be useful to obtain information about eigenvalues and eigenvectors given only partial information concerning the entries of the matrix. In particular, it may be known which components of the system must wait for input from which other components, while the waiting times are unknown. It will then be known where the finite entries of the matrix of interest occur, but their magnitudes will be unknown; that is, only the "pattern" of the matrix will be specified. In Section 4, we obtain results concerning eigenvalues and eigenvectors which depend only on the pattern of the given matrix.

Finally, in Section 5 we present inequalities concerning the maximal circuit mean of a matrix over the max algebra. Most of these are motivated by known corresponding inequalities for the spectral radius of a nonnegative matrix.

2. Eigenvalues and Eigenvectors. The max algebra consists of the set $\boldsymbol{M} = \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the set of real numbers, equipped with two binary operations, addition and multiplication, denoted by $\oplus$ and $\otimes$ respectively. The operations are defined as follows

$$a \oplus b = \max(a, b), \text{ the maximum of } a \text{ and } b$$

and

$$a \otimes b = a + b.$$

Clearly, $-\infty$ and 0 serve as identity elements for the operations $\oplus$ and $\otimes$ respectively. We denote $x_1 \oplus \ldots \oplus x_n$ by $\sum_{i=1}^{n}{}_{\oplus}\, x_i$, or by $\sum_{\oplus} x_i$ when the range of summation of the index i is clear from the context.

We deal with vectors and matrices over the max algebra. Basic operations on matrices are defined in the natural way. Thus, if $A = [a_{ij}]$, $B = [b_{ij}]$ are $m \times n$ matrices over M, then $A \oplus B$ is the $m \times n$ matrix with (i,j)-entry $a_{ij} \oplus b_{ij}$. If $c \in M$, then $c \otimes A$ is the matrix $[c \otimes a_{ij}] = [c + a_{ij}]$. If A is $m \times n$ and B is $n \times p$, then $A \otimes B$ is the $m \times p$ matrix with (i,j)-entry

$$\sum_{k=1}^{n}{}_{\oplus}\, a_{ik} \otimes b_{kj} = \max_{k} (a_{ik} + b_{kj}).$$

It is easily verified that matrix multiplication is associative and that it distributes over matrix addition. If A is $n \times n$, we denote $A \otimes \ldots \otimes A$ taken k times by A^k.

The transpose of the matrix A is denoted by A^T. The $n \times n$ matrix with each diagonal entry zero and each off-diagonal entry $-\infty$ is the identity matrix over the max algebra. If we permute the rows (and/or columns) of the identity matrix then we obtain a permutation matrix over the max algebra. If A, B are $m \times n$ matrices over M, then $A \geq B$ means that $a_{ij} \geq b_{ij}$ for all i, j. Similarly $A > B$ means that $a_{ij} > b_{ij}$ for all i, j.

A column or row vector x over M is said to be finite if each component x_i of the vector is finite. A vector is called partly infinite if it has a finite component as well as an infinite component. A matrix or vector with each component $-\infty$ is called infinite and we denote it by $-\infty$ as well; this should not cause any confusion.

Let A be an $n \times n$ matrix over $\mathbb{M}$. We associate a directed graph (digraph) $G(A)$ with A as follows. The vertices of $G(A)$ are $1, 2, \ldots, n$. There is an edge from vertex i to vertex j, denoted by (i,j), if a_{ij} is finite and in that case we say that a_{ij} is the weight of the edge (i,j). We use standard terminology from the theory of digraphs. Thus a path of length ℓ in a digraph is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_\ell, i_{\ell+1})$, also denoted by $i_1 \to i_2 \to \ldots \to i_\ell \to i_{\ell+1}$. The weight of a path is the sum of the weights of the edges in the path. The average weight of the path $i_1 \to i_2 \to \ldots \to i_{\ell+1}$ is defined as

$$\frac{a_{i_1 i_2} + a_{i_2 i_3} + \ldots + a_{i_\ell i_{\ell+1}}}{\ell}.$$

A circuit of length ℓ is a closed path $i_1 \to i_2 \to \ldots \to i_\ell \to i_1$, where $i_1, \ldots, i_\ell$ are distinct. We write this circuit as $(i_1\ i_2\ \ldots\ i_\ell)$ and while doing so we make the convention that i_1 be the least among the integers $i_1, \ldots, i_\ell$. A circuit of length one is a loop. We denote the set of circuits in $G(A)$, or in A, by $C(A)$. If $\tau \in C(A)$ then the average weight of τ is called the mean of the circuit τ, denoted by $M_A(\tau)$. We define the maximal circuit mean of A, denoted by $\mu(A)$, as

$$\mu(A) = \max_{\tau \in C(A)} M_A(\tau)$$

if $C(A) \neq \phi$, and we set $\mu(A) = -\infty$ otherwise. A circuit $\tau \in C(A)$ is called a critical circuit if $M_A(\tau) = \mu(A)$. The set of all critical circuits in A is denoted by $\tilde{C}(A)$.

A digraph is strongly connected if there exists a path from any vertex to any other vertex. We say that the matrix A is irreducible if $G(A)$ is strongly connected. If A is not irreducible then we say that it is reducible. If A is an $n \times n$ matrix over $\boldsymbol{M}$ then clearly A is irreducible if and only if $\left[e^{a_{ij}}\right]$ is a nonnegative, irreducible matrix in the usual sense (see e.g. [3]). We also remark that A is reducible if and only if either A is 1×1 containing $-\infty$ or there exists a permutation matrix Q over the max algebra such that

$$Q \otimes A \otimes Q^T = \begin{bmatrix} A_{11} & -\infty \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square matrices of order at least one.

We now define the concepts of eigenvalues and eigenvectors over the max algebra. Let A be an $n \times n$ matrix over $\boldsymbol{M}$. We say that $\lambda \in \boldsymbol{M}$ is an eigenvalue of A if there exists a vector $x \neq -\infty$ such that

$$A \otimes x = \lambda \otimes x. \tag{2.1}$$

In this case we say that x is an eigenvector of A corresponding to the eigenvalue λ. Furthermore, we call (λ, x) an eigenpair of A. Note that (λ, x) is an eigenpair of A if and only if $x \neq -\infty$ and $\max_j (a_{ij} + x_j) = \lambda + x_i$, $i = 1, 2, \ldots, n$. For example, if $A = \begin{bmatrix} 3 & -\infty \\ 2 & 4 \end{bmatrix}$, then $A \otimes \begin{bmatrix} -\infty \\ 0 \end{bmatrix} = 4 \otimes \begin{bmatrix} -\infty \\ 0 \end{bmatrix}$, thus 4 is an eigenvalue of A. It can be checked that 4 is the only eigenvalue of A. Note that A^T has both 3 and 4 as eigenvalues.

If Q is a permutation matrix over the max algebra and $\lambda \in \boldsymbol{M}$ then (λ, x) is an eigenpair of A if and only if $(\lambda, Q \otimes x)$ is an eigenpair of $Q \otimes A \otimes Q^T$. In particular, A and $Q \otimes A \otimes Q^T$ have the same eigenvalues. In view of these observations we often find it convenient to deal with $Q \otimes A \otimes Q^T$ with a suitable permutation matrix Q instead of the matrix A itself. Note that $G(A)$ and $G(Q \otimes A \otimes Q^T)$ are identical except for the labelling of the vertices.

The first three results deal with the occurrence of $-\infty$ as an eigenvalue.

LEMMA 2.1. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$ and suppose $\boldsymbol{C}(A) = \phi$. Then $-\infty$ is the unique eigenvalue of A.*

Proof. Since $\boldsymbol{C}(A)$ has no circuit, we can relabel the vertices of $G(A)$ such that there is no edge from i to j if $i \leq j$ (see e.g. [6, p. 50]). Thus we may assume, without loss of generality, that

$$A = \begin{bmatrix} -\infty & \cdots & -\infty \\ a_{21} & -\infty & \vdots \\ \vdots & \vdots \ddots & \\ a_{n1} & a_{n2} \cdots & -\infty \end{bmatrix}.$$

The vector $x = [-\infty \ldots -\infty\, 0]^T$ satisfies $A \otimes x = -\infty \otimes x$ and hence $-\infty$ is an eigenvalue of A. Now suppose (λ, y) is an eigenpair of A, so that

$$A \otimes y = \lambda \otimes y. \tag{2.2}$$

Suppose $\lambda \neq -\infty$. Then the first equation in the system (2.2) gives $y_1 = -\infty$. The second equation in (2.2) is $a_{21} + y_1 = \lambda + y_2$, and since $y_1 = -\infty$, we have $y_2 = -\infty$. Continuing in this way we conclude that all components of y are $-\infty$, which is a contradiction. Thus $\lambda = -\infty$ and the proof is complete. □

We remark here that the converse of Lemma 2.1 is also true, thus $\boldsymbol{C}(A) = \phi$ if $-\infty$ is the only eigenvalue of A. This will follow from Theorem 2.8.

LEMMA 2.2. *Let* A *be an* $n \times n$ *matrix over* $\boldsymbol{M}$. *Then* $-\infty$ *is an eigenvalue of* A *if and only if* A *has an infinite column.*

In fact, if $(-\infty, x)$ is an eigenpair of A and if x_j is finite, then the j-th column of A is $-\infty$. In particular, if $(-\infty, x)$ is an eigenpair of A and x is finite, then $A = -\infty$. The "only if" part of Lemma 2.2 is contained in [11, p. 201]. The next result is an immediate consequence of Lemma 2.2.

COROLLARY 2.3. *If* A *is an irreducible matrix then any eigenvalue of* A *is finite.*

We will see later in this section that if A is irreducible, then $\mu(A)$ is the only eigenvalue of A.

LEMMA 2.4. *Let* A *be an* $n \times n$ *matrix over* $\boldsymbol{M}$. *If* A *has a partly infinite eigenvector then* A *is reducible.*

Proof. Let (λ, x) be an eigenpair of A with x partly infinite. If $\lambda = -\infty$ then by Lemma 2.2 we conclude that A has an infinite column and hence A is reducible. So suppose that λ is finite. We assume, after permuting the rows and columns of A in an identical fashion if necessary, that $x = [-\infty^T \ y^T]^T$ where y is finite. Partition A conformally so that we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \otimes \begin{bmatrix} -\infty \\ y \end{bmatrix} = \lambda \otimes \begin{bmatrix} -\infty \\ y \end{bmatrix}.$$

Then $A_{12} \otimes y = \lambda \otimes -\infty$. Since y is finite, $A_{12} = -\infty$ and hence A is reducible. □

COROLLARY 2.5. *If A is an $n \times n$ irreducible matrix over M then any eigenvector of A is finite.*

We now state a simple fact about digraphs.

LEMMA 2.6. *Let G be a digraph in which every vertex has out-degree at least one. Then G has a circuit.*

Proof. If G has no circuit, then, as noted in the proof of Lemma 2.1, we can relabel the vertices of G so that there is no edge from i to j if $i \le j$. But then vertex 1 has out-degree zero which is a contradiction. Therefore G must have a circuit. □

THEOREM 2.7. *Let A be an $n \times n$ matrix over M, let (λ, x) be an eigenpair of A and suppose x is finite. Then $\lambda = \mu(A)$.*

Proof. If $\lambda = -\infty$ then $A = -\infty$ (see the remark following Lemma 2.2) and hence $\lambda = \mu(A)$. So we can assume that λ is finite. We have

$$\max_j (a_{ij} + x_j) = \lambda + x_i, \quad i = 1, 2, \ldots, n \tag{2.3}$$

and hence, in particular

$$a_{ij} + x_j - x_i \le \lambda, \quad i, j = 1, 2, \ldots, n. \tag{2.4}$$

Since λ is finite, we must have $C(A) \neq \phi$ by Lemma 2.1. Let $\tau = (i_1 i_2 \dots i_k) \in C(A)$. Apply (2.4) to the entries of A in positions $(i_1, i_2), (i_2, i_3), \dots, (i_k, i_1)$, and add the resulting inequalities to conclude that $M_A(\tau) \leq \lambda$.

Construct a digraph H with vertices $1, 2, \dots, n$ and an edge (i,j) if and only if there is equality in (2.4), i.e., $a_{ij} + x_j - x_i = \lambda$. It follows from (2.3) that each vertex in H has out-degree at least one and hence by Lemma 2.6, H has a circuit, say, τ. Since λ is finite, H is clearly a subgraph of $G(A)$. Thus $\tau \in C(A)$ and furthermore, $M_A(\tau) = \lambda$. Therefore, $\lambda = \mu(A)$ and the proof is complete. □

Let A be an $n \times n$ matrix over M with $C(A) \neq \phi$ and let B be the $n \times n$ matrix with $b_{ij} = a_{ij} - \mu(A)$ for all i, j. Define $\Gamma(B)$ by

$$\Gamma(B) = B \oplus B^2 \oplus \dots \oplus B^n. \tag{2.5}$$

First observe that the (i,j)-entry of B^k is precisely the maximal weight of a path of length k from vertex i to vertex j in $G(B)$. Any path in $G(B)$ of length more than n must contain a circuit and since any circuit mean in $C(B)$ is at most zero, we have

$$\Gamma(B) \geq B^k, \quad k \geq n+1. \tag{2.6}$$

It follows from (2.5), (2.6) that the (i,j)-entry of $\Gamma(B)$, denoted by $\Gamma_{ij}(B)$, is the maximal weight of a path (of any length) from i to j in $G(B)$.

The next result is known in the case that the matrix is irreducible, see [11, p. 185-190]; our proof technique is similar. The result for an irreducible matrix will be stated as a corollary.

THEOREM 2.8. *Let* A *be an* $n \times n$ *matrix over* M. *Then* $\mu(A)$ *is an* *eigenvalue of* A.

Proof. If $C(A) = \phi$, *then* $\mu(A) = -\infty$ and it is an eigenvalue of A by Lemma 2.1. So we assume that $C(A) = \phi$, in which case $\mu(A)$ must be finite. Let B be the matrix with $b_{ij} = a_{ij} - \mu(A)$ for all i, j (thus $A = \mu(A) \otimes B$) and let $\Gamma(B)$ be as in (2.5). Let $j \in \{1,2,...,n\}$ be a vertex which belongs to a critical circuit in $C(B)$. Any path from j to itself in $G(B)$ must have weight at most zero and since j belongs to a critical circuit, there is at least one path from j to itself of weight zero. Thus $\Gamma_{jj}(B) = 0$ and, in particular, $\Gamma_j(B)$, the j-th column of $\Gamma(B)$, is not $-\infty$.

Now observe that for any i, the maximal weight of a path, of any length, from i to j may also be expressed as

$$\sum_{\ell=1}^{n}{}_{\oplus}\; b_{i\ell} \otimes \Gamma_{\ell j}(B) \tag{2.7}$$

since there exists such a path (in which a vertex may occur more than once) of length at least two, in view of the fact that j is on a critical circuit. However, (2.7) is precisely the i-th entry of $B \otimes \Gamma_j(B)$. Therefore $B \otimes \Gamma_j(B) = \Gamma_j(B) = 0 \otimes \Gamma_j(B)$ and hence $(0, \Gamma_j(B))$ is an eigenpair of B. Now $A \otimes \Gamma_j(B) = \mu(A) \otimes B \otimes \Gamma_j(B) = \mu(A) \otimes \Gamma_j(B)$ and therefore $\mu(A)$ is an eigenvalue of A. □

COROLLARY 2.9. *Let* A *be an irreducible* $n \times n$ *matrix over* $\mathbb{M}$. *Then* $\mu(A)$ *is the only eigenvalue of* A *and every eigenvector corresponding to* $\mu(A)$ *is finite.*

Proof. By Theorem 2.8, $\mu(A)$ is an eigenvalue of A. Suppose (λ, x) is an eigenpair of A. Since A is irreducible, by Corollary 2.3, $\lambda \neq -\infty$. By Corollary 2.5, x is finite and then it follows from Theorem 2.7 that $\lambda = \mu(A)$. Any eigenvector corresponding to $\mu(A)$ is finite by Corollary 2.5 and the proof is complete. □

Let A be an $n \times n$ matrix over $\boldsymbol{M}$. The critical graph of A is a digraph with vertices $1, 2, ..., n$, defined as follows. For $i,j \in \{1,2,...,n\}$, (i,j) is an edge in the critical graph of A if and only if it belongs to a critical circuit in $\boldsymbol{C}(A)$. The next result is known (see [11, p. 191]) and will be used in Sections 4 and 5.

THEOREM 2.10. *Let* A *be an* $n \times n$ *irreducible matrix over* $\boldsymbol{M}$. *Then the following conditions are equivalent.*

(i) *The critical graph of* A *is strongly connected.*

(ii) *If* x *and* y *are both eigenvectors of* A *then* $x = \alpha \otimes y$ *for some* $\alpha \in \mathbb{R}$.

According to Theorem 2.10, for an irreducible matrix A, the critical graph of A is strongly connected if and only if A has a unique eigenvector, up to a scalar multiple over the max algebra.

3. Reducible Matrices. Let A be an $n \times n$ matrix over $\boldsymbol{M}$. It is well-known (see e.g. [3, p. 39]) that by permuting the rows and columns of A in an identical fashion we can bring A to the Frobenius Normal Form. More precisely, there exists a permutation matrix Q over the max algebra such that $Q \otimes A \otimes Q^T$ equals

$$\begin{bmatrix} A_{11} & -\infty & \cdots & -\infty \\ A_{21} & A_{22} & \cdots & -\infty \\ \vdots & \vdots & \ddots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{bmatrix} \tag{3.1}$$

where each A_{ii} is either square and irreducible or is 1×1 containing $-\infty$.

Let A be an $n \times n$ matrix over $\boldsymbol{M}$ and let $G(A)$ be the digraph associated with A as in Section 2. If $i,j \in \{1,2,...,n\}$ then we say that i is equivalent to j if $i = j$

or if in $G(A)$ there exists a path from i to j and a path from j to i. This is clearly an equivalence relation.

Now suppose that A is in Frobenius Normal Form (3.1). The following notation will be used throughout this section. For $k = 1, 2, ..., q$ let V_k denote the set of indices of rows in A that intersect the diagonal block A_{kk}, and let α_k be the number of elements in V_k. The sets V_k are called the classes of A. Observe that $i, j \in \{1, 2, ..., n\}$ are equivalent if and only if there is a class containing both i and j.

LEMMA 3.1. *Let A be an $n \times n$ matrix over M which is in Frobenius Normal Form* (3.1) *and let (λ, x) be an eigenpair of A. Suppose x is partitioned as*

$$x = \left[x_{(1)}^T \dots x_{(q)}^T\right]^T \tag{3.2}$$

where $x_{(i)}$ is of order α_i, $i = 1, 2, ..., q$. Then for any i, either $x_{(i)} = -\infty$ or $x_{(i)}$ is finite.

Proof. If x is finite then there is nothing to prove so suppose that x is partly infinite. Fix $k \in \{1, 2, ..., q\}$. If $\alpha_k = 1$ then the result is obvious, so suppose $\alpha_k \geq 2$. Let

$$W_k = \{i : \ i \in V_k, \ x_i = -\infty\}, \quad Z_k = \{i : \ i \in V_k, \ x_i \neq -\infty\}.$$

If $x_i = -\infty$ and $x_j \neq -\infty$ for some i, j in V_k, then we must have $a_{ij} = -\infty$, for otherwise, the i-th equation in $A \otimes x = \lambda \otimes x$ is not satisfied. Thus if $i \in W_k$, $j \in Z_k$ then $a_{ij} = -\infty$. Since A_{kk} is irreducible, we must have either $W_k = \phi$ or $Z_k = \phi$. Thus either $x_{(k)}$ is finite or $x_{(k)} = -\infty$ and the proof is complete. □

THEOREM 3.2. *Let A be an $n \times n$ matrix over M which is in Frobenius*

Normal Form (3.1). *Then each eigenvalue of* A *is in the set* $\{\mu(A_{11}), \ldots, \mu(A_{qq})\}$.

Proof. Let (λ, x) be an eigenpair of A and let x be partitioned as in (3.2). By Lemma 3.1 there exists $F \subset \{1, 2, \ldots, q\}$ such that $x_{(k)}$ is finite if $k \in F$ and $x_{(k)} = -\infty$ otherwise. Let B be the principal submatrix of A formed by the rows and columns indexed by $\bigcup_{i \in F} V_i$. It follows from $A \otimes x = \lambda \otimes x$ that the finite part of x (which is precisely the subvector of x formed by $x_{(k)}$, $k \in F$) is an eigenvector of B corresponding to the eigenvalue λ. It follows from Theorem 2.7 that $\lambda = \mu(B)$. Clearly $\mu(B) = \max_{i \in F} \mu(A_{ii})$ and therefore $\mu(B) \in \{\mu(A_{11}), \ldots, \mu(A_{qq})\}$. □

The following two theorems are the main results of this section. The case $q = 2$ is stated in [1, p. 115; 10, p. 56], and Theorem 3.4 is given in [15, Ch. 4, Cor. 2.2.5].

THEOREM 3.3. *Let* A *be an* $n \times n$ *matrix over* M *which is in Frobenius Normal Form* (3.1). *Let* $i \in \{1, 2, \ldots, q\}$, *let* $T_1 = \{j\colon\ \mu(A_{jj}) < \mu(A_{ii})\}$ *and let* $T_2 = \{j\colon\ \mu(A_{jj}) = \mu(A_{ii})\}$. *Then* $\mu(A_{ii})$ *is an eigenvalue of* A *if and only if there exists* $T \subset T_1 \cup T_2$ *such that* $T \cap T_2 = \phi$ *and such that* $A_{jk} = -\infty$ *for any* $j \notin T$, $k \in T$.

Proof. First suppose that there exists $T \subset T_1 \cup T_2$ with $T \cap T_2 = \phi$ and suppose $A_{jk} = -\infty$ whenever $j \notin T$, $k \in T$. Let F be the principal submatrix of A indexed by the rows and columns in $\bigcup_{k \in T} V_k$. Then $\mu(F) = \mu(A_{ii})$. It follows by Theorem 2.8 that $\mu(A_{ii})$ is an eigenvalue of F and we let z be a corresponding eigenvector. Index the coordinates of z by $\bigcup_{j \in T} V_j$. For $j \in T$, let $z_{(j)}$ be the subvector of z with coordinates indexed by V_j. Now construct the vector x of order n, partitioned as in (3.2) as follows. If $j \in T$ then set $x_{(j)} = z_{(j)}$ and if $j \notin T$ then set $x_{(j)} = -\infty$. We then claim that $A \otimes x = \mu(A_{ii}) \otimes x$. To verify the claim we must show that

$$\begin{bmatrix} A_{11} & -\infty & \cdots & -\infty \\ A_{21} & A_{22} & \cdots & -\infty \\ \vdots & \vdots & \ddots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{bmatrix} \otimes \begin{bmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(q)} \end{bmatrix} = \mu(A_{ii}) \otimes \begin{bmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(q)} \end{bmatrix}$$

which is equivalent to

$$\sum_{\ell=1}^{j} A_{j\ell} \otimes x_{(\ell)} = \mu(A_{ii}) \otimes x_{(j)}, \quad j = 1, 2, \ldots, q. \tag{3.3}$$

If $j \in T$ then (3.3) reduces to $\sum_{\ell \in T}^{\oplus} A_{j\ell} \otimes z_{(\ell)} = \mu(A_{ii}) \otimes z_{(j)}$, which holds since z is an eigenvector of F corresponding to $\mu(A_{ii})$. If $j \notin T$ then $A_{jk} = -\infty$ for any $k \in T$. Since $x_{(j)} = -\infty$ if $j \notin T$, (3.3) reduces to $-\infty = -\infty$, which is obvious. Thus the claim is proved and $\mu(A_{ii})$ is an eigenvalue of A.

To prove the converse, suppose $\mu(A_{ii})$ is an eigenvalue of A. First suppose that $\mu(A_{ii}) = -\infty$. Then by Lemma 2.2 A must have a column equal to $-\infty$. Thus there exists $\ell \in \{1, 2, \ldots, q\}$ such that $A_{\ell\ell}$ is 1×1, $\alpha_\ell = 1$ and the column of A indexed by V_ℓ (which is a singleton) is $-\infty$. Now $T = \{\ell\}$ satisfies the required conditions and the result is proved. So we assume that $\mu(A_{ii})$ is finite and let x be an eigenvector corresponding to $\mu(A_{ii})$ partitioned as in (3.2) so that (3.3) holds.

If x is finite then by Theorem 2.7, $\mu(A_{ii}) = \mu(A)$ and hence

$T_1 \cup T_2 = \{1, 2, ..., q\}$. Set $T = \{1, 2, ..., q\}$ so that $T \cap T_2 \neq \phi$ and the condition $A_{jk} = -\infty$ for any $j \notin T$, $k \in T$ is vacuously true. Finally, suppose x is partly infinite. By Lemma 3.1 there exists $T \subset \{1, 2, ..., q\}$ such that $x_{(\ell)}$ is finite if $\ell \in T$ and $x_{(\ell)} = -\infty$ otherwise. Let F be the principal submatrix of A determined by the rows and columns in $\bigcup_{j \in T} V_j$. The finite part of x is an eigenvector of F with eigenvalue $\mu(A_{ii})$ and by Theorem 2.7, $\mu(A_{ii}) = \mu(F)$. In particular, $\mu(A_{ii}) \geq \mu(A_{jj})$ for any $j \in T$. Therefore $T \subset T_1 \cup T_2$. Since $\mu(F) = \max_{\ell \in T} \mu(A_{\ell\ell})$, we have $T \cap T_2 \neq \phi$. Now suppose that for some $j \notin T$, $k \in T$, we have $A_{jk} \neq -\infty$. Since $x_{(k)}$ is finite, $\sum_{\ell=1}^{j}{}_{\oplus} A_{j\ell} \otimes x_{(\ell)} \neq -\infty$, whereas $\mu(A_{ii}) \otimes x_{(j)} = -\infty$. Thus (3.3) is not satisfied, which is a contradiction. Therefore for any $j \notin T$, $k \in T$ we have $A_{jk} = -\infty$ and the proof is complete. □

If V_i and V_j are classes, we say V_j has access to V_i provided either $i = j$ or there is a $u \in V_j$ and a $v \in V_i$ such that there is a path from u to v in $G(A)$. Since each A_{jj} is either irreducible or $[-\infty]$, the relation "has access to" is reflexive and transitive. If $\mu(A_{jj}) > \mu(A_{ii})$ then we say that class V_j dominates class V_i.

THEOREM 3.4. *Let A be an $n \times n$ matrix over M which is in Frobenius Normal Form* (3.1), *and let $\lambda \in M$. Then λ is an eigenvalue of A if and only if there is an i such that $\mu(A_{ii}) = \lambda$ and no class which dominates V_i has access to V_i.*

Proof. Suppose i exists as described. Let $T_1 = \{j: \ \mu(A_{jj}) < \mu(A_{ii})\}$, $T_2 = \{j: \ \mu(A_{jj}) = \mu(A_{ii})\}$, and let $T = \{j: \ V_j \text{ has access to } V_i\}$. If $m \in T$ then V_m does not dominate V_i, so $\mu(A_{mm}) \leq \mu(A_{ii})$ and $m \in T_1 \cup T_2$. Hence $T \subset T_1 \cup T_2$.

Since $i \in T$, $T \cap T_2 \neq \phi$. Let $j \notin T$ and $k \in T$ and suppose $A_{jk} \neq -\infty$. Then V_j has access to V_k, and since V_k has access to V_i, V_j has access to V_i, contradicting the fact that $j \notin T$. Thus if $j \notin T$ and $k \in T$, we have $A_{jk} = -\infty$, and so by Theorem 3.3, $\lambda = \mu(A_{ii})$ is an eigenvalue of A.

Now suppose λ is an eigenvalue of A. By Theorem 3.2, there is a p such that $\lambda = \mu(A_{pp})$. Let $T_1 = \{j: \ \mu(A_{jj}) < \mu(A_{pp})\}$, $T_2 = \{j: \ \mu(A_{jj}) = \mu(A_{pp})\}$. By Theorem 3.3 there is a $T \subset T_1 \cup T_2$ with $T \cap T_2 \neq \phi$ such that if $j \notin T$ and $k \in T$ then $A_{jk} = -\infty$. Let $i \in T \cap T_2$. Then $\mu(A_{ii}) = \mu(A_{pp}) = \lambda$. Suppose that V_j dominates V_i but that V_j has access to V_i. Let $u \in V_j$ and $v \in V_i$ such that there is a path $u = x_1 \to x_2 \to \cdots \to x_r = v$ in $G(A)$. Let m be the smallest index such that $x_m \in V_t$ for some $t \in T$. Now $x_1 \in V_j$ and $j \notin T$ since $\mu(A_{jj}) > \mu(A_{ii}) = \mu(A_{pp})$. Hence $m > 1$, and $x_{m-1} \in V_s$ for some $s \notin T$. But (x_{m-1}, x_m) is an edge in $G(A)$ from $x_{m-1} \in V_s$ to $x_m \in V_t$, so $A_{st} \neq -\infty$, contradicting the choice of T. Hence if V_j dominates V_i, then V_j does not have access to V_i, and the theorem is proved. □

As an example, consider A in Frobenius Normal Form with five diagonal blocks

$$A = \begin{bmatrix} 0 & 3 & -\infty & -\infty & -\infty & -\infty & -\infty \\ 1 & -5 & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 5 & -\infty & -\infty & -\infty & -\infty \\ 2 & -3 & 6 & 4 & -1 & -\infty & -\infty \\ 0 & 1 & 2 & 1 & 1 & -\infty & -\infty \\ -\infty & -\infty & 1 & -\infty & -\infty & 3 & -\infty \\ 7 & -4 & 2 & 3 & -\infty & 1 & -\infty \end{bmatrix}.$$

For this matrix, $\mu(A_{11}) = 2$, $\mu(A_{22}) = 5$, $\mu(A_{33}) = 4$, $\mu(A_{44}) = 3$, $\mu(A_{55}) = -\infty$. Thus, by the above results, the eigenvalues of A are $-\infty$, 3, 4, 5.

We now present some applications of Theorems 3.3 and 3.4.

COROLLARY 3.5. *Let* A *be an* $n \times n$ *matrix over* $\mathbf{M}$ *in Frobenius Normal Form* (3.1). *Then the following assertions are true.*

(i) $\max_j \mu(A_{jj})$ *is an eigenvalue of* A.

(ii) *Let* $j \in \{1,2,...,q\}$ *and suppose* $A_{ij} = -\infty$ *for* $i \neq j$, *then* $\mu(A_{jj})$ *is an eigenvalue of* A. *In particular,* $\mu(A_{qq})$ *is an eigenvalue of* A.

(iii) *Suppose* $k \in \{1,2,...,q\}$ *and suppose* $\mu(A_{jj}) \leq \mu(A_{kk})$ *for* $k \leq j \leq q$. *Then* $\mu(A_{kk})$ *is an eigenvalue of* A.

Proof. Apply Theorem 3.3 with $T = \{1,2,...,q\}$ for (i), $T = \{j\}$ for (ii), and $T = \{k, k+1, ..., q\}$ for (iii). □

COROLLARY 3.6. *Let* A *be an* $n \times n$ *matrix over* $\mathbf{M}$ *in Frobenius Normal Form* (3.1). *Suppose* $q > 1$, $\mu(A_{qq}) > \mu(A_{ii})$ *and* $A_{qi} \neq -\infty$ *for* $i = 1,2,...,q-1$. *Then* $\mu(A_{qq})$ *is the only eigenvalue of* A. *Furthermore any eigenvector of* A *is partly infinite.*

Proof. By Theorem 3.2 each eigenvalue of A is $\mu(A_{ii})$ for some i. Suppose $i < q$ then V_q dominates V_i and has access to V_i. Hence, by Theorem 3.4, $\mu(A_{ii})$ is not an eigenvalue, and thus $\mu(A_{qq})$ is the only eigenvalue of A. If x is an eigenvector of A, partition x as in (3.2). Then (2.1) implies that $x_{(1)} = x_{(2)} = \cdots = x_{(q-1)} = -\infty$. □

A description of the eigenspaces of A is given in [15, Ch. 4, Section 2.3; 24, Th. 1]. Here we characterize matrices which admit a finite eigenvector. It is easily checked that if $\mu(A) = -\infty$, A has a finite eigenvector if and only if $A = -\infty$. We now consider the case $\mu(A) > -\infty$. For $k \in \{1,2,...,q\}$, V_k is said to be a basic class provided

$\mu(A_{kk}) = \mu(A)$.

THEOREM 3.7. *Let* A *be an* $n \times n$ *matrix over* M *which is in Frobenius Normal Form* (3.1), *and suppose* $\mu(A)$ *is finite. Then* A *has a finite eignevector if and only if each class has access to a basic class.*

Proof. If we subtract $\mu(A)$ from each entry in A, we obtain a matrix with the same eigenvectors and accessibility properties as A, but with maximal circuit mean 0. Hence we may assume without loss of generality that $\mu(A) = 0$.

Suppose each class has access to a basic class. Let $I = \{i\colon\ V_i \text{ is basic}\}$. Define $\Gamma = \Gamma(A)$ as in (2.5), and for $i \in I$ let $\beta_i \in V_i$ such that β_i lies on a critical circuit of A. As in the proof of Theorem 2.8, Γ_{β_i}, the β_i^{th} column of Γ, is an eigenvector of A corresponding to 0. We will show that $x = \sum_{i \in I}^{\oplus} \Gamma_{\beta_i}$ is finite, and hence is a finite eigenvector of A.

Suppose $1 \leq k \leq n$. There exists j such that $k \in V_j$, and there exists $i \in I$ such that V_j has access to V_i. Let $u \in V_j$ and $v \in V_i$ such that $G(A)$ has a path from u to v. Since A_{jj} is irreducible or $[-\infty]$ and A_{ii} is irreducible, either $u = k$ or there is a path in $G(A)$ from k to u, and there is a path in $G(A)$ from v to β_i. Hence there is a path in $G(A)$ from k to β_i. Now $\Gamma_{k\beta_i} = -\infty$ only if there is no such path in $G(A)$, so the k^{th} entry of Γ_{β_i} is finite. Hence $x = \sum_{i \in I}^{\oplus} \Gamma_{\beta_i}$ is finite, and x is a finite eigenvector of A.

To prove the converse, suppose x is a finite eigenvector of A. Let $B = [b_{ij}]$ be defined by $b_{ij} = a_{ij} + x_j - x_i$. Since x corresponds to $\mu(A) = 0$ by Theorem 2.7, it can be seen that the maximal entry in each row of B is 0; that is, B is row stochastic in the max algebra. Furthermore since x is finite, B has the same classes and the same accessibility properties as A. Hence we may assume without loss of generality that A is row stochastic in the max algebra. We must show that each class has access to a basic class. Let $1 \le k \le q$. If V_k is basic, we are done. Otherwise, we claim there is an m with $m < k$ such that V_k has access to V_m. There is a row in A_{kk} whose maximal entry is negative. Let i index this row in the matrix A, and let j be a column index such that $a_{ij} = 0$. Then $j \in V_m$ for some $m < k$ and the claim is proved. If V_m is basic, we are done; otherwise we repeat the process. Since V_1 must be basic, we eventually arrive at a basic class to which V_k has access, and the theorem is established. $\square$

We remark that matrix B can be written as $\mathbf{diag}(-x) \otimes A \otimes \mathbf{diag}(x)$. It follows that in the max algebra, analogous to the case of nonnegative matrices [3, p. 49], A has a finite eigenvector if and only if it is diagonally similar to a scalar multiple of a row stochastic matrix.

4. Pattern Properties in Max Algebra. In this section we investigate spectral properties that depend only on the placement of finite and infinite entries in the matrix, and not on the magnitudes of the finite entries. Such properties are called "pattern properties" of the matrix.

A (square) pattern is an $n \times n$ array $P = [p_{ij}]$ of symbols chosen from $\{*, -\infty\}$. If A is an $n \times n$ matrix over $\mathbb{M}$, we write $A \in P$ provided

$$a_{ij} \in \mathbb{R} \ \ if \ \ p_{ij} = *, \quad a_{ij} = -\infty \ \ if \ \ p_{ij} = -\infty.$$

Following [18], a pattern P is said to allow a particular property if there is a matrix $A \in P$ which has the property. P is said to require the property if every matrix $A \in P$

has the property. We determine which patterns allow and which require various spectral properties in the max algebra.

The digraph $G(P)$ of an $n \times n$ pattern P has vertices $\{1, 2, \ldots, n\}$, and an edge from i to j if and only if $p_{ij} = *$. We denote the set of circuits in $G(P)$ by $C(P)$. The concept of reducibility of a square matrix, introduced in Section 2, extends in an obvious way to patterns. P is irreducible if and only if $G(P)$ is strongly connected. It follows that P is reducible if and only if P is 1×1 containing $-\infty$, or if by an identical permutation of rows and columns P can be brought to the form $\begin{bmatrix} P_{11} & -\infty \\ P_{21} & P_{22} \end{bmatrix}$, where P_{11} and P_{22} are square with order at least one. We will also deal with the Frobenius Normal Form of the pattern, defined analogously to that of a matrix, see Section 3.

We first discuss properties of the eigenvalues of a matrix determined by its pattern.

LEMMA 4.1. *Let* P *be a pattern. The following are equivalent.*

(i) P *requires a finite eigenvalue;*

(ii) P *allows a finite eigenvalue;*

(iii) $C(P)$ *is not empty.*

Proof. The proof follows easily from Lemma 2.1 and Theorem 2.8. □

LEMMA 4.2. *Let* P *be a pattern. The following are equivalent.*

(i) P *requires* $-\infty$ *as an eigenvalue;*

(ii) P *allows* $-\infty$ *as an eigenvalue;*

(iii) P *has an infinite column.*

Proof. The proof follows immediately from Lemma 2.2. □

The following corollary is an immediate consequence of Lemmas 4.1 and 4.2.

COROLLARY 4.3. *Let* P be a pattern.

(i) P *requires that* $-\infty$ *be the only eigenvalue if and only if* P *allows the same*

property, and this occurs if and only if $C(P)$ *is empty.*

(ii) P *requires that all eigenvalues be finite if and only if* P *allows the same property, and this occurs if and only if* P *has no infinite column.*

THEOREM 4.4. *Let* P *be a pattern.*

(i) P *requires a unique and finite eigenvalue if and only if* P *has no infinite column and the Frobenius Normal Form of* P *has exactly one irreducible diagonal block.*

(ii) P *allows a unique and finite eigenvalue if and only if* P has no infinite column.

Proof. *(i)*. We may assume without loss of generality that P is in Frobenius Normal Form. Suppose P requires a unique and finite eigenvalue. By Lemma 4.2, P has no infinite column. If P had a 1×1 diagonal block $[-\infty]$ in the lower right corner, P would have an infinite column. Hence the lower right diagonal block is irreducible. If P had another irreducible diagonal block, a matrix $A \in P$ could be constructed with the lower right diagonal block having eigenvalue 0 and another irreducible diagonal block having a positive eigenvalue. It follows from Corollary 3.4 that A would have two eigenvalues, one 0 and one positive, violating the fact that P requires a unique eigenvalue.

Now suppose that P has no infinite column and exactly one irreducible block, which then must be P_{qq}, the lower right block. Let $A \in P$. By Lemma 2.2, $-\infty$ is not an eigenvalue of A. By Theorem 2.8, A has an eigenvalue which, by Theorem 3.2, is $\mu(A_{ii})$ for some diagonal block A_{ii} in A. Since A_{qq} is the only irreducible diagonal block in A, $\mu(A_{qq}) > -\infty$ is the only eigenvalue of A.

(ii) If P allows a unique and finite eigenvalue, then P does not require $-\infty$ as an eigenvalue, so by Lemma 4.2 P has no infinite column. Conversely, if P has no infinite column, then the matrix $A \in P$ which has 0 in all the * positions will have the unique and finite eigenvalue 0. □

We now turn to pattern properties concerning the eigenvectors of a matrix.

We obtain necessary and sufficient conditions on a pattern that it allow (or require) all (or some) eigenvectors to be finite (or partly finite). Some of the results parallel those concerning partly zero eigenvectors in the conventional algebra presented in [20]. Note that the eigenpairs of the matrix (pattern) with each entry $-\infty$ are of the form $(-\infty, x)$ with $x \neq -\infty$. We exclude that pattern from consideration in the following.

THEOREM 4.5. *Let P be a pattern with at least one *. Then P requires that all eigenvectors be partly infinite if and only if P has an infinite row.*

Proof. First suppose P has no infinite row. Let $A \in P$ be obtained by replacing all *'s with 0's. Then the vector of all 0's is a finite eigenvector of A corresponding to the eigenvalue 0. Hence P does not require that all eigenvectors be partly infinite.

Now suppose that row i of P is infinite, but that $A \in P$ has a finite eigenvector x corresponding to eigenvalue λ. Then entry i of $A \otimes x$ is $-\infty$, so $\lambda \otimes x_i$ is $-\infty$. Since x_i is finite, $\lambda = -\infty$. Now if a_{jk} is finite, then entry j of $A \otimes x$ is finite, whereas entry j of $\lambda \otimes x = -\infty$. Hence $A = -\infty$, so $P = -\infty$, a contradiction. Therefore if P has an infinite row, then P requires that all eigenvectors be partly infinite. □

COROLLARY 4.6. *Let P be a pattern with at least one *. Then P allows a finite eigenvector if and only if P has no infinite row.*

THEOREM 4.7 *Let P be a pattern with at least one *. The following are equivalent.*

(i) *P is irreducible;*

(ii) *P requires that all eigenvectors be finite;*

(iii) *P allows all eigenvectors to be finite.*

Proof. *(i)* $\Rightarrow$ *(ii)*: If $A \in P$ then A is irreducible, so by Corollary 2.5, all eigenvectors of A are finite. Therefore *(i)* $\Rightarrow$ *(ii)*.

(ii) $\Rightarrow$*(iii)* is trivial.

(iii) $\Rightarrow$*(i)*: Suppose that P is reducible, so that without loss of generality we may assume $P = \begin{bmatrix} P_{11} & -\infty \\ P_{21} & P_{22} \end{bmatrix}$. Let $A = \begin{bmatrix} A_{11} & -\infty \\ A_{21} & A_{22} \end{bmatrix} \in P$ be partitioned as P is. Let $x_{(2)}$ be an eigenvector of A_{22} corresponding to $\mu(A_{22})$, and let $x = \begin{bmatrix} -\infty \\ x_{(2)} \end{bmatrix}$. Then $A \otimes x = \mu(A_{22}) \otimes x$, so x is an eigenvector of A which is partly infinite. Hence P does not allow all eigenvectors to be finite. Therefore *(iii)* $\Rightarrow$ *(i)*. □

COROLLARY 4.8. *Let* P *be a pattern with at least one *. The following are equivalent.*

(i) P *is reducible;*

(ii) P *allows a partly infinite eigenvector;*

(iii) P *requires a partly infinite eigenvector.*

Proof. The equivalence of *(i)* through *(iii)* in Theorem 4.7 implies the corollary. □

THEOREM 4.9. *Let* P *be a pattern with at least one *. Then* P *requires a finite eigenvector if and only if* P *has no infinite row and the Frobenius Normal Form of* P *has exactly one irreducible diagonal block.*

Proof. We may assume without loss of generality that P is in Frobenius Normal Form. Suppose P requires a finite eigenvector. By Theorem 4.5, P has no infnite row. Therefore the upper left diagonal block P_{11} in P is irreducible. Suppose there is a $k > 1$ such that P_{kk} is irreducible. We will construct a matrix $A \in P$ with all eigenvectors partly infinite, contradicting the hypothesis on P. To do this, let $U_1 = \begin{bmatrix} P_{21} \\ P_{31} \\ \cdot \\ \cdot \\ P_{q1} \end{bmatrix}$, and $U_2 = \begin{bmatrix} P_{22} & -\infty & -\infty & \cdots & -\infty \\ P_{32} & P_{33} & -\infty & \cdots & -\infty \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{q2} & P_{q3} & \cdots & \cdots & P_{qq} \end{bmatrix}$. Then $P = \begin{bmatrix} P_{11} & -\infty \\ U_1 & U_2 \end{bmatrix}$, and P_{kk} is one of the diagonal

blocks in U_2. Since P_{kk} is irreducible, P_{kk} has a circuit. Select a circuit in P_{kk} and set all its * entries equal to 1. Set the other * entries in U_2 to 0 to create a matrix $A_2 \in U_2$. Set all * entries in P_{11} and U_1 to 0 to complete $A = \begin{bmatrix} A_{11} & -\infty \\ A_1 & A_2 \end{bmatrix} \in P$ with $\mu(A_{11}) = 0$ and $\mu(A) = \mu(A_2) = 1$. Suppose A has a finite eigenvector $x = \begin{bmatrix} x_{(1)} \\ x_{(2)} \end{bmatrix}$

partitioned to conform to the partition of A above. Since x is finite, the corresponding eigenvalue must be $\mu(A_{11})$ by Theorem 2.7. But then $A_{11} \otimes x_{(1)} = 1 \otimes x_{(1)}$, which is impossible because the only eigenvalue of A_{11} is 0. Hence A cannot have a finite eigenvector and the desired contradiction is reached.

Now suppose P has no infinite row and has exactly one irreducible diagonal block, which must then be P_{11}. Let $A \in P$ be partitioned as P is. We construct a finite eigenvector of A inductively as follows. Since A_{11} is irreducible, A_{11} has a finite eigenvector $x_{(1)}$ corresponding to its eigenvalue $\mu(A_{11})$. Let q be the number of digaonal blocks in P and suppose that, for $2 \le i < q$, $x_2, \ldots, x_i$ have been selected from M. Let $x_{i+1} = [A_{i+1,1} A_{i+i,2} \cdots A_{i+1,i}] \otimes [x_{(1)}^T x_2 \cdots x_i]^T - \mu(A_{11})$, a finite member of M. It then follows that $x = [x_{(1)}^T x_2 \cdots x_q]^T$ is a finite eigenvector of A corresponding to

$\mu(A_{11})$, so P requires a finite eigenvector. □

COROLLARY 4.10. *Let* P *be a pattern with at least one* *. *Then* P *allows all eigenvectors to be partly infinite if and only if* P *has an infinite row or the Frobenius Normal Form of* P *has two irreducible diagonal blocks.*

Proof. Upon observing that a pattern with no infinite row must have a Frobenius Normal Form with the upper left diagonal block irreducible, the corollary follows immediately from Theorem 4.9. □

THEOREM 4.11. *Let* P *be a pattern. Then* P *requires a unique and finite eigenvector if and only if* P *is irreducible and the directed graph* $G = G(P)$ *does not contain two vertex-disjoint circuits.*

Proof. Assume P is irreducible and G does not have two vertex-disjoint circuits. Let $A \in P$. Then A has a unique eigenvalue which is $\mu(A)$, each eigenvector of A is finite, and A has a unique eigenvector if and only if the critical graph C of A is strongly connected. Now C is a subgraph of G, and is a union of circuits. Since G does not have two vertex-disjoint circuits, C does not have two vertex-disjoint circuits. If i and j are vertices in C, then i lies on a circuit C_i and j lies on a circuit C_j. If $C_i = C_j$ there are paths from i to j and from j to i in C. Otherwise C_i and C_j share a vertex, and again there are paths from i to j and from j to i in C. Hence C is strongly connected and the eigenvector of A is unique up to scalar multiples in the max algebra. Therefore P requires a unique and finite eigenvector.

Now assume P requires a unique and finite eigenvector. If P were reducible, then by Corollary 4.8 P would allow a partly infinite eigenvector. Hence P is irreducble. Suppose G has two vertex-disjoint circuits. Then we may select two vertex-disjoint circuits in G and construct a matrix $A \in P$ which has 1 in the positions belonging to either of the two circuits and 0 and $-\infty$ elsewhere. Then the circuit means are 1 on each of the two circuits and less than 1 on each other circuit, so the critical graph of A is the union of the two disjoint circuits and is not strongly connected. Hence the eigenvector of A is not unique, contradicting the assumption on P. Hence G does not have two vertex-disjoint circuits. □

5. Inequalities. The first few results in this section are motivated by known inequalities for the spectral radius (or the Perron root) $\rho(B)$ of a nonnegative matrix B. Thus, Lemmas 5.1, 5.2 and Corollary 5.3 are analogs of well-known bounds for the Perron root, see, for example, [3, p. 28; 23, p. 31]. Theorem 5.4 is the max algebra version of a result due to Birkhoff and Varga [4]. The parallels between inequalities for $\mu(A)$, where A is a matrix over $\boldsymbol{M}$ and $\rho(B)$, where B is a nonnegative matrix, are quite striking and remain to be fully explored. Theorem 5.12 is yet another result in this direction.

Let A be an $n \times n$ matrix over $\boldsymbol{M}$, then $\mu(A)$ is an eigenvalue of A and there is a vector $x \neq -\infty$ such that $A \otimes x = \mu(A) \otimes x$. We refer to x as a right eigenvector of A corresponding to $\mu(A)$. Since $\mu(A) = \mu(A^T)$, there is a vector $y \neq -\infty$ such that $A^T \otimes y = \mu(A) \otimes y$, or equivalently $y^T \otimes A = \mu(A) \otimes y^T$. We refer to y as a left eigenvector of A. If A is irreducible, then x and y are finite and $\mu(A)$ is the only eigenvalue of A. The following two lemmas appear in a stronger form in [15, Ch. 4, Lemmas 1.3.8, 1.3.9].

LEMMA 5.1. *Let A be an $n \times n$ irreducible matrix over $\boldsymbol{M}$. Suppose there exist $\tau \in \boldsymbol{M}$ and a vector $z \neq -\infty$ such that $A \otimes x \geq \tau \otimes z$. Then $\mu(A) \geq \tau$.*

Proof. Let y be a left eigenvector of A. Since $A \otimes z \geq \tau \otimes z$, we have

$$y^T \otimes A \otimes z \geq \tau \otimes y^T \otimes z. \tag{5.1}$$

However, $y^T \otimes A = \mu(A) \otimes y^T$ and therefore, by (5.1),

$$\mu(A) \otimes y^T \otimes z \geq \tau \otimes y^T \otimes z. \tag{5.2}$$

Since A is irreducible, y is finite. Also, $z \neq -\infty$, and hence $y^T \otimes z \neq -\infty$. It follows from (5.2) that $\mu(A) \geq \tau$. $\square$

The proof of the next result is similar.

LEMMA 5.2. *Let* A *be an* $n \times n$ *irreducible matrix over* $\boldsymbol{M}$. *Suppose there exist* $\eta \in \boldsymbol{M}$ *and a vector* $w \neq -\infty$ *such that* $A \otimes w \leq \eta \otimes w$. *Then* $\mu(A) \leq \eta$.

COROLLARY 5.3. *Let* A *be an* $n \times n$ *matrix over* $\boldsymbol{M}$. *Then*

$$\min_{i} \max_{j} a_{ij} \leq \mu(A) \leq \max_{i,j} a_{ij}.$$

Proof. We prove the result when A is irreducible. The general result can then be obtained by a continuity argument. Let $\alpha = \min_{i} \max_{j} a_{ij}$ and let 0 denote the vector with each component zero. Then $A \otimes 0 \geq \alpha \otimes 0$. It follows from Lemma 5.1 that $\mu(A) \geq \alpha$. It is easy to see that for any $\sigma \in C(A)$; $M_A(\sigma) \leq \max_{i,j} a_{ij}$, and hence $\mu(A) \leq \max_{i,j} a_{ij}$, giving the second inequality. □

THEOREM 5.4. *Let* A *be an* $n \times n$ *irreducible matrix over* $\boldsymbol{M}$. *Then the following assertions hold.*

(i) $\mu(A) = \max_{x > -\infty} \min_{y > -\infty} \left(y^T \otimes A \otimes x - y^T \otimes x\right)$

(ii) $\mu(A) = \min_{y > -\infty} \max_{x > -\infty} \left(y^T \otimes A \otimes x - y^T \otimes x\right)$.

Proof. For any finite vectors x, y, we have

$$y^T \otimes A \otimes x = \max_{i,j}(a_{ij} + y_i + x_j)$$
$$= \max_{i,j}(a_{ij} + x_j - x_i + y_i + x_i)$$
$$\geq \min_i \max_j (a_{ij} + x_j - x_i) + y^T \otimes x.$$

Therefore,

$$y^T \otimes A \otimes x - y^T \otimes x \geq \min_i \max_j (a_{ij} + x_j - x_i). \tag{5.3}$$

Suppose $\min_i \max_j (a_{ij} + x_j - x_i) = \max_j (a_{kj} + x_j - x_k)$. Let z be the vector with $z_k = -x_k$, with the remaining components chosen finite so that $z^T \otimes x = 0$ and satisfying $\max_j (a_{ij} + z_i + x_j) \leq \max_j (a_{kj} + z_k + x_j)$, $i = 1, 2, ..., n$. When we set $y = z$, equality holds in (5.3) and hence we have shown that for any finite x,

$$\min_{y > -\infty} (y^T \otimes A \otimes x - y^T \otimes x) \tag{5.4}$$

exists. Thus by (5.3)

$$\min_{y > -\infty} (y^T \otimes A \otimes x - y^T \otimes x) = \min_i \max_j (a_{ij} + x_j - x_i).$$

Let

$$S = [a_{ij} + x_j - x_i]. \tag{5.5}$$

Then $\mu(A) = \mu(S)$ and by Corollary 5.3

$$\min_i \max_j (a_{ij} + x_j - x_i) \le \mu(S). \tag{5.6}$$

Therefore, we conclude that

$$\mu(A) \ge \sup_{x > -\infty} \min_{y > -\infty} (y^T \otimes A \otimes x - y^T \otimes x). \tag{5.7}$$

When we set x to be a right eigenvector of A, we see that for any finite y, $y^T \otimes A \otimes x - y^T \otimes x = \mu(A)$. Thus, *(i)* follows from (5.7). The proof of *(ii)* is similar. □

In the next few results we will be concerned with additive properties of μ in both the conventional and the max algebra.

THEOREM 5.5. *Let* $X_1, ..., X_m$ *be* $n \times n$ *matrices over* $\boldsymbol{M}$. *Then*

$$\mu\left(\sum_{i=1}^{m} X_i\right) \le \sum_{i=1}^{m} \mu(X_i). \tag{5.8}$$

Furthermore, if $C\left(\sum_{i=1}^{m} X_i\right) \ne \phi$, *then equality holds in* (5.8) *if and only if*

$$\tilde{C}\left(\sum_{i=1}^{m} X_i\right) \subset \bigcap_{i=1}^{M} \tilde{C}(X_i). \tag{5.9}$$

Proof. First suppose $C(\sum X_i) = \phi$. Then by Lemma 2.1, $\mu(\sum X_i) = -\infty$ and (5.8) is obvious. Note that in this case equality holds in (5.8) if and only if $C(X_k) = \phi$ for some $k \in \{1, 2, ..., m\}$.

Now suppose $C(\sum X_i) \ne \phi$, so that $C(X_i) \ne \phi$, $i = 1, 2, ..., m$. Let $\sigma \in \tilde{C}(\sum X_i)$. Then

$$\mu\left(\sum X_i\right) = M_{\sum X_i}(\sigma) = \sum M_{X_i}(\sigma) \le \sum \mu(X_i). \tag{5.10}$$

Hence (5.8) holds. If (5.9) holds, then $\sigma \in \tilde{C}(X_i)$, $i = 1, 2, ..., m$. It follows that $M_{X_i}(\sigma) = \mu(X_i)$, $i = 1, 2, ..., m$ and equality must hold in (5.10) and hence in (5.8). Conversely, if equality holds in (5.8) and if (5.9) is not true, then there exists $\sigma \in \tilde{C}(\sum X_i)$, such that $\sigma \in \tilde{C}(X_k)$ for some k. Then $M_{X_k}(\sigma) < \mu(X_k)$. Thus the inequality in (5.10) must be strict. Then (5.8) is strict as well, which is a contradiction. That completes the proof. □

LEMMA 5.6. *Let* X, Y *be* $n \times n$ *matrices over* $\mathcal{M}$ *such that* $X \ge Y$. *Then* $\mu(X) \ge \mu(Y)$.

Proof. The result is obvious if $C(Y) = \phi$, since in that case, $\mu(Y) = -\infty$. So suppose $C(Y) \ne \phi$. For any $\sigma \in \tilde{C}(Y)$,

$$\mu(Y) = M_Y(\sigma) \le M_X(\sigma) \le \mu(X)$$

and the proof is complete. □

Observe that Lemma 5.6 shows that if Z is a principal submatrix of X, then $\mu(X) \ge \mu(Z)$.

To discuss the case of equality in Lemma 5.6, we now introduce some notation. Suppose σ is the circuit $(i_1 i_2 \dots i_k)$. Recall that in this notation we assume i_1 to be the least integer among $i_1, i_2, ..., i_k$ and this convention makes the representation of the circuit uniquely determined. If X is an $n \times n$ matrix and if $\sigma = (i_1 i_2 \dots i_k) \in C(X)$, then we define $X(\sigma)$ as the vector $\left[x_{i_1 i_2} x_{i_2 i_3} \cdots x_{i_k i_1}\right]^T$.

LEMMA 5.7. *Let* X, Y *be* $n \times n$ *matrices over* $\mathcal{M}$ *such that* $X \ge Y$ *and suppose* $\mu(Y)$ *is finite. Then the following conditions are equivalent.*

(i) $\mu(X) = \mu(Y)$;

(ii) *there exists* $\sigma \in \tilde{C}(X) \cap \tilde{C}(Y)$ *such that* $M_X(\sigma) = M_Y(\sigma)$;

(iii) *there exists* $\sigma \in \tilde{C}(X)$ *such that* $M_X(\sigma) = M_Y(\sigma)$;

(iv) $\tilde{C}(Y) \subset \tilde{C}(X)$ *and for all* $\sigma \in \tilde{C}(Y)$, $X(\sigma) = Y(\sigma)$.;

(v) $\tilde{C}(Y) \subset \tilde{C}(X)$ *and there exists* $\sigma \in \tilde{C}(Y)$ *such that* $X(\sigma) = Y(\sigma)$.

Proof. First observe that since $\mu(Y)$ is finite, and $X \geq Y$, $\mu(X)$ is finite and $C(Y)$, $C(X)$ are nonempty.

(i) → *(ii)*. Let $\sigma \in \tilde{C}(Y)$. Then

$$\mu(Y) = M_Y(\sigma) \leq M_X(\sigma) \leq \mu(X) \tag{5.11}$$

and since $\mu(X) = \mu(Y)$, equality holds throughout in (5.11). It follows that $\sigma \in \tilde{C}(X) \cap \tilde{C}(Y)$ and $M_X(\sigma) = M_Y(\sigma)$.

Assertion *(ii)* ⇒ *(iii)* is trivial.

(iii) ⇒ *(i)*. Let $\sigma \in \tilde{C}(X)$ such that $M_X(\sigma) = M_Y(\sigma)$. Then

$\mu(X) = M_X(\sigma) = M_Y(\sigma) \leq \mu(Y) \leq \mu(X)$, and hence $\mu(X) = \mu(Y)$.

(i) ⇒ *(iv)*. Let $\sigma \in \tilde{C}(Y)$ so that (5.11) holds. As we claimed in the proof of *(i)* ⇒ *(ii)*, since $\mu(X) = \mu(Y)$, equality holds throughout in (5.11). It follows that $\sigma \in \tilde{C}(X)$ and $M_X(\sigma) = M_Y(\sigma)$. Since $X \geq Y$, we have $X(\sigma) \geq Y(\sigma)$. If $X(\sigma) \neq Y(\sigma)$, then it will follow, after taking the sum of the entries in $X(\sigma)$, $Y(\sigma)$, that $M_X(\sigma) > M_Y(\sigma)$, which is a contradiction. Thus $X(\sigma) = Y(\sigma)$.

It is easy to see that (*iv*) ⇒ (*v*) and (*v*) ⇒ (*i*). That completes the proof. □

THEOREM 5.8. *Let* $X_1, \ldots, X_m$ *be* $n \times n$ *matrices and let* $X = \sum_{\oplus} X_i$. *Then*

$$\mu(X) \geq \sum\nolimits_{\oplus} \mu(X_i). \tag{5.12}$$

Furthermore, equality holds in (5.12) *if and only if one of the following conditions is satisfied.*

(i) $\mu(X) = -\infty$;

(ii) $\mu(X)$ *is finite and there exists* $\sigma \in \tilde{C}(X)$ *and* $k \in \{1,2,\cdots,m\}$ *such that* $X_k(\sigma) \geq X_i(\sigma),\ i = 1,2,\cdots,m.$

Proof. If $\mu(X) = -\infty$, then $\mu(X_i) = -\infty$, $i = 1,2,\cdots,m$ and both sides in (5.12) are $-\infty$. So we assume that $\mu(X)$ is finite. Since $X \geq X_i$, $i = 1,2,\cdots,m$, by Lemma 5.6, we have $\mu(X) \geq \mu(X_i)$, $i = 1,2,\cdots,m$ and hence (5.12) holds.

If equality holds in (5.12) then there exists $k \in \{1,2,\cdots,m\}$ such that $\mu(X) = \mu(X_k)$. Thus $\mu(X_k)$ is finite. By Lemma 5.7 (see *(i)* → *(v)*), there exists $\sigma \in \tilde{C}(X)$ such that $X(\sigma) = X_k(\sigma)$. It follows that $X_k(\sigma) \geq X_i(\sigma)$, $i = 1,2,\cdots,m$.

Conversely, if *(iii)* holds, then $X(\sigma) = X_k(\sigma)$. Thus $\mu(X_k)$ is finite. Set $Y = X_k$ and use implication *(iii)* → *(i)* of Lemma 5.7 to conclude that equality holds in (5.10). □

We say that a square matrix D is a diagonal matrix over the max algebra if $d_{ij} = -\infty$ for all $i \neq j$. A well-known result due to Cohen [7], see also [17, p. 364], asserts that the Perron root of a nonnegative matrix B is a convex function of the diagonal entries of B. In this context the next result is somewhat surprising since it says that $\mu(A)$, considered as a function of the diagonal entries of A, is linear over the max algebra.

THEOREM 5.9. *Let* X *be an* $n \times n$ *matrix over* $\boldsymbol{M}$ *and let* $D_1, \cdots, D_m$ *be* $n \times n$ *diagonal matrices over the max algebra. Then*

$$\mu\left(X \oplus \sum\nolimits_{\oplus} D_j\right) = \sum\nolimits_{\oplus} \mu\left(X \oplus D_j\right). \tag{5.13}$$

Proof. Let $X_j = X \oplus D_j$, $j = 1, 2, \cdots, m$. Then $X \oplus \sum_{\oplus} D_j = \sum_{\oplus} X_j$. If $\mu\left(X \oplus \sum_{\oplus} D_j\right) = -\infty$, then (5.13) is true by Theorem 5.8. So let $\mu\left(X \oplus \sum_{\oplus} D_j\right)$ be finite. If there exists $\sigma \in \tilde{C}\left(X \oplus \sum_{\oplus} D_j\right)$ of length more than one, then $\sigma \in \tilde{C}\left(X \oplus D_j\right)$ $j = 1, 2, \cdots, m$ and (5.13) is proved, in view of *(ii)* $\Rightarrow$ *(i)* of Lemma 5.7. So suppose that every circuit in $\tilde{C}\left(X \oplus \sum_{\oplus} D_j\right)$ is of length one, and let σ be one such. Clearly, there exists $k \in \{1, 2, \cdots, m\}$ such that $D_k(\sigma) \geq D_j(\sigma)$, and hence $X_k(\sigma) \geq X_i(\sigma)$, $i = 1, 2, \cdots, m$. Thus *(ii)*, Theorem 5.8 is satisfied and (5.13) holds. □

Let B be an $n \times n$ nonnegative, irreducible matrix. Then it is known, see [13], that

$$f^T B g \geq \rho(B) f^T g \tag{5.14}$$

where f and g are right and left Perron eigenvectors of B respectively. We now obtain a max algebra analog of (5.14). We first introduce some notation and prove certain preliminary results. Let $C \neq -\infty$ be an $n \times n$ matrix over $\mathcal{M}$ and $\Omega(C) = \left\{(i,j): \ c_{ij} = \max_{k,\ell} c_{k\ell}\right\}$. Construct a $(0, 1)$ matrix $\hat{C} = \left[\hat{c}_{ij}\right]$ by setting $\hat{c}_{ij} = 1$ if $(i,j) \in \Omega(C)$ and $\hat{c}_{ij} = 0$ otherwise. Let $\gamma = \sum_{s=1}^{n} \sum_{t=1}^{n} \hat{c}_{st}$, and for $i, j = 1, 2, \cdots, n$, let $\alpha_i(C) = \frac{1}{\gamma} \sum_{t=1}^{n} \hat{c}_{it}$ and $\beta_j(C) = \frac{1}{\gamma} \sum_{s=1}^{n} \hat{c}_{sj}$. With this notation, we have the following result, which is the max algebra analog of [2 Th. 3].

LEMMA 5.10. *Let* A *be an* $n \times n$ *matrix over* M, *with* $A \neq -\infty$, *let* u, v, w, z *be vectors over* M *with* w *and* z *finite, and let* $C = [a_{ij} \otimes z_i \otimes w_j]$. *Then*

$$v^T \otimes A \otimes u - z^T \otimes A \otimes w \geq \sum_{i=1}^{n} \alpha_i(C)(v_i - z_i) + \sum_{j=1}^{n} \beta_j(C)(u_j - w_j).$$

Proof. For any i, j, we have

$$a_{ij} \otimes v_i \otimes u_j - a_{ij} \otimes z_i \otimes w_j = v_i - z_i + u_j - w_j. \tag{5.15}$$

If $(i,j) \in \Omega(C)$, then $a_{ij} \otimes z_i \otimes w_j = z^T \otimes A \otimes w$. Apply (5.15) to each $(i,j) \in \Omega(C)$ and add the resulting equations to get

$$\sum_{(i,j) \in \Omega(C)} a_{ij} \otimes v_i \otimes u_j - \gamma(z^T \otimes A \otimes w) = \sum_{(i,j) \in \Omega(C)} (v_i - z_i) + \sum_{(i,j) \in \Omega(C)} (u_j - w_j). \tag{5.16}$$

Now
$$\sum_{(i,j) \in \Omega(C)} (v_i - z_i) = \sum_{i=1}^{n} \left\{ (v_i - z_i) \sum_{j=1}^{n} \hat{c}_{ij} \right\} = \gamma \sum_{i=1}^{n} \alpha_i(C)(v_i - z_i),$$

and similarly
$$\sum_{(i,j) \in \Omega(C)} (u_j - w_j) = \gamma \sum_{j=1}^{n} \beta_j(C)(u_j - w_j).$$

Since $\sum_{(i,j) \in \Omega(C)} a_{ij} \otimes v_i \otimes u_j \leq \gamma(v^T \otimes A \otimes u)$ the result follows from (5.16) after a trivial simplification. □

LEMMA 5.11. *Let* G *be a digraph with vertices* $\{1, 2, ..., k\}$ *such that the in-degree and the out-degree of each vertex is at least one. Suppose* G *has a unique circuit.*

Then G *itself is a circuit of length* k.

Proof. Let σ be the circuit in G and suppose that the length of σ is less than k. Let v be a vertex not on σ. Using the degree requirements, we can construct paths of arbitrary length leading out of and into v. But then there must be a circuit containing v. This contradicts the uniqueness of σ. Thus the length of σ must be k. Again, using the uniqueness of σ, G cannot have edges other than those in σ. □

THEOREM 5.12. *Let* A *be an* $n \times n$ *irreducible matrix over* $\boldsymbol{M}$ *and suppose* $G(A)$ *has a unique critical circuit. Let* x *and* y *be right and left eigenvectors of* A *respectively. Let* u, v *be vectors over* $\boldsymbol{M}$ *such that* $u_i \otimes v_i = x_i \otimes y_i$, $i = 1, 2, \ldots, n$. *Then*

$$v^T \otimes A \otimes u \geq \mu(A) \otimes y^T \otimes x.$$

In particular,

$$x^T \otimes A \otimes y \geq \mu(A) \otimes y^T \otimes x$$

Proof. Let C be the $n \times n$ matrix with $c_{ij} = a_{ij} \otimes y_i \otimes x_j$, and define $\hat{C}$, $\alpha_i(C)$, $\beta_j(C)$ as before. Note that

$$\max_j c_{ij} = \max_j (a_{ij} \otimes y_i \otimes x_j) = \mu(A) \otimes y_i \otimes x_i, \quad i = 1, 2, \cdots, n$$

$$\max_i c_{ij} = \max_i (a_{ij} \otimes y_i \otimes x_j) = \mu(A) \otimes y_j \otimes x_j, \quad j = 1, 2, \cdots, n.$$

We assume, without loss of generality, that

$$y_1 \otimes x_1 = \cdots = y_k \otimes x_k > y_{k+1} \otimes x_{k+1} \geq \cdots \geq y_n \otimes x_n.$$

If $i > k$, then the maximum entry in the i-th row of C is $\mu(A) \otimes y_i \otimes x_i < \mu(A) \otimes y_1 \otimes x_1$. Thus $\hat{c}_{ij} = 0$. Similarly, $\hat{c}_{ij} = 0$ if $j > k$.

We now claim that there exists a circuit of length k on $\{1, 2, ..., k\}$ such that $\hat{C}$ has 1 precisely at these positions and 0 elsewhere. To prove the claim, first note that for $1 \le i, j \le k$, $\hat{c}_{ij} = 1$ if and only if $a_{ij} \otimes y_i \otimes x_j = \mu(A) \otimes y_i \otimes x_i$, which is equivalent to $a_{ij} \otimes x_j = \mu(A) \otimes x_i$, since y_i is finite. Thus, circuits in $G(\hat{C})$ with positive edge weights correspond to critical circuits in $G(A)$.

Let H be the digraph defined on $\{1, 2, ..., k\}$ as follows: (i, j) is an edge in H if and only if $\hat{c}_{ij} = 1$. Then, using the assumption that $G(A)$ has a unique critical circuit, we see that H satisfies the hypotheses of Lemma 5.11. Therefore the claim is proved by Lemma 5.11. It follows that

$$\alpha_i(C) = \beta_i(C) = \begin{cases} \dfrac{1}{k}, & i = 1, 2, ..., k \\ 0, & i = k+1, ..., n. \end{cases}$$

Now by Lemma 5.10, we have

$$v^T \otimes A \otimes u - y^T \otimes A \otimes x \ge \sum_{i=1}^{k} \frac{1}{k}(v_i - y_i + u_i - x_i). \tag{5.17}$$

Since $u_i \otimes v_i = x_i \otimes y_i$, $i = 1, 2, ..., k$, the result follows from (5.17). The second inequality in the theorem follows by setting $v = x$, $u = y$. □

We conjecture that Theorem 5.12 holds for arbitrary matrices. Thus suppose A is an $n \times n$ matrix over $\boldsymbol{M}$ and let x and y be right and left eigenvectors of A

(corresponding to $\mu(A)$). Then we conjecture that $x^T \otimes A \otimes y \geq \mu(A) \otimes y^T \otimes x$. Let A be an $n \times n$ matrix over M and let B be the Hadamard exponential of A, i.e., $b_{ij} = e^{a_{ij}}$ for all i,j. Then $e^{\mu(A)}$ is the maximal circuit geometric mean of the nonnegative matrix B. We remark that the maximal circuit geometric mean of a nonnegative matrix has been considered in the literature, see, e.g. [12, 14, 19].

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