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Third-Order Hankel and Toeplitz Determinants for Starlike Functions Connected with the Sine Function

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Abstract: Let $\mathcal{S}_s^*$ be the class of normalized functions f defined in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ such that the quantity $\frac{zf'(z)}{f(z)}$ lies in an eight-shaped region in the right-half plane and satisfying the condition $\frac{zf'(z)}{f(z)} \prec 1 + \sin z$ $(z \in \mathbb{D})$. In this paper, we aim to investigate the third-order Hankel determinant $H_3(1)$ and Toeplitz determinant $T_3(2)$ for this function class $\mathcal{S}_s^*$ associated with sine function and obtain the upper bounds of the determinants $H_3(1)$ and $T_3(2)$.

Keywords: starlike function; Toeplitz determinant; Hankel determinant; sine function; upper bound

MSC: 30C45; 30C50; 30C80

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (z \in \mathbb{D}) \tag{1}$$

and let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions.

Suppose that $\mathcal{P}$ denotes the class of analytic functions p normalized by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

and satisfying the condition

$$\Re(p(z)) > 0 \quad (z \in \mathbb{D}).$$

We easily see that, if $p(z) \in \mathcal{P}$, then a Schwarz function $\omega(z)$ exists with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that (see [1])

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \quad (z \in \mathbb{D}).$$

Very recently, Cho et al. [2] introduced the following function class $\mathcal{S}_s^*$, which are associated with sine function:

$$S_s^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \sin z \ \ (z \in \mathbb{D}) \right\}, \tag{2}$$

where "$\prec$" stands for the subordination symbol (for details, see [3]) and also implies that the quantity $\frac{zf'(z)}{f(z)}$ lies in an eight-shaped region in the right-half plane.

The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ of functions f was stated by Noonan and Thomas [4] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

This determinant has been considered by several authors, for example, Noor [5] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f(z)$ given by Equation (1) with bounded boundary and Ehrenborg [6] studied the Hankel determinant of exponential polynomials.

In particular, we have

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \quad (n = 1,\ q = 3).$$

Since $f \in \mathcal{S}$, $a_1 = 1$,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

We note that $|H_2(1)| = |a_3 - a_2^2|$ is the well-known Fekete-Szego functional (see, for example, [7–9]).

On the other hand, Thomas and Halim [10] defined the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix} \quad (n \geq 1,\ q \geq 1).$$

The Toeplitz determinants are closely related to Hankel determinants. Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. For a good summary of the applications of Toeplitz matrices to the wide range of areas of pure and applied mathematics, we can refer to [11].

As a special case, when $n = 2$ and $q = 3$, we have

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

In recent years, many authors studied the second-order Hankel determinant $H_2(2)$ and the third-order Hankel determinant $H_3(1)$ for various classes of functions (the interested readers can see, for instance, [12–25]). However, apart from the work in [10,21,26,27], there appears to be little literature dealing with Toeplitz determinants. Inspired by the aforementioned works, in this paper, we aim to investigate the third-order Hankel determinant $H_3(1)$ and Toeplitz determinant $T_3(2)$ for the above function class $\mathcal{S}_s^*$ associated with sine function, and obtain the upper bounds of the above determinants.

2. Main Results

To prove our desired results, we need the following lemmas.

Lemma 1. *If $p(z) \in \mathcal{P}$, then exists some x, z with $|x| \leq 1$(see [28]), $|z| \leq 1$, such that*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2c_1 x(4 - c_1^2) - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - |x|^2)z.$$

Lemma 2. *Let $p(z) \in \mathcal{P}$ (see [29]), then*

$$|c_n| \leq 2, \ n = 1, 2, \cdots.$$

We now state and prove the main results of our present investigation.

Theorem 1. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form Equation (1), then*

$$|a_2| \leq 1, \ \ |a_3| \leq \frac{1}{2}, \ \ |a_4| \leq \frac{5}{9}, \ \ |a_5| \leq \frac{47}{72}. \tag{3}$$

Proof. Since $f(z) \in \mathcal{S}_s^*$, according to subordination relationship, so there exists a Schwarz function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$\frac{zf'(z)}{f(z)} = 1 + \sin(\omega(z)).$$

Now,

$$\frac{zf'(z)}{f(z)} = \frac{z + \sum_{n=2}^{\infty} n a_n z^n}{z + \sum_{n=2}^{\infty} a_n z^n}$$

$$= (1 + \sum_{n=2}^{\infty} n a_n z^{n-1})[1 - a_2 z + (a_2^2 - a_3)z^2 - (a_2^3 - 2a_2 a_3 + a_4)z^3$$

$$+ (a_2^4 - 3a_2^2 a_3 + 2a_2 a_4 + a_3^2 - a_5)z^4 + \cdots]$$

$$= 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2 a_3 + 3a_4)z^3$$

$$+ (4a_5 - a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2)z^4 + \cdots. \tag{4}$$

Define a function

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots.$$

Clearly, we have $p(z) \in \mathcal{P}$ and

$$\omega(z) = \frac{p(z) - 1}{1 + p(z)} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}. \tag{5}$$

On the other hand,

$$1 + \sin(\omega(z)) = 1 + \frac{1}{2}c_1 z + (\frac{c_2}{2} - \frac{c_1^2}{4})z^2 + (\frac{5c_1^3}{48} + \frac{c_3 - c_1 c_2}{2})z^3$$

$$+ (\frac{c_4}{2} + \frac{5c_1^2 c_2}{16} - \frac{c_2^2}{4} - \frac{c_1 c_3}{2} - \frac{c_1^4}{32})z^4 + \cdots. \tag{6}$$

Comparing the coefficients of $z,\ z^2,\ z^3,\ z^4$ between Equations (4) and (6), we obtain

$$a_2 = \frac{c_1}{2},\ a_3 = \frac{c_2}{4},\ a_4 = \frac{c_3}{6} - \frac{c_1 c_2}{24} - \frac{c_1^3}{144},\ a_5 = \frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32}. \tag{7}$$

By using Lemma 2, we thus know that

$$|a_2| \leq 1,\ \ |a_3| \leq \frac{1}{2},\ \ |a_4| \leq \frac{5}{9},\ \ |a_5| \leq \frac{47}{72}.$$

The proof of Theorem 1 is completed. □

Theorem 2. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form in Equation (1), then we have*

$$|a_3 - a_2^2| \leq \frac{1}{2}. \tag{8}$$

Proof. According to Equation (7), we have

$$|a_3 - a_2^2| = \left| \frac{c_2}{4} - \frac{c_1^2}{4} \right|.$$

By applying Lemma 1, we get

$$|a_3 - a_2^2| = \left| \frac{x(4 - c_1^2)}{8} - \frac{c_1^2}{8} \right|.$$

Let $|x| = t,\ t \in [0,1],\ c_1 = c,\ c \in [0,2]$. Then, using the triangle inequality, we obtain

$$|a_3 - a_2^2| \leq \frac{t(4 - c^2)}{8} + \frac{c^2}{8}.$$

Suppose that

$$F(c,t) = \frac{t(4 - c^2)}{8} + \frac{c^2}{8},$$

then $\forall t \in (0,1),\ \forall c \in (0,2)$,

$$\frac{\partial F}{\partial t} = \frac{4 - c^2}{8} > 0,$$

which shows that $F(c,t)$ is an increasing function on the closed interval [0,1] about t. Therefore, the function $F(c,t)$ can get the maximum value at $t = 1$, that is, that

$$\max F(c,t) = F(c,1) = \frac{(4 - c^2)}{8} + \frac{c^2}{8} = \frac{1}{2}.$$

Thus, obviously,

$$|a_3 - a_2^2| \leq \frac{1}{2}.$$

The proof of Theorem 2 is thus completed. □

Theorem 3. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form in Equation (1), then we have*

$$|a_2 a_3 - a_4| \leq \frac{1}{3}. \tag{9}$$

Proof. From Equation (7), we have

$$\begin{aligned}|a_2a_3 - a_4| &= |\tfrac{c_1c_2}{8} + \tfrac{c_1^3}{144} - \tfrac{c_3}{6} + \tfrac{c_1c_2}{24}| \\ &= |\tfrac{c_1c_2}{6} - \tfrac{c_3}{6} + \tfrac{c_1^3}{144}|.\end{aligned}$$

Now, in view of Lemma 1, we get

$$|a_2a_3 - a_4| = \left|\frac{7c_1^3}{144} + \frac{(4-c_1^2)c_1x^2}{24} - \frac{(4-c_1^2)(1-|x|^2)z}{12}\right|.$$

Let $|x| = t,\ t \in [0,1],\ c_1 = c,\ c \in [0,2]$. Then, using the triangle inequality, we deduce that

$$|a_2a_3 - a_4| \leq \frac{7c^3}{144} + \frac{(4-c^2)ct^2}{24} + \frac{(4-c^2)(1-t^2)}{12}.$$

Assume that

$$F(c,t) = \frac{7c^3}{144} + \frac{(4-c^2)ct^2}{24} + \frac{(4-c^2)(1-t^2)}{12}.$$

Therefore, we have, $\forall t \in (0,1),\ \forall c \in (0,2)$

$$\frac{\partial F}{\partial t} = \frac{(4-c^2)t(c-2)}{12} < 0,$$

namely, $F(c,t)$ is an decreasing function on the closed interval [0,1] about t. This implies that the maximum value of $F(c,t)$ occurs at $t = 0$, which is

$$\max F(c,t) = F(c,0) = \frac{(4-c^2)}{12} + \frac{7c^3}{144}.$$

Define

$$G(c) = \frac{(4-c^2)}{12} + \frac{7c^3}{144},$$

clearly, the function $G(c)$ has a maximum value attained at $c = 0$, also which is

$$|a_2a_3 - a_4| \leq G(0) = \frac{1}{3}.$$

The proof of Theorem 3 is completed. □

Theorem 4. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form in Equation (1), then we have*

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}. \tag{10}$$

Proof. Suppose that $f(z) \in \mathcal{S}_s^*$, then from Equation (7), we have

$$|a_2a_4 - a_3^2| = \left|\tfrac{c_1c_3}{12} - \tfrac{c_1^2c_2}{48} + \tfrac{c_1^4}{48} - \tfrac{c_2^2}{16}\right|.$$

Now, in terms of Lemma 1, we obtain

$$\begin{aligned}|a_2a_4 - a_3^2| &= \left|\tfrac{c_1c_3}{12} - \tfrac{c_1^2c_2}{48} - \tfrac{c_1^4}{288} - \tfrac{c_2^2}{16}\right| \\ &= \left|-\tfrac{5c_1^4}{576} - \tfrac{x^2c_1^2(4-c_1^2)}{48} - \tfrac{x^2(4-c_1^2)^2}{64} + \tfrac{c_1(4-c_1^2)(1-|x|^2)z}{24}\right|.\end{aligned}$$

Let $|x| = t,\ t \in [0,1],\ c_1 = c,\ c \in [0,2]$. Then, using the triangle inequality, we get

$$|a_2a_4 - a_3^2| \leq \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576}.$$

Putting

$$F(c,t) = \frac{t^2c^2(4-c^2)}{48} + \frac{(1-t^2)c(4-c^2)}{24} + \frac{t^2(4-c^2)^2}{64} + \frac{5c^4}{576},$$

then, $\forall t \in (0,1),\ \forall c \in (0,2)$, we have

$$\frac{\partial F}{\partial t} = \frac{t(c^2-8c+12)(4-c^2)}{96} > 0,$$

which implies that $F(c,t)$ increases on the closed interval [0,1] about t. That is, that $F(c,t)$ have a maximum value at $t = 1$, which is

$$\max F(c,t) = F(c,1) = \frac{c^2(4-c^2)}{48} + \frac{(4-c^2)^2}{64} + \frac{5c^4}{576}.$$

Setting

$$G(c) = \frac{c^2(4-c^2)}{48} + \frac{(4-c^2)^2}{64} + \frac{5c^4}{576},$$

then we have

$$G'(c) = \frac{c(4-c^2)}{24} - \frac{c^3}{24} - \frac{c(4-c^2)}{16} + \frac{5c^3}{144}.$$

If $G'(c) = 0$, then the root is $c = 0$. In addition, since $G''(0) = -\frac{1}{12} < 0$, so the function $G(c)$ can take the maximum value at $c = 0$, which is

$$|a_2a_4 - a_3^2| \leq G(0) = \frac{1}{4}.$$

The proof of Theorem 4 is completed. □

Theorem 5. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form in Equation (1), then we have*

$$|a_2^2 - a_3^2| \leq \frac{5}{4}. \tag{11}$$

Proof. Suppose that $f(z) \in \mathcal{S}_s^*$, then, by using Equation (7), we have

$$|a_2^2 - a_3^2| = |\tfrac{c_1^2}{4} - \tfrac{c_2^2}{16}|.$$

Next, according to Lemma 1, we obtain

$$|a_2^2 - a_3^2| = \left|\tfrac{c_1^2}{4} - \tfrac{c_2^2}{16}\right|$$

$$= \left|\tfrac{c_1^2}{4} - \tfrac{c_1^4}{64} - \tfrac{xc_1^2(4-c_1^2)}{32} - \tfrac{x^2(4-c_1^2)^2}{64}\right|.$$

Let $|x| = t,\ t \in [0,1],\ c_1 = c,\ c \in [0,2]$. Then, by applying the triangle inequality, we get

$$|a_2^2 - a_3^2| \leq \frac{c^2}{4} + \frac{c^4}{64} + \frac{tc^2(4-c^2)}{32} + \frac{t^2(4-c^2)^2}{64}.$$

Taking

$$F(c,t) = \frac{c^2}{4} + \frac{c^4}{64} + \frac{tc^2(4-c^2)}{32} + \frac{t^2(4-c^2)^2}{64}.$$

Then, $\forall t \in (0,1),\ \forall c \in (0,2)$, we have

$$\frac{\partial F}{\partial t} = \frac{c^2(4-c^2)}{32} + \frac{t(4-c^2)^2}{32} > 0,$$

which implies that $F(c,t)$ increases on the closed interval [0,1] about t. Namely, the maximum value of $F(c,t)$ attains at $t = 1$, which is

$$\max F(c,t) = F(c,1) = \frac{c^2}{4} + \frac{c^4}{64} + \frac{c^2(4-c^2)}{32} + \frac{(4-c^2)^2}{64}.$$

Let

$$G(c) = \frac{c^2}{4} + \frac{c^4}{64} + \frac{c^2(4-c^2)}{32} + \frac{(4-c^2)^2}{64},$$

then

$$G'(c) = \tfrac{c}{2} > 0,\ \forall c \in (0,2).$$

Therefore, the function $G(c)$ is an increasing function on the closed interval [0,2] about c, and thus $G(c)$ has a maximum value attained at $c = 2$, which is

$$|a_2^2 - a_3^2| \le G(2) = \frac{5}{4}.$$

The proof of Theorem 5 is completed. □

Theorem 6. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form in Equation (1), then we have*

$$|a_2a_3 - a_3a_4| \le \frac{13}{12}. \tag{12}$$

Proof. Assume that $f(z) \in \mathcal{S}_s^*$, then from Equation (7), we obtain

$$|a_2a_3 - a_3a_4| = |\tfrac{c_1c_2}{8} + \tfrac{c_1^3c_2}{576} - \tfrac{c_2c_3}{24} + \tfrac{c_1c_2^2}{96}|.$$

Now, by using Lemma 1, we see that

$$|a_2a_3 - a_3a_4| = \left|\tfrac{c_1c_2}{8} + \tfrac{c_1^3c_2}{576} - \tfrac{c_2c_3}{24} + \tfrac{c_1c_2^2}{96}\right|$$

$$= \left|\tfrac{c_1^3}{16} - \tfrac{c_1^5}{576} - \tfrac{11xc_1^3(4-c_1^2)}{1152} + \tfrac{xc_1(4-c_1^2)}{16} + \tfrac{x^2c_1(4-c_1^2)[c_1^2+x(4-c_1^2)]}{192} + \tfrac{c_1x^2(4-c_1^2)^2}{128} + \tfrac{(1-|x|^2)z(4-c_1^2)[x(4-c_1^2)+c_1^2]}{96}\right|.$$

If we let $|x| = t,\ t \in [0,1],\ c_1 = c,\ c \in [0,2]$, then, using the triangle inequality, we have

$$|a_2a_3 - a_3a_4| \le \tfrac{c^3}{16} + \tfrac{c^5}{576} + \tfrac{11tc^3(4-c^2)}{1152} + \tfrac{t(4-c^2)}{8} + \tfrac{t^2[c^2+t(4-c^2)](4-c^2)}{96} + \tfrac{t^2(4-c^2)^2}{64} + \tfrac{(4-c^2)[t(4-c^2)+c^2]}{96}.$$

Setting

$$F(c,t) = \tfrac{c^3}{16} + \tfrac{c^5}{576} + \tfrac{11tc^3(4-c^2)}{1152} + \tfrac{t(4-c^2)}{8} + \tfrac{t^2[c^2+t(4-c^2)](4-c^2)}{96} + \tfrac{t^2(4-c^2)^2}{64} + \tfrac{(4-c^2)[t(4-c^2)+c^2]}{96}.$$

Then, we easily see that, $\forall t \in (0,1),\ \forall c \in (0,2)$,

$$\tfrac{\partial F}{\partial t} = \tfrac{11c^3(4-c^2)}{1152} + \tfrac{(4-c^2)}{8} + \tfrac{t[c^2+t(4-c^2)](4-c^2)}{48} + \tfrac{t^2(4-c^2)^2}{96} + \tfrac{t(4-c^2)^2}{32} + \tfrac{(4-c^2)^2}{96} > 0,$$

which implies that $F(c,t)$ is an increasing function on the closed interval [0,1] about t. That is, that the maximum value of $F(c,t)$ occurs at $t = 1$, which is

$$\max F(c,t) = F(c,1) = \frac{c^3}{16} + \frac{c^5}{576} + \frac{11c^3(4-c^2)}{1152} + \frac{(4-c^2)}{8} + \frac{(4-c^2)}{24} + \frac{(4-c^2)^2}{64} + \frac{(4-c^2)}{24}.$$

Taking

$$G(c) = \frac{c^3}{16} + \frac{c^5}{576} + \frac{11c^3(4-c^2)}{1152} + \frac{(4-c^2)}{8} + \frac{(4-c^2)}{24} + \frac{(4-c^2)^2}{64} + \frac{(4-c^2)}{24},$$

then

$$G'(c) = \frac{3c^2}{16} + \frac{5c^4}{576} + \frac{11c^2(4-c^2)}{384} - \frac{11c^4}{576} - \frac{c(4-c^2)}{16} - \frac{c}{12},$$

$$G''(c) = \frac{3c}{8} + \frac{5c^3}{144} + \frac{11c(4-2c^2)}{192} - \frac{11c^3}{144} - \frac{(4-c^2)}{16} + \frac{c^2}{8} - \frac{1}{12}.$$

We easily find that $c = 0$ is the root of the function $G'(c) = 0$, since$G''(0) < 0$, which implies that the function $G(c)$ can reach the maximum value at $c = 0$, also which is

$$|a_2a_3 - a_3a_4| \le G(0) = \frac{13}{12}.$$

The proof of Theorem 6 is completed. □

Theorem 7. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form in Equation (1), then we have*

$$|H_3(1)| \le \frac{275}{432} \approx 0.637. \tag{13}$$

Proof. Since

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

by applying the triangle inequality, we get

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{14}$$

Now, substituting Equations (3), (8), (9) and (10) into Equation (14), we easily obtain the desired assertion (Equation (13)). □

Theorem 8. *If the function $f(z) \in \mathcal{S}_s^*$ and of the form in Equation (1), then we have*

$$|T_3(2)| \le \frac{139}{72} \approx 1.931. \tag{15}$$

Proof. Because

$$T_3(2) = a_2(a_2^2 - a_3^2) - a_3(a_2a_3 - a_3a_4) + a_4(a_3^2 - a_2a_4),$$

by using the triangle inequality, we obtain

$$|T_3(2)| \le |a_2||a_2^2 - a_3^2| + |a_3||a_2a_3 - a_3a_4| + |a_4||a_3^2 - a_2a_4|. \tag{16}$$

Next, from Equations (3), (10), (11) and (12), we immediately get the desired assertion (Equation (15)). □

Finally, we give two examples to illustrate our results obtained.

Example 1. *If we take the function* $f(z) = e^z - 1 = z + \sum_{n=2}^{\infty} \frac{z^n}{n!} \in \mathcal{S}_s^*$, *then we obtain*

$$\begin{aligned} |H_3(1)| &\le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \\ &= \frac{1}{3!} \times |\frac{1}{2!} \times \frac{1}{4!} - \frac{1}{3!} \times \frac{1}{3!}| + \frac{1}{4!} \times |\frac{1}{4!} - \frac{1}{2!} \times \frac{1}{3!}| + \frac{1}{5!} \times |\frac{1}{3!} - \frac{1}{2!} \times \frac{1}{2!}| \\ &\approx 0.004 < 0.637. \end{aligned}$$

Example 2. *If we set the function* $f(z) = -\log(1-z) = z + \sum_{n=2}^{\infty} \frac{z^n}{n} \in \mathcal{S}_s^*$, *then we get*

$$\begin{aligned} |T_3(2)| &\le |a_2||a_2^2 - a_3^2| + |a_3||a_2a_3 - a_3a_4| + |a_4||a_3^2 - a_2a_4| \\ &= \frac{1}{2} \times |\frac{1}{2} \times \frac{1}{2} - \frac{1}{3} \times \frac{1}{3}| + \frac{1}{3} \times |\frac{1}{2} \times \frac{1}{3} - \frac{1}{3} \times \frac{1}{4}| + \frac{1}{4} \times |\frac{1}{3} \times \frac{1}{3} - \frac{1}{2} \times \frac{1}{4}| \\ &\approx 0.107 < 1.931. \end{aligned}$$

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