GLOBAL EXISTENCE AND VALIDITY FOR THE BOLTZMANN HIERARCHY

By

A.E. HURD

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ON GLOBAL EXISTENCE AND VALIDITY FOR THE BOLTZMANN HIERARCHY

A. E. Hurd

University of Victoria

Victoria, Canada

§1 INTRODUCTION

A celebrated result of Oscar Lanford [16, 17] shows that the rescaled correlation functions associated with the motion of finitely many hard spheres approach solutions of the Boltzmann hierarchy in the Boltzmann-Grad limit, at least for sufficiently short times. The importance of this result is that it settles rigorously the long controversial discussion as to whether statistical mechanical irreversibility occurs in the Boltzmann-Grad limit or not, and establishes the relevance of the Boltzmann equation in statistical mechanics.

Because of the restriction to short times, we cannot use Lanford's result to obtain a rigorous proof of approach to equilibrium, probably the central problem in non-equilibrium statistical mechanics. The purpose of this paper is to provide a global (i.e., for all time) existence and validity result for the Boltzmann hierarchy. We show that the set of all weak limits of finite particle reduced correlation evolutions in the Boltzmann-Grad limit is nonempty, and that each such weak limit can be lifted to reduced correlation evolutions on a nonstandard space which satisfy an integrated nonstandard version of the Boltzmann hierarchy. The result is thus of the same type (for the Boltzmann hierarchy) as recent results for the Boltzmann equation established by Arkeryd [5, 6, 7].

Lanford's proof begins with a form of the BBGKY hierarchy for hard spheres, and unfortunately the usual derivation of this hierarchy (see e.g. [9, 10]) is formal. That there was a problem with the formal derivation was first pointed out by H. Spohn, who then attempted to provide a rigorous proof in the unpublished (and apparently incomplete) paper [20]. In this paper we first present as a lemma an integrated version of the hierarchy which suffices for our results. A few remarks are perhaps in order concerning the proof of this lemma. In the usual derivation of the BBGKY hierachy for hard spheres, the collision integral arises from integration by parts, and it is not at all clear how the (infinite) interparticle forces enter into it. In [11, I.2] H. Grad makes some relevant comments in this regard. In the proof presented here, it is made clear that the collision integral arises from collisions, or, more precisely, from discontinuities in the momenta in collisions. After the research on the lemma reported here was completed, the author learned of an independent and different derivation of the hierarchy by Illner and Pulvirenti [14].

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§2. THE RESULTS

Throughout the paper, Λ will denote a bounded domain in $\mathbb{R}^3$ with a C^1 boundary $\partial\Lambda$ and volume $|\Lambda|$, and N will denote the set of natural number. The one particle phase space is the set $\Lambda \times S$, where $S = \mathbb{R}^3$ equipped with the usual Borel structure. Points in the one particle phase space will be denoted by (q, p), where q is the position in Λ and p is the momentum of the particle. Throughout we assume for convenience that each particle is of unit mass. We let $\pi_n = (\Lambda \times S)^n$ (n-fold cartesian product), where we define $\pi_\emptyset$ to be a single point. On π_n we can define an equivalence relation ~ by putting $(x_1,\ldots,x_n) \sim (y_1,\ldots,y_n)$ if there exists a permutation ξ of $\{1,2,\ldots,n\}$ with $x_i = y_{\xi(i)}$ for $1 \le i \le n$. The quotient space of π_n under ~ is denoted by X_n, with a resulting map $\xi_n : \pi_n \longrightarrow X_n$. The grand canonical phase space is then, as usual, defined to be the set $X_o = \bigcup X_n$ $(n \in N)$, and is thus the set of all collections of finitely many particles (with repetitions) in Λ. If we define $\pi = \bigcup \pi_n (n \ge \emptyset)$, then there is a map $\xi : \pi \longrightarrow X_o$ which reduces to ξ_n on each summand. Points in X_o and π will be denoted (somewhat loosely considering the repetitions) by $x = \{x_1, x_2, \ldots, x_n\} = \{x^n\}$ and $x = (x_1,\ldots, x_n) = (x^n)$ respectively, where $x_i = (q_i, p_i)$, $q_i \in A$, $p_i \in S$ and we use the notation $x^n = (x_1,\ldots,x_n)$.

Presently we will be considering particles which are hard spheres, so it will be necessary to discuss subsets of X_o in which points cannot come too close together, or too near the boundary $\partial\Lambda$ of Λ. Thus for $\delta \ge \emptyset$ in $\mathbb{R}$, we define the set X^δ to be the set of points $x \in X_o$ such that if $x = \{x_1, x_2, \ldots, x_n\}$, and $x_i = (q_i, p_i)$, then $\inf\{|q_i - q_j| : 1 \le i < j \le n\} > \delta$, and $\inf\{|q_i - q| : 1 \le i \le n,\ q \in \partial\Lambda\} > \delta/2$, where $|.|$ denotes the norm in $\mathbb{R}^3$. The set X^δ inherits its Borel structure from X_o. In the same way, we may define $\pi_n^\delta = \{x \in \pi_n : \inf\{|q_i - q_j| : 1 \le i < j \le n\} > \delta\}, \inf\{|q_i - q| : 1 \le i \le n,\ q \in \partial\Lambda\} > \delta\}\}$, and $\pi^\delta = \bigcup \pi_n^\delta (n \ge \emptyset)$, and then ξ maps π^δ onto X^δ.

We will now use the constructions of the previous paragraph to define measures on X_o. For us the natural measure to use on π is $d\sigma = \sum d\sigma^n(x^n)/n!$ $(n \ge \emptyset)$, where $d\sigma^n$ is the n-fold normalized Maxwellian product measure on $(\Lambda \times S)^n$ ($d\sigma^\emptyset$ being point mass on the single point $\pi_\emptyset$), defined by

$$(2.1) \qquad d\sigma^n(x^n) = \prod_{i=1}^{n} d\sigma_i$$

($d\sigma^{\emptyset}$ being point mass on the single point $\pi_{\emptyset}$) where

$$(2.2) \qquad d\sigma_i = k\, e^{-\frac{\beta}{2} p_i^2}\, dx_i, 1 \le i \le n,$$

$k = |\Lambda|^{-1} (\beta / 2\pi)^{3/2}$ being the normalizing constant, and $dx_i = dq_i dp_i$. Since ξ is Borel measurable, we may put $\zeta = \sigma \circ \xi^{-1}$, the measure induced on X_o by σ. Also, ζ_n is ζ restricted to X_n. This use of Maxwellians is for convenience only, and will alter slightly the form of some standard expressions, e.g., the BBGKY hierarchy.

Our states (probability measures) μ on X_o are obtained from measurable densities f on X_o, so that

$$(2.3) \qquad \mu(F) = \int_F f\, d\zeta,$$

Such states can also be written as

$$(2.4) \qquad \mu(F) = \int_{\overline{F}} \overline{f}\, d\sigma = \sum 1/n! \int_{\overline{F}_n} \overline{f}_n d\sigma_n \quad (n \ge 0)$$

where $\overline{f} = f \circ \xi$, $\overline{F} = \xi^{-1}(F)$, $\overline{f}_n = \overline{f} \mid \pi_n(\Lambda)$, $\overline{F}_n = \overline{F} \cap \pi_n$. Using this representation, the k^{th} correlation measure ρ_k of μ is by definition a measure on π with a density (correlation function) which takes the form

$$(2.5) \qquad \rho_k(x^k) = \sum 1/(n-k)! \int \overline{f}_n(x^n)\, d\sigma_{k+1} \ldots d\sigma_n \quad (n \ge k)$$

and the integration is over π_{n-k}.

Suppose now that $\{T^t\}$ is a family (usually a group) of measurable transformations $T^t\colon X_o \to X_o$. We define $\mu(t)$ by $\mu(t)(F) = \mu(T^{-t}(F))$. Clearly, $\mu(t)(F) = \int_F \bar{f}(t)d\sigma$, and the correlation function $\rho_k(t)$ of $\mu(t)$ is given by

$$\text{(2.6)} \qquad \rho_k(t)(x^n) = \sum 1/(n-k)!\int \bar{f}_n(t)(x^n)\, d\sigma_{k+1}\ldots d\sigma_n \quad (n \geq k)$$

where here and throughout, $f(t) = f\circ T^{-t}$. We will also use the notation $f(t)(x^n) = f(t,x^n)$.

We are now going to consider a family of structures on X_o indexed by $m \in N$. The structures will include a family $\{T_m^t\}$ of hard sphere flows on X_o, a family of Borel probability measures $\{\mu_k^m\}$, with an associated family of correlation measures $\{\rho_k^m\}$, and finally an associated family $\{\psi_k^m\}$ of reduced correlation functions to be defined shortly. Throughout the discussion we will assume given a function $\delta = \delta(m)$ which satisfies the condition

$$\text{(2.7)} \qquad \lim_{m\to\infty} m[\delta(m)]^2 = \kappa$$

First we define the dynamical flows. For each $m \in N$, we imagine each point in $X^\delta(\Lambda)$ surrounded by a particle of diameter $\delta = \delta(m)$. The particles will move according to the hard sphere dynamics; that is, each particle moves in a straight line until it encounters another particle or the boundary. In a collision with the boundary, the particles reflect elastically. In a collision between two particles having positions and momenta (q_i,p_i) and (q_j,p_j), the ingoing momenta p_i and p_j are abruptly changed to the outgoing momenta p_i' and p_j', where

$$p_i' = p_i - k_i(p_j,\ \omega),$$

(2.8)

$$p_j' = p_j + k_i(p_j,\ \omega),$$

$\omega = (q_j-q_i)/|q_j-q_i|$ is the unit vector giving the direction between the centers, and $k_i(p,\ \omega) = \omega\cdot(p_i-p)$. After this interchange of momenta, the particles continue in rectilinear motion.

There are several problems with this definition of the dynamics. Firstly, it is not clear that, beginning with given initial conditions, we can find the positions and momenta of the particles at any later time, and indeed, multiple and grazing collisions can occur. But Alexander [1] has shown that these cases occur only on a set of measure zero in X^δ. Thus we may assume that there exists a measurable subset $\hat{X}^\delta$ of X^δ of full measure, and a group $\{T_m^t\colon \hat{X}^\delta \to \hat{X}^\delta : t \in \mathbb{R}\}$ of measurable transformations, where $T_m^t(x)$ is obtained by evolving the initial conditions $x \in \hat{X}^\delta$ over a time t using the hard sphere dynamics associated with particles of diameter $\delta = \delta(m)$. Similarly, there is a group, which we also denote by $\{T_m^t\}$, of maps $T_m^t\colon \hat{\pi}^\delta \to \hat{\pi}^\delta$, $\hat{\pi}^\delta = \xi^{-1}(\hat{\pi}^\delta)$, obtained by labelling the particles and using the hard sphere dynamics; the flows T_m^t commute with the map ξ. Note that the flows are so far defined only on $\hat{X}^\delta$ and $\hat{\pi}^\delta$, which may even be empty if $\delta(m)$ is too large: for convenience we thus define the T_m^t everywhere by putting $T_m^t(x) = x$ for $x \in X_o - \hat{X}^\delta$ or $x \in \pi - \hat{\pi}$.

The second and intrinsic problem is that the maps T_m^t, though measurable, are not even continuous, let alone differentiable. Due to the abrupt changes in momentum, the trajectories are only piecewise continuous. To be definite we impose the condition that the trajectories are continuous from the left, that is,

$$(2.9) \qquad \lim_{t\to\tau-0} T_m^t(x) = T_m^\tau(x).$$

Of most concern to us is the fact that the measures $d\sigma$ and $d\zeta$ are invariant under T_m^t.

We want to consider a special class $\{\mu^m\}$ of states having (symmetric) densities $\overline{f}^m$ on π of the form

$$(2.10) \qquad \begin{aligned} \overline{f}_n^m &= m!\, g^m, \quad n = m \\ &= 0, \quad n \neq m \end{aligned}$$

where g^m satisfies the following conditions:

2.1 (i) $\int g^m \, d\sigma_m = 1$ for all $m \in \mathbb{N}$;

(ii) $\lim_{t\to\tau} g^m(t,x^m) = g^m(\tau,x^m)$, i.e., the g^m are continuous along trajectories;

(iii) g^m is concentrated on π_m, and is zero on $\pi_m - \hat{\pi}_m^\delta$;

(iv) Given $\varepsilon > 0$ in $\mathbb{R}$, there are k, $l \in \mathbb{N}$ so that $\int_{|g^m| \geq k} g^m \, d\sigma_m < \varepsilon$ and $\int_{|g^m|\geq k} g^m \sum_1^m p_i^2 \, d\sigma_m < \varepsilon$ for all $m \geq 1$.

The condition 2.1 (iv) is a kind of uniform integrability condition on the initial conditions g^m.

The associated time evolving correlation functions are

$$(2.11) \qquad \begin{aligned} \rho_k^m(t,x^k) &= \frac{m!}{(m-k)!} \int g^m(t,x^m)\, d\sigma_{k+1} \ldots d\sigma_m \quad \text{for } k < m. \\ &= m!\, g^m(t,x^m), \quad k = m \end{aligned}$$

$= \emptyset, \; k > m.$

Since these correlation functions must diverge as $m \to \infty$, one defines the rescaled correlation functions ψ_k by

$$\psi_k^m(t,x^k) = m^{-k} \rho_k^m(t,x^k). \tag{2.12}$$

The important fact for us is that, for a certain class of smooth functions $\varphi\colon \mathbb{R} \times \pi_k^\delta(\Lambda) \to \mathbb{R}$, the function $\int\varphi(t,x^k)\psi_k^m(t,x^k)d\sigma_1\ldots d\sigma_k$ is absolutely continuous and hence differentiable almost everywhere as a function of t, and satisfies an integrated form of the BBGKY hierarchy.. The functions φ which will be allowed are specified in the following definition.

2.2 Definition

The set Φ_k consists of those functions $\varphi \in C^1(\mathbb{R} \times \pi_k^\delta(\Lambda))$ which are symmetric and for which

(i) $\varphi(t)$ has compact support in $\pi_k^\delta(\Lambda)$ for each $t \in \mathbb{R}$,

(ii) Whenever $|q_i - q_j| = \delta$, we have

$$\varphi(t,\ldots,q_i,p_i,\ldots,q_j,p_j,\ldots,x_k) = \varphi(t,\ldots,q_i,p_i',\ldots,q_j,p_j,\ldots,x_k)$$
$$= \varphi(t,\ldots,q_i,p_i',\ldots,q_j,p_j,\ldots,x_k),$$

where p_i' is obtained from (2.1) with p_i and p_j replacing p_1 and p_2.

Note that any function φ which vanishes in a neighborhood of the collision set $\pi_k(\Lambda) - \pi_k^\delta(\Lambda)$ for all t satisfies condition 2.1(ii), and hence Φ_k is dense in $L^1(\pi_k^\delta(\Lambda))$.

In the following, we use the notation $\psi_k^m(t)[\varphi(t)] = \int\varphi(t,x^k)\psi_k^m(t,x^k)d\sigma_1\ldots d\sigma_k$. The BBGKY hierarchy now takes the form

2.3 Lemma

Let $\tau \in \mathbb{R}^+$, For any $\varphi \in \Phi_k$ and $1 \le k < m$, we have

$$(2.13) \qquad \psi_k^m(t)[\varphi(t)] = \psi_k^m(0)[\varphi(0)] + m\delta^2 \int_0^\tau \{\psi_k^m(t)[H\varphi(t)] + C_{k+1}^m \psi_{k+1}^m(t)[\varphi(t)]\}\, dt,$$

where

$$(2.14) \qquad H\varphi = \frac{\partial \varphi}{\partial t} + \sum_{i=1}^{k} p_i \frac{\partial \varphi}{\partial q_i},$$

$$(2.15) \qquad C_{k+1}^m \psi_{k+1}[\varphi] = \delta^2 \sum_{i=1}^{s} \int_A \varphi(x^k) \{ \psi_{k+1}(t,\ldots,q_i,p_i',\ldots,q_i-\delta\omega,p) - \psi_{k+1}(t,x^k,q_i+\delta\omega,p)\} k_i(p,\omega)\, dx^k\, d\omega dp,$$

$\delta = \delta(m)$, and $A = \{ (x^k,\omega,p) : \omega \in S_2,\ \omega\cdot(p_i-p) \ge 0\}$. The operator C_{k+1}^m is called the collision integral.

It will be convenient for us to regard the sequence $\{\psi_k^m;\ k \in N\}$ as an element of a space of sequences of functions, and hence we write $\underline{\psi}^m = \{\psi_k^m:\ k \in N\}$ The object of this paper is to show that the set of cluster points of the net $\{\underline{\psi}^m(t)\}$ of time evolving rescaled correlation functions $\underline{\psi}^m(t)$ in the natural weak topology is non-empty, and that each such cluster point $\underline{\psi}\ (t)$ is associated with a nonstandard ${}^{\#}\psi$ which satisfies a nonstandard version of the Boltzmann hierarchy, which is a hierarchy of the same form as (2.15 except that C_{k+1}^m is replaced by a similar operator with $\delta = 0$. For the nonstandard analysis, the reader is referred to [12, 19, 21].

We now specify the topologies in which the cluster points are to be taken.

2.4 Definition

Let $L^1_{loc}(\mathbb{R} \times \pi_k)$ denote the set of measurable functions $\psi(t,x^k) = \psi(t)(x^k)$ on $\mathbb{R} \times \pi_k$ for which $\|\psi(t)\| \in L^1[a,b]$ for each $a,b \in \mathbb{R}$ ($\|\cdot\|$ denotes the norm in $L^1(\pi_k)$), and let B denote the set of sequences $\Psi = \{\psi^m_k(t,x^k)$: $k \in \mathbb{N}\}$ for which $\psi_k \in L^1_{loc}(\mathbb{R} \times \pi_k)$ for each $k \in \mathbb{N}$. A neighborhood subbase of $\hat{\Psi} \in B$ in the weak topology is given by sets of the form

$$(2.16) \qquad N_\alpha[\hat{\Psi}] = \{ \Psi : \int \varphi(t)[\hat{\psi}_k(t) - \psi_k(t)]d\sigma^k < \varepsilon, \int_a^b \varphi(s)[\hat{\psi}_k(s) - \psi_k(s)]d\sigma^k \, ds < \delta, \},$$

where $\alpha = (\varepsilon, \delta, \varphi, k, a, b)$ and $\varphi \in \Phi_k$.

We may now state our main theorem.

2.5 Theorem

The set Ω_∞ of all cluster points in the weak topology of the family $\Omega = \{\Psi^m$: $m \in \mathbb{N}\}$ in B is nonempty. For any $\hat{\Psi} = \{\hat{\psi}_k$: $k \in \mathbb{N}\} \in \Omega_\infty$, the functions ${}^\#\psi_k = \hat{\psi}_k \circ st$ (where st is the standard part map from $ns({}^*\mathbb{R} \times {}^*\pi_k)$ to $\mathbb{R}\times\pi_k$, ns denoting the near standard points) satisfy the Boltzmann hierarchy

$$(2.17) \quad {}^\#\psi_k(t)[\overset{o}{\varphi}(t)] = {}^\#\psi_k(0)[\overset{o}{\varphi}(0)] + \kappa\int_0^t \{{}^\#\psi_k(s)[\overset{o}{A}\varphi(s)] + C_{k+1}\,{}^\#\psi_{k+1}(s)[\overset{o}{\varphi}(s)]\}\; {}^o ds, \; k \in \mathbb{N},$$

where ${}^{\circ}\mu$ denotes the Loeb measure associated with μ, $\overset{\circ}{f} = {}^{\circ}({}^{*}f)$, ${}^{\#}\psi_k(s)[f] = \int {}^{\#}\psi_k(s, x^k) f(x^k) {}^{\circ}d\sigma^k$, and

$$(2.18) \qquad H\varphi = \frac{\partial\varphi}{\partial t} + \sum_{i=1}^{k} p_i \frac{\partial\varphi}{\partial q_i},$$

$$(2.19) \quad C_{k+1} {}^{\#}\psi_{k+1}[\varphi] = \sum_{i=1}^{s} \int_A \varphi(x^k) \{ {}^{\#}\psi_{k+1}(t,\ldots,q_i,p_i',\ldots,q_i,p) - \hat{\psi}_{k+1}(t,x^k,q_i,p)\} k_i(p, \omega) \, {}^{\circ}dx^k \, {}^{\circ}d\omega \, {}^{\circ}dp,$$

and $A = \{ (x^k,\omega,p) \in ns(\pi_k \times {}^{*}S_2 \times {}^{*}\mathbb{R}^3) : k_i(p, \omega) \geq 0\}$.

§3 PROOF OF THE LEMMA

As in [13] the proofs to follow use nonstandard analysis. The analysis will be carried out in a sufficiently saturated enlargement $V({}^{*}\mathbb{R})$ of the superstructure $V(\mathbb{R})$ over $\mathbb{R}$. We write $r \simeq s$ if r and s in ${}^{*}\mathbb{R}$ are infinitesimally close. The standard part of $r \in {}^{*}\mathbb{R}$ is denoted by ${}^{\circ}r$ or st(r), with the convention that ${}^{\circ}r = +\infty$ or $-\infty$ if r is positive or negative infinite.

In the ensuing discussion we will need to consider the set of points in $\hat{\pi}^{\delta}_m$ which involve collisions between particles. To this end we define the set

$$(3.1) \qquad K_{ij} = \{x \in \hat{\pi}^{\delta}_m : q_j = q_i + \delta\omega_{ij},\ k_i(p_j, \omega) \geq 0,\ \omega \in S_2\}$$

(S_2 being the unit sphere in $\mathbb{R}^3$) consisting of those points representing an ingoing collision between the i^{th} and the j^{th} particles, and put

$$(3.2) \qquad K = \bigcup K_{ij} \ (1 \leq i < j \leq n).$$

K inherits its topology from $\hat{\pi}^{\delta}_m$, and on K there is defined the natural measure

ν given by $d\nu = \sum d\nu_{ij}$ ($1 \le i < j \le n$), where $d\nu_{ij}$ is the measure on K_{ij} given by

$$(3.3) \quad d\nu_{ij} = \delta^2 \, d\sigma_1 \ldots d\sigma_{j-1} \, d\sigma_{j+1} \ldots d\sigma_n \, k e^{-\frac{\beta}{2} p_j^2} \, k_i(p_j, \omega) \, d\omega \, dp_j .$$

Here $d\nu_{ij}$ represents the measure on the unit sphere with center q_i in q_j space. Note that $K_{ij} \cap K_{ik} = \emptyset$ since there are no multiple collisions. In the same way we may define the set K^+, representing outgoing collisions by replacing the condition $k_i(p_j, \omega) \ge 0$ by $k_i(p_j, \omega) < 0$ in (3.1). There is a measure preserving map $\xi : K^+ \longrightarrow K$, where $\xi(y)$ is obtained from $y \in K^+$ by replacing outgoing by ingoing momenta using (2.8).

We now let $\Xi = \{ x \in \pi_m^\delta : T^t(x) \in K$ for some $t > 0 \}$ denote the set of points which undergo a collision at some time $t > 0$. The set Ξ has a representation as a special flow with base K [2, 3, 20]. To do so, we define the function τ on K by

$$(3.4) \qquad \tau(y) = \min \{ t \in \mathbb{R}^+ : T^{-t}(y) \in K^+ \}.$$

The function τ is never zero and may take the value ∞, but elsewhere is continuous on K and hence measurable since we have ruled out grazing collisions for $x \in \hat{\pi}_m^\delta$. For any $x \in \Xi$, the set $\{t \in [0, \infty) : T^t(x) \in K \}$ has no finite point of accumulation since we have eliminated multiple collisions. We now let

$$(3.5) \qquad \tilde{\Xi} = \{ (s,y) : 0 \le s < \tau(y), \; y \in K \}.$$

$\tilde{\Xi}$ inherits a topology and measure from $K \times [0, \infty)$. Next we define a 1-1 bicontinuous measure-preserving map $\Phi : \Xi \longrightarrow \tilde{\Xi}$ by letting $\Phi(x) = (s, y)$, where $s = s(x) = \min \{t : T^t(x) = y \in K\}$. We can use the rules (2.8) to define a measurable map $S : K \longrightarrow K$ by the prescription $S(y) = \xi(T^{-\tau(y)})$. Then corresponding to the semigroup $\{T^{-t}: t \geq 0\}$ of maps on Ξ there is the semigroup $\{S^{-t}: t \geq 0\}$ of maps on $\tilde{\Xi}$ defined by

$$S^{-t}(s, y) = (t+s, y) \text{ for } 0 \leq t < \tau(y) - s,$$

$$(3.6) \quad S^{-t}(y, s) = (t + s - \tau(y) - \dots - \tau(s^{n-1}y), S^n y)$$

$$\text{for } t(y) + \dots + t(S^{n-1}y) - s \leq t < \tau(y) + \dots + \tau(S^n y) - s.$$

T^{-t} and S^{-t} are related by the equation $\Phi \circ T^{-t} = S^{-t} \circ \Phi$. We can replace integration of a function g on Ξ by integration of $\tilde{g} = g \circ \Phi^{-1}$ on $\tilde{\Xi}$. The collision integral will involve integration on $\tilde{\Xi}$.

Throughout the following discussion we will suppress the index m on functions and dynamics. Suppose then that g is a symmetric function in $L^1(\hat{\pi}_m^\delta)$ with corresponding time evolved correlation functions $\rho_k(x^k)$.

Any function $\varphi \in \Phi_k$ can be regarded as a function on $\hat{\pi}_m^\delta(\Lambda)$ which is independent of the variables x_n, $k+1 \leq n \leq m$. In the proof of the lemma we use the fact that for $\varphi \in \Phi_k$,

$$(3.70) \qquad \int \varphi(t, x^k) g(t, x^m) d\sigma_1 \dots d\sigma_m = \int \varphi(t, T^t(x^m)) f(x^m) d\sigma_1 \dots d\sigma_m.$$

Note that on the right-hand side we regard x^k as being a point in $\hat{\pi}_m^\delta(\Lambda)$ by the obvious embedding, and the map T^t, though it acts only on x^k, involves the effects of all m particles. In the following, we denote $T^t(x)$ by $x(t) = (x_1(t), \dots, x_m(t))$, where $x_i(t) = (q_i(t), p_i(t))$. Also we will denote

$\varphi(t,T^t(x_m))$ by $\varphi(t, x_k(t))$, remembering that $x_k(t)$ depends on $x_1,\ldots,x_m$, as well as t.

The continuity assumption 2.1(ii) is used in the following way. Suppose that D is a compact subset of K such that $D \times [0,\tau] \subseteq \tilde{\Xi}$ for some $\tau \in \mathbb{R}^+$; we want to consider integrals of the function $\tilde{g} = g\circ\Phi^{-1}$ on $D \times [0,\tau]$, where g satisfies condition 2.1(ii) and is integrable on $D \times [0,\tau]$. Since no collisions have taken place in the time interval $[0,\tau]$, i.e., $S^{-t}(y) \in K^+$ for $y \in D$ and $t \in [0,\tau]$, we see that the function $\tilde{g}(s.y) \in L^1([o,\tau] \times D)$ is continuous as a function of $s \in [0,\tau]$. By the Fubini theorem, $\tilde{g}(s,\cdot)$ is in $L^1(D)$ for a.e. $s \in [0,\tau]$, and $\int\tilde{g}(s,y)d\sigma(y)$ is continuous as a function of s, so we conclude that $\int\tilde{g}(0,y)d\sigma(y)$ is finite, and that

$$(3.8) \qquad \lim_{s\to 0} \int \tilde{g}(s,y)d\sigma(y) = \int \tilde{g}(0,y)d\sigma(y).$$

With the preliminaries over, we are now ready to begin the proof of the lemma. We first assume that the function g is supported in a compact set of the form $\Sigma_s = \Lambda \times S$, where $S = \{ p = (p_1,\ldots,p_m) : \sum p_i^2 \ (1 \le i \le m) \le s$, s a positive real number }; the proof will be completed by a limiting argument. Note that $\Sigma_s \cap \hat{\pi}_m^\delta(\Lambda)$ is invariant under T^t since $\sum p_i^2$ is conserved under collisions..

The proof will use Keisler's Infinite Sum Theorem [12, 15], which in our case can be stated as follows: In order to show that (2.13) is true, we need only show that

$$(3.9) \qquad d\rho_k(t)[\varphi(t)] = \rho_k(t+dt)[\varphi(t+dt)] - \rho_k(t)[\varphi(t)]$$
$$\simeq dt\ \{\rho_k(t)[H\varphi(t)] + C_{k+1}\rho_{k+1}(t)[\varphi(t)]\}$$

for $t \geq 0$ in ${}^*\mathbb{R}$, where dt is a positive infinitesimal increment in time and $\simeq$ denotes the relation of being infinitesimally close [here and in the following we make no notational distinction between standard functions and their * - transforms in the nonstandard model; the context will make clear what is meant].

Let $t \in {}^*\mathbb{R}^+$, and dt be an infinitesimal, positive increment in time. Using (3.7), we have

$$(3.10)\qquad \frac{(m-k)!}{m!}\, d\rho_k(t)[\varphi(t)] = \int[\varphi(t+dt,x^k(t+dt)) - \varphi(t,x^k(t))]g(x^m)d\sigma^m.$$

$$= \int[\varphi(t+dt,x^m(t+dt)) - \varphi(t,x^m(t+dt))]g(x^m)d\sigma^m$$

$$+\sum_{i=1}^{k} \int\{\varphi(t,x_1(t),\dots,x_{i-1}(t),q_i(t+dt),p_i(t+dt),\dots,x_k(t+dt))$$

$$-\varphi(t,x_1(t),\dots,x_{i-1}(t),q_i(t),p_i(t+dt),\dots,x_k(t+dt))\}g(x^m)d\sigma^m$$

$$+\sum_{i=1}^{k} \int\{\varphi(t,x_1(t),\dots,x_{i-1}(t),q_i(t),p_i(t+dt),\dots,x_k(t+dt))$$

$$-\varphi(t,x_1(t),\dots,x_{i-1}(t),q_i(t),p_i(t),\dots,x_k(t+dt))\}g(x^m)d\sigma^m.$$

Using the mean value theorem and the fact that $\varphi \in C^1$, we see that the first integral is infinitesimally close to

$$(3.11)\qquad dt \int \frac{\partial\varphi}{\partial t}(t,x^k(t))g(x^m)d\sigma^m = dt \int \frac{\partial\varphi}{\partial t}(t,x^k)\ g(t,x^m)d\sigma^m = dt\ \rho_k(t)[\frac{\partial\varphi}{\partial t}].$$

Next we consider the terms in the two sums in (3.10). We write each integral in the sums as the sum of integrals, depending on whether the i^{th} particle does or does not undergo a collision with the j^{th} particle in the time interval [t, t+dt]. More explicitly, we let $A_{ij}(t) = T^{-t}(A_{ij})$, where

(3.12) $A_{ij} = \{ x = (x_1,\dots,x_m) \in \Sigma \cap \hat{\pi}^{\delta}_m(\Lambda) \ : \ T^s(x) \in K_{ij},\ 0 \le s < dt\}.$

$A_{ij}(t)$ is thus the set of initial conditions $x = (x_1,\dots,x_m) \in \Sigma \cap \hat{\pi}^{\delta}_m(\Lambda)$ for which there is a collision between the i^{th} and j^{th} particles in the time interval $[t, t+dt)$. It is important to note that the momenta p_i in $x_i = (q_i,p_i)$, $1 \le i \le m$ are all finite since $x \in \Sigma$, and the same is true of the momenta in $T^{-t}(x_1,\dots,x_m)$. From this fact it follows that there is only one collision in the infinitesimal time interval $[t, t+dt)$. Because of the fact that there are no multiple collisions, it is easy to see that the A_{ij} are disjoint. We now let $A_i(t) = \bigcup A_{ij}(t)(1 \le j \le m)$. It is important to remark at this point that we do not need to consider collisions with the boundary since φ has compact support.

On $A_i'(t) = [A_i(t)]'$ there are no collisions involving the i^{th} particle in $[t, t+dt)$. Thus we have $q_i(t+dt) - q_i(t) = p_i(t)dt$ and $p_i(t+dt) = p_i(t)$, and using the mean value theorem and the fact that $\varphi \in C^1$, we get

$$\begin{aligned}(3.13)\qquad & \int_{A_i'(t)} [\varphi(\dots,q_i(t+dt),p_i(t+dt),\dots) - \\ & \qquad -\varphi(\dots,q_i(t),p_i(t+dt),\dots)]\ g(x^m)\ d\sigma^m \\ & \simeq dt \int_{A_i'(t)} p_i \frac{\partial\varphi}{\partial q_i}(x^k(t))\ g(x^m)\ d\sigma^m\end{aligned}$$

On the other hand, since the dynamics is continuous in q_i and φ is continuous, $|\varphi(\dots,q_i(t+dt),p_i(t+dt),\dots) - \varphi(\dots,q_i(t),p_i(t+dt),\dots)| \le \eta$ where η is infinitesimal, and so

$$(3.14)\qquad \Big| \int_{A_i(t)} [\varphi(\dots,q_i(t+dt),P_i(t+dt),\dots) -$$

$$- \varphi(\ldots,q_i(t),p_i(t+dt),\ldots)]\, g(x^m)\, d\sigma^m \,|$$

$$\leq \eta \sum_{j=1}^{m} \int_{A_{ij}(t)} g(x^m)\, d\sigma^m$$

$$= \eta \sum_{j=1}^{m} \int_{A_{ij}} g(t,x^m)\, d\sigma^m.$$

Now note that

$$(3.15) \qquad \int_{A_{ij}} g^t(x^m)\, dx^m = \int_0^{dt} \int_{K_{ij}} \tilde{g}(S^{-t+s}(s,y)\, d\nu(y)\, ds$$
$$= \int_0^{dt} \int_{K_{ij}} \tilde{g}^{t-s}(y,s)\, d\nu(y) ds$$

where here and later we use the notation $\tilde{g}(t) = g(t)\circ\Phi^{-1}$. Using (2.12) and the mean-value theorem of the integral calculus, we see that the right-hand side of (3.14) is infinitesimally close to $\eta B dt$ for some positive constant B, and as a result the first sum in (3.10) is infinitesimally close to

$$(3.16) \qquad dt \sum_{i=1}^{k} \int p_i \frac{\partial\varphi}{\partial q_i}(t,x^k)\, g(t,x^m)\, d\sigma^m = dt \sum_{i=1}^{k} \int p_i \frac{\partial\varphi}{\partial q_i}\, \rho_k(t)\, d\sigma^k.$$

We now consider a typical term in the second sum in (3.10). First we note that $p_i(t+dt) = p_i(t)$ in $A_i'(t)$, so that

$$(3.17) \qquad \int_{A_i'(t)} [\varphi(\ldots,q_i(t),p_i(t+dt),\ldots) -$$
$$- \varphi(\ldots,q_i(t),p_i(t),\ldots)]\, g(x^m)\, d\sigma^m = 0$$

Now in $A_{ij}(t)$, the i^{th} particle collides with the j^{th} particle in the time interval [t, t+dt). The (outgoing) velocity $p_i(t+dt)$ of the i^{th} particle after such a collision is given by

$$(3.18)\qquad p_i(t) - \left[\omega_{ij}(t+\alpha dt)\cdot[p_i(t) - p_j(t)]\right] \omega_{ij}(t+\alpha dt)$$
$$\simeq p_i(t) - \left[\omega_{ij}(t)\cdot[p_i(t) - p_j(t)\right] \omega_{ij}(t),$$
$$= p'_{ij}(t)$$

where $0 < \alpha < 1$ and $\omega_{ij}(t) = [q_j(t) - q_i(t)]/|q_j(t) - q_i(t)|$. Thus, using 2.1(ii), we have

$$(3.19)\qquad \int \{\varphi(\dots,q_i(t),p_i(t+dt),\dots) -$$
$$-\varphi(\dots,q_i(t),p_i(t),\dots)\}\, g(x^m)\, d\sigma^m$$
$$\simeq \sum_{j=k+1}^{m} \int_{A_{ij}(t)} \{\varphi\dots,q_i(t),p'_{ij}(t),\dots) -$$
$$-\varphi(\dots,x_i(t),\dots)\}\, g(x^m)\, d\sigma^m$$
$$\simeq \sum_{j=k+1}^{m} \int_{A_{ij}} \{\varphi(\dots,q_i,p'_{ij},\dots) -$$
$$-\varphi(\dots,x_i,\dots)\}\, g(t,x^m) d\sigma^m$$
$$\simeq (m-k) \int_{A_{i,k+1}} \{\varphi(t,\dots,q_i,p'_i,\dots) - \varphi(t,x^k)\}\, g(t,x^m)\, d\sigma^m,$$

the last by symmetry, where $p'_i = p'_{i,k+1}$. The last integral in (3.19) can again be treated as an integral over $\tilde{\Xi}$. Using the definition of the surface measure $d\sigma_{ij}$ and the continuity as before, we have

$$(3.20)\quad (m-k)\int_{A_{i,k+1}} \varphi(t,x^k) g(t,x^m) d\sigma^m = (m-k)\int_{o}^{dt} \int_{K_{i,k+1}} \tilde{\varphi}(t)(\tau,y)\ \tilde{g}(t)(\tau,y) d\nu_{i,k+1} d\tau$$
$$= (m-k)\ dt\ \delta^2 \int k_i(p,\ \omega)\ \varphi(t,x^k)\ g(t,x^m)\ d\sigma^k\ d\sigma_{k+2}\dots d\sigma_m\ ke^{-\frac{\beta}{2}p^2}\ d\omega dp$$

$$= dt\ \delta^2\ \frac{(m-k)!}{m!} \int_A k_i(p,\ \omega)\ \varphi(t,x^k)\ \rho_{k+1}(t,x^k,q_i+\ d\omega,p)d\sigma^k\ ke^{-\frac{\beta}{2}p^2}\ d\omega dp,$$

where the integration is over the set $A = \{(x^k,\ \omega,\ p) : \omega \in S_2,\ k_i(p,\ \omega) \geq 0\}$. To deal with the integral $\int \varphi(t,\ldots,q_i,p_i',\ldots)\ g(t,x^m)\ d\sigma^m$ over $A_{i,k+1}$ we first use the fact that $\omega\cdot(p_i-p) = -\omega\cdot(p_i'-p')$, and the above argument to obtain

$$(3.21)\qquad (m-k) \int \varphi(t,\ldots,q_i,p_i',\ldots)g(t,x^m)\ d\sigma^m$$
$$\simeq -\ dt\ \delta^2\ \frac{(m-k)!}{m!}\int_A \omega\cdot(p_i'-p')\varphi(t,\ldots,q_i,p_i',\ldots)\ \times$$
$$\times\ \rho_{k+1}(t,x^k,q_i+d\omega,p)d\sigma^k ke^{-\frac{\beta}{2}p^2}\ d\omega dp$$

Since the transformation (2.8) is orthogonal, and hence $dp_i dp = dp_i'\ dp'$, we may replace ω by $-\omega$, and we see that the last line in (3.21) is

$$(3.22)\quad \simeq dt\ \delta^2\ \frac{(n-k)!}{n!} \int_A k_i(p,\ \omega)\ \varphi(x^k)\ \times$$
$$\times\ \rho_{k+1}(t,\ldots,q_i,p_i',\ldots,q_i-d\omega,p)\ d\sigma^k\ ke^{-\frac{\beta}{2}p^2} d\omega dp.$$

Putting all of the above together with Keisler's Infinite Sum Theorem, we obtain the desired result for functions of compact support.

To prove the result for general $f \in L^1(\pi_n^\delta)$ is now simply a matter of approximating f in L^1 by $f_s = f\chi_{\Sigma_s}$. The one point to notice in this connection is that the collision integral, when integrated over t, correspoonds to a full volume integral on $\check{\Xi}$, as we have seen from the calculations above.

§ 4. PROOF OF THE THEOREM

The set Ω_∞ of cluster points will be obtained by considering the elements in ${}^*\Omega = \{\psi^m : m \in {}^*N\}$ for which $m \in {}^*N_\infty$, the set of infinite natural numbers. Throughout the following discussion, m will denote a fixed element in ${}^*N_\infty$.

We first note that an immediate argument by transfer using 2.1(iv) shows that $g^m(x^m)$ is S-integrable [12]. Next we show that $g^m(t,x^m)$ is S-integrable as a function of x^m for any $t \in {}^*\mathbb{R}$. Using the invariance of $d\sigma$ under T_m^t we see that for any internally measurable subset A,

$$(4.1) \qquad \int_A g^m(t,x^m)d\sigma^m = \int_{T_m^{-t}(A)} g^m(x^m)d\sigma^m$$

and the S-integrability of $g^m(t,x^m)$ follows from that of $g^m(x^m)$ on putting $A = \{x \in \pi_m : |g^m(t,x^m)| \geq \omega\} = T_m^t(\{x \in \pi_m : |g_m(x^m)| \geq \omega\})$ for $\omega \in {}^*N_\infty$.

Now we show that the functions $\psi_k^m(t,x^k)$ are S-integrable as functions of x^k for any $t \in {}^*\mathbb{R}$, $k \in N,$. To do so, we need only check condition (iii) of Definition 3 in [4]. Let A be an internally meaurable subset of ${}^*\pi_k$ with $\int_A d\sigma^k \simeq 0$. Then $\int_A \psi_k^m(t,x^k)\ d\sigma^k = \frac{m!}{m^k(m-k)!} \int_{A\times\pi_{m-k}} g^m(t,x^m)\ d\sigma^m \simeq 0$ since $\int_{A\times\pi_{m-k}} d\sigma^m \simeq \int_A d\sigma^k \simeq 0$ and $\frac{m!}{m^k(m-k)!} \simeq 0$, and the condition is satisfied. It follows from the fact that $\int \psi_k^m(t,x^k)\ d\sigma_k$ is bounded (by one) that it is S-integrable as a function of $t \in {}^*\mathbb{R}$ on any finite interval $[a,b] \subseteq ns\ ({}^*\mathbb{R})$, the near-standard points in ${}^*\mathbb{R}$.

For any $k \in N$, the measures $d\sigma_k$ are near-standardly concentrated. It follows from this fact and the fact that the ψ_k^m are S-integrable that the functions ${}^o\psi_k^m$ are integrable with respect to ${}^o d\sigma_k$ and

$$(4.2)\quad {}^{\circ}\!\int_{{}^*\pi_k} {}^*\varphi(t,x^k)\psi_k^m(t,x^k)\,{}^*d\sigma^k \simeq \int_{{}^*\pi_k} {}^{\circ}({}^*\varphi(t,x^k)\ {}^{\circ}\psi_k^m(t,x^k)\ {}^{\circ}d\sigma_k,$$

$$\simeq \int_{ns({}^*\pi_k)} {}^{\circ}({}^*\varphi(t,x^k)\ {}^{\circ}\psi_k^m(t,x^k)\ {}^{\circ}d\sigma_k,$$

and

$$(4.3)\quad {}^{\circ}\!\int_a^b {}^*dt \int_{{}^*\pi_k} {}^*\varphi(t,x^k)\psi_k^m(t,x^k)\,{}^*d\sigma^k \simeq \int_a^b {}^{\circ}dt\ {}^{\circ}\!\!\int_{{}^*\pi_k} ({}^*\varphi(t,x^k)\ \psi_k^m(t,x^k)\ d\sigma_k$$

$$\simeq \int_a^b {}^{\circ}dt \int_{ns({}^*\pi_k)} {}^{\circ}\!\varphi(t,x^k)\ {}^{\circ}\psi_k^m(t,x^k)\,{}^{\circ}d\sigma_k$$

for any continuous φ, where $a,b \in ns({}^*\mathbb{R})$.

We now proceed to characterize the set Ω_∞. By §10.2 in [8] there are, for any $m \in {}^*N_\infty$ and $k \in N$, functions ${}^{\#}\psi_k^m(t,\ x^k)$ and $\hat{\psi}_k^m(t,x^k)$, defined on $ns({}^*\mathbb{R}\times{}^*\pi_k)$ and $\mathbb{R}\times\pi_k$ respectively, so that $E^{st}({}^{\circ}\psi_k^m) = {}^{\#}\psi_k^m = \hat{\psi}_k^m \circ st$, where E^{st} denotes conditional expectation with respect to the σ-algebra generated by $st\colon ns({}^*\mathbb{R}\times{}^*\pi_k) \to \mathbb{R}\times\pi_k$. If $a,\ b \in \mathbb{R}$ and $\varphi \in \Phi_k$, $k \in N$, we have, using (4.2) and (4.3) that

$$(4.4)\quad \int {}^*\varphi(t)\psi_k^m(t)\,{}^*d\sigma_k \simeq \int \varphi(t)\hat{\psi}_k^m(t)d\sigma_k,$$

and

$$(4.5)\quad \int_a^b {}^*ds \int {}^*\varphi(s)\ \psi_k^m(t)\ {}^*d\sigma_k \simeq \int_a^b ds \int \varphi(t)\ \hat{\psi}_k^m(s)\ d\sigma_k.$$

By Theorem 4.2.6 in [19], the nonempty set $\{\hat{\Psi}^m\colon m \in {}^*N_\infty\} = \Omega_\infty$.

Finally we show that for any $\hat{\Psi}^m = \{\hat{\psi}_k^m\colon k \in N\} \in \Omega_\infty$, the functions ${}^{\#}\psi_k = \hat{\psi}_k \circ st$ satisfy the Boltzmann hierarchy. By transfer from (2.11) we have

$$(4.6)\qquad \int {}^*\varphi(t,x^k)\psi_k^m(t,x^k)\,{}^*d\sigma^k - \int {}^*\varphi(\emptyset,x^k)\psi_k^m(\emptyset,x^k)\,{}^*d\sigma^k$$
$$= m\delta^2\int_o^t\{\psi_k^m(s)[{}^*H\varphi(s)] + C_{k+1}^m\psi_{k+1}^m(s)[{}^*\varphi(t)]\}\,{}^*ds.$$

Taking standard parts on both sides of (4.6), there results

$$(4.7)\qquad {}^o\psi_k^m(t)[\overset{o}{\varphi}(t)] - {}^o\psi_k^m(\emptyset)[\overset{o}{\varphi}(\emptyset)]$$
$$= \kappa\ {}^o\int_o^t\{\psi_k^m(s)[{}^*H\varphi(s)] + C_{k+1}^m\psi_{k+1}^m(s)[{}^*\varphi(t)]\}\,{}^*ds.$$

We can replace ${}^o\psi_k^m$ on the left by ${}^\#\psi_k^m$ since $\overset{o}{\varphi}$ is measurable with respect to the σ-algebra generated by st.

We now deal with the first term on the right. First note that

$$(4.8)\qquad \psi_k^m(s)[{}^*H\varphi(s)] = \frac{m!}{m^k(m-k)!}\int g^m(s,x^m)[\frac{\partial\varphi}{\partial t} + \sum_1^k p_i\frac{\partial\varphi}{\partial q_i}]\ {}^*d\sigma^m.$$

The integrand on the right is in absolute value $\leq B\ |g^m|\sum_1^m p_i^2$ where B depends on φ and, applying the arguments above with $d\sigma^m$ replaced by the invariant and near-standardly concentrated measure $[\sum_1^m\ p_i^2]\ d\sigma^m$, we see that that it is S-integrable, and

$$(4.9)\qquad {}^o\int_o^t\ \psi_k^m(s)[{}^*H\varphi(s)]\,{}^*ds = \int_o^t\ {}^o\psi_k^m(s)[\overset{o}{H}\varphi(s)]\,{}^o ds.$$

Again, we can replace ${}^o\psi_k^m$ by ${}^\#\psi_k^m$.

There remains to consider the collision term. Notice from (3.19) and

(3.20) that $\int_o^t C^m_{k+1}\ \psi^m_{k+1}[{}^*\varphi(s)]\ {}^*ds$ is a finite sum of terms of the form

$$(4.10)\qquad \frac{m!(m-k-1)!}{m!}\int_o^t {}^*\tilde{F}(t)(s,x)\tilde{g}^m(S^{-t+s}(s,y))\}d\nu_{i,k+1}\,{}^*ds,$$

where ${}^*F(t)(x^m) = {}^*\varphi(t,\ldots,q_i,p_i',\ldots) - {}^*\varphi(t,x^k)$. The integration on the right is over a set of positive measure in $\tilde{\Xi}$, so the integrand over this set is S-integrable since g^m is S-integrable. We conclude that

$$(4.11)\qquad {}^o\int_o^t C^m_{k+1}\psi^m_{k+1}[{}^*\varphi(s)]\ {}^*ds = \int_o^t C^m_{k+1}\ {}^o\psi_{k+1}[\overset{o}{\varphi}]\ {}^o ds.$$

Finally, ${}^o\psi^m_k$ can be replaced by ${}^\#\psi^m_k$ as above. After doing so, we note that C^m_{k+1} can be replaced by the collision term C_{k+1} since, e.g., ${}^\#\psi^m_{k+1}(t,\ldots,q_i,p_i',\ldots,q_i-\delta\omega,p) = {}^\#\psi^m_{k+1}(t,\ldots,q_i,p_i',\ldots,q_i,p)$ because ${}^\#\psi^m_{k+1}$ is constant on monads and $\delta = \delta(m)$ is infinitesimal. This completes the proof.

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University of Victoria,

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