INDEPENDENCE COUNTEREXAMPLES

by

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Events $A_1, A_2, \cdots, A_n$ are independent provided

$$P(A_{i_1}A_{i_2}\cdots A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r})$$

for every subcollection $A_{i_1}, A_{i_2}, \cdots, A_{i_r}$ of two or more A–events. Since there are $2^n - n - 1$ equations in this definition, the task of verifying independence is laborious when n is large. It is natural for students to wonder whether or not checking an appropriate subset of these equations would suffice. For example, perhaps pairwise independence would imply independence. Most textbooks give examples to show there are no such shortcuts, but usually limit the scope of these examples to cases where $n = 3$. (See [1], [2], and [3].)

The purpose of this note is to show, for arbitrary n, that all $2^n - n - 1$ defining equations are needed to insure the independence of A_1, $A_2, \cdots, A_n$. Specifically, we will show how to construct examples where only one arbitrarily chosen equation is not satisfied while the remaining $2^n - n - 2$ equations are satisfied.

We begin with n independent events $A_1, A_2, \cdots, A_n$, where $0 < P(A_i) < 1$ for all $i = 1, 2, \cdots, n$. Suppose we decide to destroy independence by invalidating just one particular k–fold intersection equation for some fixed $k \in \{2,3,\cdots,n\}$. We relabel, if necessary, so that the equation to be invalidated is

$$P(A_1A_2\cdots A_k) = P(A_1)P(A_2)\cdots P(A_k).$$

We fix positive constant ϵ smaller than $\min\{P(B_i): i = 1, 2, \cdots, 2^n\}$, where $\{B_1, B_2, \cdots, B_{2^n}\}$ is the partition of the sample space determined by the events $A_1, A_2, \cdots, A_n$. That is, each event B_i has the form $A_1^{\delta_1}A_2^{\delta_2}\cdots A_n^{\delta_n}$ where each $A_j^{\delta_j}$ is either A_j of A_j^c. Now, we change the probabilities of each of those 2^k B–events that have the form

$$A_1^{\delta_1}A_2^{\delta_2}\cdots A_k^{\delta_k}A_{k+1}^c\cdots A_n^c$$

as follows. For each $j \in \{0,1,\cdots,k\}$, the probability of $A_1^{\delta_1}\cdots A_k^{\delta_k}A_{k+1}^c\cdots A_n^c$ is changed by the amount $(-1)^j\epsilon$ if and only if exactly j of the first k A–events are complemented. The remaining $2^n - 2^k$ B–event probabilities are left unchanged.

These changes give a valid assignment of probability because all B–event probabilities remain nonnegative and the net change in the sum of all B–event probabilities is

$$\sum_{j=0}^{k} \binom{k}{j}(-1)^j\epsilon = (-1+1)^k\epsilon = 0.$$

Since each A–event and each intersection of A–events can be expressed as the disjoint union of B–events, we can check the defining equations for independence by expressing each $P(A_{i_1}A_{i_2}\cdots A_{i_r})$ as the sum of probabilities of the B–event components of $A_{i_1}A_{i_2}\cdots A_{i_r}$. In this way we will show that all A–event probabilities remain unchanged as well as all defining equations for independence except for one.

The probability of $A_1A_2\cdots A_k$ has been increased by ϵ, because $P(A_1\cdots A_kA_{k+1}^c\cdots A_n^c)$ has been changed by $(-1)^0\epsilon$ and no other B–event component probabilities of $A_1A_2\cdots A_k$ have been changed. If $\{i_1,i_2,\cdots,i_r\} \subset \{1,2,\cdots,k\}$ with $1 \le r < k$, the sum of B–event component probabilities of $A_{i_1}A_{i_2}\cdots A_{i_r}$ has been changed by

$$\sum_{j=0}^{k-r} \binom{k-r}{j} (-1)^j \epsilon = (-1+1)^{k-r}\epsilon = 0,$$

so $P(A_{i_1}A_{i_2}\cdots A_{i_r})$ has not been changed. Finally, if $\{i_1,i_2,\cdots,i_r\}$ contains at least one integer larger than k, then $P(A_{i_1}A_{i_2}\cdots A_{i_r})$ remains unchanged because no B–event component probabilities of $A_{i_1}A_{i_2}\cdots A_{i_r}$ have been changed. Hence, the changes have created the desired probability structure.

We conclude this note by applying the modification procedure to construct an example where independence of A_1, A_2, A_3 fails only because

$$P(A_1A_2) \neq P(A_1)P(A_2).$$

Beginning with A_1, A_2, A_3 independent such that

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{2} ,$$

we fix positive constant ϵ smaller than $\frac{1}{8}$. Now, we modify the B-event probabilities according to the described procedure, obtaining

$$\begin{aligned}
P(A_1A_2A_3^c) &= \frac{1}{8} + (-1)^0\epsilon = \frac{1}{8} + \epsilon, \\
P(A_1A_2^cA_3^c) &= \frac{1}{8} + (-1)^1\epsilon = \frac{1}{8} - \epsilon, \\
P(A_1^cA_2A_3^c) &= \frac{1}{8} + (-1)^1\epsilon = \frac{1}{8} - \epsilon, \\
P(A_1^cA_2^cA_3^c) &= \frac{1}{8} + (-1)^2\epsilon = \frac{1}{8} + \epsilon.
\end{aligned}$$

The remaining four B-event probabilities are left unchanged at $\frac{1}{8}$ each. Then

$$\begin{aligned}
P(A_1) &= P(A_1A_2A_3) + P(A_1A_2A_3^c) + P(A_1A_2^cA_3) + P(A_1A_2^cA_3^c) \\
&= \frac{1}{8} + (\frac{1}{8} + \epsilon) + \frac{1}{8} + (\frac{1}{8} - \epsilon) = \frac{1}{2} , \\
P(A_2) &= P(A_1A_2A_3) + P(A_1A_2A_3^c) + P(A_1^cA_2A_3) + P(A_1^cA_2A_3^c) \\
&= \frac{1}{8} + (\frac{1}{8} + \epsilon) + \frac{1}{8} + (\frac{1}{8} - \epsilon) = \frac{1}{2} , \\
P(A_3) &= P(A_1A_2A_3) + P(A_1A_2^cA_3) + P(A_1^cA_2A_3) + P(A_1^cA_2^cA_3) \\
&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} , \\
P(A_1A_2A_3) &= \frac{1}{8} = P(A_1)P(A_2)P(A_3), \\
P(A_2A_3) &= P(A_1A_2A_3) + P(A_1^cA_2A_3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = P(A_2)P(A_3), \\
P(A_1A_3) &= P(A_1A_2A_3) + P(A_1A_2^cA_3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = P(A_1)P(A_3);
\end{aligned}$$

however,

$$P(A_1A_2) = P(A_1A_2A_3) + P(A_1A_2A_3^c) = \frac{1}{8} + (\frac{1}{8} + \epsilon) = \frac{1}{4} + \epsilon \neq P(A_1)P(A_2).$$

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