ACHROMATIC NUMBERS AND GRAPH OPERATIONS

by

PAVOL HELL AND D.J. MILLER

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Pavol Hell
Simon Fraser University, Burnaby, B.C.

Donald J. Miller
University of Victoria, Victoria, B.C.

ABSTRACT

In this paper we investigate the achromatic number of the disjoint union of graphs and the achromatic number of the categorical product of graphs. We obtain the best possible upper and lower bounds for the union of graphs and the best possible lower bound for the product of graphs. Finally we show that the achromatic number of the product of graphs is bounded above.

1. INTRODUCTION

A colouring C on a graph X is a partition of the vertex set into independent sets, i.e., C is an equivalence relation on $V(X)$ such that no two equivalent vertices are adjacent. Such a partition is also known as a congruence on X; whenever it is convenient we will use "congruence" terminology. The quotient graph X/C is defined as follows: $V(X/C)$ is the set of equivalence classes of C and two equivalence classes are adjacent if there is a vertex in one class adjacent to some vertex in the other class. If X/C is the complete graph on n vertices then C is called a complete n–colouring of X. The chromatic number of X, denoted chr X, is the smallest n for which X admits a complete n–colouring and the achromatic number of X, denoted achr X, is the largest n for which X admits a complete n–colouring. The reducing congruence R on a graph is defined as follows: two vertices of X belong to the same congruence class of R if and only if they have the same set of neighbours. X/R is called the reduced graph. X is irreducible if each congruence class of R consists of a single vertex, i.e., if distinct vertices have distinct neighbourhoods.

Let G and H be two graphs. A homomorphism $f : G \to H$ is a vertex mapping $f : V(G) \to V(H)$ such that $gg' \in E(G)$ implies $f(g)f(g') \in E(H)$, i.e., f preserves adjacency. f is called an epimorphism if $f(V(G)) = V(H)$ and in addition for every $hh' \in E(H)$ there exists $gg' \in E(G)$ with $f(g) = h$ and $f(g') = h'$. If f is an epimorphism we write $f : G \twoheadrightarrow H$ and $H = f(G)$. It is clear that X admits an n–colouring C if and only if there is a homomorphism $f : X \longrightarrow K_n$, and a complete n–colouring C if and only if there is an epimorphism $f : X \twoheadrightarrow K_n$. If $f : G \to H$ then chr $G \leq$ chr H and if $f : G \twoheadrightarrow H$ then achr $G \geq$ achr H.

If G and H are graphs we define the product $G \times H$ as follows:

$V(G\times H) = V(G) \times V(H)$ and $(g,h)(g',h') \in E(G\times H)$ if and only if $gg' \in E(G)$ and $hh' \in E(H)$. Since $G \times H \twoheadrightarrow G$, we have $\text{chr}(G\times H) \leq \text{chr}\ G$ and $\text{achr}(G\times H) \geq \text{achr}\ G$. Hence

$$\text{chr}(G\times H) \leq \min\{\text{chr}\ G, \text{chr}\ H\}$$

and

$$\text{achr}(G\times H) \geq \max\{\text{achr}\ G, \text{achr}\ H\}.$$

It has been a longstanding conjecture of S. Hedetniemi that $\text{chr}(G\times H) = \min\{\text{chr}\ G, \text{chr}\ H\}$ [6]. This interesting conjecture has resulted in a number of mathematical developments, e.g., [1], [2], [3].

In this paper we investigate the achromatic number of $G \times H$. This leads in a natural way to a consideration of the achromatic number of the union of graphs as well.

2. UNIONS

In this section we first give best possible upper and lower bounds for the achromatic number of the disjoint union $G \cup H$ of two graphs G and H. Then we discuss the case of an arbitrary number of graphs.

Theorem 1. $\max\{\text{achr}\ G, \text{achr}\ H\} \leq \text{achr}(G\cup H) \leq \text{achr}\ G \cdot \text{achr}\ H.$

Proof. In order to show that $\text{achr}(G\cup H) \leq \text{achr}\ G \cdot \text{achr}\ H$ assume the contrary, that is, that there exist graphs G and H such that $\text{achr}(G\cup H) > \text{achr}\ G \cdot \text{achr}\ H$. Let $\text{achr}\ G = a$ and $\text{achr}\ H = b$. Hence there exists an epimorphism $\varphi : G \cup H \twoheadrightarrow K_s$ with $s > ab$. Let G' be the spanning subgraph of K_s with edge set equal to $\varphi(E(G))$ and let H' be the spanning subgraph of K_s with edge set equal to $\varphi(E(H))$. Let $\text{achr}\ G' = a'$ and achr $H' = b'$. Note that $a' \leq a$ and $b' \leq b$. Let c_1 be an a'–colouring of G' and let c_2 be a b'–colouring of H'. Consider the colouring c of K_s defined by $c(v) = (c_1(v), c_2(v))$ for $v \in V(K_s)$. Then as each edge of K_s is in G' or in H', c is a proper colouring of K_s with $ab < s$ colours; a contradiction.

Note that the upper bound holds regardless of whether the graphs G and H are disjoint. In order to prove the upper bound best possible consider

$$G = K_{\underbrace{b,b,\cdots,b}_{a \text{ parts}}} \quad \text{and} \quad H = K_{\underbrace{a,a,\cdots,a}_{b \text{ parts}}}.$$

Let $V(G) = V(H) = V(K_{ab})$ be the set $\{(x,y) \mid 1 \leq x \leq a,\ 1 \leq y \leq b\}$. The edges of G join all pairs of vertices with different first coordinate and the edges of H join all pairs of vertices with different second coordinate. Hence $G \cup H = K_{ab}$ and consequently $\text{achr}(G \cup H) = ab = \text{achr } G \cdot \text{achr } H$. Clearly by taking $G' \cong G$ and $H' \cong H$ we can make G' and H' disjoint.

To prove the lower bound, we need to assume that G and H are disjoint. Indeed, it is easy to see that $K_{n,n}$ can be viewed as the union of two "cocktail party graphs" G and H, both isomorphic to $K_{n,n} - M$ (where M is a perfect matching of $K_{n,n}$):

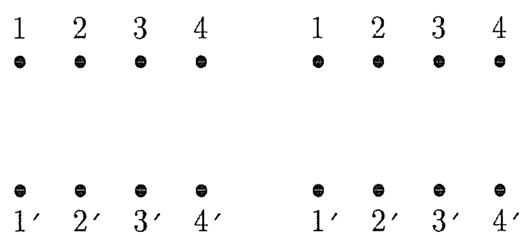


Figure 1. Two "cocktail party graphs" whose union is $K_{4,4}$.

Then achr $G = $ achr $H = n$ but achr$(G \cup H) = 2$. Now consider G and H to be disjoint, and achr $G = a \le b = $ achr H. Since there is an epimorphism $f : H \twoheadrightarrow K_b$, and there are homomorphisms $g : G \to K_a$ and $h : K_a \to K_b$, it is easy to construct an epimorphism $\tilde{f} : G \cup H \twoheadrightarrow K_b$. Thus achr$(G \cup H) \ge b$.

To prove the lower bound best possible take $G = K_a$ and $H = K_b$ with $a \le b$. Clearly achr$(G \cup H) = b = \max\{\text{achr } G, \text{achr } H\}$.

Theorem 2. *Let* achr $G_i = a$ *for* $i = 1,2,\cdots,d$. *Then*

$$\text{achr } dK_a \le \text{achr}(G_1 \cup \cdots \cup G_d) \le a^d.$$

Moreover for any fixed a *and all sufficiently large* d,

$$\text{achr } dK_a \approx \sqrt{a^2 - a} \cdot \sqrt{d}\ .$$

In particular, for large a (i.e., *for* $1 << a << d$) *we have*

$$\text{achr } dK_a \approx a\sqrt{d}\ .$$

Proof. In order to establish the lower bound we first note that $G_1 \cup \cdots \cup G_d \twoheadrightarrow dK_a$. This implies that achr$(G_1 \cup \cdots \cup G_d) \ge$ achr dK_a.

For a fixed a there are infinitely many values of d, say $d = d_1, d_2, \cdots$, such that

$$\text{achr } dK_a = \frac{1}{2} + \sqrt{\frac{1}{4} + d(a^2 - a)}\ .$$

To see this, recall Wilson's theorem [7]: if $\binom{a}{2}$ divides $\binom{m}{2}$ and $a-1$ divides $m-1$, then K_m admits an edge–decomposition into $d = \dfrac{\binom{m}{2}}{\binom{a}{2}}$ copies of K_a, provided m is sufficiently large. Thus for all sufficiently large values m which satisfy the two divisibility conditions, achr $dK_a = m$. (Here $d = \binom{m}{2}/\binom{a}{2}$ and so $m = \frac{1}{2} + \sqrt{\frac{1}{4} + d(a^2 - a)}$.) Note that the two divisibility conditions are equivalent to a congruence condition on m modulo $a^2 - a$. Therefore there is an infinite sequence of values of m, say $m_1 < m_2 < \cdots$, such that $m_{i+1} - m_i \le a^2 - a$ and

$$\mathrm{achr}\, d_i K_a = m_i$$

where $d_i = \binom{m_i}{2} / \binom{a}{2}$. Note that $\lim_{i\to\infty} \frac{d_{i+1}}{d_i} = 1$, since

$$1 \le \frac{d_{i+1}}{d_i} = \frac{m_{i+1}(m_{i+1}-1)}{m_i(m_i-1)} \le \frac{(m_i+a^2-a)(m_i+a^2-a-1)}{m_i(m_i-1)}$$

$$= 1 + \frac{2(a^2-a)}{m_i - 1} + \frac{(a^2-a)(a^2-a-1)}{m_i^2 - m_i}$$

and $m_i \to \infty$ as $i \to \infty$. For d with $d_i \le d < d_{i+1}$ we obtain

$$m_i = \mathrm{achr}\, d_i K_a \le \mathrm{achr}\, dK_a \le \mathrm{achr}\, d_{i+1} K_a = m_{i+1} .$$

Hence

$$\frac{m_i}{\sqrt{a^2-a} \cdot \sqrt{d}} \le \frac{\mathrm{achr}\, dK_a}{\sqrt{a^2-a} \cdot \sqrt{d}} \le \frac{m_{i+1}}{\sqrt{a^2-a} \cdot \sqrt{d}} .$$

Moreover,

$$\frac{m_{i+1}}{\sqrt{a^2-a} \cdot \sqrt{d}} = \frac{\frac{1}{2} + \sqrt{\frac{1}{4} + d_{i+1}(a^2-a)}}{\sqrt{a^2-a} \cdot \sqrt{d}} \le \frac{1 + \sqrt{a^2-a}\cdot\sqrt{d_{i+1}}}{\sqrt{a^2-a}\cdot\sqrt{d}} \le \frac{1}{\sqrt{a^2-a}\cdot\sqrt{d_i}} + \sqrt{\frac{d_{i+1}}{d_i}}$$

and

$$\frac{m_i}{\sqrt{a^2-a} \cdot \sqrt{d}} = \frac{\frac{1}{2} + \sqrt{\frac{1}{4} + d_{i+1}(a^2-a)}}{\sqrt{a^2-a} \cdot \sqrt{d}} \geq \sqrt{\frac{d_i}{d_{i+1}}}$$

both have a limit of 1 is $i \to \infty$. Therefore $\text{achr } dK_a \approx \sqrt{a^2-a} \cdot \sqrt{d}$.

The proof of the upper bound is a straightforward generalization of the proof of Theorem 1 with $a = b$. Moreover the upper bound can be generalized to a union of graphs with arbitrary achromatic numbers as follows:

$$\text{achr}(G_1 \cup \cdots \cup G_d) \leq \text{achr } G_1 \cdot \text{achr } G_2 \cdot \cdots \cdot \text{achr } G_d .$$

This is best possible, as can be seen by taking $G_i = K_{\underbrace{b_i, b_i, \cdots, b_i}_{a_i \text{ parts}}}$ where $b_i = \prod_{\substack{j=1 \\ j \neq i}}^{d} a_j$ for $i = 1, \cdots, d$. Then $\text{achr } G_i = a_i$ and by generalizing the proof of Theorem 1, we have $G_1 \cup G_2 \cup \cdots \cup G_d \twoheadrightarrow K_{a_1 a_2 \cdots a_d}$. Hence

$$\text{achr}(G_1 \cup \cdots \cup G_d) = a_1 a_2 \cdots a_d = \text{achr } G_1 \cdot \text{achr } G_2 \cdot \cdots \cdot \text{achr } G_d.$$

3. PRODUCTS

In this section we first give the best possible lower bound for the product of two graphs. We show that in almost all cases the achromatic number of the product is bounded below by the sum of the achromatic numbers of the factors, and we give a complete list of the exceptional cases. Next an example is given to show that the achromatic number of the product can grow exponentially. Finally we establish that there is an upper bound for the achromatic number of the product of two graphs. Products of several graphs are also discussed.

Theorem 3. *For graphs* G *and* H

$$\mathrm{achr}(G\times H) \geq \mathrm{achr}\, G + \mathrm{achr}\, H$$

unless (*assuming without loss of generality that* $\mathrm{achr}\, H \leq \mathrm{achr}\, G$)

1. $\mathrm{achr}\, H = 3$ and $\mathrm{achr}\, G \leq 5$, in which case $\mathrm{achr}(G\times H) \geq \mathrm{achr}\, G + \mathrm{achr}\, H - 1$.
2. $\mathrm{achr}\, H = 2$, in which case $\mathrm{achr}(G\times H) \geq \mathrm{achr}\, G + \mathrm{achr}\, H - 2$.

Proof. Let $\mathrm{achr}\, G = m$ and $\mathrm{achr}\, H = n$. Then $G \twoheadrightarrow K_m$ and $H \twoheadrightarrow K_n$ and therefore $G \times H \twoheadrightarrow K_m \times K_n$. Hence $\mathrm{achr}(G\times H) \geq \mathrm{achr}(K_m\times K_n)$. It remains to show that

$$\mathrm{achr}(K_m\times K_n) = \begin{cases} m + n - 2 & \text{if } n = 2 \\ m + n - 1 & \text{if } n = 3 \text{ and } 3 \leq m \leq 5 \\ m + n & \text{otherwise} \end{cases}$$

Let $V(K_m) = \{1,2,\cdots,m\}$ and $V(K_n) = \{1,2,\cdots,n\}$. Then

$$V(K_m\times K_n) = \{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

We consider the vertices of $K_m \times K_n$ to be arranged in rows and columns. The vertex (i,j) is in column i and row j, where the columns are counted from the left and the rows from the bottom. Any two vertices in $K_m \times K_n$ in different rows and in different columns are adjacent and no two vertices in the same row or the same column are adjacent, i.e., (i,j) is adjacent to (i',j') if and only if $i \neq i'$ and $j \neq j'$. Hence in any complete colouring of $K_m \times K_n$ each colour–class must belong to either a row or a column. Moreover no two colour–classes can belong to the same row or the same column. Consequently any complete colouring of $K_m \times K_n$ can use at most $m + n$ colours and therefore

$$\mathrm{achr}(K_m \times K_n) \leq m + n.$$

Hence in order to show that $\mathrm{achr}(K_m \times K_n) = m + n$ we need only exhibit a complete $(m+n)$–colouring. If $m \geq n \geq 4$ then the complete colouring illustrated in Figure 2 (we only draw the vertices of $K_m \times K_n$ the edges are understood) has $m + n$ colour–classes. Hence in this case $\mathrm{achr}(K_m \times K_n) = m + n$.

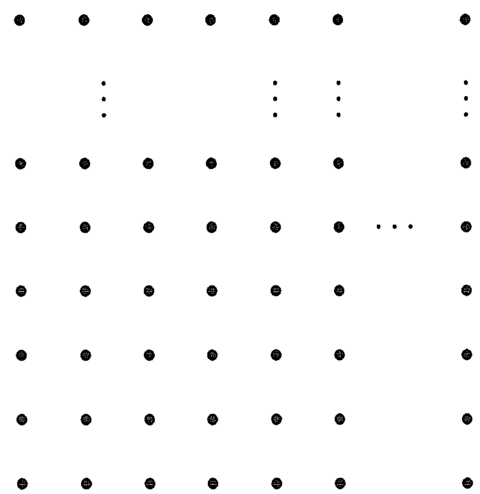


Figure 2. A complete (n+m)–colouring of $K_m \times K_n$, where $m,n \geq 4$.

If $m \geq 6$ and $n = 3$ then the complete $(n+m)$–colouring shown in Figure 3 establishes that in this case $\mathrm{achr}(K_m \times K_n) = m + n$.

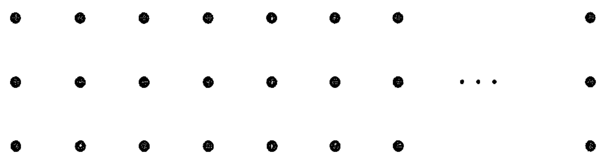


Figure 3. A complete (m+n)–colouring of $K_m \times K_n$ where $m \geq 6$, $n = 3$.

Next we show that if $3 \leq m \leq 5$ and $n = 3$ then $\mathrm{achr}(K_m \times K_n) = m + n - 1$. Assume instead that there is an (m+n)–colouring of $K_m \times K_n$ for $3 \leq m \leq 5$ and $n = 3$. There are mn vertices and $m + n$ colour–classes. Hence for $m = 3$ or 4 there must be at least two singleton colour–classes and for $m = 5$ at least one singleton colour–class. If $\{(1,1)\}$ and $\{(2,2)\}$ are the two singleton colour–classes, for the case $m = 3$ or 4, then the vertex (1,2) cannot belong to a colour–class in the first column nor in the second row. This is a contradiction. Now consider the case where $m = 5$ and $\{(2,2)\}$ say is the only singleton class (hence all other classes must be doubletons) and without loss of generality we may assume that $\{(1,1),(2,1)\}$ and $\{(3,1),(3,2)\}$ are two colour–classes. It follows that the first three columns of $K_5 \times K_3$ must be coloured as in Figure 4. Since the three

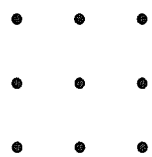
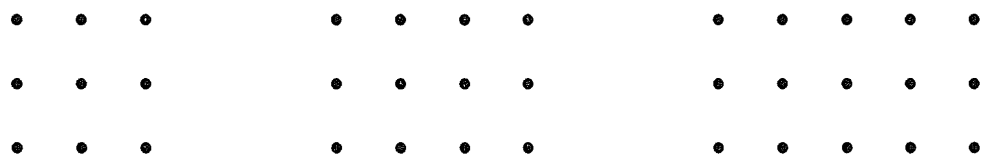


Figure 4. A colouring of the first three columns of $K_5 \times K_3$.

rows already contain colour–classes this forces columns four and five each to be colour–classes. This contradicts the fact that the other classes are doubletons.

Hence if $3 \le m \le 5$ and $n = 3$ then $achr(K_m \times K_n) \le n + n - 1$. In Figure 5 we exhibit an $(m+n-1)$–colouring of $K_m \times K_n$ for $3 \le m \le 5$ and $n = 3$.



$K_3 \times K_3$ $K_4 \times K_3$ $K_5 \times K_3$

Figure 5. An $(m+n-1)$–colouring of $K_m \times K_n$ for $3 \le m \le 5$ and $n = 3$.

Finally for the case $n = 2$ it is easy to establish that there can be no $m + 2$ or $m + 1$ colouring of $K_m \times K_2$. An m colouring of $K_m \times K_2$ is established by taking the m columns as colour classes.

The following rather surprising example gives two graphs G and H both with achromatic number equal to a such that $achr(G \times H) = a^{2\binom{a+1}{2}-2} \approx a^{a^2}$. Let

$$b = \left[\begin{matrix} a+1 \\ 2 \end{matrix}\right] - 1, \quad G = b\,K_2 \quad \text{and} \quad H = K_{\underbrace{a^{2b-1}, a^{2b-1}, \ldots, a^{2b-1}}_{a \text{ parts}}}.$$

Since $G = \left\{\left[\begin{matrix} a+1 \\ 2 \end{matrix}\right] - 1\right\} K_2$ there are insufficiently many edges for G to map onto K_{a+1}; but obviously $G \twoheadrightarrow K_a$ hence $achr\ G = a$. To show that $achr\ H = a$ consider two vertices that belong to the same part of H and suppose they are assigned different colours. Then any vertex in another part could not receive either of these two colours and consequently these two colour classes could not have an edge between them. Therefore H does not map onto K_{a+1} but obviously $H \twoheadrightarrow K_a$ and hence achr $H = a$.

$G \times H = 2b\, K_{\underbrace{a^{2b-1},\cdots,a^{2b-1}}_{a \text{ parts}}}$ and by Theorem 2 and the proof of Theorem 1

we have

$$\text{achr } 2b\, K_{\underbrace{a^{2b-1},\ \cdots,\ a^{2b-1}}_{a \text{ parts}}} = a^{2b} = a^{2\binom{a+1}{2}-2} .$$

The preceding example shows that the achromatic number of the product of two graphs can grow exponentially. The following result shows however that the achromatic number of the product is bounded above.

Theorem 4. *Let* achr G = achr $H = a$ *then* achr $G \times H$ *is bounded above by a function which only depends on* a.

Proof. For any integer a there exists a constant k_a such that $|V(X)| \leq k_a$ for all irreducible graphs X with achr $X = a$ [4], [5]. The reduced graphs G/R and H/R are irreducible and hence $|V(G/R)| \leq k_a$ and $|V(H/R)| \leq k_a$. Since $(G\times H)/R \cong G/R \times H/R$ we have $|V((G\times H/R))| \leq k_a^2$ and therefore by [5] Cor. 2.5 we have that $\text{achr}(G\times H) \leq \left\{3^{(k_a)^2}\right\}^{1/3}$.

Remark. Theorem 4 can be generalized as follows: For any positive integer $k \geq 2$, there exists a k–variable function f_k such that achr $G_1 = a_1$, achr $G_2 = a_2, \cdots,$ achr $G_k = a_k$ implies $\text{achr}(G_1 \times \cdots \times G_k) \leq f_k\,(a_1,\cdots,a_k)$.

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