Asymptotic Existence Results on Specific Graph Decompositions

by

Justin Chan
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Supervisory Committee

Dr. Peter Dukes, Supervisor
(Department of Mathematics and Statistics)

Dr. Gary MacGillivray, Departmental Member
(Department of Mathematics and Statistics)

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Dr. Peter Dukes, Supervisor
(Department of Mathematics and Statistics)

Dr. Gary MacGillivray, Departmental Member
(Department of Mathematics and Statistics)

Abstract

This work examines various asymptotic edge-decomposition problems on graphs. A *G-group divisible design* of type $[g_1, \ldots, g_u]$ and index λ is a decomposition of the edges of the complete λ-fold multipartite graph H, with *groups* (maximal independent sets) $G_1, \ldots, G_n$, $|G_i| = g_i$, into graphs (*blocks*) isomorphic to G. A *G-frame* is a G-GDD where the blocks can be partitioned into classes where the vertices of the blocks in each class do not intersect and their union is $V(H) \setminus G_i$, for some $1 \leq i \leq n$. A k-RGDD is a K_k-GDD of index 1 where the blocks can be partitioned into classes where the vertices of the blocks in each class cover all of $V(H)$ exactly once.

These structures are called *uniform* if $g_i = g$ is the same for all i. We shall prove that, given all parameters except u, G-GDDs and G-frames exist for all asymptotically large u satisfying the necessary conditions. Our primary technique is to invoke a useful theorem of Lamken and Wilson on edge-colored graph decompositions. The basic construction for k-RGDDs shall be outlined at the end of the thesis.

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Notation and Terminology

A *design*, in the broadest sense, is an edge decomposition of a graph. The word design usually refers to a 2-design decomposition of a complete graph of some index $\lambda > 0$. We shall focus only on decompositions into *blocks* isomorphic to a given graph G (G*-blocks*); such a decomposition is called a G*-design*.

We shall always assume that the graph G is nontrivial. A graph is *nontrivial* if it contains at least one edge.

Designs are often simply designated by three parameters $(1, 2, 3)$, where 1 is the graph being decomposed, 2 the graph(s) into which it decomposes, and 3 the index (a positive integer representing multiplicity of edges in the graph being decomposed). Parameters 1 and 2, when replaced by positive integers, are understood to refer to complete graphs of those orders. For example, a (v, G, λ)-design is a decomposition of K_v^λ into blocks isomorphic to G.

An *ideal in* $\mathbb{Z}$, henceforth called an *ideal*, is a set of integers which is closed under addition and integer multiplication. It is easily proven in elementary number theory that all ideals in $\mathbb{Z}$ are generated from a single element, either 0 (the *zero ideal*) or the least positive integer in the set (a *nonzero ideal*).

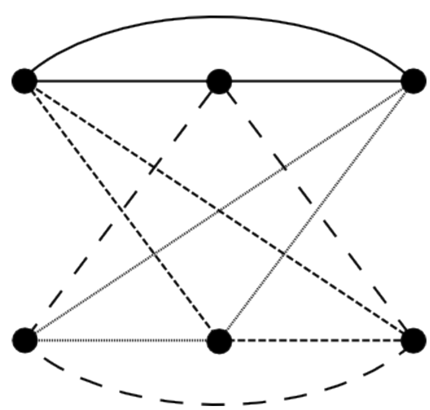
The notation $a \equiv b$, without a modulus, shall mean $a - b \in \mathbb{Z}$. Like any congruence, we can add and subtract rationals and multiply by integers on both sides, without changing the congruence.

A directed edge from x to y is given the notation $[x \to y]$, and an undirected edge between x and y is given the notation $\{x, y\}$. Undirected graphs are considered as directed graphs with directed edges $[x \to y]$ and $[y \to x]$ for each $\{x, y\} \in E(G)$.

The brackets () shall be used for vectors, [ ] for collections, and {} for sets. The notation $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors, denotes the vector which is the concatenation (join) of the two vectors. The vector $\mathbf{j}_n$ denotes the all-ones vector of length n. The coordinates of a vector may be indexed rather than explicitly ordered. For example, the coordinates of a vector may be indexed by the set of all $\binom{n}{2}$ unordered pairs of an n-set.

Designs may be represented in different forms. Take, for example, the design $[\{1_1, 2_1, 3_1\}, \{1_1, 2_2, 3_2\}, \{1_2, 2_1, 3_2\}, \{1_2, 2_2, 3_1\}]$, which is a decomposition of the multipartite graph $H_{[2,2,2]}$ into blocks of size 3 (K_3-blocks), where $\{a, b, c\}$ represents a block.

The graph decomposition form of this design is given below, where the blocks are marked with edge colors:



The notational form is given as $[1_1 2_1 3_1, 1_1 2_2 3_2, 1_2 2_1 3_2, 1_2 2_2 3_1]$.

Acknowledgments

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Chapter 1

Introduction

Consider the complete λ-fold multipartite graph $H = K^{\lambda}_{[g_1,g_2,\ldots,g_n]}$ with the vertices of H partitioned into sets $G_1, G_2, \ldots, G_u$, $|G_i| = g_i$, called *groups*, and with λ edges between every two vertices not in the same group, and no edges between any two vertices in the same group.

A *group divisible G-design* (G-GDD) of type $g_1, \ldots, g_u$ and index λ, denoted $([g_1, \ldots, g_u], G, \lambda)$-GDD, is an edge-decomposition of H into a collection $\mathcal{B}$ of graphs isomorphic to G (called *G-blocks* or *blocks*). An example of a $([2, 2, 2], K_3, 1)$-GDD is shown in figure 1.1.

A *G-frame* of type $g_1, \ldots, g_u$ and index λ, denoted $([g_1, \ldots, g_u], G, \lambda)$-FD, is a $([g_1, \ldots, g_u], G, \lambda)$-GDD such that the block collection can be partitioned into partial resolution classes $\mathcal{N}_1, \ldots, \mathcal{N}_m$, such that, for each partial resolution class, for some G_i, the vertices of the blocks cover the vertices of $H \setminus G_i$ exactly once, and do not cover any vertex in G_i.

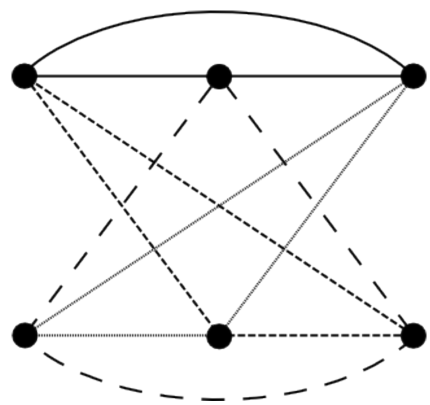

Figure 1.1: A $([2, 2, 2], K_3, 1)$-GDD

Example 1.1. $[[1_1 2_1 3_1, 1_2 2_2 3_2], [1_1 2_2 4_1, 1_2 2_1 4_2],$
$[2_1 3_2 4_1, 2_2 3_1 4_2], [1_2 3_1 4_1, 1_1 3_2 4_2]]$ is a $([2, 2, 2, 2], K_3, 1)$-FD.

The graph H which G-GDDs decompose should be thought of as an "inflation" of a complete graph K_n which has each of its vertices replaced by groups. Different groups have edges between them but there are no edges between vertices on the same group. This expansion was realized by Wilson in proving asymptotic existence of structures known as pairwise balanced designs, which we will talk about later. Thus a G-GDD is an analogue of a G-design, and a G-frame is an analogue of a near-resolvable (classes partition all but one vertex) G-design. In particular, the "holes" of a G-GDD can be filled in with G-designs to produce larger G-designs, and the "holes" of a G-frame can be filled with resolvable G-designs to produce larger resolvable G-designs. This is the primary reason for interest in these structures.

In general, G-GDDs and G-frames have complicated necessary conditions. The focus of this thesis is on uniform G-GDDs and G-frames, which not only have well-formed necessary conditions, but are linked to edge-colored designs. We say that a G-GDD or G-frame, as well as the graph H which it decomposes, is *uniform* or of type g^u if all the groups are the same size; that is, $|G_1| = \ldots = |G_u| = g$. We denote them as $([g]^u, G, \lambda)$-GDD, $([g]^u, G, \lambda)$-FD, and $K^{\lambda}_{[g]^u}$, respectively.

The definition, as given here, does not exclude designs where a block contains two or more vertices in the same group G_i. Since we will be relating uniform G-GDDs and G-frames to edge-colored graph decompositions, and we see $H = K_{[g]^u}$ as an inflation of K_u, we shall assume that the vertices of every block lie in different groups. This follows naturally if G is a complete graph K_k. Even without this assumption, uniform G-GDDs and G-frames in general will still satisfy the necessary conditions below.

1.1 Necessary Conditions for uniform G-GDDs

As with any other design, we first find the necessary conditions for the parameters. Every 2-design has two inherent necessary conditions:

- A global condition. The edges of the graph to be decomposed must be split into the edges of the graph(s) into which it decomposes.
- A local condition. The edges of each vertex of the graph to be decomposed must be split into the edges of each vertex of the graph(s) into which it decomposes.

For uniform G-GDDs and G-frames, we are interested in varying the parameter u (the number of groups). Because $H = K^{\lambda}_{[g]^u}$ is uniform, the degree of each vertex is the same. Each vertex is joined to $g(u-1)$ others; hence the degree, including multiplicity, is $\lambda g(u-1)$. There are gu vertices. Counting undirected edges as two directed edges in opposite direction, there are $\lambda g(u-1)gu = \lambda g^2 u(u-1)$ directed edges.

Let e be the number of edges of G. The graph G has $2e$ directed edges; hence we have the global condition:

$$\lambda g^2 u(u-1) \equiv 0 \pmod{2e}.$$

Suppose G has vertices with degrees $d_1, \ldots, d_n$. Let D be the greatest common divisor of these degrees. This gives us the local condition:

$$\lambda g(u-1) \equiv 0 \pmod{D}.$$

1.2 Necessary Conditions for uniform G-frames

Let n be the number of vertices of G. Each partial resolution class covers all vertices of H except for one particular group. There are $g(u-1)$ such vertices, so one necessary condition is:

$$g(u-1) \equiv 0 \pmod{n}$$

Furthermore, uniform G-frames are examples of equireplicate designs. An *equireplicate design* is a decomposition of a simple nontrivial regular graph F into blocks isomorphic to G such that each vertex of F is contained in the same number of blocks. Let E be the number of edges in F, N the number of vertices, δ the degree of each vertex and r the number of blocks on each point. Then $\delta = 2E/N$, $r = (2E/2e)(n/N)$, so $r/\delta = n/2e$.

Consider a vertex of F in a design, and let d_i be the degrees of G. Each block on that vertex contributes d_i edges for some i to the total of δ edges on that vertex, and 1 block to the total of r blocks on that vertex. Thus in such a design,

$$\sum s_i \begin{bmatrix} d_i \\ 1 \end{bmatrix} = \begin{bmatrix} \delta \\ r \end{bmatrix} = \delta \begin{bmatrix} 1 \\ r/\delta \end{bmatrix} = \delta \begin{bmatrix} 1 \\ n/2e \end{bmatrix}$$

for some non-negative integers s_i. The set of all integers c such that $c(1, n/2e)$ is an integral linear combination of elements in $\{(d_i, 1) : 1 \leq i \leq n\}$ is an ideal. This ideal is nonzero since the sum of all $(d_i, 1)$ is $(2e, n) = 2e(1, n/2e)$. Therefore the least positive integer in the ideal, denoted γ, always exists, and generates the ideal. In particular, γ divides δ.

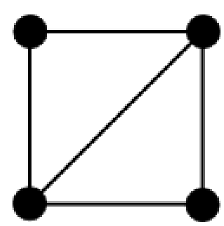

Figure 1.2: $K_4 \setminus e$

Example 1.2. G is the graph shown in figure 1.2. We have $n = 4$, $e = 5$, so $(1, n/2e) = (1, 2/5)$. Since the linear combination of the $(d_i, 1)$ is integral, $\gamma(1, n/2e)$ must be integral, so $\gamma \geq 5$. The set of $(d_i, 1)$ in this case is $\{(2, 1), (3, 1)\}$, whereupon $(2, 1)+(3, 1) = (5, 2) = 5(1, 2/5)$. Therefore $\gamma = 5$. If there is any equireplicate G-design, then 5 divides δ, the degree of the regular graph being decomposed.

In the case of a $([g]^u, G, \lambda)$-FD , $\delta = \lambda g(u-1)$ (the degree of each vertex in H) so we have another necessary condition:

$$\lambda g(u-1) \equiv 0 \pmod{\gamma}$$

For any group G_i, let m be the number of partial resolution classes missing G_i in the $([g]^u, G, \lambda)$-FD . Then $r = m(u-1)$, and using $r/\delta = n/2e$, we have $m = \lambda gn/2e$, so

$$\lambda gn \equiv 0 \pmod{2e}.$$

One important property is that m does not depend on u. The identity $m = \lambda gn/2e$ will appear frequently within proofs involving uniform G-frames.

1.3 Links to Edge-Colored Designs

In this section, we refer to directed edges, and an undirected edge is composed of two directed edges in opposite directions. Consider the multipartite multigraph H of index λ and groups $G_1, \ldots, G_u$ with $|G_1| = \ldots = |G_u| = g$, and the complete multigraph $K = K_u^\lambda$. We can think of H as K, but with each vertex in K replaced with the g vertices forming a group. Then instead of an edge from one vertex to another in K, we now have g^2 edges from one group to another in H. Thus, because we assumed that every block has vertices in different groups, every block in the decomposition of H can be "projected" onto K.

This approach alone, however, loses information about where, of the g vertices in the starting group, the edge begins, and where, of the g vertices in the ending group, the edge ends. To encode this information, consider, instead of K, a complete graph with multiple edge colors in $S \times S$, where $S = \{1, \ldots, g\}$, and with λ of each color from each vertex to each other vertex. The colors $S \times S$ indicate the starting and ending vertices. Thus the edges of H correspond to the edges of $K^* = K_u^{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda} = (\lambda, \lambda, \ldots, \lambda) = \lambda \mathbf{j}_{g^2}$, a vector of length g^2. The superscript vector indicates multiplicity of different colors.

Consider a block in a decomposition of H forming a group divisible design. By assumption, the vertices of the block lie in different groups. We can correspond this block with an edge-colored block in K^* as follows: Color the vertices of the edge-colored block with S. That is, assign a mapping κ from the vertices of G into S. This corresponds with choosing one of the g vertices of a group for the vertices of the block in H.

Then in the edge-colored block, an edge from a vertex of color i to a vertex of color j has edge color (i, j). Call each edge-colored block G_κ depending on mapping κ. If the block in H is isomorphic to G, then the edge-colored blocks consist of (directed) edges of some colors forming the edges of G. Then a decomposition of H into blocks isomorphic to G corresponds to a decomposition of K^* into the G_κ.

Proposition 1.1. A $([g]^u, G, \lambda)$-GDD , with each block having vertices in different groups, is equivalent to a decomposition of $K_u^{\boldsymbol{\lambda}}$ of color multiplicity $\lambda \mathbf{j}_{g^2}$ into edge-colored blocks G_κ defined above.

1.4 Partial resolution class

Now suppose we want to add the requirement of partial resolution classes covering the vertices of $H \setminus G_i$ to create a G-frame. To do this, we draw a parallel with the case of near-resolvable designs.

A (v, k, λ)-*near-resolvable design* is a decomposition of $K = K_v^\lambda$ into blocks K_k such that the collection of blocks can be partitioned into partial resolution classes each of which partition the vertices of $K \setminus x$ for some $x \in V(K)$. We shall assume that each vertex x is such that $K \setminus x$ is partitioned by exactly one partial resolution class. In this case, $\lambda = k - 1$.

To establish a correspondence with edge-colored graph decompositions, consider K^* as the complete edge-colored graph on two colors with multiplicity λ and 1. So $K^* = K_u^{\boldsymbol{\lambda}}$ where $\boldsymbol{\lambda} = (\lambda, 1)$. Each block in the near-resolvable design on K, as well as its association to a particular partial resolution class, corresponds to an edge-colored block as follows:

As the edge-colored block, take the block in the near-resolvable design and give it the color which is of multiplicity λ, and add directed edges in the other color, from each vertex in the original block, to a special point denoted ∞. This special point represents the vertex which the partial resolution class does not cover. The directed edges pointing to ∞ ensure that the blocks which have these edges do not intersect vertices but cover all vertices other than ∞, and thus form a partial resolution class. Therefore a near-resolvable design on K corresponds to a decomposition of K^* into the edge-colored blocks above.

In a similar manner, we can establish a correspondence between G-frames and edge-colored graph decompositions. Suppose we have a decomposition of H into G. In the previous section, we corresponded this decomposition to a decomposition of K^* into edge-colored graphs G_κ. Remember that each group of H corresponds to a vertex in K^*.

To add the partial resolution condition, we create edge-colored blocks with directed edges from the vertices of the corresponding edge-colored block, to a special vertex ∞. For colors corresponding to these directed edges, we need to distinguish the vertex in the group that it belongs, so there is an element of $S = \{1, \ldots, g\}$ involved. However, in this case, there are m classes missing each group. To ensure that the classes missing a particular group are distinguished from each other (so that we don't end up having one "superclass") we must have $M = \{1^*, \ldots, m^*\}$ to distinguish the m classes. Then each edge color for these directed edges is $S \times M$.

Let κ be the mapping from the vertices of G into $S = \{1, \ldots, g\}$, as before. Let $r \in M$. Then, an edge from a vertex of color i to a vertex of color j has edge color $(i, j) \in S \times S$. In addition, from a vertex of color i, add a directed edge from it to ∞ of color $(i, r) \in S \times M$. Call these edge-colored blocks $G_{\kappa r}$, depending on κ and r. Then a decomposition of H into

G with the partial resolution condition corresponds to a decomposition of $K^* = K_u^{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda} = (\lambda, \lambda, \ldots, \lambda, 1, 1, \ldots, 1) = (\lambda \mathbf{j}_{g^2}, \mathbf{j}_{gm})$, into edge-colored blocks $G_{\kappa r}$.

Proposition 1.2. A $([g]^u, G, \lambda)$-FD , with each block having vertices in different groups, is equivalent to a decomposition of $K_u^{\boldsymbol{\lambda}}$ of color multiplicity $(\lambda \mathbf{j}_{g^2}, \mathbf{j}_{gm})$ into edge-colored blocks $G_{\kappa r}$ defined above.

Our main goal is to prove asymptotic existence (in u) of uniform G-frames which satisfy the necessary conditions.

Theorem 1.3. Let G be a simple nontrivial graph with e undirected edges and n vertices, γ be defined as above, $g \in \mathbb{Z}$, $g > 0$, and $\lambda \in \mathbb{Z}, \lambda > 0$ such that $m = \lambda ng/2e \in \mathbb{Z}$. Then there exists u_0 such that for all $u \geq u_0$ satisfying

$$g(u-1) \equiv 0 \pmod{n}$$
$$\lambda g(u-1) \equiv 0 \pmod{\gamma},$$

a $([g]^u, G, \lambda)$-FD exists

In Chapter 2, we will give the needed background on designs. In Chapter 3, we will introduce the Lamken-Wilson Theorem, the tool used to prove asymptotic existence. In Chapter 4, we will show how the Lamken-Wilson Theorem proves asymptotic existence in the case of uniform G-GDDs. Chapter 5 will explain resolvable designs. We finish off with Chapter 6, a proof of asymptotic existence for uniform G-frames, and show how frames apply to structures such as resolvable designs.

Chapter 2

Background

2.1 Block Designs

Definition. Given a set S of order v, a collection $\mathcal{B}$ of *blocks* (subsets of S) of order k, and a positive integer λ, we say that the collection of blocks forms a *balanced incomplete block design*, or (v, k, λ)-*design*, if every pair of distinct elements in S are contained together in exactly λ blocks of the collection.

This is the original definition, although it can be defined as a decomposition of the multigraph K_v^λ into K_k, as in the previous section.

Commonly studied block designs include Steiner triple systems ($(v, 3, 1)$-designs), finite projective and affine planes.

Remark. Projective and affine planes are named for the geometries from which they are derived. A finite projective plane is a design with every pair of blocks intersecting exactly once. All finite projective planes are $(k^2 + k +$

$1, k+1, 1)$-designs and vice versa.

A finite affine plane is a design such that all blocks can be partitioned into sets of mutually disjoint blocks, each set covering all elements, with every pair of blocks that are not in the same set intersecting exactly once. The sets are known as resolution classes. All finite affine planes are $(k^2, k, 1)$-designs and vice versa.

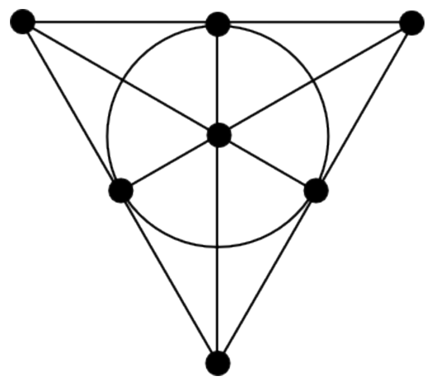

Figure 2.1: The Fano Plane

Example 2.1. $[124, 235, 346, 450, 561, 602, 013]$ is a $(7, 3, 1)$-design isomorphic to the projective plane of order 2 (the Fano plane), and is the smallest non-trivial design (that is, the smallest v for which a $(v, k, 1)$-design exists with more than one block and $k \geq 3$). Note that this design can also be obtained by cycling the difference set $\{1, 2, 4\} \pmod 7$. See the Fano plane block diagram in figure 2.1, where the dots are the elements, and the lines are the six geometric line segments along with the geometric circle.

Two useful values in describing designs are b, the number of blocks, and r, the common number of blocks on each element. The b blocks cover all

$\lambda v(v-1)/2$ pairs and each block covers $k(k-1)/2$ pairs. Therefore $b = \lambda v(v-1)/k(k-1)$. On each element, the r blocks cover all $\lambda(v-1)$ pairs with that element, and each block covers $k-1$ pairs with that element. Therefore, $r = \lambda(v-1)/(k-1)$.

Necessary Conditions 2.1. For a (v, k, λ)-design:

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)},$$
$$\lambda(v-1) \equiv 0 \pmod{k-1}.$$

These necessary conditions are not sufficient in general. For instance, Fisher's Inequality is stated here without proof.

Proposition 2.2. (Fisher's Inequality) In any (v, k, λ)-design, $b \geq v$.

Example 2.2. Non-existence of a $(96, 20, 1)$-design. The necessary conditions are satisfied, and we have $b = 24$ and $r = 5$. However, $b < v$. Therefore, a $(96, 20, 1)$-design does not exist.

Even with Fisher's Inequality considered, the conditions are still not sufficient.

Proposition 2.3. (Corollary to Bruck-Ryser Theorem) If there exists an affine plane of order k with $k \equiv 1, 2 \pmod 4$, then k is a sum of two squares.

Example 2.3. Non-existence of a $(36, 6, 1)$-design. The necessary conditions are satisfied, and we have $b = 42$ and $r = 7$, and $b > v$. If a (36,6,1)-design exists, then it is an affine plane of order 6. But $6 \equiv 2 \pmod 4$ and is not a sum of two squares. Therefore a (36,6,1)-design does not exist.

Through design theory, an important goal is to determine whether or not the necessary conditions are asymptotically sufficient; that is, given all other parameters, is there a v_0 such that designs exist for all $v \geq v_0$ satisfying the necessary conditions?

Specific values of k and λ were often considered. For instance, $(v, 2, 1)$-designs exist for all v (since the blocks can be taken as all possible pairs). Also well known are Steiner triple systems ($(v, 3, 1)$-designs) which exist for all $v \equiv 1, 3 \pmod 6$.

2.2 Wilson's Theory on Pairwise Balanced Designs

In 1972, R.M. Wilson proved ([8]) that, given k and λ, (v, k, λ)-designs exist for all sufficiently large admissible v. It follows from an extensive theory of pairwise balanced designs.

Definition. Given a set S of order v, a collection $\mathcal{B}$ of blocks which are subsets of S of any order $k \in K$, where K is a set of positive integers, and a positive integer λ, we say that the collection of blocks forms a *pairwise balanced design*, or (v, K, λ)-*design* or *PBD*(v, K, λ), if every pair of distinct elements in S are contained together in exactly λ blocks of the collection.

The necessary conditions are much like above for BIBDs, but since there are different block sizes, we take the GCD of the moduli.

For a PBD(v, K, λ):

$$\lambda v(v-1) \equiv 0 \pmod{\gcd\{k(k-1)|k \in K\}}$$
$$\lambda(v-1) \equiv 0 \pmod{\gcd\{k-1|k \in K\}}$$

Theorem 2.4. A PBD(v, K, λ) exists for all sufficiently large v satisfying

$$\lambda v(v-1) \equiv 0 \pmod{\gcd\{k(k-1)|k \in K\}}$$
$$\lambda(v-1) \equiv 0 \pmod{\gcd\{k-1|k \in K\}}$$

The key part of the proof is showing how larger designs can be constructed from smaller ones. One main ingredient in the proof is the concept of group divisible designs. We can construct large designs by taking together smaller designs and linking them with a group divisible design.

Definition. Given a set $S = \{s_1, s_2, \ldots, s_v\}$ of order v, a weight function w mapping the elements of S into the positive integers, a collection $\mathcal{B}$ of blocks which are subsets of $\{(s_i, m) : 1 \leq m \leq w(s_i)\}$ of any order $k \in K$, where K is a set of positive integers, and a positive integer λ, the collection of blocks forms a *group divisible design* of type $w(s_1), w(s_2), \ldots, w(s_v)$, block sizes in K, and index λ if:

- In every block, for any two elements $(s_i, m), (s_j, n)$ in the block, $s_i \neq s_j$.

- Every pair $\{(s_i, m), (s_j, n)\}$, $s_i \neq s_j$, is contained in exactly λ blocks.

The collection of weights $[w(s) : s \in S]$ become the *group sizes* of the GDD. Pairwise balanced designs themselves are group divisible designs with group sizes all 1. Commonly used notation for group sizes is $w(1), w(2), \ldots,$ $w(s)$. If $w(1) = w(2) = \ldots = w(s) = g$, it is often written g^n. In this thesis, we shall surround group sizes with [ ] to avoid confusion.

There are necessary conditions for group divisible designs, but for non-uniform cases, the conditions are too unwieldy to state, and are not necessary for this purpose.

The goal is to construct large group divisible designs, and fill the holes with existing designs. This forms a recursive construction.

A well-known method of expanding group divisible designs is known as Wilson's Fundamental Construction. Basically, the construction replaces vertices with multiple vertices and blocks with GDDs. It can be stated as follows:

Theorem 2.5. Let $\mathcal{B}$ be a GDD with groups $G_1, G_2, \ldots, G_n$ and index λ_1. Take a weight function w mapping all vertices of the GDD to non-negative integers and suppose that, for each block of the GDD, K-GDDs of type $[w(t_1), w(t_2), \ldots, w(t_k)]$ and index λ_2 exist, where $t_1, t_2, \ldots, t_k$ are the vertices in the block. Then there exists a K-GDD of type $[a_1, a_2, \ldots, a_n]$ and index $\lambda_1\lambda_2$, where $a_i = \sum_{s \in G_i} w(s)$.

Typically, $\lambda_1 = \lambda_2 = 1$ and the weight function is a constant number greater than 1.

Finally, the basis for constructing GDDs that cover a broad range of values is formed by transversal designs.

Definition. A *transversal design* (TD) is a $([m]^k, k, 1)$-GDD; such a design is denoted $TD(k, m)$. A TD is *resolvable* if its blocks can be partitioned into sets of disjoint blocks, each set covering all vertices.

Transversal designs have some interesting properties. In a transversal design, the blocks are largest in the sense that they intersect every group exactly once. The name 'transversal' is derived from Latin squares, where a transversal is a set of n coordinates in $\mathbb{Z}_n \times \mathbb{Z}_n$ with all coordinates on different rows and columns. The existence of a $TD(k, m)$ is equivalent to the existence of a resolvable $TD(k-1, m)$, and a set of $k-2$ mutually orthogonal Latin squares of order m.

In addition, the existence of a $TD(r+1, r)$ is equivalent to the existence of an affine plane of order r, and a projective plane of order r. In particular, by using finite field constructions, a $TD(r+1, r)$, and thus a $TD(k, r), k \leq r+1$, exists for all prime powers r. We can also "multiply" a $TD(k, m)$ and a $TD(k, n)$ using Wilson's Fundamental Construction to form a $TD(k, mn)$.

In particular, it is easily shown that a $TD(3, m)$ exists for all m and a $TD(4, m)$ exists for all $m \not\equiv 2 \pmod 4$. Note that a $TD(4, m)$ exists for all $m \equiv 2 \pmod 4$ where $m \geq 10$, but it is harder to prove. More generally, Chowla, Erdös, and Strauss proved ([5]) that, given $k \geq 2$, a $TD(k, m)$ exists for all sufficiently large m.

To cover a range of possible design sizes, we take a $TD(k,m)$ and remove some vertices in one group. This leaves us with a $\{k-1,k\}$-GDD of type $[m]^{k-1}[a]$, where $0 \le a \le m$.

The solution of all v such that a $(v,K,1)$-design exists is called the *PBD-closure* and is denoted $\mathbf{B}(K)$. A set K is called *PBD-closed* if $\mathbf{B}(K) = K$. In particular, $\mathbf{B}(K)$ is PBD-closed. The goal is to find the PBD-closure of a given set.

Example 2.4. Let $K = \{3,4\}$. We know a $TD(4,m)$ exists for all $m \not\equiv 2 \pmod 4$. Take a $TD(4,m)$ and remove some vertices from one group, giving us a K-GDD of type $[m]^3[a]$, $0 \le a \le m$. Thus, if there exist a $(m,K,1)$-PBD and $(a,K,1)$-PBD, then these can fill the holes of the GDD to produce a $(3m+a,K,1)$-PBD.

To build the construction, we produce a sufficiently large number of base designs. The necessary condition is $v \equiv 0,1 \pmod 3$. It can be shown, from trial and from using these methods, that $(v,K,1)$-designs exist for all admissible v up to 51, excluding 6 (there is no $(6,K,1)$-design). We want to show that $\mathbf{B}(K) = \{v : v \equiv 0,1 \pmod 3\} \setminus \{6\}$. Any $v \ge 52$ can be written as $3m+a$, where $m < v, m \not\equiv 2 \pmod 4, m \equiv 0,1 \pmod 3$ and $a \le m, a \equiv 0,1 \pmod 3), a \ne 6$. This is because, for each m, all admissible values between $3m+7$ and $4m$ are covered, consecutive possible values for m differ by no more than 3, and $4m \ge 3(m+3)+6$ if and only if $m \ge 15$. By induction using the GDD construction above, $\mathbf{B}(K) = \{v : v \equiv 0,1 \pmod 3\} \setminus \{6\}$.

Example 2.5. $K = \{3\}$. The necessary condition is $v \equiv 1, 3 \pmod 6$. We know from above that $(u, \{3, 4\}, 1)$-designs exist for $u \equiv 0, 1 \pmod 3), u \neq 6$. Now one can easily find $\{3\}$-GDDs of type $[2]^3$ and $[2]^4$ by trial. Giving each vertex of the $\{3, 4\}$-PBD weight 2 and using Wilson's Fundamental Construction, we have a $\{3\}$-GDD of type $[2]^u$. Filling in the holes with 3-blocks, all joined to an outside vertex, gives a $(2u + 1, 3, 1)$-design. This covers all admissible v except 13. However, one can find a $(13, 3, 1)$-design. Therefore $\mathbf{B}(K) = \{v : v \equiv 1, 3 \pmod 6\}$.

For larger values in K, the effective bounds become much greater, and there will be increasingly many exceptions to admissible v. The CRC Handbook of Combinatorial Designs ([6]) contains a table of PBD-closures for some small K.

Group divisible G-designs (where G is a graph) will be discussed later, where such designs can be derived from edge-colored graph decompositions.

2.3 Viewing designs as graph decompositions

A (v, k, λ)-design is equivalent to a decomposition of K_v^λ, the complete multigraph of index λ, into blocks K_k. A similar decomposition occurs for pairwise balanced designs, and for a group divisible design, the graph decomposed is $K^\lambda_{g_1,g_2,\ldots,g_n}$, the complete multipartite multigraph of index λ and group sizes $g_1, g_2, \ldots, g_n$.

From this view, it is natural to ask whether complete graphs can be decomposed into graphs (known as G-blocks or blocks) isomorphic to some simple nontrivial graph G. Once again, R. M. Wilson was able to extend his result on asymptotic existence of block designs to graph decompositions. The necessary conditions in this case are:

$$\lambda v(v-1) \equiv 0 \pmod{2e},$$
$$\lambda(v-1) \equiv 0 \pmod{D}.$$

where e is the number of edges of G and D is the greatest common divisor of the degrees of G. For small examples of G, it is sufficient to find small complete graphs which G decomposes, and use PBD-closure to combined them.

A further generalization, described in the next section, considers not only multiple graphs, but also edge colors (a form of edge distinction). The resulting theorem is a powerful tool that will be used in this thesis.

Chapter 3

The Lamken-Wilson Theorem

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be the vector representing color multiplicity; that is, λ_i edges of color i between each vertex. Lamken and Wilson's paper ([1]) describes decompositions of edge-colored complete graphs, that is, a decomposition of $K_v^{\boldsymbol{\lambda}}$ into fixed directed edge-colored graphs on r colors. It is a generalization and unification of classical design theory, and thus has many applications. The theorem was originally shown for $\boldsymbol{\lambda} = (1, 1, \ldots, 1)$ and extended to color multiplicity.

For an edge-colored directed graph F and a vertex x, let $\mu(F)$ be the vector, of length the same as $\boldsymbol{\lambda}$, representing the number of edges of each color in F, and let $\tau(F, x)$ be the vector, of length twice that of $\boldsymbol{\lambda}$, representing the indegrees and outdegrees of $x \in V(F)$ for each color in F. If $\tau(F, x)$ does not depend on x, we write $\tau(F, *)$. As discussed in Chapter 1, we shall denote two of the same vector $\boldsymbol{\lambda}$, concatenated together, as $(\boldsymbol{\lambda}, \boldsymbol{\lambda})$.

Let $\mathcal{G}$ be a collection of edge-colored simple directed graphs (no loops or multiple edges). For a decomposition of the edge-colored complete graph $K_v^{\boldsymbol{\lambda}}$ into graphs in $\mathcal{G}$, there are two necessary conditions:

- $\mu(K_v^{\boldsymbol{\lambda}}) = v(v-1)\boldsymbol{\lambda}$ is an integral linear combination of vectors $\mu(G), G \in \mathcal{G}$,
- $\tau(K_v^{\boldsymbol{\lambda}}, *) = (v-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ is an integral linear combination of vectors $\tau(G, x), G \in \mathcal{G}, x \in V(G)$.

The set of all m such that $m\boldsymbol{\lambda}$ is an integral linear combination of the $\mu(G)$ is an ideal. Assuming the first necessary condition above, the ideal is nonzero, so the least positive integer in the set, denoted $\beta(\mathcal{G}, \boldsymbol{\lambda})$, exists, and generates the ideal. Similarly, the set of all t such that $t(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ is an integral linear combination of the $\tau(G, x)$ is an ideal. Assuming the second necessary condition above, the ideal is nonzero, so the least positive integer in the set, denoted $\alpha(\mathcal{G}, \boldsymbol{\lambda})$, exists, and generates the ideal. We then have the equivalent necessary conditions:

$$\begin{aligned} v(v-1) &\equiv 0 \pmod{\beta(\mathcal{G}, \boldsymbol{\lambda})}, \\ v-1 &\equiv 0 \pmod{\alpha(\mathcal{G}, \boldsymbol{\lambda})}. \end{aligned}$$

We shall also suppose that $\mathcal{G}$ is admissible. A collection of edge-colored graphs $\mathcal{G}$ is *admissible* if $\boldsymbol{\lambda}$ is a positive rational linear combination of $\mu(G), G \in \mathcal{G}$.

The main theorem of Lamken and Wilson in [1] asserts that these conditions are asymptotically sufficient. It is as follows:

Theorem 3.1. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mathcal{G}$ be an admissible family of simple edge-colored digraphs on r colors. Then $\alpha(\mathcal{G}, \boldsymbol{\lambda})$ and $\beta(\mathcal{G}, \boldsymbol{\lambda})$, as defined above, exist, and there exists a constant v_0 such that $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all $v \geq v_0$ satisfying:

$$\begin{aligned} v(v-1) &\equiv 0 \pmod{\beta(\mathcal{G}, \boldsymbol{\lambda})}, \\ v-1 &\equiv 0 \pmod{\alpha(\mathcal{G}, \boldsymbol{\lambda})}. \end{aligned}$$

We can equivalently state the theorem as follows:

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mathcal{G}$ be an admissible family of simple edge-colored digraphs on r colors. Then there exists a constant v_0 such that $\mathcal{G}$-decompositions of $K_v^{\boldsymbol{\lambda}}$ exist for all $v \geq v_0$ such that:

- $\mu(K_v^{\boldsymbol{\lambda}}) = v(v-1)\boldsymbol{\lambda}$ is an integral linear combination of vectors $\mu(G), G \in \mathcal{G}$,
- $\tau(K_v^{\boldsymbol{\lambda}}, *) = (v-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ is an integral linear combination of vectors $\tau(G, x), G \in \mathcal{G}, x \in V(G)$.

Example 3.1. Let $\boldsymbol{\lambda} = (1, 1)$ and let $\mathcal{G}$ consist of this one graph G shown in Figure 3.1. $\mu(G) = (10, 10)$ and $\tau(G) = (2, 2, 2, 2)$. Clearly $\mathcal{G}$ is admissible, and $\beta(\mathcal{G}, \boldsymbol{\lambda}) = 10$ and $\alpha(\mathcal{G}, \boldsymbol{\lambda}) = 2$. The theorem says that for all sufficiently large u satisfying:

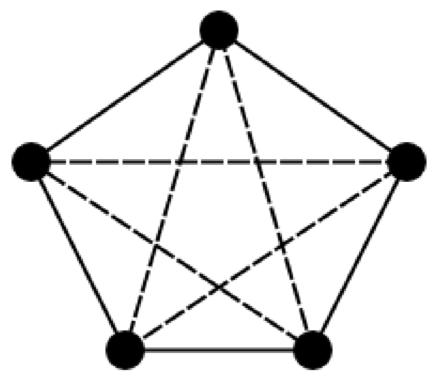

Figure 3.1: Example for complementing 5-cycles

$$v(v-1) \equiv 0 \pmod{10}$$
$$v-1 \equiv 0 \pmod{2}$$

a decomposition of $K_v^{(1,1)}$ exists.

The conditions imply $v \equiv 1, 5 \pmod{10}$. In actual fact, these designs exist for all $v \equiv 1, 5 \pmod{10}$, $v \neq 15$ ([1]).

For the purposes of proving theorems with general $\mathcal{G}$, it is easier to show that the last two necessary conditions are satisfied. The following lemma, one which was used to prove the theorem, will be useful:

Lemma 3.2. *Given an $m \times n$ rational matrix M and a rational vector $\boldsymbol{c}$ of length m, the equation $M\boldsymbol{x} = \boldsymbol{c}$ has an integral solution $\boldsymbol{x}$ if and only if for all rational vectors $\boldsymbol{y}$ such that $\boldsymbol{y}^T M$ is integral, $\boldsymbol{y}^T \boldsymbol{c}$ is integral.*

In the context of the theorem, we let $\mathbf{c}$ be the μ-vector (or τ-vector) of the complete edge-color graph and M be the matrix of all possible μ-vectors (or τ-vectors). By the lemma, to prove that $\mathbf{c}$ is a linear combination of vectors in M, we need only check that if we are given a vector of rationals $\mathbf{y}$ such that the dot product of $\mathbf{y}$ with every vector in M is integral, then the dot product of $\mathbf{y}$ with $\mathbf{c}$ is integral.

The elements of the vector $\mathbf{y}$ are associated with each color in any μ-vector (or a color and either the in or out direction in the case of a τ-vector). Rather than specifying the order of colors specifically, the elements of $\mathbf{y}$ will be labeled to reflect the color association. For example X_{23} denotes the color $(2, 3) \in S$.

As stated in [1], the theorem has many applications. Some of them simply verify the asymptotic solutions of already-solved problems with specific $\mathcal{G}$. Perhaps the more important applications are ones which apply to general structures such as uniform group divisible designs. We give two such applications in the remainder of the thesis.

Chapter 4

Group divisible designs

In the introductory chapter, we established the following necessary conditions for a uniform G-GDD of type $[g]^u$ and index λ:

$$\text{(Global)} \qquad \lambda g^2 u(u-1) \equiv 0 \pmod{2e},$$

$$\text{(Local)} \qquad \lambda g(u-1) \equiv 0 \pmod{D}.$$

Suppose we want to show a $([g]^u, G, \lambda)$-GDD exists for sufficiently large u satisfying these necessary conditions. The idea is to form a $([g]^u, G, \lambda)$-GDD from the decomposition of a $K_u^{\boldsymbol{\lambda}}$, where the vertices of $K_u^{\boldsymbol{\lambda}}$ correspond to groups in H, each edge in $K_u^{\boldsymbol{\lambda}}$ corresponds to the edges between two groups in H, and the colors in $K_u^{\boldsymbol{\lambda}}$ correspond to which vertices are selected in the beginning and ending group for each edge in H. Not all possible $([g]^u, G, \lambda)$-

GDD may be formed though, such as a $([g]^u, G, \lambda)$-GDD with at least one block covering two or more vertices in the same group.

In [3], K. I. Chang proved the following theorem for $G = K_k$ and $\lambda = 1$, and in [1], Lamken and Wilson used their theorem to prove it again, and for general λ. With the Lamken-Wilson Theorem, this paper shall prove the analogous statement for general G.

Theorem 4.1. Let G be a simple nontrivial graph with e undirected edges, D be the greatest common divisor of the degrees of G, $g \in \mathbb{Z}$, $g > 0$, and $\lambda \in \mathbb{Z}, \lambda > 0$. Then there exists u_0 such that for all $u \geq u_0$ satisfying

$$\lambda g^2 u(u-1) \equiv 0 \pmod{2e}, \tag{4.1}$$

$$\lambda g(u-1) \equiv 0 \pmod{D}, \tag{4.2}$$

a $([g]^u, G, \lambda)$-GDD exists.

Proof. Let $S = \{1, \ldots, g\}$ and let the colors be elements of $S \times S$. Take the edge-colored complete graph $K_u^{\boldsymbol{\lambda}}$ where u satisfies the conditions above and between each vertex there are λ of each color in $S \times S$ ($\boldsymbol{\lambda} = \lambda \mathbf{j}_{g^2}$). Let κ be a vertex-coloring map from $V(G)$ into S. Define G_κ as follows: $V(G_\kappa) = V(G)$, and (directed) edge $[x \to y] \in E(G)$ is in $E(G_\kappa)$ and has color $(\kappa(x), \kappa(y))$. Let $\mathcal{G}$ be the collection of all G_κ over all possible κ.

A decomposition of $K_u^{\boldsymbol{\lambda}}$ into graphs in $\mathcal{G}$ yields a $([g]^u, G, \lambda)$-GDD :

- The vertices of $K_u^{\boldsymbol{\lambda}}$ represent the groups of H.

- Each edge of $K_u^{\boldsymbol{\lambda}}$ represents edges from one group to another of H.
- S represents the vertices in each group.
- The colors $S \times S$ represent the vertices for the beginning and ending groups for each edge.

We now use the Lamken-Wilson Theorem. Let μ and τ be defined as in Chapter 3. We only need to show that the conditions above imply the following:

(a) $\boldsymbol{\lambda}$ is a positive rational linear combination of all $\mu(G_\kappa)$.

(b) $u(u-1)\boldsymbol{\lambda}$ is an integral linear combination of all $\mu(G_\kappa)$.

(c) $(u-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ is an integral linear combination of all $\tau(G_\kappa, x)$.

4.1 Admissibility

To show $\boldsymbol{\lambda}$ is a positive rational linear combination of all $\mu(G_\kappa)$, find the sum of all $\mu(G_\kappa)$. For $(i,j) \in S \times S$, take an edge $[u \to v] \in E(G)$ and let $\kappa(u) = i, \kappa(v) = j$. Then there are g possible vertex colors for each of $n-2$ remaining vertices. Doing this over all edges, we have $2eg^{n-2}$ for each $(i,j) \in S \times S$, so the sum is $2eg^{n-2}\mathbf{j}_{g^2}$. Hence $\boldsymbol{\lambda} = \lambda \mathbf{j}_{g^2}$ is a positive rational linear combination of all $\mu(G_\kappa)$.

4.2 Global condition

Using Lemma 3.2, let the matrix M consist of all column vectors $\mu(G_\kappa)$ and $\mathbf{c} = u(u-1)\boldsymbol{\lambda}$. Suppose we have $\mathbf{y}$, a vector of rationals X_{ij}, indexed by color $(i,j) \in S \times S$, so that $\mathbf{y}^T M$ is integral. Then $\mathbf{y}^T \mu(G_\kappa)$ is integral for all G_κ, so for all vertex colorings κ:

$$\sum_{i,j} \left| \{(b,c) : [b \stackrel{(i,j)}{\to} c]\} \right| X_{ij} \equiv 0,$$

We want to show $\mathbf{y}^T \mathbf{c}$ is integral. To do this, we choose particular κ. We will color the vertices using only two colors i, j. Because G is nontrivial, there exists some $\{x_0, y_0\} \in E(G)$. Choosing the set of vertices which have color j as the following, and using $Z_{ij} = X_{ij} + X_{ji}$ we end up with the equations:

$$\emptyset : \qquad 2eX_{ii} \equiv 0, \tag{4.3}$$

$$\{x\} : \qquad \deg(x) Z_{ij} + (2e - 2\deg(x)) X_{ii} \equiv 0, \tag{4.4}$$

$$\{x_0\} : \qquad \deg(x_0) Z_{ij} + (2e - 2\deg(x_0)) X_{ii} \equiv 0, \tag{4.5}$$

$$\{y_0\} : \qquad \deg(y_0) Z_{ij} + (2e - 2\deg(y_0)) X_{ii} \equiv 0, \tag{4.6}$$

$$\{x_0, y_0\} : \qquad (\deg(x_0) + \deg(y_0) - 2) Z_{ij} + 2X_{jj} + (2e - 2\deg(x_0) - 2\deg(y_0) + 2) X_{ii} \equiv 0. \tag{4.7}$$

Now add (4.3) and (4.7) and subtract (4.5) and (4.6):

$$2(X_{ij} + X_{ji}) \equiv 2(X_{ii} + X_{jj}). \tag{4.8}$$

Already, since $u(u-1)$ is even, we have:

$$\mathbf{y}^T\mathbf{c} = \lambda u(u-1)\sum_{i,j} X_{ij} \equiv \lambda g u(u-1)\sum_{i} X_{ii}. \tag{4.9}$$

Subtract (4.3) from (4.4):

$$\deg(x)(X_{ij} + X_{ji}) \equiv 2\deg(x)X_{ii}.$$

Recall from elementary number theory that D, the greatest common divisor of the degrees of G, is a linear combination of those degrees. Applying the linear combination to the previous equation, and simplifying, gives

$$D(X_{ij} + X_{ji}) \equiv 2DX_{ii}.$$

By necessary condition (4.2), $\lambda g u(u-1)/D$ is an integer, so

$$\lambda g u(u-1)(X_{ij} + X_{ji}) \equiv 2\lambda g u(u-1)X_{ii}.$$

Since $u(u-1)$ is even, use (4.8) and we have

$$\lambda g u(u-1)X_{ii} \equiv \lambda g u(u-1)X_{jj}.$$

Applying this equation to (4.9):

$$\mathbf{y}^T\mathbf{c} \equiv \lambda g u(u-1)\sum_i X_{ii} \equiv \lambda g^2 u(u-1)X_{11}.$$

Finally, by necessary condition (4.1), $\lambda g^2 u(u-1)/2e$ is an integer. Using this fact on (4.3) ($2eX_{ii} \equiv 0$) and the last equation, we end up with $\mathbf{y}^T\mathbf{c} \equiv 0$.

4.3 Local Condition

Using Lemma 3.2, let the matrix M consist of all column vectors $\tau(G_\kappa, x)$ and $\mathbf{c} = (u-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$. Suppose we have $\mathbf{y}$, a vector of rationals X_{ij}, Y_{ij}, indexed by in-colors and out-colors, respectively, of color $(i,j) \in S \times S$, so that $\mathbf{y}^T M$ is integral. Then $\mathbf{y}^T\tau(G_\kappa, x)$ is integral for all G_κ and $x \in G_\kappa$, so for all vertex colorings κ and vertices x:

$$\sum_i \left|\{b : [b \overset{(i,j)}{\to} x]\}\right| (X_{ij} + Y_{ji}) \equiv 0, \quad \kappa(x) = j$$

We want to show $\mathbf{y}^T\mathbf{c}$ is integral. To do this, we choose particular κ. We will color the vertices using only two colors i, j. Because G is nontrivial, there exists some $\{x_0, y_0\} \in E(G)$. Choosing the set of vertices which have color j

as the following, and using $Z_{ij} = X_{ij} + Y_{ji}$ we end up with the equations:

$$\emptyset: \quad \deg(x)Z_{ii} \equiv 0, \tag{4.10}$$

$$\{x\}: \quad \deg(x)Z_{ij} \equiv 0, \tag{4.11}$$

$$\{x_0\}: \quad \deg(x_0)Z_{ij} \equiv 0, \tag{4.12}$$

$$\{x_0, y_0\}: \quad (\deg(x_0) - 1)Z_{ij} + Z_{jj} \equiv 0, \tag{4.13}$$

Subtract (4.12) from (4.13):

$$Z_{ij} \equiv Z_{jj}.$$

Already, we have:

$$\mathbf{y}^T\mathbf{c} = \lambda(u-1)\sum_{i,j} Z_{ij} \equiv \lambda g(u-1)\sum_{i} Z_{ii}.$$

As in the case for $\mu(G_\kappa)$, take (4.10) and use the linear combination for D in terms of the degrees of G to get $DZ_{ii} \equiv 0$. By necessary condition (4.1), $\lambda g(u-1)/D$ is an integer. Thus we have $\mathbf{y}^T\mathbf{c} \equiv 0$. □

With $g = 1$ we have:

Corollary 4.2. Let G be a simple nontrivial graph with e undirected edges, D be the greatest common divisor of the degrees of G, and $\lambda \in \mathbb{Z}, \lambda > 0$.

Then there exists v_0 such that for all $v \geq v_0$ satisfying

$$\lambda v(v-1) \equiv 0 \pmod{2e}$$
$$\lambda(v-1) \equiv 0 \pmod{D},$$

a (v, G, λ)-design exists.

This is a restatement of Wilson's graph decomposition result and can also be derived directly from the Lamken-Wilson Theorem.

Chapter 5

Resolvable Designs

A *resolvable block design* with parameters v, k, λ, denoted (v, k, λ)-RBD, is a (v, k, λ)-design on a set S of order v where all the blocks can be partitioned into collections, called *resolution classes*, such that every element of S is contained in exactly one block in each resolution class. Necessary conditions for RBDs are the same as for block designs, except that one of those necessary conditions is replaced with the resolvability condition:

$$v \equiv 0 \pmod{k}$$
$$\lambda(v-1) \equiv 0 \pmod{k-1}$$

Ex. $[[\{1,2\},\{3,4\}],[\{1,3\},\{2,4\}],[\{1,4\},\{2,3\}]]$ is a $(4,2,1)$-RBD and the affine plane of order 2.

Ex. A $(v, 2, 1)$-RBD exists for all even integers v (the necessary condition). To see this, take the set $\mathbb{Z}_{2n+1} \cup \{\infty\}$, take one resolution class $[\{0, \infty\}, \{1, -1\}, \{2, -2\}, \ldots, \{n, -n\}]$, and cycle it mod $2n+1$ to produce a $(2n+2, 2, 1)$-RBD. The $(4, 2, 1)$-RBD above can be formed with $n = 1$.

A close relative to resolvable designs are near-resolvable designs (NRBD), which is defined as in resolvable designs above, except that the classes are called "near-resolution classes" and that every element except one in S is contained in exactly one block in each near-resolution class, and the exception is not contained in any block in that collection. Near-resolvable designs tend to be larger than resolvable designs in the sense that there are at least v near-resolution classes. Necessary conditions are:

$$v - 1 \equiv 0 \pmod{k}$$
$$\lambda(v - 1) \equiv 0 \pmod{k - 1}$$

In particular, $\lambda \equiv 0 \pmod{k-1}$.

Ex. The resolvable design in the previous example can be reduced to a $(2n+1, 2, 1)$-NRBD by removing the blocks containing ∞.

Resolvable designs and near-resolvable designs are a particularly useful model in the scheduling of tournaments. Particular interest in resolvable designs began outside of design theory, in the form of combinatorial puzzles. Kirkman's Schoolgirl Problem (a combinatorics puzzle equivalent to finding a $(15, 3, 1)$-RBD) is one example.

Study of resolvable designs in the context of design theory began in 1969 when Ray-Chaudhuri and Wilson generalized Kirkman's Schoolgirl Problem and proved that a $(v, 3, 1)$-RBD exists for all $v \equiv 3 \pmod 6$, which is also the necessary condition [4]. Further generalization led to the existence of a $(v, 4, 1)$-RBD for all necessary v and a $(v, k, 1)$-RBD for all sufficiently large v satisfying the necessary conditions.

Ray-Chaudhuri and Wilson's proof of asymptotic existence relies on the fact that the set of r (the number of blocks on a vertex, and the number of resolution classes) for which resolvable designs exist is PBD-closed, and thus, a PBD-closure of some subset of this set.

A $(v, k, 1)$-RBD is equivalent to a special $(v + r, \{k + 1, r\}, 1)$-pairwise balanced design called the $\{k, r\}$-*completed design* with one block, called the *base block*, of size r, and all other blocks of size $k + 1$. The RBD forms a completed design by adding the base block of r vertices, with each vertex joined to one resolution class, and all blocks of size k in that class extended to blocks of size $k + 1$ to cover that vertex.

Likewise, removing the base block in the completed design yields an RBD. This is because in any block design with index 1, the set of all blocks on a vertex partition all other vertices. So if a vertex in the base block is chosen, all blocks on that vertex partition all other vertices. One of the blocks is the base block, and the others form a resolution class.

With the same argument, we can choose a point not in the base block, which has r blocks on it, partitioning all other vertices, and each one containing exactly one vertex of the base block. This particular point is called the *point at infinity* and is denoted ∞.

Given RBDs with various small r-values, we construct these $\{k, r\}$-completed designs. We can now construct a $\{k, R\}$-completed design for any R in the PBD-closure of the set of r-values above as follows: Take a base block of size R and an arbitrary point at infinity ∞ and R blocks through ∞. Now the base block can be decomposed into smaller base blocks in the set of r-values. On these base blocks we place the smaller completed designs with ∞ as the point at infinity, and blocks through ∞ coinciding with the blocks through ∞ of the large design.

Thus the base block of size R, along with the blocks through ∞ counted once each, and all the other blocks from the smaller designs, form the $\{k, R\}$-completed design. Removing the base block yields an RBD with an r-value of R.

Ex. Existence of $(v, 3, 1)$-RBDs. The necessary condition is $v \equiv 3 \pmod 6$. Designs can be found for $(9, 3, 1)$-RBD and $(15, 3, 1)$-RBD with r-values 4 and 7, respectively. From the CRC Handbook of Combinatorial Designs ([6]), the PBD-closure of $\{4, 7\}$ is $\mathbf{B}(\{4, 7\}) = \{r : r \equiv 1 \pmod 3\} \setminus \{10, 19\}$. This covers all admissible v except 21 and 39. Since RBDs exist for these values, a $(v, 3, 1)$-RBD exists for all $v \equiv 3 \pmod 6$.

Naturally, resolvable block designs can be extended to the subject of resolvable G-designs. In 2007, Dukes and Ling showed the asymptotic existence of resolvable G-designs. However, there was a flaw in proving the asymptotic existence of a type of design known as a uniform G-frame. G-frames are a partial resolution extension of GDDs and a group divisible extension of near-resolvable designs. They allow recursive methods of constructing resolvable designs by filling in the holes of the G-frame with smaller resolvable designs. The main goal of this paper is to fix the flaw and prove the asymptotic existence of uniform G-frames.

Chapter 6

G-frames and Resolvability

In the introductory chapter, we established the following necessary conditions for a uniform G-frame of type $[g]^u$ and index λ:

$$\text{(Resolution)} \qquad g(u-1) \equiv 0 \pmod{n},$$

$$\text{(Equireplicate)} \qquad \lambda g(u-1) \equiv 0 \pmod{\gamma}.$$

Theorem 6.1. Let G be a simple nontrivial graph with e undirected edges and n vertices, γ be defined as above, $g \in \mathbb{Z}$, $g > 0$, and $\lambda \in \mathbb{Z}, \lambda > 0$ such that $m = \lambda gn/2e \in \mathbb{Z}$. Then there exists u_0 such that for all $u \geq u_0$

satisfying

$$g(u-1) \equiv 0 \pmod{n}, \tag{6.1}$$

$$\lambda g(u-1) \equiv 0 \pmod{\gamma}, \tag{6.2}$$

a $([g]^u, G, \lambda)$-FD exists.

Proof. Let $S = \{1, \dots, g\}, M = \{1^*, \dots, m^*\}$ and let the colors be elements of $(S \times S) \cup (S \times M)$. Take the edge-colored complete graph $K_u^{\boldsymbol{\lambda}}$ where u satisfies the conditions above and between each vertex there are λ of each color in $S \times S$ and 1 of each color in $S \times M$ ($\boldsymbol{\lambda} = (\lambda \mathbf{j}_{g^2}, \mathbf{j}_{gm})$). Let κ be a vertex-coloring map from $V(G)$ into S, and $r \in M$. Define $G_{\kappa r}$ as follows: $V(G_{\kappa r}) = V(G) \cup \{\infty\}$, (directed) edge $[x \to y] \in E(G)$ is in $E(G_{\kappa r})$ and has color $(\kappa(x), \kappa(y)) \in S \times S$, and for $x \in V(G)$, edge $[x \to \infty]$ is in $E(G_{\kappa r})$ and has color $(\kappa(x), r) \in S \times M$. Let $\mathcal{G}$ be the collection of all $G_{\kappa r}$ over all possible κ and r.

A decomposition of $K_u^{\boldsymbol{\lambda}}$ into graphs in $\mathcal{G}$ yields a $([g]^u, G, \lambda)$-FD :

- The vertices of $K_u^{\boldsymbol{\lambda}}$ represent the groups of H.
- Each edge of $K_u^{\boldsymbol{\lambda}}$ represents edges from one group to another of H.
- S represents the vertices in each group.
- The colors $S \times S$ represent the vertices for the beginning and ending groups for each edge.

- M represents the partial resolution classes missing a given group.
- The colors $S \times M$ represent the vertex for a group and the class in which the block belongs.

We now use the Lamken-Wilson Theorem. Let μ and τ be defined as in Chapter 3. We only need to show that the conditions above imply the following:

(a) $\boldsymbol{\lambda}$ is a positive rational linear combination of all $\mu(G_{\kappa r})$.

(b) $u(u-1)\boldsymbol{\lambda}$ is an integral linear combination of all $\mu(G_{\kappa r})$.

(c) $(u-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$ is an integral linear combination of all $\tau(G_{\kappa r}, x)$.

6.1 Admissibility

To show $\boldsymbol{\lambda}$ is a positive rational linear combination of all $\mu(G_{\kappa r})$, find the sum of all $\mu(G_{\kappa r})$. For $(i,j) \in S \times S$, take an edge $[u \to v] \in E(G)$ and let $\kappa(u) = i, \kappa(v) = j$. Then there are g possible vertex colors for each of $n-2$ remaining vertices. Doing this over all edges and all $r \in M$, we have $2emg^{n-2}$ for each $(i,j) \in S \times S$.

For $(i,r) \in S \times M$, let $r \in M$, take a vertex $u \in V(G)$ and let $\kappa(u) = i$. Then there are g possible vertex colors for each of $n-1$ remaining vertices. Doing this over all vertices, we have ng^{n-1} for each $(i,r) \in S \times M$.

Hence the sum of all $\mu(G_{\kappa r})$ is $(2emg^{n-2}\mathbf{j}_{g^2}, ng^{n-1}\mathbf{j}_{gm})$. Dividing by ng^{n-1} and using $m = \lambda gn/2e$ gives us $(\lambda\mathbf{j}_{g^2}, \mathbf{j}_{gm})$. Hence $\boldsymbol{\lambda} = (\lambda\mathbf{j}_{g^2}, \mathbf{j}_{gm})$ is a positive rational linear combination of all $\mu(G_{\kappa r})$.

6.2 Global condition

Using Lemma 3.2, let the matrix M consist of all column vectors $\mu(G_{\kappa r})$ and $\mathbf{c} = u(u-1)\boldsymbol{\lambda}$. Suppose we have $\mathbf{y}$, a vector of rationals X_{ij}, indexed by color $(i,j) \in S \times S$ and U_{ir} indexed by color $(i,r) \in S \times M$, so that $\mathbf{y}^T M$ is integral. Then $\mathbf{y}^T \mu(G_{\kappa r})$ is integral for all $G_{\kappa r}$, so for all vertex colorings κ and all $r \in M$:

$$\sum_{i,j} \left| \{(b,c) : [b \stackrel{(i,j)}{\to} c]\} \right| X_{ij} + \sum_i |\{b : \kappa(b) = i\}| \, U_{ir} \equiv 0.$$

We want to show $\mathbf{y}^T\mathbf{c}$ is integral. To do this, we choose particular κ, and keep $r \in M$ general. We will color the vertices using only two colors i, j. Because G is nontrivial, there exists some $\{x_0, y_0\} \in E(G)$. Choosing the set of vertices which have color j as the following, and using $Z_{ij} = X_{ij} + X_{ji}$ we end up with the equations:

$$\begin{aligned}
\emptyset: &\quad 2eX_{ii} + nU_{ir} \equiv 0, && (6.3)\\
\{x\}: &\quad \deg(x)Z_{ij} + (2e - 2\deg(x))X_{ii} + (n-1)U_{ir} + U_{jr} \equiv 0, && (6.4)\\
\{x_0\}: &\quad \deg(x_0)Z_{ij} + (2e - 2\deg(x_0))X_{ii} + (n-1)U_{ir} + U_{jr} \equiv 0, && (6.5)\\
\{y_0\}: &\quad \deg(y_0)Z_{ij} + (2e - 2\deg(y_0))X_{ii} + (n-1)U_{ir} + U_{jr} \equiv 0, && (6.6)\\
\{x_0, y_0\}: &\quad (\deg(x_0) + \deg(y_0) - 2)Z_{ij} + 2X_{jj} + (n-2)U_{ir} + 2U_{jr} &&\\
&\quad +(2e - 2\deg(x_0) - 2\deg(y_0) + 2)X_{ii} \equiv 0. && (6.7)
\end{aligned}$$

Now add (6.3) and (6.7) and subtract (6.5) and (6.6):

$$2(X_{ij} + X_{ji}) \equiv 2(X_{ii} + X_{jj}). \quad (6.8)$$

Already, since 2 divides $u(u-1)$, we have:

$$\begin{aligned}
\mathbf{y}^T\mathbf{c} &= \lambda u(u-1)\sum_{i,j} X_{ij} + u(u-1)\sum_{i,r} U_{ir}\\
&\equiv \sum_i \left(\lambda g u(u-1)X_{ii} + u(u-1)\sum_r U_{ir}\right). \quad (6.9)
\end{aligned}$$

Subtract (6.3) from (6.4):

$$\deg(x)(X_{ij} + X_{ji}) + U_{jr} \equiv 2\deg(x)X_{jj} + U_{ir}.$$

Recall that $\gamma(1, n/2e) = (\gamma, \gamma n/2e)$ is an integral linear combination of the $(d_i, 1)$. Furthermore, the above equation holds for all $r \in M$ so we can vary the r while adding and subtracting. Using the linear combination while varying r gives:

$$\gamma(X_{ij} + X_{ji}) + \sum_{h=1}^{\gamma n/2e} U_{ja_h} \equiv 2\gamma X_{ii} + \sum_{h=1}^{\gamma n/2e} U_{ia_h}.$$

where the a_i are arbitrary numbers in M.

By necessary condition (6.2), $\lambda gu(u-1)/\gamma$ is an integer. Taking the last equation $\lambda gu(u-1)/\gamma$ times while varying the r, and using $m = \lambda gn/2e$, gives

$$\lambda gu(u-1)(X_{ij} + X_{ji}) + \sum_{h=1}^{mu(u-1)} U_{ja_h} \equiv 2\lambda gu(u-1)X_{ii} + \sum_{h=1}^{mu(u-1)} U_{ia_h}.$$

Since $u(u-1)$ is even, use (6.8) and we have

$$\lambda gu(u-1)X_{ii} + \sum_{h=1}^{mu(u-1)} U_{ia_h} \equiv \lambda gu(u-1)X_{jj} + \sum_{h=1}^{mu(u-1)} U_{ja_h}.$$

Set the a_i so that there are $u(u-1)$ of each of the m elements in M. Then

$$\lambda gu(u-1)X_{ii} + u(u-1)\sum_{r \in M} U_{ir} \equiv \lambda gu(u-1)X_{jj} + u(u-1)\sum_{r \in M} U_{jr}.$$

Applying this equation to (6.9):

$$\mathbf{y}^T\mathbf{c} \equiv \sum_i \left(\lambda g u(u-1) X_{ii} + u(u-1) \sum_r U_{ir} \right) \equiv \lambda g^2 u(u-1) X_{11} + g u(u-1) \sum_r U_{1r}.$$

From (6.3) ($2eX_{ii} + nU_{ir} \equiv 0$), varying r gives $nU_{ir} \equiv nU_{is}$. By necessary condition (6.1), $g(u-1) \equiv 0 \pmod{n}$. Applying this congruence to the last term in the previous equation gives

$$\mathbf{y}^T\mathbf{c} \equiv \lambda g^2 u(u-1) X_{11} + mgu(u-1) U_{11}.$$

Finally, by (6.1), $mgu(u-1)/n$ is an integer. Multiplying (6.3) by $mgu(u-1)/n$ and using $m = \lambda gn/2e$ gives $\mathbf{y}^T\mathbf{c} \equiv 0$.

6.3 Local condition

Using Lemma 3.2, let the matrix M consist of all column vectors $\tau(G_{\kappa r}, x)$ and $\mathbf{c} = (u-1)(\boldsymbol{\lambda}, \boldsymbol{\lambda})$. Suppose we have $\mathbf{y}$, a vector of rationals X_{ij}, Y_{ij}, indexed by in-colors and out-colors, respectively, of color $(i,j) \in S \times S$, and U_{ir}, V_{ir} indexed by in-colors and out-colors, respectively, of color $(i,r) \in S \times M$, so that $\mathbf{y}^T M$ is integral. Then $\mathbf{y}^T \tau(G_{\kappa r}, x)$ is integral for all $G_{\kappa r}$ and $x \in V(G_{\kappa r})$. For $x \in V(G)$, we have for all vertex colorings κ and all $r \in M$:

$$\sum_i \left| \{b : [b \overset{(i,j)}{\to} x]\} \right| (X_{ij} + Y_{ji}) + V_{jr} \equiv 0, \quad \kappa(x) = j$$

and for $x = \infty$:

$$\sum_i |\{b : \kappa(b) = i\}| \, U_{ir} \equiv 0. \tag{6.10}$$

We want to show $\mathbf{y}^T\mathbf{c}$ is integral. To do this, we choose particular κ. We will color the vertices using only two colors i, j. Because G is nontrivial, there exists some $\{x_0, y_0\} \in E(G)$. Choosing the set of vertices which have color j as the following, and using $Z_{ij} = X_{ij} + Y_{ji}$ we end up with the equations for the case $x \in V(G)$:

$$\emptyset : \qquad \deg(x)Z_{ii} + V_{ir} \equiv 0, \tag{6.11}$$

$$\{x\} : \qquad \deg(x)Z_{ij} + V_{jr} \equiv 0, \tag{6.12}$$

$$\{x_0\} : \qquad \deg(x_0)Z_{ij} + V_{jr} \equiv 0, \tag{6.13}$$

$$\{x_0, y_0\} : \qquad (\deg(x_0) - 1)Z_{ij} + Z_{jj} + V_{jr} \equiv 0. \tag{6.14}$$

and for the case $x = \infty$:

$$\emptyset : \qquad nU_{ir} \equiv 0, \tag{6.15}$$

$$\{x\} : \qquad (n-1)U_{ir} + U_{jr} \equiv 0, \tag{6.16}$$

Subtract (6.13) from (6.14):

$$Z_{ij} \equiv Z_{jj}.$$

By subtracting (6.15) from (6.16), $U_{ir} \equiv U_{jr}$. Varying r in (6.11) gives $V_{ir} \equiv V_{is}$. We then have

$$\begin{aligned}\mathbf{y}^T\mathbf{c} &= \lambda(u-1)\sum_{i,j} Z_{ij} + (u-1)\sum_{i,r} V_{ir} + (u-1)\sum_{i,r} U_{ir} \\ &\equiv \sum_i (\lambda g(u-1)Z_{ii} + m(u-1)V_{i1}) + \sum_r g(u-1)U_{1r}\end{aligned}$$

Since $nU_{ir} \equiv 0$ and, by necessary condition (6.1), $g(u-1)/n$ is an integer, the last term vanishes, so

$$\mathbf{y}^T\mathbf{c} \equiv \sum_i (\lambda g(u-1)Z_{ii} + m(u-1)V_{i1}) \tag{6.17}$$

As in the case for $\mu(G_{\kappa r})$, take (6.11) and use the linear combination for $(\gamma, \gamma n/2e)$ in terms of the $(d_i, 1)$ to obtain

$$\gamma Z_{ii} + (\gamma n/2e)V_{ir} \equiv 0$$

But by necessary condition (6.2), $\lambda g(u-1) \equiv 0 \pmod{\gamma}$. Hence by multiplying by $\lambda g(u-1)/\gamma$ and using $m = \lambda gn/2e$,

$$\lambda g(u-1)Z_{ii} + m(u-1)V_{ir} \equiv 0.$$

Applying this equation to (6.17) results in $\mathbf{y}^T\mathbf{c} \equiv 0$. □

With $g = 1$ we have the following specialization of near-resolvable block designs.

Corollary 6.2. Let G be a simple nontrivial graph with n vertices and e undirected edges, γ be defined as above, and $\lambda \in \mathbb{Z}, \lambda > 0$, such that $\lambda n/2e \in \mathbb{Z}$. Then there exists v_0 such that for all $v \geq v_0$ satisfying

$$v - 1 \equiv 0 \pmod{n}$$
$$\lambda(v-1) \equiv 0 \pmod{\gamma},$$

a near-resolvable (v, G, λ)-design exists.

6.4 Applications to resolvable designs

The following theorem was first proved by Ray-Chaudhuri and Wilson ([4]):

Theorem 6.3. For $k \geq 2$, there exists a constant v_0 such that a $(v, k, 1)$-RBD exists for all $v \geq v_0$ satisfying $v \equiv k \pmod{k(k-1)}$.

Proof. Put $G = K_k$, $g = k-1$, and $\lambda = 1$ in Theorem 6.1. Now $n/2e = 1/(k-1)$ and $\gamma = k-1$, so $m = \lambda ng/2e = 1$. The conditions reduce to $(k-1)(u-1) \equiv 0 \pmod{k}$ and $\lambda(k-1)(u-1) \equiv 0 \pmod{k-1}$, the second of which is true by default. Since $(k-1, k) = 1$, $u \equiv 1 \pmod{k}$. So for sufficiently large u satisfying $u \equiv 1 \pmod{k}$, there is a G-frame of type $(k-1)^u$ and index 1. Now since $m = 1$, there is only one resolution class missing each group of size $k-1$. For each partial resolution class missing a group, add a K_k covering that group and a special vertex ω. This results in a $((k-1)u+1, k, 1)$-RBD. □

This motivates the following observation:

Proposition 6.4. Suppose there exists a $(g+1, G, \lambda)$-RGD and a (g^u, G, λ)-frame. Then there exists a $(gu+1, G, \lambda)$-RGD.

Proof. For each partial resolution class in the frame missing a group, add one resolution class from the RGD covering that group and a special vertex ω. The number of partial resolution classes in the frame missing a given group is $m = \lambda gn/2e = \lambda((g+1)-1)n/2e = r$, which is the number of resolution classes in the RGD. □

A *k-RGDD of type* $[g]^u$ is a $([g]^u, k, 1)$-GDD that is resolvable into resolution classes. Here, we focus on $G = K_k$ and $\lambda = 1$. There are two necessary conditions:

$$gu \equiv 0 \pmod{k}$$
$$g(u-1) \equiv 0 \pmod{k-1}$$

We present a construction:

Let there be a k-GDD of type $[g]^u$ and a $(v, k, 1)$-RBD. Together, they form the basis of a k-RGDD of type $[g]^{uv}$. Now let there be a k-frame of type

$[g(uv-1)]^t$. Then the k-RGDD of type $[g]^{uv}$ fills the holes of the frame to form a k-RGDD of type $[g]^{t(uv-1)+1}$.

These lead to the following necessary (and asymptotically sufficient) conditions:

$$\begin{aligned}
g^2u(u-1) &\equiv 0 \pmod{k(k-1)} \\
g(u-1) &\equiv 0 \pmod{k-1} \\
v &\equiv 0 \pmod{k} \\
v-1 &\equiv 0 \pmod{k-1} \\
g(uv-1)(t-1) &\equiv 0 \pmod{k} \\
g(uv-1)(t-1) &\equiv 0 \pmod{k-1} \\
g(uv-1) &\equiv 0 \pmod{k-1}
\end{aligned}$$

The second and fourth equations imply $g(uv-1) \equiv 0 \pmod{k-1}$, so the last two equations are redundant.

The goal now is to prove that, for all sufficiently large x satisfying the necessary conditions, there is a k-RGDD of type $[g]^x$. We consider finding examples for each admissible congruence class mod $k(k-1)$. Because of the construction above, we find examples where $x = t(uv-1)+1$, where t, u, v satisfy the necessary conditions for the construction.

For simplicity, we shall assume that $gu \equiv 0 \pmod{k}$, so that u satisfies the first condition above. We shall also assume that $x \equiv u \pmod{k(k-1)}$, since u and x satisfy the same conditions. We know that $x = t(uv-1)+1$. From properties 3 and 4 above, $v \equiv k \pmod{k(k-1)}$. So we solve the equation:

$$u \equiv t(uk-1)+1 \pmod{k(k-1)}$$

whereupon taking mod k and mod $k-1$, we have:

$$t \equiv 1-u \pmod{k}$$

$$(u-1)t \equiv x-1 \pmod{k-1}$$

The second of the two can be solved by letting $t \equiv 1 \pmod{k-1}$. By the Chinese Remainder Theorem, there is a solution satisfying the two equations for t mod $k(k-1)$.

In summary, we can prove:

Proposition 6.5. Given g, k, there exists a k-RGDD of type $[g]^x$ for some x in every admissible congruence class $\pmod{k(k-1)}$.

This is an important ingredient in ongoing work on asymptotic existence of k-RGDDs.

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