

THE LINEAR SHALLOW WATER THEORY:  
A MATHEMATICAL JUSTIFICATION

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1. Introduction.

It is well known that a boundary-value problem involving an elliptic partial differential equation plays a central role in shallow water theory [see 6, 10]. Besides its many important applications, the theory is of considerable mathematical interest growing from the curious instance of the approximation of the solution of a boundary-value problem for an elliptic partial differential equation by the solution of an initial-value problem for a hyperbolic partial differential equation [10]. This explains, for example, the appearance of the initial-value problem for the wave equation

$$u_{tt} = \alpha(u_{xx} + u_{yy})$$

in the mathematical model for waves on the surface of an infinite ocean of uniform depth.

The many conventional derivations of the shallow water theory that exist in the literature [10] are open to several objections which include (a) only first order approximations to the solution can be obtained, and (b) there is ambiguity about what parameter determines the accuracy of the approximation. Friedrichs [3] was able to remove these objections by deriving the shallow water theory by a formal perturbation procedure in powers of a parameter associated with the problem. A mathematical justification of the shallow water theory would require a proof that this perturbation series is convergent, or is asymptotically valid.

Despite its importance in the applications and its considerable mathematical interest, no mathematical justification has been given for the theory other than Shinbrot's results [8] for the case of simple harmonic motion of two-dimensional linear flows.

It is our intention here to provide a mathematical justification for the shallow water theory for time-dependent two-dimensional flows of an inviscid, irrotational, incompressible fluid, moving under the force of gravity, and having a free surface. In doing so, we present another derivation of the shallow water equations.

## 2. Formulation of Problem.

Let us choose the  $X$ -axis to coincide with the free surface of the fluid in its undisturbed position, and the  $Y$ -axis pointing upward. The water now flows between two surfaces: the bottom described by the equation

$$Y = -\epsilon B(X, \epsilon)$$

and the free surface described by

$$Y = \eta(X, t, \epsilon)$$

where  $\epsilon$  is a positive parameter assumed to be small. (In general this parameter represents some physical quantity associated with the flow, for example the ratio of the mean depth of the ocean to the radius of curvature of waves on its surface.)

We make the following additional assumptions:

- (i)  $B$  is a bounded function in its arguments and has continuous derivatives through the second order with respect to  $X$ ,
- (ii) There exist positive constants  $b_1$  and  $b_2$  such that

$$b_1 \leq B(X, \epsilon) \leq b_2 \quad \text{for all } X \in \mathbb{R}, \text{ and}$$

- (iii)  $B_X(x, \epsilon) \in L^2(\mathbb{R})$ .

Assumption (ii) provides a real limitation on our results since it says that the flow takes place in an infinite ocean with no shores. But it is this condition which guarantees existence and uniqueness of gravity waves in the linear theory. See [2, 4].

The normalized equations of water waves in two space dimensions are:  
Laplace's equation

$$\bar{\phi}_{XX} + \bar{\phi}_{YY} = 0,$$

the condition

$$\bar{\phi}_{\nu} = 0$$

on the bottom signifying that no fluid flows through the bottom, and the non-linear Bernoulli equation

$$\eta + \bar{\phi}_t + \frac{1}{2}|\nabla\bar{\phi}|^2 = \text{constant}$$

and the kinematic equation

$$\dot{Y} = \eta_t + \eta_X \dot{X}$$

on the free surface of the fluid. Here  $\bar{\phi}_{\nu}$  denotes the outward normal derivative of  $\bar{\phi}$ ,  $\nabla$  is the gradient operator  $\left[ \frac{\partial}{\partial X} \hat{i} + \frac{\partial}{\partial Y} \hat{j} \right]$ , and  $\dot{f}(\ ) \equiv df/dt$ . Here the density of the fluid and the acceleration due to gravity are set equal to 1. The functions  $\eta$  and  $\bar{\phi}$  are assumed known at time  $t = 0$ .

The problem of determining the velocity potential  $\bar{\phi}$  and the free surface  $\eta$  is very difficult because of the non-linearities in the boundary conditions prescribed on a surface which itself is unknown. For this reason we adopt the

assumptions of the linear theory: namely  $\eta$ ,  $\eta_x$  and  $|\nabla\bar{\phi}|$  are small. These assumptions permit us to drop the nonlinear terms in the boundary conditions, and replace the free surface  $Y = \eta$  by the fixed surface  $Y = 0$  to obtain the boundary-value problem

$$(2.1) \quad \bar{\phi}_{XX} + \bar{\phi}_{YY} = 0 \quad \text{in } \Omega^* = \{(X, Y): X \in \mathbb{R}, -\epsilon B(X, \epsilon) < Y < 0\},$$

$$(2.2) \quad \bar{\phi}_\nu = 0 \quad \text{on } \Gamma_1 = \{(X, Y): X \in \mathbb{R}, Y = -\epsilon B(X, \epsilon)\},$$

and

$$(2.3a) \quad \eta + \bar{\phi}_t = 0, \quad \bar{\phi}_Y = \eta_t \quad \text{on } \Gamma = \{(X, Y): X \in \mathbb{R}, Y = 0\},$$

or equivalently,

$$(2.3b) \quad \bar{\phi}_y + \bar{\phi}_{tt} = 0 \quad \text{on } \Gamma,$$

$$(2.4) \quad \bar{\phi}(X, 0, 0) = f_0(X), \quad \bar{\phi}_t(X, 0, 0) = f_1(X) \quad \text{on } \Gamma.$$

It is this problem that will receive our attention in this paper.

For flows in more general domains  $\Omega^*$  (normal domains) Friedman and Shinbrot [1] proved existence and uniqueness theorems results for the boundary-value problem (2.1-4), in fact for its  $n$ -dimensional analogue, showing that (2.1-2) and (2.4) are satisfied classically, but that (2.3) was satisfied in general, only in a weak sense to be defined below. Let  $H$  be a Hilbert space with inner-product  $(\cdot, \cdot)_H$ . Let  $k$  be a non-negative integer. We denote by  $C^k([0, \infty), H)$  the space of  $H$ -valued functions defined on  $[0, \infty)$  with strongly continuous strong derivatives through order  $k$ .

We shall denote by  $H^k(\Omega^*)$  the space of functions which, together with their distributional derivatives through order  $k$  are in  $L^2(\Omega^*)$ . The norm in this space is defined by

$$\|f\|_{H^*(\Omega^*)} = \left\{ \sum_{|\alpha| \leq k} \int_{\Omega^*} |D_\alpha f|^2 dXdY \right\}^{1/2}$$

where  $\alpha = (\alpha_1, \alpha_2)$ ,  $\alpha_1, \alpha_2 \geq 0$ ;  $|\alpha| = \alpha_1 + \alpha_2$  and  $D_\alpha = \frac{\partial^{\alpha_1 + \alpha_2}}{\partial X^{\alpha_1} \partial Y^{\alpha_2}}$ .

Let  $\bar{\phi}$  and  $\bar{\psi}$  denote the restriction of  $\bar{\phi}$  and  $\bar{\psi}$  to  $\Gamma$ , respectively. We say that a function  $\bar{\phi}(X, Y, t) \in C([0, \infty); H^1(\Omega^*))$  is a strong solution of (2.1-4) if

$$(2.5) \quad \bar{\phi} \in C([0, \infty), L^2(\Gamma));$$

$$(2.6) \quad \text{the strong derivatives } \bar{\phi}_t \text{ and } \bar{\phi}_{tt} \text{ exists and belong to } L^2(\Gamma) \text{ for } t \geq 0;$$

$$(2.7) \quad (\nabla \bar{\psi}, \nabla \bar{\phi})_{L^2(\Omega^*)} + (\bar{\psi}, \bar{\phi}_{tt})_{L^2(\Gamma)} = 0 \text{ for all } \bar{\psi} \in H^1(\Omega^*); \text{ and}$$

$$(2.8) \quad \bar{\phi}(X, 0) = f_0(X), \bar{\phi}_t(X, 0) = f_1(X).$$

For normal domains Friedman and Shinbrot [1] were able to prove existence of strong solutions of (2.1-4). However for standard domains (domains such that  $d(\Gamma, \Gamma_1) \geq \delta > 0$ ,  $d(A, B)$  denoting the distance between the two sets  $A$  and  $B$ ) Garipov [4], and Friedman and Shinbrot [2] have proved the existence of smooth solutions of (2.1-4) satisfying all conditions classically given sufficient smoothness on the initial data  $f_0(X)$  and  $f_1(X)$ .

Thus the condition in assumption (ii) ensures that  $\Omega^*$  is a standard domain, and consequently that (2.1-4) has a smooth classical solution.

Let  $\bar{\phi}(X, t)$  be the restriction to  $\Gamma$  of the solution  $\bar{\Phi}(X, Y, t)$  of (2.1-4). Then  $\bar{\Phi}(X, Y, t)$  satisfies the boundary value problem

$$\bar{\Phi}_{XX} + \bar{\Phi}_{YY} = 0 \quad \text{in } \Omega^*,$$

$$\bar{\Phi} = \bar{\phi} \quad \text{on } \Gamma,$$

and

$$\frac{\partial \bar{\Phi}}{\partial \nu} = 0 \quad \text{on } \Gamma_1.$$

Since  $\bar{\Phi}$  depends uniquely and linearly on the function  $\bar{\phi}$  we may define a linear operator  $K$  by the equation

$$K\bar{\phi} = \bar{\Phi}_Y \quad \text{on } \Gamma.$$

The operator  $K$ , in general, is a pseudo-differential operator which is positive and has a self-adjoint extension to  $L^2(\Gamma)$  which we denote also by  $K$ .

The boundary conditions (2.3b) and (2.4) yield the abstract initial value problem in  $L^2(\Gamma)$

$$(2.9) \quad \bar{\phi}_{tt} + K\bar{\phi} = 0,$$

$$(2.10) \quad \bar{\phi}(X, 0) = f_0(X),$$

$$(2.11) \quad \bar{\phi}_t(X, 0) = f_1(X).$$

The relation of the abstract initial value problem (2.9-2.11) to the boundary-value problem (2.1-4) is provided by the following result.

**Theorem (2.1).** [2]. If  $\bar{\phi}(X, Y, t)$  is a strong solution of (2.1-4), then its restriction  $\bar{\phi}$  to  $\Gamma$  is a solution of (2.9-2.11), and conversely, if  $V(t) = \bar{\phi}(X, t)$  is a solution of (2.9-11) with values in  $L^2(\Gamma)$  and having two strong derivatives in  $L^2(\Gamma)$ , then it is the restriction to  $\Gamma$  of a strong solution of (2.1-4).

It is an approximation of the solution to the abstract problem (2.9-11) which leads to a justification of the shallow water theory for two-dimensional linear flows, and to a new derivation of the shallow water equations.

### 3. An Equivalent Problem in a Region Independent of $\epsilon$ and $B$ .

For each fixed  $\epsilon > 0$ ,  $\Omega^*$ , which depends upon  $\epsilon$ , is a standard domain. It is useful to introduce new independent and dependent variables as follows. In place of the space variables  $X$  and  $Y$  we introduce

$$x = X, \quad y = Y/(-\epsilon B(X, \epsilon))$$

and for the dependent variable we introduce  $\Phi$  defined by

$$\bar{\Phi}(X, Y, t) = X + \epsilon\Phi(x, y, t).$$

Introduction of these variables in (2.1-4) then yields

$$(3.1) \quad L\Phi = \Phi_{yy} + \epsilon^2 \{ \nabla \cdot \vec{F}_{\Phi} - BB_{xx}\Phi \} = 0 \quad \text{in } \Omega, \quad t \geq 0;$$

$$(3.2) \quad -\Phi_y + \epsilon^2 \vec{F}_{\Phi} \cdot (-\hat{j}) = -\epsilon BB_x \quad \text{on } \Gamma_1 = \{(x, y): x \in \mathbb{R}, y = -1\}, \quad t \geq 0;$$

$$(3.3) \quad \Phi_y + \epsilon B\Phi_{tt} = 0 \quad \text{on } \Gamma = \{(x, y): x \in \mathbb{R}, y = 0\}, \quad t \geq 0;$$

$$(3.4) \quad \epsilon\Phi(x, 0, 0) = \bar{f}_0(x); \quad \epsilon\Phi_t(x, 0, 0) = f_1(x) \quad \text{on } \Gamma$$

where  $\Omega = \{(x, y): x \in \mathbb{R}, -1 < y < 0\}$ ;  $\nabla \cdot \vec{F}_{\Phi}$  is the divergence of  $\vec{F}_{\Phi}$ ;  $\bar{f}_0(x) = f_0(x) - x$ ; and  $\vec{F}_{\Phi} = (B_{xx}^2\Phi - BB_{xy}\Phi)\hat{i} + (B_{xy}^2\Phi - BB_{yx}\Phi)\hat{j}$ . We denote by  $F$  the transformation sending  $(X, Y, t; \Phi)$  into  $(x, y, t; \Phi)$ . This transformation is invertible and we denote its inverse by  $F^{-1}$ .

We associate with (3.1-4) the boundary-value problem

$$(3.5) \quad \Psi_{yy} + \epsilon^2 (\nabla \cdot \vec{F}_{\Psi} - BB_x\Psi) = 0 \quad \text{in } \Omega, \quad t \geq 0;$$

$$(3.6) \quad -\Psi_y - \epsilon^2 \vec{F}_{\Psi} \cdot \hat{j} = -\epsilon BB_x \quad \text{on } \Gamma_1, \quad t \geq 0;$$

$$(3.7) \quad \Psi_y + \alpha B \Psi = Bf \quad \text{on } \Gamma_0, \quad t \geq 0; \quad \alpha > 0;$$

$$(3.8) \quad \Psi \in H^1(\Omega).$$

Now a solution  $\Psi$  of (3.5-7) satisfies (2.1-4) if  $f$  is chosen so that the initial value problem

$$(3.9) \quad -\epsilon \Psi_{tt} + \alpha \Psi = f \quad \text{on } \Gamma, \quad t \geq 0;$$

$$(3.10) \quad \epsilon \Psi(x, 0, 0) = \bar{f}_0(x), \quad \epsilon \Psi_t(x, 0, 0) = f_1(x) \quad \text{on } \Gamma, \quad t \geq 0$$

is satisfied.

Essentially what has happened here is a separation of variables. Now (3.9-10) is an unusual initial-value problem in that  $\Psi$  and  $f$  are both unknown functions which are to be determined.

**Theorem 3.1.** For each fixed  $\epsilon > 0$  the problem (3.5-8) has a unique solution for each  $f \in C_0^\infty(\Gamma)$ .

*Proof.* Applying the transformation  $F^{-1}$  to the independent and dependent variables of (3.5-8) one obtains a boundary value problem for which existence and uniqueness results were established in [2].

We consider the associated boundary value problem obtained from (3.5-8) by replacing (3.6) with the condition

$$(3.12) \quad \Psi_y + \epsilon \frac{2-\hat{j}}{F_\Psi} \cdot \hat{j} = 0.$$

This problem has a unique solution for each  $f \in C_0^\infty(\Gamma)$ . We define an operator  $\bar{T}$  as follows

$$(3.13) \quad \bar{T}f(x, t) = \Psi(x, 0, t)$$

where  $f \in C_0^\infty(\Gamma)$  and  $\Psi$  is the unique solution of (3.5, 3.7-8, 3.12).

**Theorem 3.2.** The operator  $\bar{T}$  has an extension to a bounded linear operator  $T$  on  $L^2(\Gamma)$ . For each  $\delta > 0$  ( $\delta < \alpha$ ) there exists  $\epsilon_0 > 0$  such that

$$\|Tf\|_{L^2(\Gamma)} \leq \frac{1}{\alpha - \delta} \|f\|_{L^2(\Gamma)} \quad \text{whenever } \epsilon < \epsilon_0.$$

The operator  $T$  is positive, and has no non-trivial null space.

**Proof.** Let  $f \in C_0^\infty(\Gamma)$ . Multiply each side of equation (3.5) by  $(\Psi/B)$  and integrate the resulting expression over  $\Omega$  to obtain, after applying the divergence theorem,

$$(3.14) \quad \int_{\Omega} \left[ \frac{\Psi^2}{B} + \epsilon^2 \left\{ \nabla \left[ \frac{\Psi}{B} \right] \cdot \vec{F}_{\Psi} - B_{,X} \Psi_{,X} \right\} \right] dV =$$

$$\int_{\Gamma} [-\alpha \Psi^2 + f\Psi] dS,$$

or equivalently,

$$\begin{aligned}
(3.15) \quad & \int_{\Omega} \left[ \frac{\Psi^2}{B} + \epsilon^2 \left\{ \overline{\left[ \frac{\Psi}{B} \right]} \cdot \vec{F}_{\Psi} - B_{x_i} \Psi_{x_i} \right\} \right] dV + \delta \int_{\Gamma} \Psi^2 dS \\
& = \int_{\Gamma} [\alpha - \delta] \Psi^2 + f\Psi dS,
\end{aligned}$$

where  $\delta$  is an arbitrarily select positive constant less than  $\alpha$ . Given  $\delta$  there exists  $\epsilon_0$  such that the left side of (3.15) can be made non-negative for  $0 < \epsilon < \epsilon_0$ . Choose  $\epsilon < \epsilon_0$ . Then

$$\begin{aligned}
(3.16) \quad \langle f, \bar{T}f \rangle_{L^2(\Gamma)} & = \langle f, \Psi \rangle_{L^2(\Gamma)} = \int_{\Gamma} f\Psi dS \geq (\alpha - \delta) \int_{\Gamma} \Psi^2 dS \\
& = (\alpha - \delta) \langle \Psi, \Psi \rangle_{L^2(\Gamma)} = (\alpha - \delta) \|\bar{T}f\|_{L^2(\Gamma)}^2,
\end{aligned}$$

or

$$\|\bar{T}f\|_{L^2(\Gamma)} \leq \frac{1}{\alpha - \delta} \|f\|_{L^2(\Gamma)}.$$

Since  $C_0^\infty(\Gamma)$  is dense in  $L^2(\Gamma)$   $\bar{T}$  has an extension  $T$  to  $L^2(\Gamma)$  with the same bound. From (3.16) it is readily seen that  $T$  is a positive operator. To show that  $T$  has no non-trivial nullspace, we employ (3.14). Suppose  $Tf = 0 = \Psi(x, 0, t)$  on  $\Gamma$ . Then

$$\begin{aligned}
(3.17) \quad 0 & = \langle f, Tf \rangle_{\Gamma} = \langle f, \Psi \rangle_{\Gamma} = \alpha \langle \Psi, \Psi \rangle_{\Gamma} + \\
& \int_{\Omega} \left[ \frac{\Psi^2}{B} + \epsilon^2 \left\{ \overline{\left[ \frac{\Psi}{B} \right]} \cdot \vec{F}_{\Psi} - B_{x_i} \Psi_{x_i} \right\} \right] dV \\
& \geq \int_{\Omega} \left[ \frac{\Psi^2}{B} + \epsilon^2 \left\{ \right\} + \frac{1}{2b_1} \Psi^2 \right] dV
\end{aligned}$$

$$\geq \nu \int_{\Omega} \left\{ \Psi^2 + \Psi_x^2 + \Psi_y^2 \right\} dV \quad \text{for some } \nu > 0$$

when  $\epsilon$  is sufficiently small. Thus  $\Psi = \Psi_x = \Psi_y = 0$  in  $\Omega$ . That  $f = 0$  follows from the equality

$$\begin{aligned} 0 &= \int_{\Omega} \left[ \frac{\Psi_y \phi}{B} + \epsilon^2 \left\{ \bar{F}_{\Psi} \cdot \nabla \left[ \frac{\phi}{B} \right] - B_{xx} \Psi_x \phi \right\} \right] dV + \alpha \int_{\Gamma} \Psi \phi \, dS \\ &= \int_{\Gamma} f \phi \, dS \quad \text{for all } \phi \in H^1(\Omega). \end{aligned}$$

Thus  $T$  has an inverse which we denote by  $A$ .

**Lemma 3.1.**  $T$  is not self-adjoint.

To see this, it suffices to show that  $\bar{T}$  is not symmetric.

**Proof.** Let  $\phi$  and  $\psi$  both satisfy (3.5) and (3.12). On the boundary  $\Gamma$  let  $\phi$  and  $\psi$  satisfy, respectively,

$$\phi_y + \alpha B \phi = Bg$$

and

$$\psi_y + \alpha B \psi = Bh$$

where  $g, h \in C_0^{\infty}(\Gamma)$ . Thus

$$\begin{aligned} \langle Tg, h \rangle_{\Gamma} &= \int_{\Gamma} \phi h \, dS = \int_{\Gamma} \frac{\phi}{B} [\psi_y + \alpha B \psi] \, dS \\ &\neq \int_{\Gamma} \frac{\psi}{B} [\phi_y + \alpha B \phi] \, dS = \int_{\Gamma} \psi g \, dS = \langle g, Th \rangle_{\Gamma}. \end{aligned}$$

Thus  $\bar{T}$  is not symmetric, and has no self-adjoint extension on  $L^2(\Gamma)$ . The non-self adjointness of  $T$  makes necessary the consideration of an associated boundary value problem whose solution approximates the solution of the boundary value problem consisting of (3.5), (3.7-8) and (3.12).

#### 4. A Boundary Value Problem for $L_0$ .

We consider

$$(4.1) \quad L_0 \phi = \phi_{yy} + \epsilon^2 B(B\phi_x)_x = 0 \quad \text{in } \Omega,$$

$$(4.2) \quad \phi_y + \alpha B\phi = Bf \quad \text{on } \Gamma,$$

$$(4.3) \quad \phi_y = 0 \quad \text{on } \Gamma,$$

$$(4.4) \quad \phi \in H^2(\Omega).$$

That this problem can have at most one solution, and that it has a solution is given by the following two results.

**Lemma 4.1. (Uniqueness).** The boundary-value problem (4.1-4) has at most one solution.

**Lemma 4.2. (Existence).** If  $f \in C_0^\infty(\Gamma)$  then the boundary-value problem has a solution.

**Proof of Lemma 4.1.** Suppose  $\phi_1$  and  $\phi_2$  are two solutions of (4.1-4). Then  $\bar{\phi} = \phi_1 - \phi_2$  satisfies the homogeneous boundary-value problem

$$(4.5) \quad L_0 \bar{\phi} = 0 \quad \text{in } \Omega,$$

$$(4.6) \quad \bar{\phi}_y = 0 \quad \text{on } \Gamma,$$

$$(4.7) \quad \bar{\phi}_y = 0 \quad \text{on } \Gamma',$$

$$(4.8) \quad \bar{\phi} \in H^1(\Omega).$$

Take the inner-product of each side of the equation (4.5) with  $\bar{\phi}/B$  and apply the divergence theorem to obtain

$$\int_{\Omega} \left[ \frac{\bar{\phi}_y^2}{B} + \epsilon^2 B \bar{\phi}_x^2 \right] dV + \alpha \int_{\Gamma} \bar{\phi}^2 ds = 0.$$

This implies

$$\int_{\Omega} \left[ \frac{\bar{\phi}_y^2}{B} + \epsilon^2 B \bar{\phi}_x^2 \right] dV + \alpha \int_{\Omega} \left[ \frac{1}{2} \bar{\phi}^2 - \bar{\phi}_y^2 \right] dV \leq 0$$

or

$$(4.9) \quad \int_{\Omega} \left\{ \frac{\alpha}{2} |\bar{\phi}|^2 + \epsilon^2 B \bar{\phi}_x^2 + \left[ \frac{1}{B} - \alpha \right] |\bar{\phi}_y|^2 \right\} dV \leq 0$$

where we have used the inequality

$$\int_{\Gamma} \bar{\phi}^2 ds \leq \int_{\Omega} \left\{ \frac{1}{2} |\bar{\phi}|^2 - |\bar{\phi}_y|^2 \right\} dV.$$

The coefficients of the  $|\bar{\phi}|^2$ ,  $|\bar{\phi}_x|^2$  and  $|\bar{\phi}_y|^2$  can be made positive by choosing  $\epsilon > 0$  and  $0 < \alpha < \frac{1}{b_2}$ . It follows that for  $\epsilon > 0$  and  $0 < \alpha < \frac{1}{b_2}$  that  $\bar{\phi} = 0 \in H^1(\Omega)$ .

The proof of Lemma (4.2) is a modification of the proof given for the special case  $B(x, \epsilon) \equiv -\frac{1}{\epsilon}$  and will not be given here.

For  $f \in C_0^\infty(\Gamma)$  let  $\phi(x, y) = \phi(x, y; f)$  be the solution of (4.1-4). We define an operator  $\bar{T}$  on  $C_0^\infty(\Gamma)$  by the formula

$$(4.10) \quad \bar{T}f = \phi(x, 0; f) = \phi(x, 0), \quad f \in C_0^\infty(\Omega).$$

**Theorem 4.1.** The operator  $\bar{T}_0$  defined by (4.10) has an extension to a bounded linear operator on  $T_0$  on  $L^2(\Gamma)$ . The operator  $T_0$  is positive, self-adjoint, and has no non-trivial null space.

**Proof.** The proof that  $\bar{T}_0$  has an extension to a bounded operator  $T_0$  on  $L^2(\Gamma)$ , and that  $T_0$  is positive, and has no nontrivial nullspace is essentially the same, except for slight modifications, as the proof of Theorem 3.2. That  $T_0$  is self adjoint follows from the identity

$$\begin{aligned} \langle \bar{T}_0 f, g \rangle_\Gamma &= \int_\Gamma \psi g \, dS = \int_\Gamma \frac{\psi}{B} [\phi_y + \alpha B \phi] dS \\ &= \int_\Gamma \frac{\psi}{B} \phi_y \, dS + \alpha \int_\Gamma \psi \phi \, dS \\ &= \int_\Omega \left[ \frac{\psi_y \phi}{B} + \epsilon^2 B \psi_x \psi_y \right] dV + \alpha \int_\Gamma \psi \phi \, dS \\ &= \int_\Gamma \frac{\phi}{B} \psi_y \, dS + \alpha \int_\Gamma \psi \phi \, dS = \int_\Gamma f \phi \, dS = \langle f, \bar{T}_0 g \rangle \end{aligned}$$

which shows that  $\bar{T}_0$  is symmetric. Since  $\bar{T}_0$  is also semi-bounded it has an

extension to a semi-bounded self-adjoint operator  $T_0$  with the same bound as  $\bar{T}_0$ . [See 7, page 330.]

Thus  $T_0$  has a self-adjoint inverse which we denote by  $A_0$ . Let  $v(t)$  be a  $L^2(I)$ -valued function defined for all non-negative  $t$ . Furthermore, let  $v(t)$  have strongly continuous strong derivatives through order two, and let  $v_0 \in \mathfrak{D}(A)$  and  $v_1 \in \mathfrak{D}(A_0^{1/2})$ . Of importance in the derivation of the linear shallow water theory is the abstract Cauchy problem

$$(4.13) \quad \epsilon v''(t) + (A_0 - \alpha I)v(t) = 0, \quad t \geq 0$$

$$(4.14) \quad v(0) = v_0 \quad v'(0) = v_1$$

whose solution is given explicitly by the formula

$$(4.15) \quad v(t) = \int_{\alpha}^{\infty} \cos(t(\lambda - \alpha)\epsilon^{-1})^{1/2} dE_{\lambda} v_0 + \\ \int_{\alpha}^{\infty} [\sin(t(\lambda - \alpha)\epsilon^{-1})^{1/2} / ((\lambda - \alpha)\epsilon^{-1})^{1/2}] dE_{\lambda} v_0$$

where  $\{E_{\lambda}\}$  is the spectral family of  $A_0$ . We remark that the solution of (4.13-14) with initial data  $\epsilon v_0 = \bar{f}_0(x)$  and  $\epsilon v_1 = f_1(x)$  is an approximation of the solution of the abstract Cauchy problem (2.9-11) of section 2.

We obtain an explicit representation for the operator  $A_0 - \alpha I$ . It is natural to begin with the boundary-value problem (4.1-4) since the inverse  $T$  of the operator  $A$  is defined as the restriction to  $I$  of its solution. We obtain a representation for its solution by employing a generalized Fourier transform.

We define the Fourier B-transform of a function  $f$  by

$$(4.16) \quad \tilde{f}(\lambda) = \mathcal{F}_B[f](\lambda) = \int_{\mathbb{R}} f(x) e^{i\lambda H(x)} dx$$

where

$$H(x) = \int_0^x \frac{ds}{B(s)},$$

and define an operator  $P$  by

$$Pf = Bf_x.$$

The following properties of the B-transform are easily verified.

**Lemma 4.3.** If  $f \in C_0^\infty(\mathbb{R})$  and  $B \in C^{m-1}(\mathbb{R})$ , then

$$(4.17) \quad \mathcal{F}_B[P^n f](\lambda) = (-i\lambda)^n \mathcal{F}_B[f](\lambda) \quad \text{for } 0 \leq n \leq m.$$

**Lemma 4.** If  $f \in L_2(\mathbb{R})$  then there exist positive constants  $c_1$  and  $c_2$  such that

$$(4.18) \quad c_1 \|f\|_{L^2(\mathbb{R})} \leq \|\tilde{f}\|_{L^2(\mathbb{R})} \leq c_2 \|f\|_{L^2(\mathbb{R})}.$$

An explicit representation for  $A_0 - \alpha I$  is provided by the following result.

**Theorem 4.2.** Let the operator  $A_0$  be defined as above and let  $f \in L^2(\Gamma)$ .

Then

$$(4.19) \quad (A_0 - \alpha I)f(x) = \frac{1}{B} \mathcal{F}_B^{-1} \left\{ \epsilon |\lambda| \tanh(\epsilon |\lambda|) \tilde{f}(\lambda) \right\}$$

where  $\mathcal{F}_B^{-1}$  is the inverse of the B-transform.

**Proof.** Apply the B-transform to (4.1-4) for  $f \in C_0^\infty(\Gamma)$  to obtain the boundary-value problem

$$\tilde{\phi}_{yy} - \epsilon^2 |\lambda|^2 \tilde{\phi} = 0, \quad -1 < y < 0,$$

$$\tilde{\phi}_y = 0, \quad y = -1,$$

$$\tilde{\phi}_y + \alpha(\widetilde{B\phi}) = \widetilde{Bf}, \quad y = 0$$

whose solution is given implicitly by

$$\tilde{\phi}(\lambda, y) = \frac{\widetilde{(B(f - \alpha\phi))}(\lambda, y)}{\epsilon |\lambda| \tanh(\epsilon |\lambda|)} [\cosh(\epsilon |\lambda| y) + \tanh(\epsilon |\lambda|) \sinh(\epsilon |\lambda| y)].$$

Now for  $y = 0$

$$\tilde{\phi}(\lambda, 0) = \frac{\widetilde{(B(f - \alpha\phi))}(\lambda, 0)}{\epsilon |\lambda| \tanh(\epsilon |\lambda|)} = \widetilde{(T_0 f)}(\lambda)$$

from which follows

$$(4.20) \quad \widetilde{(B(f - \alpha T_0 f))}(\lambda) = \epsilon |\lambda| \tanh(\epsilon |\lambda|) \widetilde{(T_0 f)}(\lambda).$$

Applying the inverse B-transform to each side of the equation resulting from the substitution of  $A_0 f$  for  $f$  in (4.20) yields the conclusion of the theorem.

We rewrite the boundary value problem (3.5), (3.7-8), (3.12) in the form

$$(4.21) \quad L_0 \psi = \epsilon^2 L_1 \psi \quad \text{in } \Omega,$$

$$(4.22) \quad \psi_y = \epsilon^2 N \psi \quad \text{on } \Gamma_1$$

$$(4.23) \quad \psi_y = -\alpha B \psi + b F \quad \text{on } \Gamma$$

$$(4.24) \quad \psi \in H^2(\Omega),$$

where

$$L_1 \psi = \nabla \cdot \vec{F}_\psi - B B_x - B(B \psi_x)_x,$$

and

$$N \psi = -\vec{F}_\psi \cdot \hat{j}.$$

Apply the Fourier B-transform to this problem to obtain the system

$$(4.25) \quad \tilde{\psi}_{yy} - \epsilon^2 |\lambda|^2 \tilde{\psi} = \epsilon^2 L_1 \tilde{\psi}, \quad -1 < y < 0;$$

$$(4.26) \quad \tilde{\psi}_y = \epsilon^2 N \tilde{\psi}, \quad y = -1,$$

$$(4.27) \quad \tilde{\psi}_y = -\alpha B \tilde{\psi} + B f, \quad y = 0.$$

The solution of this problem is given implicitly by

$$\begin{aligned}\tilde{\Psi}(\lambda, y, t) &= (\tilde{A} + \tilde{B}) \cosh(\epsilon |\lambda| y) + \tilde{A} \tanh(\epsilon |\lambda|) \sinh(\epsilon |\lambda| y) \\ &\quad + \tilde{C}(y)\end{aligned}$$

where

$$\begin{aligned}\tilde{A} &= \frac{\widetilde{B(F-\alpha\Psi)}(\lambda, y, t) - \epsilon^2 \int_{-1}^0 \cosh(\epsilon |\lambda| s) [\widetilde{L_1\Psi}(\lambda, s, t)] ds}{\epsilon |\lambda| \tanh(\epsilon |\lambda|)}, \\ \tilde{B} &= \frac{\epsilon^2 \widetilde{N\Psi}(\lambda, -1, t)}{\sinh(\epsilon |\lambda|)},\end{aligned}$$

and

$$\tilde{C}(y) = \frac{\epsilon^2 \int_{-1}^y \sinh(\epsilon |\lambda| (y-s)) [\widetilde{L_1\Psi}(\lambda, s, t)] ds}{\epsilon |\lambda|}, \quad \text{and}$$

on  $\Gamma$  it takes on the value

$$\tilde{\Psi}(\lambda, 0, t) = \tilde{A} + \tilde{B} + \tilde{C}(0)$$

$$\begin{aligned}&\widetilde{B(f-\alpha\Psi)}(\lambda, 0, t) - \epsilon^2 \int_{-1}^0 \frac{\cosh(\epsilon |\lambda| (s+1))}{\cosh(\epsilon |\lambda|)} [\widetilde{L_1\Psi}(\lambda, s, t)] ds \\ &= \frac{\widetilde{B(f-\alpha\Psi)}(\lambda, 0, t) - \epsilon^2 \int_{-1}^0 \frac{\cosh(\epsilon |\lambda| (s+1))}{\cosh(\epsilon |\lambda|)} [\widetilde{L_1\Psi}(\lambda, s, t)] ds}{\epsilon |\lambda| \tanh(\epsilon |\lambda|)} \\ &\quad - \left[ \frac{\epsilon^2 \widetilde{N\Psi}(\lambda, -1, t)}{\sinh(\epsilon |\lambda|)} \right] = \widetilde{Tf}(\lambda, t).\end{aligned}$$

From the last equality we obtain the representation

$$(4.28) \quad \widetilde{B(A-\alpha I)\Psi}(\lambda, 0, t) = \epsilon |\lambda| \tanh(\epsilon |\lambda|) \tilde{\Psi}(\lambda, 0, t) + \tilde{G}_1(\lambda, t, \epsilon)$$

where

$$\begin{aligned} \tilde{G}_1(\lambda, t, \epsilon) &= \epsilon^2 \int_{-1}^0 \frac{\cosh(\epsilon |\lambda| (s+1))}{\cosh(\epsilon |\lambda|)} \widetilde{L}_1 \Psi(\lambda, s, t) ds \\ &+ \frac{\epsilon^3 |\lambda| \widetilde{N} \Psi(\lambda, -1, t)}{\cosh(\epsilon |\lambda|)}. \end{aligned}$$

An application of the inverse Fourier B-transform to (4.28) gives the representation

$$\begin{aligned} (4.29) \quad (A - \alpha I) \Psi(x, 0, t) &= \frac{1}{B} \mathcal{F}_B^{-1} \left\{ \epsilon |\lambda| \tanh(\epsilon |\lambda|) \widetilde{\Psi}(\lambda, 0, t) \right. \\ &\quad \left. + \tilde{G}_1(\lambda, t, \epsilon) \right\}. \end{aligned}$$

### 5. Derivation of "Shallow" Water Equations.

Let  $W(x,t)$  be the restriction of  $\Psi$  to  $\Gamma$ . Then the results of the last section enable us to rewrite equation (3.8-9) in the form

$$(5.1) \quad \epsilon W_{tt} + (A - \alpha I)W = 0, \quad t > 0$$

$$(5.2) \quad \epsilon W(x,0) = \bar{f}_0(x), \quad \epsilon W_t(x,0) = f_1$$

where  $f_0, f_1 \in L^2(\Gamma)$ , and  $W$  is a  $L^2(\Gamma)$ -valued function of  $t$  which has strongly continuous strong derivatives through the second order. Thus we meet an abstract Cauchy problem for  $W$ . If  $(A - \alpha I)$  were self-adjoint, which it is not, then spectral theory could be used to write an explicit representation for  $W$ . (See [7]).

Employing the representation given in (4.29) for  $(A - \alpha I)$  permits (5.1) to be written in the form

$$(5.3) \quad \epsilon W_{tt} + KW = \frac{1}{B} G_1(x,t;\Psi), \quad t > 0,$$

where the operator  $K$ , defined by

$$(5.4) \quad KW(x,t) = \frac{1}{B} \mathcal{F}_B^{-1} \left\{ \epsilon |\lambda| \tan(\epsilon |\lambda|) \tilde{W}(\lambda, t) \right\} = (A_0 - \alpha I)W,$$

is a self-adjoint operator. (See remarks following proof of Theorem 4.2.) An equivalent, and more revealing way of writing (5.3) is

$$(5.4) \quad \epsilon W_{tt} + \frac{1}{B} \varphi_B^{-1} \left\{ (\epsilon |\lambda|)^2 \tilde{W}(\lambda, t) \right\} = \frac{1}{B} \varphi_B^{-1} \left\{ \epsilon |\lambda| (\epsilon |\lambda| - \tanh(\epsilon |\lambda|)) \tilde{W}(\lambda, t) \right\} \\ + \frac{1}{B} G_1(x, t, \varphi).$$

Since the second term on the left-side of (5.4) is a representation of  $-\epsilon^2 (BW_x)_x$ , the equation

$$(5.5) \quad \epsilon W_{tt} - \epsilon^2 (BW_x)_x = G_2(x, t, \varphi)$$

is obtained. Now if the right side of (5.5) is small enough, solutions of (5.5) will be close to solutions of the equation

$$(5.6) \quad W_{tt}^0 - \epsilon (BW_x^0)_x = 0,$$

the usual equation of the linear shallow water theory for two-dimensional flows. This derivation of (5.6) appears new.

6. An Estimate for  $\| \eta - \eta^0 \|_{L^2(r)}$ .

To complete the theory developed here an estimate must be obtained for the magnitude of the error resulting from approximating  $W_t$  by  $W_t^0$ . The estimate may be obtained from a consideration of the two abstract initial value problems

$$(6.1) \quad \epsilon W_{tt} + KW = \frac{1}{B} G_1(x, t, \mathcal{P}),$$

$$(6.2) \quad \epsilon W(x, 0) = \bar{f}_0(x), \quad \epsilon W_t(x, 0) = f_1(x)$$

and

$$(6.3) \quad \epsilon W_{tt}^0 + K_0 W^0 = 0,$$

$$(6.4) \quad \epsilon W^0(x, 0) = \bar{f}_0(x), \quad \epsilon W_t^0(x, 0) = f_1(x),$$

where  $K_0$  is the operator defined by

$$K_0 \phi(x, t) = -\epsilon^2 (B \phi_x)_x.$$

We are now able to state our major result.

**Theorem 6.1.** (i). Let  $\bar{f}_0 \in \mathcal{D}(K_0)$  and  $f_1 \in \mathcal{D}(K_0^{1/2})$ . (ii) Let  $\mathcal{P}$  satisfy (4.21-24). (iii) Let  $\eta = -\epsilon/W_t$  and  $\eta^0 = -\epsilon W_t^0$  where  $W$  and  $W^0$  are the solutions of (6.1-2) and (6.3-4), respectively. Then for  $S > 0$  the following estimate is valid.

$$\|\eta(\cdot, t) - \eta^0(\cdot, t)\|_{L^2(\Gamma)} = o(\epsilon^{1/2}),$$

for all  $t \in [0, S]$ .

The proof requires several preparatory results which are now given.

**Lemma 6.1.** If condition (ii) is satisfied, then there exists a constant  $C_1$  such that

$$\|\frac{1}{B} G(x, t, \psi)\|_{L^2(\Gamma)} \leq C_1 \epsilon^2 \|\psi\|_{H^2(\Omega)}.$$

**Lemma 6.2.** Let  $\epsilon$  be a fixed positive number. then there exists a constant  $C_2 > 0$  such that

$$|\epsilon \lambda (\tanh(\epsilon |\lambda|) - \epsilon |\lambda|)| \leq C_2 \epsilon^2 |\lambda|^2 \quad \text{for all } \lambda \in \mathbb{R}.$$

**Lemma 6.3.** Let  $W$  be the solution of the abstract Cauchy problem

$$(6.5) \quad W''(t) + LW(t) = G(t), \quad t > 0$$

$$(6.6) \quad W(0) = W'(0) = 0,$$

where  $L$  is a positive self-adjoint operator defined in a Hilbert space  $X$  with norm  $\|\cdot\|$ , and  $G(t)$  is an  $X$ -valued function of  $t$  whose domain is the non-negative real axis. Then

$$\|W'(t)\| \leq 2 \int_0^S \|G(s)\| ds \quad \text{for } 0 \leq t \leq S.$$

Proof. Take inner product (6.5) with  $W'(t)$  to obtain

$$\langle W''(t), W'(t) \rangle + \langle AW, W'(t) \rangle = \langle G, W'(t) \rangle$$

from which follows

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \|W'(t)\|^2 + \|A^{1/2}W(t)\|^2 \right\} &\leq \|G\| \|W'(t)\| \\ &\leq 2\|G(t)\| \left\{ \|W'(t)\|^2 + \|A^{1/2}W\|^2 \right\}^{1/2}. \end{aligned}$$

The solution of this inequality which satisfies the initial condition in (6.6) is given by

$$\left\{ \|W'(t)\|^2 + \|A^{1/2}W\|^2 \right\}^{1/2} \leq 2 \int_0^t \|G(s)\| ds.$$

Thus,

$$\|W'(t)\| \leq 2 \int_0^S \|G(s)\| ds.$$

Lemma 6.4. If  $f \in \mathcal{D}(K_0)$  then  $f \in \mathcal{D}(K)$ .

Proof.  $f \in \mathcal{D}(K_0)$  implies

$$\begin{aligned}
\infty > \|K_0 f\|_{L^2(\Gamma)} &= \left\| \frac{1}{B} \mathcal{F}_B^{-1} \left\{ \epsilon |\lambda|^{2\tilde{f}(\lambda)} \right\} \right\|_{L^2(\Gamma)} \\
&\geq d_1 \left\| \epsilon^2 |\lambda|^{2\tilde{f}(\lambda)} \right\|_{L^2(\Gamma)} \\
&\geq d_2 \left\| \epsilon |\lambda| \tanh(\epsilon |\lambda|) \tilde{f}(\lambda) \right\| \\
&\geq \left\| \frac{1}{B} \mathcal{F}_B^{-1} \left\{ \epsilon |\lambda| \tanh(\epsilon |\lambda|) \tilde{f}(\lambda) \right\} \right\| \\
&= d_3 \|Kf\|_{L^2(\Gamma)}
\end{aligned}$$

where lemma 4.4 has been used repeatedly.

**Lemma 6.5.** Let  $W^0(x, t)$  satisfy the initial value problem (6.3-4) for  $f_0 \in \mathcal{D}(K_0)$  and  $f_1 \in \mathcal{D}(K_0^{1/2})$ . Then there exists a constant  $C$  such that

$$\left\| (K - K_0) W^0(x, t) \right\|_{L^2(\Gamma)} \leq C(\epsilon + \epsilon^{1/2}).$$

**Proof.** Let  $E_\mu$  be the spectral family of operators associated with  $\bar{K}_0$  where  $K_0 = \epsilon^2 \bar{K}_0$ . Then an explicit representation for the solution of  $W^0(x, t)$  of (6.3-4) is given by

$$W^0(x, t) = \int_0^\infty \left[ \cos(\sqrt{\mu} \epsilon t) dE_\mu f_0 + \frac{\sin(\sqrt{\mu} \epsilon t)}{\sqrt{\mu} \epsilon} dE_\mu f_1 \right].$$

Furthermore,

$$K_0 W^0(x, t) = \epsilon^2 \int_0^\infty \mu \left[ \cos(\sqrt{\mu \epsilon} t) dE_\mu f_0 + \frac{\sin(\sqrt{\mu \epsilon} t)}{\sqrt{\mu \epsilon}} dE_\mu f_1 \right]$$

It follows from the last equation that

$$\begin{aligned} \left\| K_0 W^0(x, t) \right\|_{L^2(\Gamma)} &\leq \epsilon^2 \left\{ \int_0^\infty \epsilon^{-1} \|\mu dE_\mu f_0\|_{L^2(\Gamma)} + \int_0^\infty \epsilon^{-1/2} \|\mu^{1/2} dE_\mu f_1\|_{L^2(\Gamma)} \right\} \\ &\leq C_1 (\epsilon + \epsilon^{1/2}) \end{aligned}$$

since  $f_0 \in \mathfrak{D}(K_0)$  and  $f_1 \in \mathfrak{D}(K_0^{1/2})$ .

The inequality in the lemma follows by observing that

$$\begin{aligned} \left\| (K - K_0) W^0(x, t) \right\|_{L^2(\Gamma)} &= \left\| \frac{1}{B} \mathfrak{F}_B^{-1} \left\{ \epsilon |\lambda| (\tanh(\epsilon |\lambda|) - \epsilon |\lambda|) \tilde{W}^0(\lambda, t) \right\} \right\|_{L^2(\Gamma)} \\ &\leq C_2 \left\| \epsilon |\lambda| (\tanh(\epsilon |\lambda|) - \epsilon |\lambda|) / w^0(\lambda, t) \right\|_{L^2(\Gamma)} \\ &\quad \text{(by lemma 4.4)} \\ &\leq C_3 \left\| \epsilon^2 |\lambda|^2 \tilde{W}^0(\lambda, t) \right\|_{L^2(\Gamma)} \\ &\quad \text{(by lemma 6.2)} \\ &\leq C_4 \left\| \frac{1}{B} \mathfrak{F}_B^{-1} \left\{ \epsilon^2 |\lambda|^2 \tilde{W}^0(\lambda, t) \right\} \right\|_{L^2(\Gamma)} \\ &\quad \text{(by lemma 4.4)} \\ &= C_4 \left\| K_0 W^0 \right\|_{L^2(\Gamma)} \end{aligned}$$

where  $C_1, C_2, C_3,$  and  $C_4$  are constants.

We move now to the proof of the theorem. Let

$$(6.7) \quad Z(x,t) = \epsilon W^0(x,t) - \epsilon W(x,t)$$

where  $W(x,t)$  and  $W^0(x,t)$  are solutions of the initial value problems (6.1-2) and (6.3-4), respectively. Then  $Z$  satisfies the initial value problem

$$(6.8) \quad Z_{tt} + \frac{1}{\epsilon} KZ = -\frac{1}{\epsilon} G_1(x,t,\psi) + (K-K_0)W^0(x,t)$$

$$(6.9) \quad Z(0) = Z^1(0) = 0.$$

We apply lemma 6.3 to obtain

$$\begin{aligned} \|Z_t(x,t)\|_{L^2(\Gamma)} &\leq 2 \int_0^S \left\{ \left\| \frac{1}{\epsilon} G_1(x,s,\psi) + (K-K_0)W^0(x,s) \right\|_{L^2(\Gamma)} \right\} ds \\ &\leq 2 \int_0^S \left\{ \left\| \frac{1}{\epsilon} G_1(x,s,\psi) \right\|_{L^2(\Gamma)} \right\} ds \\ &\quad + 2 \int_0^S \left\{ \left\| (K-K_0)W^0(x,s) \right\|_{L^2(\Gamma)} \right\} ds \\ &\leq D_1 \epsilon^2 \| \psi \|_{H^2(\Omega)}^2 + 2CS(\epsilon + \epsilon^{1/2}) \\ &= O(\epsilon^{1/2}) \end{aligned}$$

But  $\|Z_t\|_{L^2(\mathcal{I})} = \|\epsilon W_t - \epsilon W_t^0\|_{L^2(\mathcal{I})} = \|\eta(\cdot, t) - \eta^0(\cdot, r)\|_{L^2(\mathcal{I})}$  and the proof is complete.

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