A UNIFIED PRESENTATION OF SOME CLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

H.M. SRIVASTAVA, H.M. HOSSEIN & M.K. AOUF

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Abstract

The authors introduce and investigate various properties of a general class

$$\mathcal{U}_k\,[p,\alpha,\beta,A,B]$$
$$(p,k\in\mathbb{N}:=\{1,2,3,\cdots\}\,;\;0\leq\alpha<p;\;\beta\geq 0;$$
$$-1\leq A<B\leq 1;\;0<B\leq 1)\,,$$

which unifies and extends several (known or new) subclasses of meromorphically multivalent functions. The properties and characteristics of this general class, which are presented here, include growth and distortion theorems; they also involve Hadamard products (or convolution) of functions belonging to the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$.

1. Introduction, Definitions, and Preliminaries

Let $\sum_{p,k}$ denote the class of functions of the form:

$$f(z)=\frac{1}{z^p}+\sum_{n=k}^{\infty}a_{n+p-1}z^{n+p-1}\quad(p,k\in\mathbb{N}:=\{1,2,3,\cdots\}), \tag{1.1}$$

which are analytic and p-valent in the *punctured* unit disk

$$\mathcal{U}^*=\{z:z\in\mathbb{C}\text{ and }0<|z|<1\}\,.$$

Many interesting families of analytic and multivalent functions were considered by earlier authors in *Geometric Function Theory* (*cf., e.g.*, [4] [8], and [11]). For a function $f(z)$ in $\sum_{p,k}$, and for fixed parameters A and B, with

$$-1\leq A<B\leq 1,\quad A+B\geq 0,\text{ and }0<B\leq 1,$$

we say that $f(z)$ is a member of the class $\mathcal{Q}_k[p,\alpha,A,B]$ if and only if it satisfies the inequality:

$$\left|\frac{\dfrac{zf'(z)}{f(z)}+p}{B\dfrac{zf'(z)}{f(z)}+[pB+(A-B)(p-\alpha)]}\right|<1\quad(z\in\mathcal{U}^*;\;0\leq\alpha<p)\,. \tag{1.2}$$

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A function $f(z) \in \sum_{p,k}$ is said to belong to the class $\mathcal{R}_k[p,\alpha,A,B]$ if and only if

$$-\frac{zf'(z)}{p} \in \mathcal{Q}_k[p,\alpha,A,B]. \tag{1.3}$$

The classes $\mathcal{Q}_1[p,\alpha,A,B]$ and $\mathcal{Q}_1[p,0,A,B]$ were introduced by Aouf [1] and Mogra [6], respectively. Some subclasses of $\sum_{p,k}$ when $k=p=1$ were considered by (for example) Miller [5], Pommerenke [9], Clunie [3], and Royster [10]. Furthermore, several subclasses of $\sum_{p,k}$ when $k=1$ were studied by (amongst others) Mogra ([6], [7]), Aouf ([1], [2]), and Uralegaddi and Ganigi [12].

Motivated essentially by many of these earlier works, we aim at investigating here various properties and characteristics of the above-defined general class

$$\mathcal{U}_k[p,\alpha,\beta,A,B]$$
$$(p,k \in \mathbb{N};\ 0 \leq \alpha < p;\ \beta \geq 0;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1)$$

of meromorphically p-valent functions in

$$\mathcal{U} := \mathcal{U}^* \cup \{0\} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

The following result can be proven fairly easily by appealing to the definition of the class $\mathcal{Q}_k[p,\alpha,A,B]$.

Lemma 1. *Let a function $f(z)$ defined by (1.1) be in the class $\sum_{p,k}$. If*

$$\sum_{n=k}^{\infty} C(p,\alpha,A,B;n)\,|a_{n+p-1}| \leq D(p,\alpha,A,B) \tag{1.4}$$

$$(0 \leq \alpha < p;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1),$$

where, for convenience,

$$C(p,\alpha,A,B;n) = (1+B)(n-1) + [2p + 2\alpha B + (B+A)(p-\alpha)]\ (n \geq k) \tag{1.5}$$

and

$$D(p,\alpha,A,B) = (B-A)(p-\alpha), \tag{1.6}$$

then $f(z) \in \mathcal{Q}_k[p,\alpha,A,B]$.

Next, by observing that

$$f(z) \in \mathcal{R}_k[p,\alpha,A,B] \iff -\frac{zf'(z)}{p} \in \mathcal{Q}_k[p,\alpha,A,B], \tag{1.7}$$

we arrive at

Lemma 2. *Let a function $f(z)$ defined by (1.1) be in the class $\sum_{p,k}$. If*

$$\sum_{n=k}^{\infty} \left(\frac{n+p-1}{p}\right) C(p,\alpha,A,B;n)\,|a_{n+p-1}| \leq D(p,\alpha,A,B) \tag{1.8}$$

$$(0 \leq \alpha < p;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1),$$

where $C(p,\alpha,A,B;n)$ and $D(p,\alpha,A,B)$ are given by (1.5) and (1.6), respectively, then $f(z) \in \mathcal{R}_k[p,\alpha,A,B]$.

In view of Lemma 1 and Lemma 2, we define the subclasses $\mathcal{Q}_k^*[p,\alpha,A,B]$ of $\mathcal{Q}_k[p,\alpha,A,B]$ and $\mathcal{R}_k^*[p,\alpha,A,B]$ of $\mathcal{R}_k[p,\alpha,A,B]$ consisting of functions which, respectively, satisfy (1.5) and (1.8).

Furthermore, we introduce and investigate the various properties and characteristics of the following general class $\mathcal{U}_k[p,\alpha,\beta,A,B]$ of functions $f(z)\in\sum_{p,k}$ which also satisfy the inequality:

$$\sum_{n=k}^{\infty} C\left(p,\alpha,A,B;n\right)\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]|a_{n+p-1}|\leq D(p,\alpha,A,B) \tag{1.9}$$

$$\left(0\leq\alpha<p;\ \beta\geq 0;\ -1\leq A<B\leq 1;\ 0<B\leq 1\right),$$

where $C(p,\alpha,A,B;n)$ and $D(p,\alpha,A,B)$ are given by (1.5) and (1.6), respectively. Clearly, we have

$$\mathcal{U}_k[p,\alpha,\beta,A,B]=(1-\beta)\mathcal{Q}_k^*[p,\alpha,A,B]+\beta\mathcal{R}_k^*[p,\alpha,A,B], \tag{1.10}$$

so that

$$\mathcal{U}_k[p,\alpha,0,A,B]=\mathcal{Q}_k^*[p,\alpha,A,B] \tag{1.11}$$

and

$$\mathcal{U}_k[p,\alpha,1,A,B]=\mathcal{R}_k^*[p,\alpha,A,B]. \tag{1.12}$$

2. Growth and Distortion Theorems

Theorem 1. *If a function $f(z)$ defined by (1.1) is in the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$, then*

$$\frac{1}{|z|^p}-\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]}|z|^{k+p-1}\leq|f(z)|$$
$$\leq\frac{1}{|z|^p}+\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]}|z|^{k+p-1} \tag{2.1}$$

$$\left(\beta\geq 0;\ z\in\mathcal{U}^*\right)$$

and

$$\frac{p}{|z|^{p+1}}-\frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]}|z|^{k+p-2}\leq|f'(z)|$$
$$\leq\frac{p}{|z|^{p+1}}+\frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]}|z|^{k+p-2} \tag{2.2}$$

$$\left(\beta\geq 0;\ z\in\mathcal{U}^*\right).$$

The bounds in (2.1) and (2.2) are attained for the function $f(z)$ given by

$$f(z)=\frac{1}{z^p}+\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]}z^{k+p-1}. \tag{2.3}$$

Proof. Noting that

$$\sum_{n=k}^{\infty}|a_{n+p-1}|\leq\frac{D\left(p,\alpha,A,B\right)}{C\left(p,\alpha,A,B;k\right)\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]} \tag{2.4}$$

for $f(z) \in \mathcal{U}_k\left[p, \alpha, \beta, A, B\right]$, we have

$$
\begin{aligned}
|f(z)| &\geq \frac{1}{|z|^p} - |z|^{k+p-1} \sum_{n=k}^{\infty} |a_{n+p-1}| \\
&\geq \frac{1}{|z|^p} - \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)\left[1 - \beta + \beta\left(\frac{k+p-1}{p}\right)\right]} |z|^{k+p-1} \qquad (2.5)
\end{aligned}
$$

$$(\beta \geq 0;\ z \in \mathcal{U}^*)$$

and

$$
\begin{aligned}
|f(z)| &\leq \frac{1}{|z|^p} + |z|^{k+p-1} \sum_{n=k}^{\infty} |a_{n+p-1}| \\
&\leq \frac{1}{|z|^p} + \frac{D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)\left[1 - \beta + \beta\left(\frac{k+p-1}{p}\right)\right]} |z|^{k+p-1} \qquad (2.6)
\end{aligned}
$$

$$(\beta \geq 0;\ z \in \mathcal{U}^*).$$

We also observe that

$$
\begin{aligned}
&\frac{C(p, \alpha, A, B; k)\left[1 - \beta + \beta\left(\frac{k+p-1}{p}\right)\right]}{k+p-1} \sum_{n=k}^{\infty} (n+p-1)\,|a_{n+p-1}| \\
&\leq \sum_{n=k}^{\infty} C(p, \alpha, A, B; n)\left[1 - \beta + \beta\left(\frac{n+p-1}{p}\right)\right] |a_{n+p-1}| \leq D(p, \alpha, A, B)\ (\beta \geq 0), \qquad (2.7)
\end{aligned}
$$

which readily yields the following distortion inequalities:

$$
\begin{aligned}
|f'(z)| &\geq \frac{p}{|z|^{p+1}} - |z|^{k+p-2} \sum_{n=k}^{\infty} (n+p-1)\,|a_{n+p-1}| \\
&\geq \frac{p}{|z|^{p+1}} - \frac{(k+p-1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)\left[1 - \beta + \beta\left(\frac{k+p-1}{p}\right)\right]} |z|^{k+p-2} \qquad (2.8)
\end{aligned}
$$

$$(\beta \geq 0;\ z \in \mathcal{U}^*)$$

and

$$
\begin{aligned}
|f'(z)| &\leq \frac{p}{|z|^{p+1}} + |z|\, k+p-2 \sum_{n=k}^{\infty} (n+p-1)\,|a_{n+p-1}| \\
&\leq \frac{p}{|z|^{p+1}} + \frac{(k+p-1)D(p, \alpha, A, B)}{C(p, \alpha, A, B; k)\left[1 - \beta + \beta\left(\frac{k+p-1}{p}\right)\right]} |z|^{k+p-2} \qquad (2.9)
\end{aligned}
$$

$$(\beta \geq 0;\ z \in \mathcal{U}^*).$$

Now it is easy to see that the bounds in (2.1) and (2.2) are attained for the function $f(z)$ given by (2.3).

Taking $\beta = 0$ in Theorem 1, we have

Corollary 1. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{Q}_k^*[p,\alpha,A,B]$, *then*

$$\frac{1}{|z|^p}-\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-1}\leq|f(z)|$$
$$\leq\frac{1}{|z|^p}+\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-1}\ (z\in\mathcal{U}^*) \tag{2.10}$$

and

$$\frac{p}{|z|^{p+1}}-\frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2}\leq|f'(z)|$$
$$\leq\frac{p}{|z|^{p+1}}+\frac{(k+p-1)D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2}\quad(z\in\mathcal{U}^*). \tag{2.11}$$

The bounds in (2.10) *and* (2.11) *are attained for the function:*

$$f(z)=\frac{1}{z^p}+\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}z^{k+p-1}. \tag{2.12}$$

Letting $\beta=1$ in Theorem 1, we have

Corollary 2. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{R}_k^*[p,\alpha,A,B]$, *then*

$$\frac{1}{|z|^p}-\frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)}|z|^{k+p-1}\leq|f(z)|$$
$$\leq\frac{1}{|z|^p}+\frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)}|z|^{k+p-1}\ (z\in\mathcal{U}^*) \tag{2.13}$$

and

$$\frac{p}{|z|^{p+1}}-\frac{pD(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2}\leq|f'(z)|$$
$$\leq\frac{p}{|z|^{p+1}}+\frac{pD(p,\alpha,A,B)}{C(p,\alpha,A,B;k)}|z|^{k+p-2}\ (z\in\mathcal{U}^*). \tag{2.14}$$

The bounds in (2.13) *and* (2.14) *are attained for the function:*

$$f(z)=\frac{1}{z^p}+\frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)}z^{k+p-1}. \tag{2.15}$$

3. Convolution Properties

For functions

$$f_j(z)=\frac{1}{z^p}+\sum_{n=k}^{\infty}a_{n+p-1,j}\,z^{n+p-1}\quad(j=1,2) \tag{3.1}$$

belonging to the class $\sum_{p,k}$, we denote by $(f_1*f_2)(z)$ the convolution (or Hadamard product) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1*f_2)(z):=\frac{1}{z^p}+\sum_{n=k}^{\infty}a_{n+p-1,1}\,a_{n+p-1,2}\,z^{n+p-1}. \tag{3.2}$$

Theorem 2. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$. Then*

$$(f_1 * f_2)(z) \in \mathcal{U}_k[p,\gamma,\beta,A,B],$$

where

$$\gamma = p - \frac{(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]+\{D(p,\alpha,A,B)\}^2}. \tag{3.3}$$

The result is sharp for the functions:

$$f_j(z) = \frac{1}{z^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]} z^{k+p-1} \quad (j=1,2). \tag{3.4}$$

Proof. In order to prove Theorem 2, we must find the largest γ such that

$$\sum_{n=k}^{\infty} \frac{C(p,\gamma,A,B;n)\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]}{D(p,\gamma,A,B)} |a_{n+p-1,1}|\,|a_{n+p-1,2}| \leq 1 \tag{3.5}$$

for $f_j(z) \in \mathcal{U}_k[p,\gamma,\beta,A,B]\,(j=1,2)$. Since $f_j(z) \in \mathcal{U}_k[p,\alpha,\beta,A,B]\,(j=1,2)$, we readily see that

$$\sum_{n=k}^{\infty} \frac{C(p,\gamma,A,B;n)\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]}{D(p,\gamma,A,B)} |a_{n+p-1,j}| \leq 1 \ (j=1,2). \tag{3.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=k}^{\infty} \frac{C(p,\gamma,A,B;n)\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]}{D(p,\gamma,A,B)} \sqrt{|a_{n+p-1,1}|\,|a_{n+p-1,2}|} \leq 1. \tag{3.7}$$

This implies that we need only show that

$$\begin{aligned} &\frac{C(p,\gamma,A,B;n)}{p-\gamma} |a_{n+p-1,1}|\,|a_{n+p-1,2}| \\ &\leq \frac{C(p,\alpha,A,B;n)}{p-\alpha} \sqrt{|a_{n+p-1,1}|\,|a_{n+p-1,2}|} \quad (n \geq k) \end{aligned} \tag{3.8}$$

or, equivalently, that

$$\sqrt{|a_{n+p-1,1}|\,|a_{n+p-1,2}|} \leq \frac{(p-\gamma)C(p,\alpha,A,B;n)}{(p-\alpha)C(p,\gamma,A,B;n)} \quad (n \geq k). \tag{3.9}$$

Hence, by the inequality (3.7), it is sufficient to prove that

$$\frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;n)\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]} \leq \frac{(p-\gamma)C(p,\alpha,A,B;n)}{(p-\alpha)C(p,\gamma,A,B;n)} \quad (n \geq k). \tag{3.10}$$

It follows from (3.10) that

$$\gamma \leq p - \frac{(B-A)(1+B)(n+p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]+\{D(p,\alpha,A,B)\}^2} \quad (n \geq k). \tag{3.11}$$

Now, defining the function $\varphi(n)$ by

$$\varphi(n) := p - \frac{(B-A)(1+B)(n+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]+\{D(p,\alpha,A,B)\}^2} \quad (n \geq k), \tag{3.12}$$

we see that $\varphi(n)$ is an increasing function of n. Therefore, we conclude that

$$\gamma \leq \varphi(k) = p - \frac{(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]+\{D(p,\alpha,A,B)\}^2}, \tag{3.13}$$

which evidently completes the proof of Theorem 2.

Letting $\beta = 0$ in Theorem 2, we arrive at

Corollary 3. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{Q}_k^*[p,\gamma,A,B]$. Then*

$$(f_1 * f_2)(z) \in \mathcal{Q}_k^*[p,\gamma,A,B],$$

where

$$\gamma = p - \frac{(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2+\{D(p,\alpha,A,B)\}^2}. \tag{3.14}$$

The result is sharp for the functions:

$$f_j(z) = \frac{1}{z^p} + \frac{D(p,\alpha,A,B)}{C(p,\alpha,A,B;k)} z^{k+p-1} \quad (j=1,2). \tag{3.15}$$

Putting $\beta = 1$ in Theorem 2, we have

Corollary 4. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{R}_k^*[p,\alpha,A,B]$. Then*

$$(f_1 * f_2)(z) \in \mathcal{R}_k^*[p,\gamma,A,B],$$

where

$$\gamma = p - \frac{p\,(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{(k+p-1)\{C(p,\alpha,A,B;k)\}^2+p\,\{D(p,\alpha,A,B)\}^2}. \tag{3.16}$$

The result is sharp for the functions:

$$f_j(z) = \frac{1}{z^p} + \frac{pD(p,\alpha,A,B)}{(k+p-1)C(p,\alpha,A,B;k)} z^{k+p-1} \quad (j=1,2). \tag{3.17}$$

Finally, we prove

Theorem 3. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $\mathcal{U}_k[p,\alpha,\beta,A,B]$. Then the function $h(z)$ defined by*

$$h(z) := \frac{1}{z^p} + \sum_{n=k}^{\infty}\left(a_{n+p-1,1}^2 + a_{n+p-1,2}^2\right) z^{n+p-1} \tag{3.18}$$

belongs to the class $\mathcal{U}_k[p,\gamma,\beta,A,B]$, *where*

$$\gamma = p - \frac{2(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]+2\{D(p,\alpha,A,B)\}^2}. \tag{3.19}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ *given by* (3.4).

Proof. Noting that

$$\sum_{n=k}^{\infty}\frac{\{C(p,\alpha,A,B;n)\}^2\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]^2}{\{D(p,\alpha,A,B)\}^2}|a_{n+p-1,j}|^2$$
$$\leq\left(\sum_{n=k}^{\infty}\frac{C(p,\alpha,A,B;n)\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]}{D(p,\alpha,A,B)}|a_{n+p-1,j}|\right)^2\leq 1\ (j=1,2) \tag{3.20}$$

for $f_j(z)\in\mathcal{U}_k\,[p,\alpha,\beta,A,B]$ $(j=1,2)$, we have

$$\sum_{n=k}^{\infty}\frac{\{C(p,\alpha,A,B;n)\}^2\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]^2}{2\{D(p,\alpha,A,B)\}^2}\left|a_{n+p-1,1}^2+a_{n+p-1,2}^2\right|\leq 1. \tag{3.21}$$

Therefore, we have to find the largest γ such that

$$\frac{C(p,\gamma,A,B;n)}{p-\gamma}\leq\frac{\{C(p,\alpha,A,B;n)\}^2\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]}{2(B-A)(p-\alpha)^2}\quad(n\geq k), \tag{3.22}$$

that is, that

$$\gamma\leq p-\frac{2(B-A)(1+B)(n+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]+2\{D(p,\alpha,A,B)\}^2}\quad(n\geq k). \tag{3.23}$$

Now, defining a function $\psi(n)$ by

$$\psi(n):=p-\frac{2(B-A)(1+B)(n+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;n)\}^2\left[1-\beta+\beta\left(\frac{n+p-1}{p}\right)\right]+2\{D(p,\alpha,A,B)\}^2}\quad(n\geq k), \tag{3.24}$$

we observe that $\psi(n)$ is an increasing function of n. Thus we conclude that

$$\gamma\leq\psi(k)=p-\frac{2(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2\left[1-\beta+\beta\left(\frac{k+p-1}{p}\right)\right]+2\{D(p,\alpha,A,B)\}^2}, \tag{3.25}$$

which completes the proof of Theorem 3.

By setting $\beta=0$, Theorem 3 leads us to

Corollary 5. *Let the functions* $f_j(z)\,(j=1,2)$ *defined by* (3.1) *be in the class* $\mathcal{Q}_k^*[p,\alpha,A,B]$. *Then the function* $h(z)$ *defined by* (3.18) *belongs to the class* $\mathcal{Q}_k^*[p,\gamma,A.B]$, *where*

$$\gamma=p-\frac{2(B-A)(1+B)(k+2p-1)(p-\alpha)^2}{\{C(p,\alpha,A,B;k)\}^2+2\{D(p,\alpha,A,B)\}^2}. \tag{3.26}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ *given by* (3.15).

Letting $\beta = 1$ in Theorem 3, we have

Corollary 6. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (3.1) be in the class $\mathcal{R}_k^*[p, \alpha, A, B]$. Then the function $h(z)$ defined by (3.18) belongs to the class $\mathcal{R}_k^*[p, \gamma, A, B]$, where*

$$\gamma = p - \frac{2p\,(B-A)\,(1+B)\,(k+2p-1)(p-\alpha)^2}{(k+p-1)\,\{C(p,\alpha,A,B;k)\}^2 + 2p\,\{D(p,\alpha,A,B)\}^2}. \tag{3.27}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by (3.17).

Many of our results in this paper (especially Corollaries 1 to 6) would simplify considerably when we set

$$A = -1 \quad \text{and} \quad B = 1.$$

The details involved in the derivation of these and other special cases of our results may be left as an exercise for the interested reader.

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H.M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada

E-Mail: HMSRI@UVVM.UVIC.CA

H.M. Hossen and M.K. Aouf
Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura
Egypt

E-Mail: SINFAC@MUM.MANS.EUN.EG