fractal and fractional

IMPACT FACTOR 3.6

CITESCORE 4.6

Article

An Application of Multiple Erdélyi–Kober Fractional Integral Operators to Establish New Inequalities Involving a General Class of Functions

Asifa Tassaddiq, Rekha Srivastava, Rabab Alharbi, Ruhaila Md Kasmani and Sania Qureshi

Special Issue
Fractional Integral Inequalities and Applications, 2nd Edition

Edited by
Dr. Artion Kashuri and Prof. Dr. Hari Mohan Srivastava

MDPI

https://doi.org/10.3390/fractalfract8080438 fractal and fractional

Article

An Application of Multiple Erdélyi–Kober Fractional Integral Operators to Establish New Inequalities Involving a General Class of Functions

Asifa Tassaddiq [1], Rekha Srivastava [2,*], Rabab Alharbi [3], Ruhaila Md Kasmani [4] and Sania Qureshi [5,6]

1 Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al Majmaah 11952, Saudi Arabia; a.tassaddiq@mu.edu.sa
2 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada
3 Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia; ras.alharbi@qu.edu.sa
4 Institute of Mathematical Sciences, Universiti Malaya, Kuala Lumpur 50603, Malaysia; ruhaila@um.edu.my
5 Department of Computer Science and Mathematics, Lebanese American University, Beirut P.O. Box 13-5053, Lebanon; sania.qureshi@faculty.muet.edu.pk
6 Department of Basic Sciences and Related Studies, Mehran University of Engineering & Technology, Jamshoro 76062, Pakistan
* Correspondence: rekhas@math.uvic.ca

Abstract: This research aims to develop generalized fractional integral inequalities by utilizing multiple Erdélyi–Kober (E–K) fractional integral operators. Using a set of j, with $(j \in \mathbb{N})$ positively continuous and decaying functions in the finite interval $a \leq t \leq x$, the Fox-H function is involved in establishing new and novel fractional integral inequalities. Since the Fox-H function is the most general special function, the obtained inequalities are therefore sufficiently widespread and significant in comparison to the current literature to yield novel and unique results.

Keywords: Fox-H function; generalized inequalities; multiple Erdélyi–Kober (E–K) fractional integral transforms; continuous functions; mathematical operators

check for updates

Citation: Tassaddiq, A.; Srivastava, R.; Alharbi, R.; Kasmani, R.M.; Qureshi, S. An Application of Multiple Erdélyi–Kober Fractional Integral Operators to Establish New Inequalities Involving a General Class of Functions. *Fractal Fract.* **2024**, *8*, 438. https://doi.org/10.3390/fractalfract8080438

Academic Editors: Artion Kashuri and Hari Mohan Srivastava

Received: 10 June 2024
Revised: 15 July 2024
Accepted: 18 July 2024
Published: 25 July 2024

1. Introduction and Motivation

The use of fractional integral inequalities in fractional calculus is critical for developing new models and techniques in the most popular branches of computer science, like artificial intelligence, machine learning, and data science. To further grasp the field of big data, artificial intelligence, and machine learning, we require both fractional dynamics and fractional-order reasoning [1]. On the other hand, fractional calculus [2,3] also covers a wide range of significant problems involving special mathematical physics functions and provides several potentially useful techniques for solving integral and differential equations. A vital tool in elementary mathematical analyses is the integral inequality [4]. The literature refers to basic inequalities by a variety of names, including Hölder, Minkowski, and Cauchy–Schwarz. Furthermore, non-integer-order integral inequalities have become one of the most useful and comprehensive tools associated with applied [5] and pure [6] mathematics.

In their sequel to [7], Srivastava and Saigo [8] applied the fractional-calculus operator with the hypergeometric function kernel to their systematic investigation of some boundary-value problems which are based upon the Euler–Darboux partial differential equation. In [9], the authors established the generalized inequalities for a set of $j \in N$ positive and decreasing functions by making use of the Saigo fractional integral operator. A few articles can be found on such applications of Marichev–Saigo–Maeda (M–S–M) operators [10–12]. The results of [9] were further explored and generalized in [13] using these operators. Furthermore, a number of researchers have investigated an array of generalized inequalities using non-integer operators [14–19]. Therefore, some modern types of fractional integrals (proportional and conformable) have been used by researchers to prove certain inequalities. Certain Chebyshev-type integral inequalities involving Hadamard's fractional operators have been studied in [20]. For example, Ref. [21] looks into a few Grüss-type inequalities involving generalized fractional integral operators. In [22], the authors establish generalized Ostrowski-type inequalities for local fractional integrals, while [23] contains generalized Hermite–Hadamard-type inequalities involving non-integer integral operators.

Different generalizations of the Erdélyi–Kober (E–K) and Riemann–Liouville (R–L) operators have been examined recently, which contain the H-, G-, or Bessel function J_ν in their integrand [24,25]. However, Srivastava's work is noteworthy for creating a thorough and comprehensive theory of fractional calculus operators. For example, Ref. [26] (see also [27]) discusses numerous generalized classes of the above-mentioned operators, such as two or more variable series expansions of the H-function. A particular class of fractional calculus operators and their applications with respect to higher transcendental functions are discussed in [28]. Moreover, Ref. [29] provides a wealth of parametric modifications to the definition of such operators, as well as associated special functions.

Taking motivation from this review of the literature, we use multiple E–K fractional operators [24,25] to investigate the new and novel inequalities that are more general than the previous research cited in [13]. These generalized operators have only [30], as far as we know, been utilized for this purpose. Thus, this article is organized as follows: Section Preliminaries and Basic Definitions provides the necessary background information and basic preliminary information before delving deeper into our new findings. We prove our main results by making use of multiple E–K integral operators in Section 2. We present a detailed derivation of a new set of inequalities based on these multiple operators and a general class of functions in Section 2. A summary of the results is given in Section 3.

Preliminaries and Basic Definitions

This subsection contains a study, which includes all prerequisites, notions, and definitions, of multiple E–K integral operators. $\mathbb{C}, \mathbb{R}$, and $\mathbb{N}$ are the symbols of the set of complex, real, and natural numbers, respectively.

Definition 1. *Consider the parameters $A_i, B_j > 0; a_i, b_j \in \mathbb{C}$, then the Fox-H function can be explored using the following definition (see, [24–26]):*

$$H^{l,m}_{p,q}(\theta) = H^{l,m}_{p,q}\left[\theta \left| \begin{matrix} (a_i, A_i)_1^p \\ (b_j, B_j)_1^q \end{matrix} \right. \right] = \frac{1}{2\pi\iota}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j, +B_j s)\prod_{i=1}^{m}\Gamma(1-a_i-A_i s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j-B_j s)\prod_{i=m+1}^{p}\Gamma(a_i+A_i s)}\theta^{-s}ds$$
$$(i = 1, \cdots, p; j = 1, \cdots, q; 0 \le m \le p; 0 \le l \le q). \tag{1}$$

This expression's numerator includes a gamma function, the poles of which are divided using the proper contour L.

Remark 1. *Furthermore, by taking into account the value of parameters A_i and B_j equal to unity, Equation (1) would be named the G-function [24]. The operators defined in this formulation are particularly important as the G-function is additionally connected to several other special functions, such as the Mittag Leffler, Hypergeometric, and Fox–Wright functions.*

Definition 2 ([24–26])**.** *Considering u as a function with the domain $[a, x], a > 0$, then multiple E–K fractional integral operators, $\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}$, are defined as*

$$\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m} u(x) = x^{-1}\int_a^x H^{m,0}_{m,m}\left[\frac{\theta}{x} \left| \begin{matrix} (\varepsilon_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\varepsilon_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{matrix} \right. \right] u(\theta)d\theta. \tag{2}$$

The order of integration in the above equation is provided by $(\eta_k \geq 0)$*, whereas* $(\varepsilon_k \geq 0)$ *is multi-weight. On the other hand, there are some extra multi-parameters* $(\beta_k > 0)$*. Additionally,* $\sum_{k=1}^{m} \varepsilon_k > 0$ *and all* $\delta_k = 0$ *are obtained from this expression, yielding* $\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(0,\ldots,0)} u(x) := u(x)$*.*

In certain cases, it is possible to decompose generalized fractional integrals into commutative products of standard operators, namely Erdélyi–Kober operators. Thus, the capabilities of special functions can be combined with the extensive use of conventional fractional calculus to create a more general calculus which has a comprehensive theoretical background and concrete applications [24,25]. Similarly, a certain number of Erdélyi–Kober (E–K) and Riemann–Liouville (R–L) operator compositions are also considered [31,32]. The kernels for these compositions are $H_{m,m}^{m,0}$ or $H_{n+n,m+n}^{m,m}$. This suggested theory has been used [24] in several areas.

Dimovski [33] discussed the spaces of good functions $C_\nu([0,\infty])$ over the set of real numbers $x > 0$. This is the space in which we prove our results.

Definition 3. *Let* $x > 0, \nu \in \mathbb{R}$ *be real numbers. If* $u(x) = x^p \tilde{u}(x), p > \nu$ *for the continuous function* $\tilde{u} \in C[0,\infty)$*, then* $u \in C_\nu$*. Comparable mapping is valid for* $u \in C_\nu^n, n \in \mathbb{N}$ *with* $\tilde{u} \in C^n([0,\infty))$*, where* $C^n([0,\infty))$ *is a space of n-times continuously differentiable functions. For* C_ν*, u is an element of* $C[0,\infty]$*, while, for* C_ν^n*, u is an element of* $C^n([0,\infty))$*. More specifically,* $C_{\mu_1}^{n_1} \subseteq C_{\mu_2}^{n_2}$*, for* $\mu_1 \geq \mu_2, n_1 \geq n_2$*.*

Additional characteristics of these spaces are shown in [24]. The power function is likewise preserved by the elements of Lebesgue integrable spaces: $(L_\nu^p(0,\infty), 1 \leq p < \infty)$ satisfies $\|u\|_{\nu,p} = \int_0^\infty [x^{\nu-1}|u(x)|^p dx]^{1/p} < \infty$. Moreover, the following presumptions govern our outcomes:

$$\begin{aligned} \eta_k \geq 0; k = 1,\ldots,m; \\ \beta_k(\varepsilon_k + 1) > -\nu, (f \in C_\nu([0,\infty)); \\ \beta_k(\varepsilon_k + 1) > \nu/p; (f \in L_\nu^p([0,\infty)). \end{aligned} \tag{3}$$

Remark 2. *The kernel of the multiple E–K fractional integral in the above Definition 2 is positive* [34–36]*.*

Moreover, these operators also preserve the class of weighted analytic functions [33]. These operators satisfy the semigroup property [33] and are bilinear, commutative, and invertible. Additionally, it is demonstrated that they behave as bounded linear operators over L_ν^p with the conditions in (3). Riemann–Liouville- and Caputo-type multiple E–K operators are also explored in [33]. If $\beta_k = \beta$, then these operators are connected to a variety of commonly used non-integer operators, as listed below [24,25], which is another intriguing characteristic of these operators:

1. A multiple E–K operator reduces to a Marichev–Saigo–Maeda (M-S-M) operator when $m = 3$ and $\beta_m = 1$ [37,38].
2. A multiple E–K operator turns into a Saigo fractional operator when $m = 2$ and $\beta_m = \beta; \sigma = t/x; \sigma = x/t$ [7,39].
3. When $m = 1$ and $\beta_m = \beta$, a Saigo fractional operator becomes an Erdélyi–Kober (E–K) operator [24,25].
4. A multiple E–K operator turns into a Riemann–Liouville (R-L) operator when $m = 1$ and $\beta_m = 1$, [24,25].

Furthermore, these operators can also be used to create a number of well-known fractional integrals, such as the Hadamard, Katugampola, Bessel, and hyper-Bessel operators and Weyl integrals (see, Ref. [40] and its references). In [26], other expansions of operator (2) are also covered. This function has been applied in relation to special functions in a number of recent papers [41–44].

Unless otherwise indicated, the conditions on the parameters established in Section 1 shall be considered typical for the purposes of this study.

2. Main Results

Here, we prove the generalization of a few classical inequalities by making use of a multiple E–K operator. We prove the following result, which will be applied to determine our primary finding:

Theorem 1. *Consider a continuous, positive, and decreasing function g over $(a,x), a>0$, and $\vartheta>0; \theta\geq\lambda>0$. Using multiple E–K fractional operator (2), we obtain*

$$\frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right]}\geq\frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right]}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\lambda}(x)\right]},\tag{4}$$

provided $m>1, \eta_k\geq 0, \beta_k>0$, and $\varepsilon_k>-1-\frac{\vartheta}{\beta_k}$.

Proof. Taking into account the conditions of this theorem, we can consider

$$\left((\omega-a)^{\vartheta}-(t-a)^{\vartheta}\right)\left(g^{\theta-\lambda}(t)-g^{\theta-\lambda}(\omega)\right)\geq 0.\tag{5}$$

According to the definition of g, points t and ω are in $(a,x), a>0, \theta\geq\lambda>0; \vartheta>0$. Using (5), we obtain

$$(\omega-a)^{\vartheta}g^{\theta-\lambda}(t)+(t-a)^{\vartheta}g^{\theta-\lambda}(\omega)-(\omega-a)^{\vartheta}g^{\theta-\lambda}(\omega)-(t-a)^{\vartheta}g^{\theta-\lambda}(t)\geq 0.\tag{6}$$

For $x>0$, let us define the function as follows:

$$\mathfrak{H}(x,t)=\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{matrix}\right]\tag{7}$$

We may note that the function $\mathfrak{H}(x,t)>0; t\in(a,x), a>0$ remains positive according to Remark 2.

Then, by multiplying (6) with the following

$$\mathfrak{H}(x,t)g^{\lambda}(t)=\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{matrix}\right]g^{\lambda}(t),$$

we obtain

$$\begin{aligned}&\mathfrak{H}(x,t)\Big[(\omega-a)^{\vartheta}g^{\theta-\lambda}(t)+(t-a)^{\vartheta}g^{\theta-\lambda}(\omega)-(\omega-a)^{\vartheta}g^{\theta-\lambda}(\omega)-(t-a)^{\vartheta}g^{\theta-\lambda}(t)\Big]g^{\lambda}(t)\\
=&(\omega-a)^{\vartheta}\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{matrix}\right]g^{\lambda}(t)g^{\theta-\lambda}(t)\\
+&(t-a)^{\vartheta}\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{matrix}\right]g^{\lambda}(t)g^{\theta-\lambda}(\omega)\\
-&(\omega-a)^{\vartheta}\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{matrix}\right]g^{\lambda}(t)g^{\theta-\lambda}(\omega)\\
-&(t-a)^{\vartheta}\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{matrix}\right]g^{\lambda}(t)g^{\theta-\lambda}(t)\geq 0.\end{aligned}\tag{8}$$

Next, we integrate (8) as follows:

$$\begin{aligned}&(\omega-a)^{\vartheta}\int_a^x \frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\end{array}\right]g^{\theta}(t)dt\\&+g^{\theta-\lambda}(\omega)\int_a^x \frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\end{array}\right](t-a)^{\vartheta}g^{\lambda}(t)dt\\&-(\omega-a)^{\vartheta}g^{\theta-\lambda}(\omega)\int_a^x \frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\end{array}\right]g^{\lambda}(t)dt\\&-\int_a^x \frac{(t-a)^{\vartheta}g^{\theta}(t)}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\end{array}\right]dt\geq 0.\end{aligned}\tag{9}$$

Afterwards, making use of (7) and (9), we obtain

$$\begin{aligned}&(\omega-a)^{\vartheta}\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]+g^{\theta-\lambda}(\omega)\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\lambda}(x)\right]\\&-(\omega-a)^{\vartheta}g^{\theta-\lambda}(\omega)\left[\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}g^{\lambda}(x)\right]-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right].\end{aligned}\tag{10}$$

Next, we multiply (10) with the following:

$$\mathfrak{H}(x,\omega)g^{\lambda}(\omega)=\frac{1}{x}H^{m,0}_{m,m}\left[\frac{\omega}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\\(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1\end{array}\right]g^{\lambda}(\omega)$$

where $\mathfrak{H}(x,\omega)$ is as provided in (7), and then we integrate the result to get

$$\begin{aligned}&\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\lambda}(x)\right]\\&-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right]\geq 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\lambda}(x)\right]\\&\geq\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right].\end{aligned}$$

Next, we divide this equation by $\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\lambda}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right]$ to obtain the required result as stated in (4). □

Remark 3. *One can reverse the inequality established in Theorem 1 using the increasing behaviour of g over* $(a,x), a>0$.

Theorem 2. *Consider a decreasing, continuous, and positive function g over* $(a,x), a>0$. *Moreover, according to the definition of function g, as points t and* ω *are in [a,b],* $\theta\geq\lambda>0;\vartheta>0$, *then, using cmultiple E–K fractional operator (2), we obtain*

$$\frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[g^{\theta}(x)]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(x-a)^{\vartheta}g^{\lambda}(x)]+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[g^{\theta}(x)]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[(x-a)^{\vartheta}g^{\lambda}(x)]}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[(x-a)^{\vartheta}g^{\theta}(x)]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[g^{\lambda}(x)]+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(x-a)^{\vartheta}g^{\theta}(x)]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[g^{\lambda}(x)]}\geq 1,\tag{11}$$

where $\eta_k\geq 0;\varepsilon_k>-1-\frac{\nu}{\beta_k}$ *for* $m>1;\beta_k>0;\varepsilon_k\geq 0.$

Proof. Multiply each side of (10) with the following:

$$\mathfrak{H}(x,\omega)g^{\lambda}(\omega) = \frac{1}{x}H^{m,0}_{m,m}\left[\frac{\omega}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1 \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1 \end{array}\right]g^{\lambda}(\omega),$$

where $\mathfrak{H}(x,\omega)$ is as defined in (7), and then integrate the result with respect to ω over (a,x) to obtain the following:

$$\begin{aligned}
&\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\lambda}(x)\right] \\
&+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\lambda}(x)\right] \\
&-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right] \\
&-\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right] \geq 0.
\end{aligned} \tag{12}$$

Then, divide each side of (12) by

$$\begin{aligned}
&\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right] \\
&+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right],
\end{aligned}$$

to obtain the results required. □

Remark 4. *When using Theorem 2, by replacing $\gamma_k = \varepsilon_k; \delta_k = \eta_k$ we obtain Theorem 1.*

Theorem 3. *For a finite number x, consider a pair of continuous and positive functions g and h over $(a,x), a > 0$ such that the behaviour of g is decreasing and that of h is increasing. Moreover, according to the definition of function g, points t and ω should be in [a,b]; $\theta \geq \lambda > 0; \vartheta > 0$. Then, using cmultiple E–K fractional operator (2), we obtain*

$$\frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)g^{\lambda}(x)\right]}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right]} \geq 1, \tag{13}$$

provided $\eta_k \geq 0; \varepsilon_k > -1 - \frac{\nu}{\beta_k}$ for $m > 1; \beta_k > 0; \varepsilon_k \geq 0$.

Proof. Using the requirements of Theorem 3, we may state that

$$\left(h^{\vartheta}(\omega) - h^{\vartheta}(t)\right)\left(g^{\theta-\lambda}(t) - g^{\theta-\lambda}(\omega)\right) \geq 0 \tag{14}$$

Using (14), we obtain

$$h^{\vartheta}(\omega)g^{\theta-\lambda}(t) + h^{\vartheta}(t)g^{\theta-\lambda}(\omega) - h^{\vartheta}(\omega)g^{\theta-\lambda}(\omega) - h^{\vartheta}(t)g^{\theta-\lambda}(t) \geq 0. \tag{15}$$

We can multiply each side of (15) with

$$\mathfrak{H}(x,t)g^{\lambda}(t) = \frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1 \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})^m_1 \end{array}\right]g^{\lambda}(t); \ \ (a \leq t \leq x),$$

where $\mathfrak{H}(x,t)$ is as provided in (7), and thus we obtain

$$\begin{aligned}
&g^{\lambda}(t)\mathfrak{H}(x,t)\Big[g^{\theta-\lambda}(\omega)h^{\vartheta}(t)+h^{\vartheta}(\omega)g^{\theta-\lambda}(t)-g^{\theta-\lambda}(t)h^{\vartheta}(t)-g^{\theta-\lambda}(\omega)h^{\vartheta}(\omega)\Big]\\
&=\frac{h^{\vartheta}(\omega)}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]g^{\theta}(t)\\
&+\frac{h^{\vartheta}(\omega)}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]g^{\theta-\lambda}(\omega)g^{\theta}(t)\\
&-\frac{h^{\vartheta}(\omega)}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]g^{\theta-\lambda}(\omega)g^{\theta}(t)\\
&-\frac{h^{\vartheta}(\omega)}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]g^{\theta}(t)\geq 0.
\end{aligned}\tag{16}$$

Next, we integrate (16), as follows:

$$\begin{aligned}
&h^{\vartheta}(\omega)\int_a^x\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]g^{\theta}(t)dt\\
&+g_p^{\theta-\gamma_p}(\omega)\int_a^x\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]h^{\vartheta}(t)g^{\lambda}(t)dt\\
&-h^{\vartheta}(\omega)g^{\theta-\lambda}(\omega)\int_a^x\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]g^{\lambda}(t)dt\\
&-\int_a^x\frac{1}{x}H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]h^{\vartheta}(t)g^{\theta}(t)dt\geq 0.
\end{aligned}\tag{17}$$

By inserting (7) into Equation (17), the following is achieved:

$$\begin{aligned}
&h^{\vartheta}(\omega)\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\Big[g^{\theta}(x)\Big]+g^{\theta-\lambda}(\omega)\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\Big[h^{\vartheta}(x)g^{\lambda}(x)\Big]\\
&-h^{\vartheta}(\omega)g^{\theta-\lambda}(\omega)\Big[\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}g^{\lambda}(x)\Big]-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\Big[h^{\vartheta}(x)g^{\lambda}(x)\Big]\geq 0.
\end{aligned}\tag{18}$$

Next, we multiply (18) with

$$\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\mathfrak{H}(x,\omega)g^{\lambda}(\omega)=\frac{1}{x}H^{m,0}_{m,m}\left[\frac{\omega}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right],$$

and integrate it as follows:

$$\begin{aligned}
&\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\Big[h^{\vartheta}(x)g^{\lambda}(x)\Big]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\Big[g^{\theta}(x)\Big]\\
&-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\Big[h^{\vartheta}(x)g^{\theta}(x)\Big]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\Big[g^{\lambda}(x)\Big]\geq 0,
\end{aligned}$$

which completes the proof as required. □

Theorem 4. *Let us suppose there are two continuous and positive functions g and h over $[a,b]$ in such a manner that g is decreasing and h is increasing over $(a,x), a>0$. Moreover, according to the definition of function g, points t and ω are in [a,b]; $\theta\geq\lambda>0$; $\vartheta>0$. Then, using the multiple E–K fractional operator (2), we obtain*

$$\frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[g^{\theta}(x)]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h^{\vartheta}(x)g^{\lambda}(x)]+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[g^{\theta}(x)]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[h^{\vartheta}(x)g^{\lambda}(x)]}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[h^{\vartheta}(x)g^{\theta}(x)]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[g^{\lambda}(x)]+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h^{\vartheta}(x)g^{\theta}(x)]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[g^{\lambda}(x)]}\geq 1,\tag{19}$$

where $m > 1, \varepsilon_k \geq 0, \beta_k > 0$, *and* $\varepsilon_k > -1 - \frac{\nu}{\beta_k}$.

Proof. Multiply each side of (18) with

$$\mathfrak{H}(x,\omega)g^{\gamma}(\theta) = \frac{1}{x} H^{m,0}_{m,m}\left[\frac{\omega}{x}\middle| \begin{matrix} (\varepsilon_k + \eta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})^m_1 \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})^m_1 \end{matrix}\right]$$

(note that $\mathfrak{H}(x,\omega)$ is as given in (7)) and then integrate the result as follows:

$$\begin{aligned} &\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)g^{\lambda}(x)\right] \\ &+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)g^{\lambda}(x)\right] \\ &-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right] \\ &-\mathfrak{I}^{\alpha,\beta,\zeta,\zeta',\lambda}_{a,x}\left[h^{\vartheta}(x)g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[g^{\lambda}(x)\right] \geq 0. \end{aligned}$$

This implies that

$$\begin{aligned} &\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)g^{\lambda}(x)\right] \\ &+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)g^{\lambda}(x)\right] \\ &\geq\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right] \\ &+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[g^{\lambda}(x)\right]. \end{aligned}$$

Next, we divide each side with the following:

$$\begin{aligned} &\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)g^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[g^{\lambda}(x)\right] \\ &+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)g^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[g^{\lambda}(x)\right], \end{aligned}$$

which can be rewritten as the statement of Theorem 4 (34). □

Remark 5. *Using Theorem 4 for* $\gamma_k = \varepsilon_k; \delta_k = \eta_k$, *we can obtain Theorem 3.*

Next, we apply the M-E–K fractional operator to establish new inequalities involving a set of n positive and decreasing functions.

Theorem 5. *Consider a set of functions* $\{g_i\}^n_{i=1}$ *defined on* $(a,x), a > 0$ *which are positive, decreasing, and continuous. Moreover, according to the definition of each function* g_i, *the points* t *and* ω *are in [a,b];* $\theta \geq \lambda_p > 0; p \in N; \vartheta > 0$. *Then, using the M-E–K fractional operator (2), we obtain*

$$\frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod^n_{i\neq p} g_i^{\lambda_i} g_p^{\theta}(x)\right]}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod^n_{i=1} g_i^{\lambda_i}(x)\right]} \geq \frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod^n_{i\neq p} g_i^{\lambda_i} g_p^{\theta}(x)\right]}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod^n_{i=1} g_i^{\lambda_i}(x)\right]}, \tag{20}$$

provided $m > 1, \varepsilon_k \geq 0, \beta_k > 0$ *and* $\varepsilon_k > -1 - \frac{\nu}{\beta_k}$.

Proof. We can state the following result because of the given properties of $\{g_i\}^n_{i=1}$ over $[a,b]$:

$$\left((\omega-a)^{\vartheta} - (t-a)^{\vartheta}\right)\left(g_p^{\theta-\lambda_p}(t) - g_p^{\theta-\lambda_p}(\omega)\right) \geq 0, \tag{21}$$

We obtain the following using (21):

$$(\omega-a)^{\vartheta}g_p^{\theta-\lambda_p}(t)+(t-a)^{\vartheta}g_p^{\theta-\lambda_p}(\omega)-(\omega-a)^{\vartheta}g_p^{\theta-\lambda_p}(\omega)-(t-a)^{\vartheta}g_p^{\theta-\lambda_p}(t)\geq 0. \quad (22)$$

We multiply each side of (22) with

$$\mathfrak{H}(x,t)\prod_{i=1}^{n}g_i^{\lambda_i}(t)=\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(t),$$

where $\mathfrak{H}(x,t)$ is as stated in (7), and therefore we obtain

$$\begin{aligned}&\mathfrak{H}(x,t)\Big[(\omega-a)^{\vartheta}g^{\theta-\lambda}(t)+(t-a)^{\vartheta}g^{\theta-\lambda}(\omega)-(\omega-a)^{\vartheta}g^{\theta-\lambda}(\omega)-(t-a)^{\vartheta}g^{\theta-\lambda}(t)\Big]\prod_{i=1}^{n}g_i^{\lambda_i}(t)\\
=&(\omega-a)^{\vartheta}\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t)\\
+&(t-a)^{\vartheta}\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(\omega)\\
-&(\omega-a)^{\vartheta}\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(\omega)\\
-&(t-a)^{\vartheta}\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t)\geq 0.\end{aligned} \quad (23)$$

We integrate the result (23) as follows:

$$\begin{aligned}&(\omega-a)^{\vartheta}\int_a^x\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t)dt\\
&+g_p^{\theta-\lambda_p}(\omega)\int_a^x\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right](t-a)^{\vartheta}\prod_{i=1}^{n}g_i^{\lambda_i}(t)dt\\
&-(\omega-a)^{\vartheta}g_p^{\theta-\lambda_p}(\omega)\int_a^x\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(t)dt\\
&-\int_a^x\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right](t-a)^{\vartheta}\prod_{i=1}^{n}g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t)dt\geq 0.\end{aligned} \quad (24)$$

Next, by inserting (7) into (24), we obtain

$$\begin{aligned}&(\omega-a)^{\vartheta}\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n}g_i^{\lambda_i}g_p^{\theta}(x)\right]+g_p^{\theta-\lambda_p}(\omega)\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i=1}^{n}g_i^{\lambda_i}(x)\right]\\
&-(\omega-a)^{\vartheta}g_p^{\theta-\lambda_p}(\omega)\left[\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\prod_{i=1}^{n}g_i^{\lambda_i}(x)\right]-\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n}g_i^{\lambda_i}g_p^{\theta}(x)\right]\geq 0.\end{aligned} \quad (25)$$

We multiply each side of (25) with the following:

$$\mathfrak{H}(x,\omega)\prod_{i=1}^{n}g_i^{\lambda_i}(\omega)=\frac{1}{x}H_{m,m}^{m,0}\left[\frac{\omega}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n}g_i^{\lambda_i}(\omega)$$

(here, we use $\mathfrak{H}(x,\omega)$ as it is given in (7)) and then we integrate this result as follows:

$$\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]$$
$$-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\geq 0$$

which leads to the inequality (20) as required. □

Remark 6. *One can reverse the inequality established in Theorem 5 for the increasing behaviour of functions* $\{g_i\}_{i=1}^n$ *over* $[a,b]$.

Theorem 6. *Consider a set of functions* $\{g_i\}_{i=1}^n$ *defined on* $[a,b]$ *which are positive, decreasing, and continuous. Moreover, according to the definition of each function* g_i*, points t and* ω *are in* $(a,x), a>0;\ \theta\geq\lambda_p>0; p\in N; \vartheta>0$*. Then, using the multiple E–K fractional operator, we obtain*

$$\frac{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]+}{\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]+}$$
$$\frac{\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]}{\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]}\geq 1, \tag{26}$$

where $\varepsilon_k>-1-\frac{\nu}{\beta_k}$ *for* $m>1; \beta_k>0; \varepsilon_k\geq 0$.

Proof. Each side of (25) is multiplied with

$$\mathfrak{H}(x,\omega)\prod_{i=1}^{n} g_i^{\lambda_i}(\omega)=\frac{1}{x}H^{m,0}_{m,m}\left[\frac{\omega}{x}\middle|\begin{array}{c}(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m\end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(\omega),$$

(here, we use $\mathfrak{H}(x,\omega)$ as it is given in (7)) and then we integrate the result as follows:

$$\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]$$
$$+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]$$
$$-\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]$$
$$-\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\geq 0. \tag{27}$$

Next, we divide (27) by

$$\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{\alpha,\beta,\zeta,\zeta',\lambda}_{a,x}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]$$
$$+\mathfrak{I}^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[(x-a)^{\vartheta}\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right].$$

Hence, the required proof is completed. □

Remark 7. *When using Theorem 6, by replacing $\gamma_k = \varepsilon_k; \delta_k = \eta_k$ we obtain Theorem 5.*

Theorem 7. *Consider $\{g_i\}_{i=1}^n$ and let h be a set of positive continuous functions over $(a,x), a > 0$. Here and in what follows, the behaviour of h is that of an increasing function, while, on the other hand, $\{g_i\}_{i=1}^n$ is a set of decaying functions on the interval $(a,x), a > 0$. Moreover, according to the definition of each function g_i, points t and ω are in [a,b]; $\theta \geq \lambda_p > 0; p \in N; \vartheta > 0$. Then, using the multiple E–K fractional, we obtain*

$$\frac{\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]}{\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\mathfrak{I}_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]} \geq 1, \tag{28}$$

provided $\varepsilon_k > -1 - \frac{\nu}{\beta_k}$ for $m > 1; \beta_k > 0; \varepsilon_k \geq 0$.

Proof. As long as the requirements of Theorem 7 are met, we can write

$$\left(h^{\vartheta}(\omega) - h^{\vartheta}(t)\right)\left(g_p^{\theta-\lambda_p}(t) - g_p^{\theta-\lambda_p}(\omega)\right) \geq 0. \tag{29}$$

Using (29), we obtain

$$h^{\vartheta}(\omega)g_p^{\theta-\lambda_p}(t) + h^{\vartheta}(t)g_p^{\theta-\lambda_p}(\omega) - h^{\vartheta}(\omega)g_p^{\theta-\lambda_p}(\omega) - h^{\vartheta}(t)g_p^{\theta-\lambda_p}(t) \geq 0. \tag{30}$$

Next, we multiply each side of (30) with

$$\mathfrak{H}(x,t)\prod_{i=1}^{n} g_i^{\lambda_i}(t) = \frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(t)$$

(where $\mathfrak{H}(x,\omega)$ is as given in (7)), to obtain

$$\begin{aligned}
&h^{\vartheta}(\omega)\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t) \\
&+h^{\vartheta}(t)\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(\omega) \\
&-h^{\vartheta}(\omega)\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(\omega) \\
&-h^{\vartheta}(t)\frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t) \geq 0.
\end{aligned} \tag{31}$$

Next, we integrate (31) over the interval $t \in (a,x)$ as follows:

$$\begin{aligned}
&h^{\vartheta}(\omega)\int_a^x \frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t)dt \\
&+g_p^{\theta-\lambda_p}(\omega)\int_a^x \frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]h^{\vartheta}(t)\prod_{i=1}^{n} g_i^{\lambda_i}(t)dt \\
&-h^{\vartheta}(\omega)g_p^{\theta-\lambda_p}(\omega)\int_a^x \frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(t)dt \\
&-\int_a^x \frac{1}{x}H_{m,m}^{m,0}\left[\frac{t}{x}\middle| \begin{array}{c} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{array}\right]h^{\vartheta}(t)\prod_{i=1}^{n} g_i^{\lambda_i}(t)g_p^{\theta-\lambda_p}(t)dt \geq 0.
\end{aligned} \tag{32}$$

This will lead to the following, together with (7):

$$h^{\vartheta}(\omega)\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right] + g_p^{\theta-\lambda_p}(\omega)\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]$$
$$-h^{\vartheta}(\omega)g_p^{\theta-\lambda_p}(\omega)\left[\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right] - \Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right] \geq 0. \tag{33}$$

Once more, we multiply (33) with

$$\mathfrak{H}(x,\omega)\prod_{i=1}^{n} g_i^{\lambda_i}(\omega) = \frac{1}{x}H_{m,m}^{m,0}\left[\frac{\omega}{x}\middle| \begin{matrix} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{matrix}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(\omega),$$

(where $\mathfrak{H}(x,\omega)$ is as referred to (7)). Next, by integrating this result over (a,x) with regard to ω, we obtain

$$\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]$$
$$-\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right] \geq 0,$$

which concludes Theorem 7. □

Theorem 8. *Consider $\{g_i\}_{i=1}^{n}$ and let h be a set of positive continuous functions over $(a,x), a>0$. Here and in what follows, the behaviour of h is that of an increasing function while, on the other hand, $\{g_i\}_{i=1}^{n}$ is a set of decaying functions on the interval $[a,b]$. Moreover, according to the definition of each function g_i, points t and ω are in [a,b]; $\theta \geq \lambda_p > 0$; $p \in N$; $\vartheta > 0$. Then, using the multiple E–K fractional operator, we obtain*

$$\frac{\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]+}{\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]+}$$
$$\frac{\Im_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]}{\Im_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]} \geq 1, \tag{34}$$

where $m>1, \varepsilon_k \geq 0, \beta_k > 0$, and $\varepsilon_k > -1 - \frac{\nu}{\beta_k}$.

Proof. Next, we multiply (33) with the following:

$$\mathfrak{H}(x,\omega)\prod_{i=1}^{n} g_i^{\lambda_i}(\omega) = \frac{1}{x}H_{m,m}^{m,0}\left[\frac{\omega}{x}\middle| \begin{matrix} (\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \\ (\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k})_1^m \end{matrix}\right]\prod_{i=1}^{n} g_i^{\lambda_i}(\omega)$$

(where (7) defines $\mathfrak{H}(x,\omega)$) and, by integrating the resulting identity over (a,x) with respect to ω, we obtain

$$
\begin{aligned}
&\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\\
&+\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\\
&-\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\\
&-\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\geq 0.
\end{aligned}
$$

Thus, it may be shown that

$$
\begin{aligned}
&\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\\
&+\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\\
&\geq\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\\
&+\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right].
\end{aligned}
$$

Dividing each side with

$$
\begin{aligned}
&\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right]\\
&+\Im^{(\varepsilon_k),(\delta_k)}_{(\beta_k),m}\left[h^{\vartheta}(x)\prod_{i\neq p}^{n} g_i^{\lambda_i} g_p^{\theta}(x)\right]\Im^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}\left[g^{\theta}(x)\right]\left[\prod_{i=1}^{n} g_i^{\lambda_i}(x)\right].
\end{aligned}
$$

results in the intended inequality (34). □

Remark 8. *By using Theorem 8 for $\gamma_k = \varepsilon_k; \delta_k = \eta_k$, Theorem 7 can be obtained.*

Remark 9. *The findings in this research generalize the findings from a few earlier studies that have been referenced.*

3. Conclusions

In the present study, we developed some novel inequalities using a multiple E–K fractional integral operator. We considered a set of j positive, decreasing, and continuous functions on the interval $[a, x]$ that satisfy certain inequalities. This study presents an extraordinary class of inequalities using multiple E–K fractional integral operators. The inequalities generated in this study are more general than the existing classical ones that are listed in it. Consequently, the obtained inequalities are deducible to several nontrivial integral inequalities for Saigo [9,17], M–S–M [11–13], R–L operators, and so on, as Section 2 of this article discusses. The authors have successfully investigated a novel special function representation using multiple E–K operators [41–44]. We have only employed these multiple E–K operators for this topic previously, but they have been utilized for other purposes in the past. This method will pave the way for future research because these operators are the most general among a well-liked class of fractional operators. We hope that the intelligent aspects of these operators will motivate readers to investigate more results involving them.

Author Contributions: Conceptualization, A.T. and R.S.; methodology, A.T. and R.A.; software, S.Q.; validation, A.T., R.S., R.A., R.M.K. and S.Q.; formal analysis, A.T. and S.Q.; investigation, A.T. and R.A.; resources, S.Q. and R.S.; writing—original draft preparation, A.T., R.S. and R.A.; writing—review and editing, R.M.K. and S.Q.; visualization, S.Q.; supervision, R.S.; project administration, A.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: The research is theoretical in nature. As a result, no data were used.

Acknowledgments: The author extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (ICR-2024-1194).

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Niu, H.; Chen, Y.; West, B.J. Why Do Big Data and Machine Learning Entail the Fractional Dynamics? *Entropy* **2021**, *23*, 297. [CrossRef] [PubMed]
2. Baleanu, D; Mustafa, O.G. On the Global Existence of Solutions to a Class of Fractional Differential Equations. *Comput. Math. Appl.* **2010**, *59*, 1835–1841. [CrossRef]
3. Kilbas, A.A. Hadamard-type fractional calculus. *J. Korean Math. Soc.* **2001**, *38*, 1191–1204.
4. Tassaddiq, A.; Kanwal, S.; Lakhani, F.; Srivastava, R. Strong and Δ-Convergence Fixed-Point Theorems Using Noor Iterations. *Axioms* **2023**, *12*, 271. [CrossRef]
5. Tassaddiq, A.; Shabbir, M.S.; Din, Q.; Naaz, H. Discretization, Bifurcation, and Control for a Class of Predator-Prey Interactions. *Fractal Fract.* **2022**, *6*, 31. [CrossRef]
6. Tassaddiq, A. General escape criteria for the generation of fractals in extended Jungck–Noor orbit. *Math. Comput. Simul.* **2022**, *196*, 1–14. [CrossRef]
7. Saigo, M. A remark on integral operators involving the Gauss hypergeometric functions. *Math. Rep. Kyushu Univ.* **1978**, *11*, 135–143.
8. Srivastava, H.M.; Saigo, M. Multiplication of fractional calculus operators and boundary value problems involving the Euler-Darboux equation. *J. Math. Anal. Appl.* **1987**, *121*, 325–369. [CrossRef]
9. Houas, M. Some integral inequalities involving Saigo fractional integral operators. *J. Interdiscip. Math.* **2018**, *21*, 681–694. [CrossRef]
10. Srivastava, H.M.; AbuJarad, E.S.A.; Jarad, F.; Srivastava, G.; AbuJarad, M.H.A. The Marichev-Saigo-Maeda fractional-calculus operators involving the (p, q)-extended Bessel and Bessel-Wright functions. *Fractal Fract.* **2021**, *5*, 210. [CrossRef]
11. Singhal, M.; Mitta, E. On new fractional integral inequalities using Marichev–Saigo–Maeda operator. *Math. Methods Appl. Sci.* **2023**, *46*, 2055–2071. [CrossRef]
12. Tassaddiq, A.; Khan, A.; Rahman, G.; Nisar, K.S.; Abouzaid, M.S.; Khan, I. Fractional integral inequalities involving Marichev–Saigo–Maeda fractional integral operator. *J. Inequal Appl.* **2020**, *2020*, 185. [CrossRef]
13. Nisar, K.S.; Rahman, G.; Khan, A.; Tassaddiq, A.; Abouzaid, M.S. Certain generalized fractional integral inequalities. *AIMS Math.* **2020**, *5*, 1588–1602. [CrossRef]
14. Dahmani, Z.; Tabharit, L. On weighted Gruss type inequalities via fractional integration. *J. Adv. Res. Pure Math.* **2010**, *2*, 31–38. [CrossRef]
15. Dahmani, Z. New inequalities in fractional integrals. *Int. J. Nonlinear Sci.* **2010**, *9*, 493–497.
16. Dahmani, Z. On Minkowski and Hermite–Hadamard integral inequalities via fractional integral. *Ann. Funct. Anal.* **2010**, *1*, 51–58. [CrossRef]
17. Chinchane, V.L.; Pachpatte, D.B. New fractional inequalities involving Saigo fractional integral operator. *Math. Sci. Lett.* **2014**, *3*, 133–139. [CrossRef]
18. Purohit, S.D.; Raina, R.K. Chebyshev type inequalities for the Saigo fractional integral and their q-analogues. *J. Math. Inequal.* **2013**, *7*, 239–249. [CrossRef]
19. Yang, H.; Qaisar, S.; Munir, A.; Naeem, M. New inequalities via Caputo-Fabrizio integral operator with applications. *AIMS Math.* **2023**, *8*, 19391–19412. [CrossRef]
20. Ntouyas, S.K.; Purohit, S.D.; Tariboon, J. Certain Chebyshev type integral inequalities involving Hadamard's fractional operators. *Abstr. Appl. Anal.* **2014**, *2014*, 249091. [CrossRef]

21. Joshi, S.; Mittal, E.; Panddey, R.M. Some Grüss type inequalities involving generalized fractional integral operator. *Bull. Transilv. Univ. Braş.* **2019**, *12*, 41–52.
22. Sarikaya, M.Z.; Budak, H. Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Am. Math. Soc.* **2017**, *145*, 1527–1538. [CrossRef]
23. Set, E.; Noor, M.A.; Awan, M.U.; Gözpinar, A. Generalized Hermite-Hadamard type inequalities involving fractional integral operators. *J. Inequal. Appl.* **2017**, *169*, 10. [CrossRef] [PubMed]
24. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Pitman Research Notes in Mathematics Series; Longman Scientific & Technical; John Wiley & Sons Inc.: New York, NY, USA, 1994; Volume 301.
25. Kiryakova, V. Unified Approach to Fractional Calculus Images of Special Functions—A Survey. *Mathematics* **2020**, *8*, 2260. [CrossRef]
26. Srivastava, H.M.; Saxena, R.K. Operators of fractional integration and their applications. *Appl. Math. Comput.* **2001**, *118*, 1–52. [CrossRef]
27. Srivastava, H.M. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. *J. Adv. Eng. Comput.* **2021**, *5*, 135–166. [CrossRef]
28. Srivastava, H.M. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry* **2021**, *13*, 2294. [CrossRef]
29. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.
30. Tassaddiq, A.; Srivastava, R.; Alharbi, R.; Kasmani, R.M.; Qureshi, S. New Inequalities Using Multiple Erdélyi–Kober Fractional Integral Operators. *Fractal Fract.* **2024**, *8*, 180. [CrossRef]
31. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematical Studies; Elsevier Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2006; Volume 204.
32. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. Fractional Integrals and Derivatives. In *Theory and Applications*; Gordon Breach Science Publishers: London, UK; New York, NY, USA, 1993.
33. Dimovski, I. Operational calculus for a class of differential operators. *CR Acad. Bulg. Sci.* **1966**, *19*, 1111–1114.
34. Karp, D.B.; Prilepkina, E.G. Completely Monotonic Gamma Ratio and Infinitely Divisible H-Function of Fox. *Comput. Methods Funct. Theory* **2016**, *16*, 135–153. [CrossRef]
35. Mehrez, K. New integral representations for the Fox–Wright functions and its applications. *J. Math. Anal. Appl.* **2018**, *468*, 650–673. [CrossRef]
36. Mehrez, K. Positivity of certain classes of functions related to the Fox H-functions with applications. *Anal. Math. Phys.* **2021**, *11*, 114. [CrossRef]
37. Marichev, O.I. Volterra equation of Mellin convolution type with a horn function in the kernel. *Vescì Akad. Navuk BSSR Ser. Fiz. Mat. Navuk* **1974**, *1*, 128–129.
38. Raina, R.K. Solution of Abel-type integral equation involving the Appell hypergeometric function. *Integral Transforms Spec. Funct.* **2010**, *21*, 515–522. [CrossRef]
39. Saigo, M.; Maeda, N. More generalization of fractional calculus. In *Transformation Methods and Special Functions, Proceedings of the 2nd International Workshop, Varna, Bulgaria, 23–30 August 1996*; Rusev, P., Dimovski, I., Kiryakova, V., Eds.; IMI-BAS: Sofia, Bulgaria, 1998; Volume 19, pp. 386–400.
40. Sousa, J.V.d.C.; Oliveira, D.S.; de Oliveira, E.C. Grüss-Type Inequalities by Means of Generalized Fractional Integrals. *Bull. Braz. Math. Soc. New Ser.* **2019**, *50*, 1029–1047. [CrossRef]
41. Tassaddiq, A.; Srivastava, R.; Kasmani, R.M.; Alharbi, R. Complex Generalized Representation of Gamma Function Leading to the Distributional Solution of a Singular Fractional Integral Equation. *Axioms* **2023**, *12*, 1046. [CrossRef]
42. Tassaddiq, A.; Srivastava, R. New Results Involving Riemann Zeta Function Using Its Distributional Representation. *Fractal Fract.* **2022**, *6*, 254. [CrossRef]
43. Tassaddiq, A.; Srivastava, R. New Results Involving the Generalized Krätzel Function with Application to the Fractional Kinetic Equations. *Mathematics* **2023**, *11*, 1060. [CrossRef]
44. Tassaddiq, A.; Cattani, C. Fractional distributional representation of gamma function and the generalized kinetic equation. *Alex. Eng. J.* **2023**, *82*, 577–586. [CrossRef]