Vlasov-Fokker-Planck Type Kinetic Models for Multilane Traffic Flow and Large Time Behavior of Kinetic Density by Entropy Methods

by

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B.Sc., Nanjing University, 2004.

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ABSTRACT

We present a class of multi-lane traffic models of Vlasov-Fokker-Planck type incorporating non-local and time-delayed braking/acceleration, diffusion and lane changing terms whose dependencies are based on empirical guidelines. By investigating the spatially homogeneous case with non-zero passing probability incorporated in the braking term, we are left with the drift diffusion equation, which leads to a multi-valued fundamental diagram. As a novelty of this thesis, we find out that the monotonicity of the quotient between the braking/acceleration and the diffusion term in average speed guarantees the single-valued fundamental diagram. We study the large time behavior of the time-dependent kinetic density by convex entropy methods based on [3]. With a positive "residual" diffusion, convergence results remain with fewer assumptions. Two simplified examples are studied to illustrate the application of entropy methods.

Contents

List of Figures

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Chapter 1

Introduction

There is no need to elaborate on the importance of transportation in our lives. Problems arising in transportation systems, especially in highway traffic, are overwhelming in districts with high population density and high car ownership. Traffic flow theories seek to describe in a precise mathematical way the interactions between the vehicles (the mobile component) and the infrastructure (the immobile component) including the highway system and operational elements, for example, traffic lights and signs. These theories consist of different mathematical models based on experiments and observations.

In this first chapter, some variables often used in traffic theories are listed in Section 1.1. A review of the evolution of the models in recent times is presented in Section 1.2, followed by an overview of the structure of this thesis in Section 1.3.

1.1 Variables of interest

Different types of models of traffic dynamics focus on different aspects of the traffic scenario. Some focus on individual vehicle behavior and others focus on the macroscopic description of traffic conditions. In general, three elements are of interest: velocity (speed), flux (flow rate) and density (concentration).

First we set

- $x \in \mathcal{R}$—**position** in a road or highway;
- $t \geq 0$—**time**.

The **speed** of an instantaneous individual vehicle is $v = x'(t)$. Vehicle speed can also be measured over short sections of distance, in which case one no longer has the instantaneous speed of the vehicle, but a close approximation to it. We are interested in the **average speed** u of a set of vehicles, specially in multi-lane traffic in which passing is allowed. The dimension of the speed is meters/hour.

The **flux**, denoted by j, is estimated through point measurements which require measurements of time. It is by definition the number of vehicles passing a position of the road x per time unit. The dimension of the flux is cars/hour.

The **density**, denoted by ρ, is the number of vehicles counted in a distance unit. However, to find the relationship among flux, density and speed we have to obtain the density at a point instead of a length of distance. Therefore, we use the definition of density at position x as below, for small fixed $\varepsilon > 0$ (large relative to the size of cars but small relative to the macroscopic scale of the road),

$$\rho(x) = \frac{\text{number of cars in}(x-\varepsilon, x+\varepsilon)}{2\varepsilon}.$$

The dimension of the density is cars/meter

The relationship between the flux, density and average speed is by definition

$$j = \rho \cdot u.$$

In kinetic type models, the kinetic traffic density f is a distribution function of speed and the macroscopic variables arise as moments of f. Note that

f can only be interpreted from a statistical point of view. Further discussion about this will be presented in Chapter 2.

1.2 Background and literature review

Before proceeding with our discussion about the Vlasov-Fokker-Planck type of traffic flow models [1],[2],[3], we mention some background and review other types of mathematical models for traffic dynamics. Over the last few decades, three types of models were developed: microscopic, macroscopic and kinetic type models.

1.2.1 Microscopic "Follow-the-Leader" models

Focusing on the individual vehicles in a single lane with passing not allowed, the microscopic types of models are systems of coupled ordinary differential delay equations in which each vehicle is represented by its own equation based on "Follow-the-Leader" theory; see [4], [5], [6]. The position $x_n(t)$ and speed $v_n(t) = x'_n(t)$ of the n-th vehicle, are time-dependent variables. Based on Newton's second law, the basic assumption of the follow-the-leader theory of traffic flow along a single lane of highway is that each car's response (braking or acceleration force) at time $t+\tau$ equals a traffic-dependent sensitivity coefficient function multiplied by the stimulus at time t. The various mathematical formulations of the ideas can be stated in the following form:

$$x_n''(t+\tau) = \frac{F_n(t+\tau)}{m} = \lambda(\ldots) \cdot (x_n^{(p)}(t) - x_{n-1}^{(p)}(t)) \qquad n = 1, \ldots, N. \quad (1.1)$$

where m is the average mass of vehicles, τ is the average reaction time of drivers to a slower leading car, $\lambda = \lambda(|x_n(t) - x_{n-1}(t)|)$ is a sensitivity coefficient function of the relative distance, and $x_n^{(p)}(t)$ denotes the $p-$th derivative of $x_n(t)$ with respect to t. Basically, see [4], the dependencies of the braking force $F_n(t+\tau)$ should not on the relative acceleration $(x_n^{(p)}(t) - x_{n-1}^{(p)}(t))$ where $p \geq 2$, since drivers cannot accurately estimate it. Besides, it has been shown that the relative distance alone, i.e., $p = 0$, will result in nonrealistic instabilities. On the other hand, a reasonable dependence is on the relative speed $(x_n'(t) - x_{n-1}'(t))$. For example, choosing the sensitivity coefficient $\lambda = \frac{A}{|x_n(t) - x_{n-1}(t)|}$, we have

$$x_n''(t+\tau) = A \frac{x_n'(t) - x_{n-1}'(t)}{|x_n(t) - x_{n-1}(t)|}, \quad (1.2)$$

where $A > 0$ is an a priori unknown positive proportionality constant obtained by observation and practical data. Sometimes a relaxation term is added to a more general equation; see [7]. This gives

$$x_n''(t+\tau) = A\frac{x_n'(t) - x_{n-1}'(t)}{|x_n(t) - x_{n-1}(t)|^{\gamma+1}} + C\frac{1}{T_r}\left[V\left(\frac{L}{x_n(t) - x_{n-1}(t)}\right) - x_n'(t)\right]$$

where $C > 0$, $\gamma \geq 0$ and the relaxation time T_r are given parameters. Here L is the length of a car and V is an equilibrium velocity which depends on macroscopic properties of the flow ahead of the driver.

In the equilibrium state in which *all cars are at equal distance d from their leading car and moving at the same speed v*, the density $\rho = 1/(L + d)$. By solving the system, this type of models allows the calculation of a **fundamental diagram** for the speed v (or flux $j = \rho \cdot v$) as a function of the density ρ in the equilibrium state. A sample of this fundamental diagram for the model given by equations (1.2) is shown in Figure 1.1, see [5].

We notice that in a typical fundamental diagram vehicles travel at a free speed v_{max} until a critical density ρ_{crit} is reached at which point the traffic will slow and eventually come to a bumper-to-bumper situation, i.e., $\rho = \rho_{max} = 1/L$. The fundamental diagram provides a qualitatively reasonable dependence of speed on density and could be used for the construction of the macroscopic type models described below.

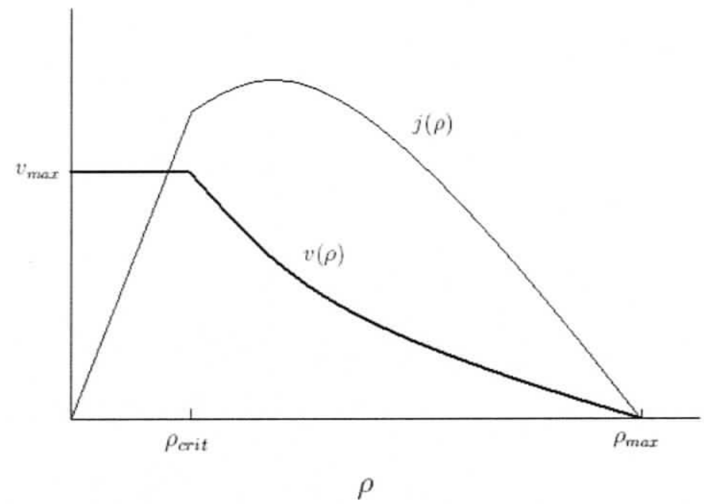

Figure 1.1: The fundamental diagram of speed as a function of density and the corresponding flux as a function of density

1.2.2 Macroscopic models: first and second order

The second type of models focus on the macroscopic traffic variables: density $\rho(x,t)$, speed $u(x,t)$ and flux $j(x,t) = \rho \cdot u$ as functions of position and time. A first order macroscopic hydrodynamic description of traffic flow was proposed by Lighthill and Whitham in 1955 [8] and Richards in 1956, therefore known as the LWR model. This is a first order partial differential equation

$$\rho_t + j_x = 0 \tag{1.3}$$

which expresses simply that there is no creation or destruction of cars on the road. A similar equation is valid in fluid dynamics and gas theory, except for the specific form of $j(\rho) = \rho \cdot u(\rho)$. As a consequence, equation (1.3) is written

$$\rho_t + j'(\rho)\rho_x = 0. \tag{1.4}$$

A natural link to microscopic models can be made by presenting $j(\rho)$ to be the fundamental diagram obtained in the equilibrated traffic in a microscopic model, e.g., Figure 1.1. Methods of characteristics are used to solve (1.4) with suitable initial values. Models of this type are actually in use for practical traffic analysis.

However, the LWR model is flawed for several reasons. First, it is based on and assumes a stationary speed-density relation. It assumes that the relationship obtained in the equilibrium state may be used under time and space dependent conditions. Second, it doesn't describe the detailed behavior of a vehicle passing through a shock caused by a discontinuity of $j'(\rho)$ or initial values. It predicts the speed changes instantaneously, see Figure 1.2. However, macroscopic models are based on temporal and spatial average values which should not lead to sharp shocks. Finally, it does not predict instabilities of the stop-and-go type. Unstable traffic flow is characterized

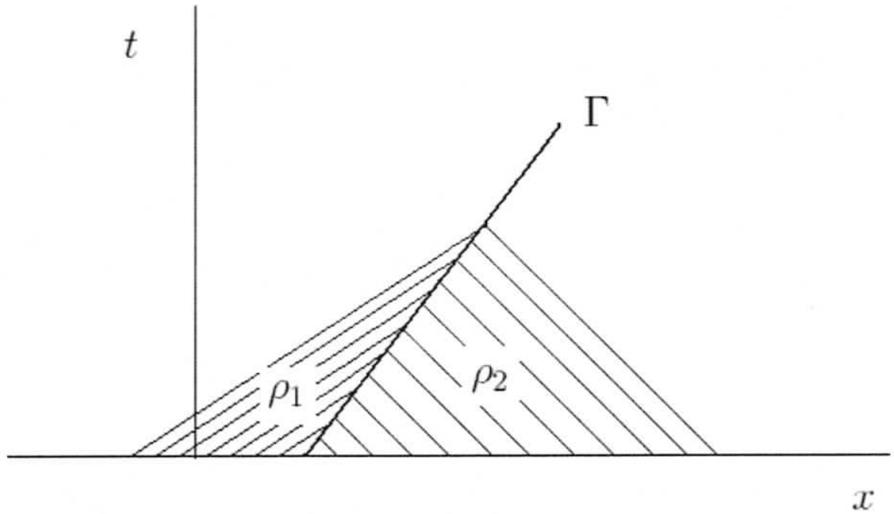

Figure 1.2: The line Γ which results from the intersection of converging characteristics ρ_1 and ρ_2 is illustrated and called *a shock wave.* Density at this intersecting point should have two values which is physically unrealizable. A shock represents a mathematical discontinuity (abrupt change) of density, see [5].

under appropriate conditions by regular stop-and-go waves with amplitude-dependent oscillation time. However, oscillatory solutions cannot be derived from the LWR model's equation. Also, the crucial instability effect is a bifurcation behavior, i.e., traffic flow becomes unstable beyond a certain density, which is not shown in the LWR model.

Two ways were developed to correct these deficiencies of the LWR model: higher-order refinements and kinetic models of traffic flow. The former way led naturally to the *second-order* macroscopic models. Well known models of this type are due to Payne [9] and Whitham [10] (the PW model) and Aw and Rascle [11].

The PW model is an attempt to investigate the structure of shocks and instability phenomena by introducing "higher-order" relations, analogous to the conservation of momentum in fluids. They consist of two equations one of which is

$$\rho_t + (\rho v)_x = 0 \tag{1.5}$$

to keep the mass conservation law. The other one, the acceleration equation, mimics the momentum equation as in one-dimensional fluid theory as

$$v_t + v v_x + \frac{p'(\rho)}{\rho}\rho_x = \nu v_{xx} \tag{1.6}$$

with a pressure law $p = p(\rho)$ inspired from gas dynamics. Here ν is a non-negative constant.

However, as pointed out by Daganzo [12], this type of argument analogous to the fluid analysis is not available because of the big difference between the dynamics of traffic and fluid flows: "*A fluid particle responds to stimuli from the front and behind, but a vehicle is an anisotropic particle that mostly responds to frontal stimuli*".

Based on Daganzo's critique about the PW model, Aw and Rascle [11] showed that the only problem about the PW model is the unrealistic dependence on the acceleration with respect to the space derivative of the "pressure". They removed the deficiencies of the PW model by simply replacing this space derivative with a *convective* derivative $\partial_t + v\partial_x$ of the "pressure" $p(\rho)$, which is taken as an *increasing* function of the density. Therefore, models of this type would be given by

$$\begin{aligned} &\rho_t + (\rho v)_x = 0 \\ &(v + p(\rho))_t + v(v + p(\rho))_x = 0 \end{aligned} \tag{1.7}$$

for the case $\nu = 0$. Aw and Rascle showed that all the inconsistencies of "second order" models resulting from the radical differences between the dynamics of traffic and fluid flows are removed and the model nicely predicts instabilities near the vacuum, i.e., for very light traffic.

1.2.3 Kinetic Models

As another way to correct the deficiencies of the LWR model, kinetic models are the third type of traffic models developed in recent decades. Such models describe the traffic **in statistical ways** by setting the kinetic density $f(x, v, t)$ as a distribution of the speed and consequently the macroscopic density $\rho(x,t) = \int_0^{v_{max}} f(x,v,t)dv$. Kinetic models provide advantages in describing multi-lane traffic scenarios. For example, it is more convenient and flexible to incorporate the hypothesis about drivers' behavior like braking, acceleration and lane-changing, based on the observation of relative speed and so on. Second, the macroscopic traffic equations derived from kinetic equations will not possess the inconsistencies as described above in the PW model. Besides, kinetic models are also easy to apply in numerical simulations.

Mathematical models of **multi-lane** traffic flow should allow some basic properties as suggested by Illner, Klar and Materne [1]:

1. The idealized scales of dependent variables, such as speed and acceleration, should be realistic and be at realistic ratios.

2. The model should (at least in some density domains) allow trivial equilibria $f^e(v) = \rho \cdot \delta_u(v)$, where $\delta_u(v)$ is a Dirac Delta function, i.e., for any pair of (ρ, v) the traffic could arrive at a steady state in which all cars travel at the same speed u, (even theoretically at a very high speed with the maximal density ρ_{max}). In certain (intermediate) density regimes it is effectively observed that traffic tends to "synchronize", [13],[14],[15],[16], while for higher density values these trivial equilibria give way to moving traffic jams and stop-and-go waves are produced, consistent with property 4 below.

3. The model should allow the calculation of the equilibrated fundamental

diagram $\rho \to u^e(\rho)$ or $\rho \to j^e(\rho)$. On multi-lane highways and within certain density regimes they should be multivalued, as suggested by traffic observations, [17], and these "bifurcations" should relate to the lane-changing behavior of drivers.

4. Finally, the model should give rise to phase transitions/stop-and-go traffic.

Basically, the microscopic follow-the-leader type of models could allow properties 1-3 at a rather simple level. For macroscopic models, specially for the first order LWR model, properties 2 and 3 are problematical since these models even need to borrow fundamental diagrams $\rho \to u^e(\rho)$ from microscopic models to illustrate the qualitative dependencies of $u(\rho)$ or $j(\rho)$. Even in Aw and Rascle's second order model of nonconservative type, which includes the relaxation term $\frac{1}{T_r(\rho)}(u^e(\rho) - v)$, the density-speed relation $\rho \to u^e(\rho)$ is needed. Besides, as mentioned in the deficiencies of the LWR model, property 4 can not be observed.

Let us present some kinetic models. The first kinetic model suggested by

Prigogine and Herman [4] is of the type

$$f_t+vf_x = (f_t)_{rel}+(f_t)_{int} = -\frac{f-f_0}{T_r}+(1-P)f_0(v)\int_0^{v_{max}} \omega(f_0(\omega)-f(x,\omega,t))d\omega. \tag{1.8}$$

where $(f_t)_{rel}$ and $(f_t)_{int}$ represent a relaxation term and an interaction term respectively; and $f_0(v)$ arises from the assumption that after a time of order T_r we will have $f(x,v,t) \approx f_0(v)$; P is a probability of passing and $u(x,t)$ is the average speed given by $\rho \cdot u = j = \int_0^{v_{max}} vf(x,v,t)dv$. As observed by Nelson and Sopasakis [17] this type of model leads to multivalued fundamental diagrams.

More recently, Klar and Wegener [18],[19],[20],[21] introduced the Enskog-type kinetic equations for the distribution function, obtained from considerations analogous to those in the kinetic theory of gases using a procedure similar to the formal derivation of Boltzmann's equation. The equation is given by

$$f_t + vf_x = C^+(f) = P_B q_B(G_B^+ - L_B^+)(f) + q_A(G_A^+ - L_A^+)(f) + \nu(G_S - L_S)(f) \tag{1.9}$$

where $C^+(f)$ denotes an interaction operator of Boltzmann-Enskog type. G_B^+ and L_B^+ denote the gain and loss terms due to braking and G_A^+ and L_A^+ those due to acceleration interactions. $P_B(\rho)$ is the probability for braking, q_B

and q_A measure the probability of finding an interaction partner for braking or accelerating and $\nu(\rho)$ is the relaxation frequency. This type of kinetic models may satisfy properties 2-4 well, but property 1 is violated because this type of equations imply that interactions and velocity adjustment of the following car are assumed instantaneous. This assumption is not acceptable in traffic flow because the braking or acceleration time is of the same order of magnitude as the "free driving time".

To address this difficulty, the Vlasov-Fokker-Planck type of models were suggested by Illner, Klar and Materne [1] to satisfy property 1. This thesis will mainly discuss and analyze a class of models of this type incorporating nonlocal and time-delayed braking or acceleration and diffusion terms with realistic time scales.

1.3 Overview of this thesis

In Chapter 2, a general introduction of Vlasov-Fokker-Planck type kinetic traffic models is presented for a simple 2-lane traffic scenario. It is shown that nonlocalities and time-delay are incorporated in the braking/acceleration, diffusion and lane-changing terms based on drivers' observations and behavior according to the leading vehicle. We also discuss the dependencies of these terms on traffic conditions, such as density, average speed, relative speed and distance. Then in Chapter 3, a spatially homogeneous case of the model in which the traffic patterns are identical in both lanes is investigated as a drift diffusion equation. We focus especially on the equilibrium state of the equation and compute the exact form of a potential equilibrium and consequently the fundamental diagram. It shows that in a certain density region the fundamental diagram is multi-valued because the non-zero lane changing probability is incorporated in the braking term. This result is further explored in Section 3.5, where we show that a single valued fundamental diagram results from the monotonicity of the quotient between the braking term B and the diffusion coefficient D with respect to the average speed u. We also introduce "residual" diffusion based on the assumption that the drivers will make errors in estimating the traffic conditions in front of them. It removes the degeneracy of the drift diffusion equation and the trivial equilibrium as well. In Chapter 4, we examine the long-time behavior of the time dependent density distribution of our VFP equation by convex entropy methods. It is shown that some assumptions on which the convergence result in [3] is based could be removed by introducing the "residual" diffusion. Two examples with assumptions on the average speeds are studied to investigate the convergence of the kinetic density by entropy methods.

Chapter 2

Vlasov-Fokker-Planck (VFP) type kinetic models

In this chapter we present a class of models of Vlasov-Fokker-Planck type for kinetic description of traffic dynamics. Real observations suggest nonlocality and time-delay in the braking/acceleration, diffusion and lane-changing terms. We assume that these terms are generally dependent on the macroscopic density and the relative speed.

2.1 Variables: kinetic density

Consider just a **2-lane** highway on which passing is allowed. We label the traffic lanes by 1 and 2, respectively. In a statistical interpretation we assume that there exists at a given time t and at a given position x a velocity distribution density function $f_i(x, v, t)$ for the i-th lane ($i = 1, 2$). Therefore, the number of cars dN that at time t are in the road interval between x and $x + dx$ and in the velocity interval between v and $v + dv$ is given by

$$dN = f(x, v, t)dxdv.$$

Once the speed distribution function is known, the macroscopic density $\rho_i(x, t)$, the flux $j_i(x, t)$ and the average speed $u_i(x, t)$ are moments of the corresponding kinetic terms with respect to the velocity v, i.e.,

$$\rho_i(x,t) = \int_0^{v_{max}} f_i(x,v,t)dv \qquad i = 1,2 \tag{2.1}$$

$$j_i(x,t) = \int_0^{v_{max}} v f_i(x,v,t)dv \qquad i = 1,2 \tag{2.2}$$

$$u_i(x,t) = \frac{j_i}{\rho_i}(x,t) = \frac{\int_0^{v_{max}} v f_i(x,v,t)dv}{\int_0^{v_{max}} f_i(x,v,t)dv} \qquad i = 1,2 \tag{2.3}$$

where v_{max} and ρ_{max} are the maximal possible velocity and density respectively, thought of as the speed limit and the limiting (bumper-to-bumper) density. Consequently, f_i can be factored as $f_i = \rho_i(x,t) \cdot F_i(x,v,t)$, where F_i is the probability density in v at (x,t).

2.2 General models

The general kinetic models of Vlasov-Fokker-Planck type for two lanes traffic flow are

$$\frac{\partial}{\partial t} f_i + v\frac{\partial}{\partial x} f_i + \frac{\partial}{\partial v}\left(B_i(...)f_i - D_i(...)\frac{\partial}{\partial v} f_i\right) = p_k(...)f_k - p_i(...)f_i \quad (2.4)$$

for $i = 1, 2$ and $k = 3 - i$. In these equations, B, D and p_i $(i = 1, 2)$ stand for the braking/acceleration force, the diffusion coefficient and the lane changing rate(s), respectively. The dependencies of these terms will be given later. At variance to the usual hypothesis that molecules move chaotically, correlation assumptions about the dependencies of these terms should reflect drivers' reactions to traffic scenarios ahead. We base our assumptions of B, D and p_i on empirical guidelines.

2.3 Nonlocality and time delay

It is reasonable to assume that drivers respond to traffic situations in front of them, basically, to the density and relative speed with respect to the average speed. Therefore, the fundamental assumption of the dependencies of braking/acceleration, diffusion and lane changing terms is that they mainly depend on the macroscopic density ρ_i, and the relative speed $v - u_i$ at a relevant threshold distance ahead of the driver.

Empirical evidence suggests that 3 different kinds of reaction time should be considered: The average individual reaction time $\tau > 0$, and "safety reaction times" T_B and T_A used for the estimation of braking and acceleration thresholds. This means that a driver at (x, t) with speed v will typically brake in reaction to slower traffic observed at $(x + H_B(v), t - \tau)$, where $H_B(v) = H_0 + T_B v$ and H_0 is the average car length plus a minimal safety distance, as shown in Figure 2.1. Similar consideration applies to acceleration, where the reference point for a driver at (x, t) with speed v is $(x + H_A(v), t - \tau)$, $H_A(v) = H_0 + T_A v$, and observations suggest that

$$0 < \tau < T_B < T_A < \infty.$$

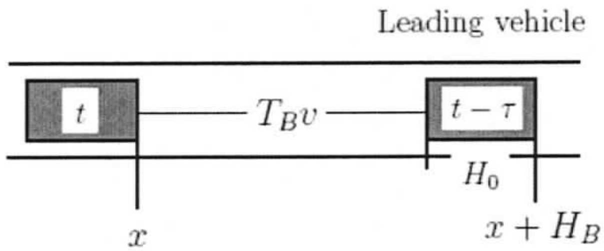

Figure 2.1: Drivers at (x,t) react to the leading slower traffic seen at $(x+H_B, t-\tau)$.

For convenience, for $i = 1, 2$, we abbreviate

$$u_i^B := u_i(x + H_B(v), t - \tau),$$

$$u_i^A := u_i(x + H_A(v), t - \tau),$$

$$\rho_i^B := \rho_i(x + H_B(v), t - \tau),$$

$$\rho_i^A := \rho_i(x + H_A(v), t - \tau).$$

From the observations above, all the terms describing drivers' behaviors actually refer to the leading traffic situations with nonlocality and time-delay. We write B, D and p_k for braking scenarios as $B(\rho_i^B, v - u_i^B)$, $D(\rho_i^B, v - u_i^B)$ and $p_k(\rho_i^B, v - u_i^B)$. Similarly, we have $B(\rho_i^A, v - u_i^A)$, $D(\rho_i^A, v - u_i^A)$ and $p_k(\rho_i^A, v - u_i^A)$ for acceleration scenarios.

Therefore, *nonlocalities* and *time-delay* are incorporated into the equations through the macroscopic density and the average speed which are coupled functions of the kinetic density as stated by (2.1) and (2.3).

2.4 Dependencies

In this section, we provide the assumptions of dependencies of the braking/acceleration, diffusion and lane-changing terms according to empirical observation. Our fundamental assumption regarding these terms is that they depend on the relative speed $v - u_i^B$ or $v - u_i^A$ and the macroscopic density ρ_i^B or ρ_i^A.

2.4.1 Passing probabilities

The lane-changing rate p_i $(i = 1, 2)$ is dependent on traffic states in the current and adjacent lanes. We take

$$p_i = P_i(...) \cdot j_i,$$

where $P_i(...) \in [0, 1]$ is a dimensionless lane changing probability, and p_i has the dimension of the flux j_i in lane i, i.e., car(s) per time unit. We shall assume that in acceleration scenarios, where $v < u_i^A$ and $v < u_i^B$, $P_i \equiv 0$. In braking scenarios where $v > u_i^B$, the driver will be motivated to change lanes, so P_i will affect the braking term B. Note that in acceleration scenarios, the two conditions $v < u_i^B$ and $v < u_i^A$ are not equivalent because $H_A > H_B$, and u_i^A and u_i^B refer to the average speed at different positions. Actually, a driver at location x and time t will consider braking or a lane change if the traffic ahead at $x + H_B(v)$ is slower than he was at time $t - \tau$. If the braking scenario does not apply and if the leading car is at $x + H_A(v)$ and moves at a speed greater than v, the driver will accelerate; otherwise, we assume that the driver will do nothing, i.e., it is possible that neither braking nor acceleration applies if $v < u_i^B$ and $v > u_i^A$.

2.4.2 Braking/acceleration force

First, we present the definition of braking/acceleration term as given in [1].

Definition 2.1

$$B_i(x, v, t) := \begin{cases} -c_B(v - u_i^B)^2 \rho_i^B (1 - P_i(...)) & \textit{if } v > u_i^B \\ c_A(v - u_i^A)^2(\rho_{max} - \rho_i^A) & \textit{if } v \leq u_i^B \textit{ and } v < u_i^A \\ 0 & \textit{otherwise.} \end{cases} \tag{2.5}$$

Remarks.

- First, it is in principle possible that neither braking nor acceleration occurs, i.e., $v < u_i^B$ and $v > u_i^A$. Hence the "otherwise" in (2.5). In this case only the diffusion term (due to the driver's inability of observing speeds with accuracy) will drive speed changes.

- Second, we notice that B_i vanishes of quadratic order in $(v-u)$ and proportional to density differences. These choices of B have the advantage of making the constants c_B and c_A dimensionless.

- Third, braking should be stronger in dense traffic, hence the simple linear dependence on ρ_i, and acceleration should be weaker in denser traffic, hence the linear dependence on $\rho_{max} - \rho_i^A$, where $\rho_{max} = 1/H_0$ is the maximal bumper-to-bumper traffic density.

- In addition, we should take $c_A < c_B$ since in most situations braking is more urgent and forced than acceleration.

- As u_i^B and u_i^A depend on v, the conditions $v > u_i^B$, $v < u_i^A$ etc. are implicit in v.

- B_i vanishes of quadratic order in $(v - u_i)$ if $u_i^B = u_i^A$, i.e., no braking and no acceleration. Later, as shown in Section 3.2, this will guarantee the existence of the trivial equilibrium which is property 2 listed in the Introduction.

- The dependency of the lane-changing probability P_i will be discussed in the next chapter, while we simply state here that $1 - P_i$ represents the braking probability.

Basically, the form (2.5) is just one possible simple guess about the dependency of the braking force. We should realize that it doesn't represent the real dependency which is probably much more complicated.

2.4.3 Diffusion

Diffusion arises from the *inability or uncertainty of a driver to estimate relative speeds with accuracy* unless speeds are 0. Therefore, a reasonable guess is setting the diffusion coefficient D proportional to a power of the relative speed $|v - u_i|$. Hence, from [1] the ansatz of D in our model is given by

Definition 2.2

$$D_i(x, v, t) := \begin{cases} \sigma(\rho_i^B, u_i^B) \cdot |v - u_i^B|^\gamma & \textit{if } v > u_i^B \\ \sigma(\rho_i^A, u_i^A) \cdot |v - u_i^A|^\gamma & \textit{if } v \leq u_i^B \textit{ and } v < u_i^A \\ 0 & \textit{otherwise}, \end{cases} \tag{2.6}$$

where $\gamma > 0$ is a parameter and the function $\sigma(\rho, u)$ satisfies

$$\sigma(\rho, u) = \sigma_c \rho_{max} u_{max}^2 m_1 \left(\frac{\rho}{\rho_{max}}\right) m_2 \left(\frac{u}{u_{max}}\right). \tag{2.7}$$

Remarks.

- As stated, D_i can be set as a proportion of a power of relative speed $v - u$. The form (2.6) implies no diffusion: $D_i = 0$ if $v = u_i^B = u_i^A$

as well. If $\gamma > 1$, this will also guarantee property 2 stated earlier. We will consider our model for general $\gamma \in (0, 2)$, although the trivial solution exists in the sense of distributions only if $\gamma > 1$; after all, the factors in the diffusion and lane changing terms are only guesses. Also we could adjust the dependence of $\sigma(\rho, u)$ such that the trivial equilibria will exist for all cases, e.g. , a factor σ worth studying would grow like $(v - u)^\gamma$ with $\gamma > 1$ for small $v - u$, but we would lose the relative simplicity of the present model.

- As for the function $\sigma(\rho, u)$, it seems reasonable to choose it to vanish rapidly enough as $\rho \to 0$ or $\rho \to \rho_{max}$, and as $u \to 0$ since in free driving ($\rho = 0$) or standing traffic ($\rho = \rho_{max} = 1/H_0$ and $u = 0$) there is no reason for diffusion.

- We choose $m_1(s)$ to vanish at 0 and 1 to enforce that no diffusion occurs at $\rho = 0$ and $\rho = \rho_{max}$. Besides, at some density, here we choose $0.3\rho_{max}$, the diffusion could be maximized as a function of ρ. This is the reason why we set a maximum for m_1 at $\rho = 0.3$. Therefore, a reasonable choice is that $m_1(s)$ is a function which consists of two linked pieces of Gaussian distributions such that $\max m_1 = m_1(0.3)$

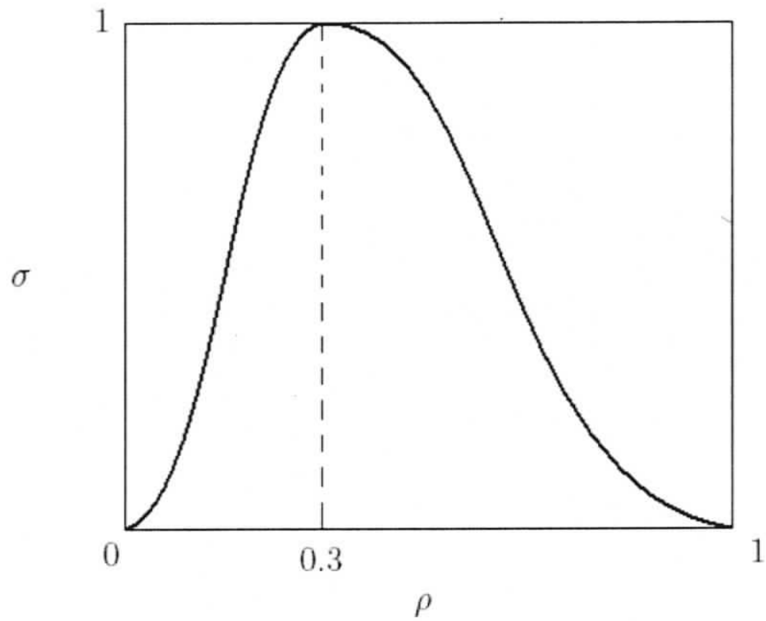

Figure 2.2: A sketch of the shape representing the dependency of the function σ on the density ρ.

and $m_1(0) = m_1(1) = 0$.

- For the function $m_2(s)$, a convenient choice is $m_2(s) = s(1-s)$. And basically, we hope to generate a shape as sketched in Figure 2.2 to represent the dependency of σ on ρ for fixed average speed u.

- σ_c is a free parameter. Furthermore, $\sigma << c_A < c_B$, as diffusion occurs on smaller scales than braking or acceleration.

- Also note that the value of $\sigma(\rho, u)$ at the boundaries of the domain $(\rho, u) \in [0, \rho_{max}] \times [0, v_{max}]$ will affect the shape of the fundamental diagrams near the endpoints $\rho = 0$ and $\rho = \rho_{max}$ significantly.

Similar to B, (2.6) is a possible simple guess of the diffusion as well based on realistic guidelines.

In the next chapter, we will discuss a spatially homogeneous case without time delay. In this case, with a simple ansatz of the lane-changing probability P, discussions will center around the analysis of equilibria of the model, the computation of the fundamental diagrams and their multi-valued properties.

Chapter 3

Spatially homogeneous case and fundamental diagrams

Our whole model is given by

$$\frac{\partial}{\partial t}f_i + v\frac{\partial}{\partial x}f_i + \frac{\partial}{\partial v}\left(B_i(...)f_i - D_i(...)\frac{\partial}{\partial v}f_i\right) = p_k(...)f_k - p_i(...)f_i \quad (3.1)$$

for $i = 1, 2$ and $k = 3 - i$, with the braking/acceleration term B and the diffusion coefficient D defined by (2.5) and (2.6). We will present the spatially homogeneous case of the model with a simple assumption of the lane-changing probability P in Section 3.1. We also set time delay $\tau = 0$ for the remainder of this thesis to obtain a simple drift diffusion equation.

Furthermore, in Section 3.2, we compute the equilibria of the spatially homogeneous case and compute the fundamental diagram which is multi-valued in a reasonable density interval because of non-zero lane changing probability P involved in the braking term. The reasons for this multi-valued property are examined in Section 3.3 and 3.5.

Positive "residual" diffusion can remove the degeneracy at $v = u$ and eliminate the trivial solution, as shown in Section 3.4.

3.1 The spatially homogeneous case: A drift diffusion equation

For simplicity of fundamental analysis, we consider the spatially homogeneous case for (3.1) in which $f_1(v,t) = f_2(v,t)$ for all (v,t), i.e., the traffic patterns are identical in both lanes and independent on x. In this case the spatial nonlocalities $H_A(v)$ and $H_B(v)$ disappear. We also set the time delay $\tau = 0$ for convenience although this is not really justified. Based on these assumptions, $v\frac{\partial}{\partial x}f$ disappears and the lane-changing rates $p_k(...)f_k - p_i(...)f_i(...)$ on the right hand side are canceled. Then (3.1) becomes a nonlinear **drift diffusion equation**

$$\frac{\partial}{\partial t}f + \frac{\partial}{\partial v}\left(B(\rho, v-u)f - D(\rho, u, v-u)\frac{\partial}{\partial v}f\right) = 0 \qquad (3.2)$$

where from (2.1) and (2.2)

$$\rho(t) = \int_0^{v_{max}} f(v,t)dv \tag{3.3}$$

and

$$u(t) = \frac{1}{\rho(t)} \int_0^{v_{max}} vf(v,t)dv. \tag{3.4}$$

Equation (3.2) is nonlinear since f enters B and D through ρ and u from (2.5) and (2.6) as follows.

$$B(\rho, v-u) = \begin{cases} -c_B(v-u)^2\rho(1-P(\rho,u,v-u)) & \text{if } v > u \\ c_A(v-u)^2(\rho_{max}-\rho) & \text{if } v \leq u \end{cases} \tag{3.5}$$

$$D(\rho,u,v-u) = \sigma_c\rho_{max}u_{max}^2 m_1\left(\frac{\rho}{\rho_{max}}\right) m_2\left(\frac{u}{u_{max}}\right) \cdot |v-u|^\gamma. \tag{3.6}$$

As mentioned in the previous chapter, the lane changing probability P_i will be involved in the braking scenario. P_i should depend on the relative speed $v - u_i^B$, on the density ρ_i^B, and on the density ρ_k and average speed u_k $(k = 3 - i)$ on the neighboring lane.

For the spatially homogeneous case here, P is set to depend only on $v - u$ and ρ. We could simply take P as a simple power of the scaled relative velocity (as it is a probability),

$$P(u, v-u) = \begin{cases} \left(\frac{v-u}{v_{max}-u}\right)^{\delta} \in [0,1] & \text{if } v > u \\ 0 & \text{if } v \leq u \end{cases} \tag{3.7}$$

Or we could take a somewhat more realistic ansatz for P,

$$P(\rho, u, v-u) = \begin{cases} h(\rho)\left(\frac{v-u}{v_{max}-u}\right)^{\delta} & \text{if } v > u \\ 0 & \text{if } v \leq u \end{cases} \tag{3.8}$$

where $\delta > 0$ and $h(0) = 1$, h decreases with ρ, and $h(\rho_{max}) = 0$. Although (3.8) is more realistic than (3.7) because it includes a density dependence, we will use (3.7) in the homogeneous situation for it catches the essence of higher motivation for lane-changing if the relative speed $(v - u)$ is large.

3.2 Equilibrium and fundamental diagrams

3.2.1 Steady solutions

We provide zero-flux boundary conditions, also known as Robin boundary conditions, to the drift diffusion equation (3.2),

$$B(\rho, v-u)f - D(\rho, u, v-u)\frac{\partial}{\partial v}f|_{v=0} = B(\rho, v-u)f - D(\rho, u, v-u)\frac{\partial}{\partial v}f|_{v=v_{max}} = 0, \tag{3.9}$$

We have the proposition of mass conservation.

Proposition 3.1 *Given zero-flux boundary conditions (3.9), the macroscopic density ρ given by (3.3) is* ***invariant in*** t*, i.e., a constant.*

Proof.

$$\begin{aligned}\frac{d\rho}{dt} &= \int_0^{v_{max}} \frac{\partial}{\partial t} f(v,t)dv \\ &= -\int_0^{v_{max}} \frac{\partial}{\partial v}\left(Bf - D\frac{\partial}{\partial v}f\right) dv = 0. \qquad \diamondsuit\end{aligned}$$

In the **steady** state, for all $v \in [0, v_{max}]$,

$$\frac{\partial}{\partial t} f(v,t) = 0.$$

We are left with

$$\frac{\partial}{\partial v}\left(Bf - D\frac{\partial}{\partial v}f\right) = 0. \tag{3.10}$$

With zero-flux boundary condition (3.9), we obtain

$$B(\rho, v-u)f = D(\rho, u, v-u)\frac{\partial}{\partial v}f \tag{3.11}$$

We can choose a $\rho \in [0, \rho_{max}]$, a $u \in [0, v_{max}]$ and then solve (3.11) subject to the normalization of $\int_0^{v_{max}} f dv = \rho$. Note that two types of solutions exist.

If $\gamma > 1$, (3.11) has the "trivial" equilibrium $f(v) = \rho \cdot \delta_u(v)$ as a solution in the sense of distribution, where $\delta_u(v)$ is a dirac function at u, which means that all the cars travel at the same speed u with the macroscopic density ρ. This should in principle be allowed in all realistic models and is in fact observed in [13],[14],[15],[16]. Actually, if we denote the 1st-order linear differential operator

$$Lf(v) = B(\rho, v-u)f(v) - D(\rho, u, v-u)f'(v)$$

for given (ρ, u), we have the **adjoint operator**

$$L^*\varphi(v) = B(\rho, v-u)\varphi(v) + D(\rho, u, v-u)\varphi'(v) + D_v(\rho, u, v-u)\varphi(v).$$

So $\rho \cdot \delta_u(v) \in L^1(0, v_{max})$ is a weak solution of (3.11) if and only if for any $\varphi(v) \in C_0^1(0, v_{max})$ we have

$$\int_0^{v_{max}} \rho \cdot \delta_u(v) L^*\varphi(v)dv = \int_0^{v_{max}} Lf(v) \cdot \varphi(v)dv = \int_0^{v_{max}} 0 \cdot \varphi(v)dv = 0,$$

i.e.,

$$\begin{aligned} \rho L^*\varphi(v)|_{v=u} &= \rho\left(B(\rho, u-u)\varphi(u) + D(\rho, u, u-u)\varphi'(u) + D'(\rho, u, u-u)\varphi(u)\right) \\ &= \rho D'(\rho, u, u-u)\varphi(u) = 0 \end{aligned}$$

This is guaranteed only if $\gamma > 1$. For the case that $\gamma \leq 1$ the diffusion coefficient $\sigma(\rho, u)$ can be adjusted such that the "trivial" equilibrium $\rho \cdot \delta_u(v)$ exists for both cases. However, we would lose the simplicity of our models.

In addition to the trivial equilibria, considering the system with $\gamma \in (0,2)$ in D and $\delta \in [0,1]$ in P, a continuous and differentiable for $v = u$ solution is obtained from (3.11),

$$f(v) = \begin{cases} C \cdot exp\left(\beta(\rho,u)(v-u)^{3-\gamma}\left[\left(\frac{v-u}{v_{max}-u}\right)^{\delta}\frac{1}{3+\delta-\gamma} - \frac{1}{3-\gamma}\right]\right) & \text{if } v > u \\ C \cdot exp\left(-\alpha(\rho,u)\frac{(u-v)^{3-\gamma}}{3-\gamma}\right) & \text{if } v \leq u. \end{cases} \tag{3.12}$$

where the constant C is determined by the normalization $\int_0^{v_{max}} f dv = \rho$ (note that choosing the same constant C for both cases is just a matter of choice, so we enforce the continuity at $v = u$), and β, α are given in terms of ρ and u by

$$\beta(\rho,u) = \frac{c_B \rho}{\sigma(\rho,u)} \geq 0, \qquad \alpha(\rho,u) = \frac{c_A(\rho_{max}-\rho)}{\sigma(\rho,u)} \geq 0$$

where $\sigma(\rho,u) = \sigma_c \rho_{max} u_{max}^2 m_1\left(\frac{\rho}{\rho_{max}}\right) m_2\left(\frac{u}{u_{max}}\right)$ as stated in Section 2.4.3.

3.2.2 Fundamental diagram

For any admissible pair of (ρ, u), (3.12) gives a potential equilibrium. However, since ρ and u are coupled with the solution f, to obtain a real equilibrium, the constraint

$$\int_0^{v_{max}} (v-u) f(v) dv = 0 \tag{3.13}$$

must be satisfied, which is the definition of the average speed u.

If we denote the left hand side $\int_0^{v_{max}}(v-u)f(v)dv$ as $R(\rho, u)$ with $f(v)$ given by (3.12), we obtain for any $\rho \in (0, \rho_{max})$, we can choose an $\epsilon > 0$ small enough such that

$$\begin{aligned} R(\rho,\epsilon) &= C\int_{\epsilon}^{v_{max}}(v-\epsilon)exp\left(\beta(\rho,\epsilon)(v-\epsilon)^{3-\gamma}\left[\left(\frac{v-\epsilon}{v_{max}-\epsilon}\right)^{\delta}\frac{1}{3+\delta-\gamma}-\frac{1}{3-\gamma}\right]\right)dv \\ &\quad -C\int_0^{\epsilon}(\epsilon-v)exp\left(-\alpha(\rho,\epsilon)\frac{(\epsilon-v)^{3-\gamma}}{3-\gamma}\right) > 0 \end{aligned}$$

and

$$\begin{aligned} R(\rho, v_{max}-\epsilon) &= -C\int_0^{v_{max}-\epsilon}(v_{max}-\epsilon-v)exp\left(-\alpha(\rho, v_{max}-\epsilon)\frac{(v_{max}-\epsilon-v)^{3-\gamma}}{3-\gamma}\right)dv \\ &\quad +C\int_{v_{max}-\epsilon}^{v_{max}}[v-(v_{max}-\epsilon)]exp(\beta(\rho, v_{max}-\epsilon)[v-(v_{max}-\epsilon)]^{3-\gamma} \\ &\quad \left[\left(\frac{v-(v_{max}-\epsilon)}{\epsilon}\right)^{\delta}\frac{1}{3+\delta-\gamma}-\frac{1}{3-\gamma}\right])dv < 0. \end{aligned}$$

Therefore, for any $\rho \in (0, \rho_{max})$ there exists at least a $u \in [0, v_{max}]$ such that $R(\rho, u) = 0$. The density and speed pair (ρ, u) with this property determine the **fundamental diagram**. For example, if we choose $\gamma = 1$, $\delta = 1$, $c_A = 5$, $c_B = 25$ and $\sigma_c = 0.25$, we obtain the fundamental diagram as shown in Figure 3.1.

Observations.

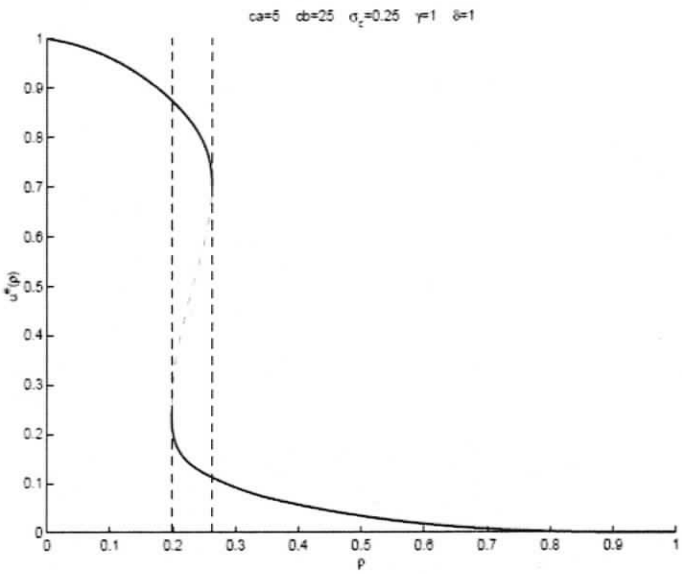

Figure 3.1: A multi-valued fundamental diagram for our spatially homogeneous model when $\gamma = 1$, $\delta = 1$, $c_A = 5$, $c_B = 25$ and $\sigma_c = 0.25$.

A remarkable numerical observation about this fundamental diagram is that it is multi-valued (three-valued) in a reasonable density interval, which is in our case between around 0.2 and 0.26. The factor which causes this phenomenon is the non-zero lane changing probability P given by (3.8), even though it is quite a simple assumption. Moreover, we will prove in the next chapter that the monotonicity of the term $\frac{B}{D}$ in u will guarantee the unique equilibrium, i.e., single-valued fundamental diagram.

In the multi-valued density interval, three different average speeds lead to different lane-changing probabilities

$$P(u, v-u) = \left(\frac{v-u}{v_{max}-u}\right)^{\delta} \quad \text{for } v > u.$$

It is obvious that for larger average speeds u, P is smaller for a fixed individual speed v because $\frac{v-u}{v_{max}-u}$ is increasing linearly in v to 1 and the slope $\frac{1}{v_{max}-u}$ is larger when u is larger. This means, in braking scenarios, drivers are less inclined to change lanes with relatively faster traffic in front of them.

We choose any pair of (ρ, u) and picture the equilibrium density distribution function $f_{\rho,u}(v)$. In the multi-valued density interval, for each density, there are three distributions corresponding to three average velocities. An example is pictured in Figure 3.2 for the case $\rho = 0.25$ in Figure 1.1. If we consider the stability of three equilibrated distributions, the two with u on stable branches (the stability has not be justified) have more realistic senses since they are the equilibrated distributions to which traffic actually converges. Combined with the lane changing probability, we could estimate the fraction of cars changing lanes given by

$$\int_0^{v_{max}} P(u, v-u) f_{\rho,u}(v) dv. \tag{3.14}$$

This expression and the density profile from Figure 3.2 can be used to ob-

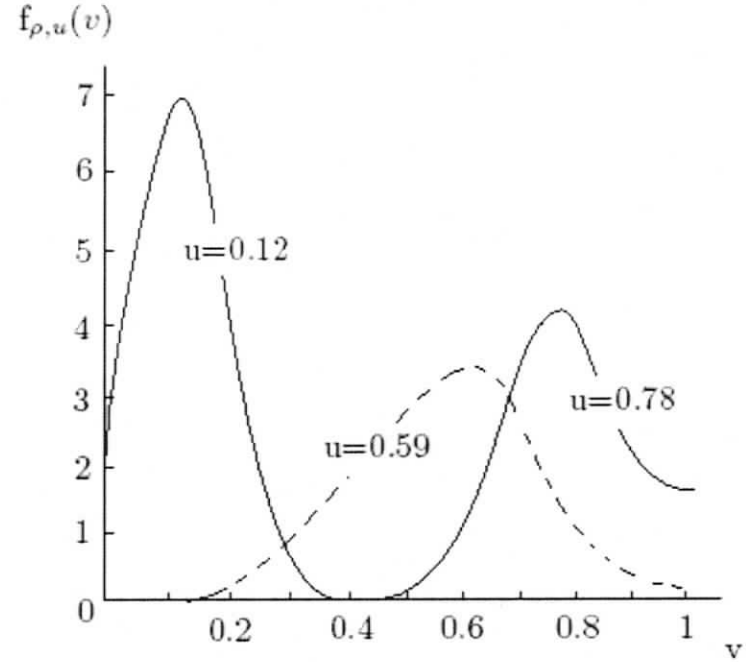

Figure 3.2: The distribution function $f_{\rho,u}(v)$ for the special choice $\rho = 0.25$ and its associated equilibrium velocities. In this case, we achieve three different distributions.

serve that there is more lane changing for the equilibrium with the largest u.

In the next section, with zero lane changing probability in the braking term, the multi-valued property of the fundamental diagram is lost. Furthermore, in section 3.5, deeper investigation reveals that the monotonicity of the quotient $\frac{B}{D}$ in u result in a single-valued fundamental diagram.

3.3 Zero lane changing probability and the Maxwellian example

To demonstrate that the multi-valued property of the fundamental diagram is due to the non-zero P (actually to the dependence on u and $v-u$), we will examine the case with zero P, and where

$$\frac{B}{D}(\rho, v-u) = \begin{cases} -\rho(v-u) & v > u \\ -c_0(1-\rho)(v-u) & v \leq u \end{cases}$$

c_0 could be thought of as the ratio between c_B and c_A, $\sigma(\rho, u)$ in D is assumed a constant and we normalize $\rho_{max} = 1$ and $v_{max} = 1$. We call this the **"asymmetric Maxwellian"** case. The equilibrium will be given by

$$f(v) = C \cdot \begin{cases} e^{-\frac{\rho}{2}(v-u)^2} & v > u \\ e^{\frac{-c_0(1-\rho)}{2}(v-u)^2} & v \leq u \end{cases}$$

where the constant C is determined by the normalization $\int_0^1 f(v)dv = \rho$. The other constraint, $\int_0^1 (v-u)f(v)dv = 0$, becomes

$$\begin{aligned} & C\left(\int_0^u (v-u)e^{-\frac{c_0(1-\rho)}{2}(v-u)^2}dv + \int_u^1 (v-u)e^{-\frac{\rho}{2}(v-u)^2}dv\right) \\ = \quad & -C\left(\frac{1}{\rho}e^{-\frac{\rho}{2}(1-u)^2} - \frac{1}{\rho} + \frac{1}{c_0(1-\rho)} - \frac{1}{c_0(1-\rho)}e^{-\frac{c_0(1-\rho)}{2}u^2}\right) = 0 \end{aligned}$$

After arrangement, we have

$$\rho e^{-\frac{c_0(1-\rho)}{2}u^2} = \rho - c_0(1-\rho) + c_0(1-\rho)e^{-\frac{\rho}{2}(1-u)^2}. \tag{3.15}$$

Consider the two sides of this equation as functions of $u \in [0,1]$. At $u = 0$, the left-hand side is ρ, the right-hand side is $\rho - c_0(1-\rho)(1 - e^{-\frac{\rho}{2}}) < \rho$; at $u = 1$, the left-hand side is $\rho e^{-\frac{c_0(1-\rho)}{2}} < \rho$, the right-hand side is ρ. Moreover, the left-hand side is strictly decreasing in u, while the right-hand side is strictly increasing in u. It follows that for each $\rho \in (0,1)$ there exists a unique $u \in [0,1]$ such that (3.15) is satisfied, which means the fundamental diagram of the "asymmetric Maxwellian" case is single valued.

Therefore, the non-zero lane changing probabilities and their dependencies are crucial for the multi-valued property of the fundamental diagram.

In the next section, a positive "residual" diffusion is introduced so that the degeneracy of the drift diffusion equation (3.2) and the existence of the trivial equilibrium $\rho \cdot \delta_u(v)$ are removed. In this case, the continuity of the steady solution $f(v)$ at $v = u$ is justified.

3.4 The "residual" diffusion

Consider the drift diffusion equation (3.2)

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial v}\left(B(\rho, v-u) f - D(\rho, u, v-u)\frac{\partial}{\partial v} f\right) = 0 \qquad (3.16)$$

where

$$D(\rho, u, v-u) = \sigma_c \rho_{max} u_{max}^2 m_1\left(\frac{\rho}{\rho_{max}}\right) m_2\left(\frac{u}{u_{max}}\right) \cdot |v-u|^{\gamma} \qquad (3.17)$$

vanishes when $v = u$. Degeneracy occurs at this point when we compute the continuous and differentiable solution $f(v)$ of

$$B(\rho, v-u) f = D(\rho, u, v-u)\frac{\partial}{\partial v} f. \qquad (3.18)$$

In section 3.2, we enforced the continuity of the steady solution at $v = u$ to obtain the exact expression of $f(v)$ as in (3.12).

One way to eliminate the degeneracy naturally is to introduce the **"residual" diffusion** *generated by drivers who assume speed u (after braking or accelerating) with a likely "overshoot"*, i.e., a driver may brake below u or accelerate above u.

We generate this residual diffusion by adding a small value $\varepsilon > 0$ to the original D to obtain a new diffusion term

$$D_\varepsilon(\rho, u, v-u) = D(\rho, u, v-u) + \varepsilon. \qquad (3.19)$$

With $D_\varepsilon > 0$, the equation

$$\frac{\partial}{\partial t} f + \frac{\partial}{\partial v}\left(B(\rho, v-u)f - D_\varepsilon(\rho, u, v-u)\frac{\partial}{\partial v} f\right) = 0 \tag{3.20}$$

is **nondegenerate** and, with the new continuous steady solution of (3.20), the multi-valued property of the fundamental diagram will not change significantly. Furthermore, as will be shown below, the trivial equilibria $\rho \cdot \delta_u(v)$ permitted by the previous model when $\gamma > 1$ are removed.

For $\gamma > 1$, the first-order linear differential operator for the stationary nondegenerate drift diffusion equation (3.20) with "residual" diffusion would be

$$Lf(v) = B(\rho, v-u)f(v) - (D(\rho, u, v-u) + \varepsilon)f'(v),$$

and the adjoint operator would be

$$L^*\varphi(v) = B(\rho, v-u)\varphi(v) + (D(\rho, u, v-u) + \varepsilon)\varphi'(v) + D_v(\rho, u, v-u)\varphi(v).$$

Hence, $\rho \cdot \delta_u(v)$ is not a solution of $Lf = 0$ anymore because

$$\begin{aligned}
&\int_0^{v_{max}} \rho \cdot \delta_u(v) L^*\varphi(v)dv \\
= \ & \rho L^*\varphi(u) \\
= \ & \rho\left(B(\rho, u-u)\varphi(u) + D(\rho, u, u-u)\varphi'(u) + D'(\rho, u, u-u)\varphi(u) + \varepsilon\varphi'(u)\right) \\
= \ & \rho\varepsilon\varphi'(u) \\
\neq \ & 0
\end{aligned}$$

for functions $\varphi(v) \in L^1(0,1)$ satisfying $\varphi'(u) \neq 0$. Therefore, when we apply "residual" diffusion to our model, we will obtain exactly continuous and differentiable steady solutions which produce multi-valued fundamental diagram.

In the next section, the "residual" diffusion is incorporated in a simple case of our model with a single-valued fundamental diagram. The reason for uniqueness of the equilibria is discussed.

3.5 An example with a single-valued fundamental diagram

In this section, choosing parameters in our model produces an example with a single-valued fundamental diagram. Deeper examination reveals that the monotonicity of $\frac{B}{D_\varepsilon}$ is crucial for the uniqueness of the equilibrium. It implies that non-zero lane changing probability P will also result in a single-valued fundamental diagram if $\frac{B}{D_\varepsilon}$ is monotone.

In the following discussion, we will assume that the maximum speed v_{max}

and density ρ_{max} are normalized to 1 without loss of generality. The spatially homogeneous drift diffusion equation of our traffic flow model with "residual" diffusion is given from (3.20) by

$$\frac{\partial}{\partial t}f = \frac{\partial}{\partial v}(D_\varepsilon(\rho, u, v-u)\frac{\partial}{\partial v}f - B(\rho, v-u)f) \qquad \forall\, t \geq 0, v \in (0,1) \quad (3.21)$$

where B and D_ε are given in terms of the density

$$\rho(t) = \int_0^1 f(t,v)dv, \qquad (3.22)$$

the average velocity

$$u(t) = \frac{1}{\rho(t)}\int_0^1 vf(t,v)dv \qquad (3.23)$$

and the relative speed $v - u$ as follows:

$$B(\rho, v-u) = \begin{cases} -c_B|v-u(t)|^2\rho(1 - \left|\frac{v-u(t)}{1-u(t)}\right|^\delta) & \text{if } v > u(t) \\ c_A|v-u(t)|^2(1-\rho) & \text{if } v \leq u(t) \end{cases} \qquad (3.24)$$

from (3.5) and (3.7) and

$$D_\varepsilon(\rho, u, v-u) = \sigma_c m_1(\rho) m_2(u(t)) \cdot |v-u(t)|^\gamma + \varepsilon \qquad (3.25)$$

from (3.17) and (3.19) for some positive constants c_A, c_B, σ_c, nonnegative exponents γ and δ and functions m_1, m_2 as given in Chapter 2. As stated in Section 3.4, the introduction of the "residual" diffusion $D_\varepsilon > 0$ removes the degeneracy of the drift diffusion equation. The modified model still leads to a multi-valued fundamental diagram.

If we choose $\delta = \infty$, $\gamma = 1$, $c_B\rho = c_A(1-\rho) := K$, and $m_2 \equiv 1$, we have

$$B = K|u - v|(u - v), \tag{3.26}$$

$$D_\varepsilon = \sigma_c m_1(\rho)|v - u| + \varepsilon, \tag{3.27}$$

and the lane changing probability $P = \left|\frac{v-u(t)}{1-u(t)}\right|^\delta$ disappears since $\left|\frac{v-u(t)}{1-u(t)}\right| < 1$ for $v > u(t)$ and $P = 0$ for $v < u(t)$.

Define F_f associated with f:

$$F_f(t,v) := -B(t,v)f(t,v) + D_\varepsilon(t,v)\frac{\partial}{\partial v}f(t,v).$$

With *zero-flux (Robin) boundary conditions:*

$$F_f(t,0) = F_f(t,1) = 0 \quad \forall\, t \geq 0, \tag{3.28}$$

the macroscopic density is preserved: $\rho \equiv const$ from Proposition 3.1.

An equilibrium (steady solution) is a solution of (3.21) such that

$$\frac{\partial}{\partial t}f = \frac{\partial}{\partial v}F_f(t,v) = 0 \qquad \forall (t,v) \in R^+ \times (0,1),$$

and the constraints (3.22) and (3.23) are satisfied. With the condition (3.28), the equilibrium satisfies

$$D_\varepsilon \frac{\partial}{\partial v}f - Bf = 0 \qquad \forall\, v \in (0,1). \tag{3.29}$$

We claim that in the case with B and D given by (3.26) and (3.27) respectively, *the steady states are unique for any* $\rho \in (0,1)$. That is, the fundamental diagram is single-valued (see Theorem 3.2) below. Note that for $\varepsilon = 0$, we obtain unique Maxwellian equilibria as in Section 3.3.

Theorem 3.2 *Given a* $\rho \in (0,1)$*, there exists a* ***unique*** $u \in (0,1)$ *such that the stationary solution* $f(v)$ *of (3.21) with* B *and* D *given by (3.26) and (3.27) satisfies the constraints (3.22) and (3.23), provided with the zero-flux boundary conditions (3.28).*

Proof:

Since

$$\frac{B}{D_\varepsilon} = \frac{K|u-v|(u-v)}{\sigma_c m_1(\rho)|u-v| + \varepsilon}$$

is dependent on the relative speed $(u-v)$, if we denote it by $\varphi(u-v)$, we have from (3.29)

$$\ln f(v) = \int_0^v \varphi(u-\omega)d\omega + \ln f(0),$$

$$f(v) = f(0)e^{\int_0^v \varphi(u-\omega)d\omega}.$$

Set

$$\Phi(u,v) = \int_0^v \varphi(u-\omega)d\omega$$

and define

$$f(u,v) = f(0)e^{\Phi(u,v)}$$

to emphasize the dependence of f on the parameter u. From constraint (3.22), we have

$$f(0) = \frac{\rho}{\int_0^1 e^{\Phi(u,v)}dv}$$

and

$$f(u,v) = \frac{\rho e^{\Phi(u,v)}}{\int_0^1 e^{\Phi(u,\omega)}d\omega}.$$

Then, let us rewrite the constraint (3.23) as

$$\int_0^1 (u-v)f(u,v)dv = 0.$$

Inserting $f(u,v)$, we have an equation for u

$$\frac{\rho}{\int_0^1 e^{\Phi(u,\omega)}d\omega}\int_0^1 (u-v)e^{\Phi(u,v)}dv = 0. \tag{3.30}$$

The only thing we need to show is that there is a unique $u \in (0,1)$ satisfying (3.30) for a given $\rho \in (0,1)$.

By a simple calculation, we see that

$$\frac{d}{du}\varphi(u-v) = \begin{cases} \frac{\lambda K(u-v)^2+2\varepsilon K(u-v)}{[\lambda(u-v)+\varepsilon]^2} > 0 & \text{if } u > v \\ \frac{\lambda K(u-v)^2-2\varepsilon K(u-v)}{[\varepsilon-\lambda(u-v)]^2} > 0 & \text{if } u < v \end{cases}$$

where $\lambda = \sigma_c m_1(\rho)$. This shows that $\varphi(u-v)$ is a **strictly increasing** function with respect to u. Consequently, $\Phi(u,v)$ and $e^{\Phi(u,v)}$ are also strictly increasing in u and $R(u) := \int_0^1 (u-v)e^{\Phi(u,v)}dv$ is also strictly increasing with respect to u. Clearly,

$$R(0) = -\int_0^1 ve^{\Phi(0,v)}dv < 0$$

$$R(1) = \int_0^1 (1-v)e^{\Phi(1,v)}dv > 0.$$

The strict monotonicity of $R(u)$ implies the **unique** existence of the parameter $u^* \in [0,1]$ such that $R(u^*) = 0$. Consequently, $f(u^*, v)$, which is **uniquely** determined by u^*, is a unique equilibrium of PDE (3.21) with the constraints (3.22) and (3.23) for a given ρ. $\diamondsuit$

Notice that the *monotonicity* of the function $\varphi(u-v) = \frac{B(u,v)}{D_\varepsilon(u,v)}$ played a key part in the uniqueness of the steady state. For the complete traffic model, which includes the passing probability $P = \left|\frac{v-u}{1-u}\right|^\delta$, the function $\varphi = \frac{B(u,v)}{D_\varepsilon(u,v)}$ is not *monotone* anymore, and that is the principal reason why *multi-valued fundamental* diagrams are possible.

In the next chapter, convergence results stated in [3] for large times to steady states are presented for our general spatially homogeneous model by using convex entropy methods. With the "residual" diffusion introduced, due to the elimination of concentrations (trivial equilibrium), some assumptions in [3] could be removed and a simpler argument could be applied to obtain the same results.

Chapter 4

Long-time behavior by entropy methods

In this chapter we examine the time asymptotics of the time-dependent kinetic density by convex entropy methods. The main result is drawn from [3] by Dolbeault and Illner. The global stability of the steady solutions is obtained in [3] provided that some entropy regimes are bounded. These assumptions could be removed by introducing the "residual" diffusion.

In Section 4.2, entropy methods are used to show the convergence of the kinetic density to the unique steady solution of two simple examples with "residual" diffusion. Although based on unrealistic assumptions on average speed, the same arguments could apply to a real coupled case if the average speed converges uniformly (which is verified in [3]).

4.1 Convex entropy methods

Recall that the full spatially homogeneous case of VFP type model without "residual" diffusion is given from (3.2) by

$$\frac{\partial}{\partial t}f + \frac{\partial}{\partial v}\left(B(\rho, v-u)f - D(\rho, u, v-u)\frac{\partial}{\partial v}f\right) = 0 \tag{4.1}$$

where the braking/acceleration force and diffusion coefficient are given from (3.5) and (3.6) respectively by

$$B(\rho, v-u) = \begin{cases} -c_B(v-u)^2\rho(1-P(\rho, u, v-u)) & \text{if } v > u \\ c_A(v-u)^2(\rho_{max}-\rho) & \text{if } v \leq u \end{cases} \tag{4.2}$$

and

$$D(\rho, u, v-u) = \sigma_c \rho_{max} u_{max}^2 m_1\left(\frac{\rho}{\rho_{max}}\right) m_2\left(\frac{u}{u_{max}}\right) \cdot |v-u|^{\gamma}. \tag{4.3}$$

The lane changing probability involved in the braking/acceleration term is given from (3.7) by

$$P(u, v-u) = \begin{cases} \left(\frac{v-u}{v_{max}-u}\right)^{\delta} \in [0,1] & \text{if } v > u \\ 0 & \text{if } v \leq u \end{cases} \tag{4.4}$$

We have showed that this model leads to a multi-valued fundamental diagram.

Usually, a powerful tool to obtain the large time behavior of the time-dependent solution $f(t,v)$ of a parabolic PDE in many situations is the maximum principle. We review it here but will see that it does not apply.

4.1.1 Why the maximum principle doesn't work

An example as follows will be enough to show why the maximum principle fails in our equation. Choosing

$$B = K|u-v|(u-v) \quad \text{and} \quad D_\varepsilon = \varepsilon,$$

the drift diffusion equation (4.1) becomes

$$\partial_t f = \varepsilon \frac{\partial^2}{\partial v^2} f - K|u-v|(u-v)\frac{\partial}{\partial v} f + 2K|v-u|f. \tag{4.5}$$

Maximum Principle, see [22]:

Let Ω be a bounded domain in $\mathcal{R}^n$ and consider inequalities of the form

$$(1) \qquad Lu \geq u_t \qquad \text{in } U_T = \Omega \times (0,T),$$

where L is a uniformly elliptic second-order differential operator; that is, we assume that

$$(2a) \qquad L = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x^i} + c(x,t)$$

satisfies the ellipticity condition

$$(2b) \qquad \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \text{for } (x,t) \in U_T$$

where $\lambda > 0$ is a constant, and the coefficients a_{ij}, b_i and c are all bounded functions on U_T.

Lemma 4.1 (Weak Parabolic Maximum Principle) *Suppose that* $u \in C^{2;1}(U_T) \bigcap C(\overline{U_T})$ *satisfies (1) with* L *given by (2a) and (2b) where* $c(x,t) \equiv 0$. *Then*

$$\max_{(x,t)\in U_T} u(x,t) \leq \max_{(x,t)\in \Gamma_T} u(x,t),$$

where $\Gamma_T = (\partial\Omega \times [0,T]) \bigcup (\Omega \times 0)$ *is the parabolic boundary.*

Unfortunately, the maximum principle is not applicable to our equation (4.5) since the term $c(t,v) = 2K|v-u|$ depends on v. Otherwise, if the term c were dependent only on t, we could multiply both sides of (4.5) by $e^{-c(t)}$ and transform the equation to be

$$\partial_t(fe^{-c(t)t}) = \widetilde{L}(fe^{-c(t)t})$$

where the $\tilde{c}$ in the new elliptic second order operator $\widetilde{L}$ vanishes. However, this kind of reduction is not possible in the present situation and thus we cannot use methods based on the Maximum Principle (e.g., the Comparison Method) to analyze the large time behavior of the solution of (4.5) as usual.

4.1.2 Entropy methods

On the other hand, convex entropy methods have been applied to parabolic equations for many years. The basic idea of convex entropy methods is to use the time-monotonicity of the entropy functional of the system to measure the distance between the solution and the equilibrium.

Consider the full model (4.1) with (4.2), (4.3) and (4.4) and normalize $v_{max} = \rho_{max} = 1$. Define the relative entropy of f with respect to g by

$$E[f|g] = \int_0^1 \Phi\left(\frac{f}{g}\right) g dv, \tag{4.6}$$

for some **convex** function Φ. For example, in [3] $\Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$ for some $\alpha > 1$ with $\Phi(x) = x \log x$ as a limit case denoted by "$\alpha = 1$" are used. Then $\Phi_\alpha(0) = 0$ and $\Psi(x) := \Phi_\alpha(x) - x\Phi_\alpha'(x) < 0$ for any $x > 0$. With

f to be the solution of (4.1) and g given by

$$g(t, v) = \kappa(t)e^{-C(t,v)} \tag{4.7}$$

where $\kappa(t)$ is chosen to satisfy

$$\begin{cases} \dot{\kappa} = \kappa \frac{\int_0^1 \Psi(\frac{f}{g})gC_t(t,v)dv}{\int_0^1 \Psi(\frac{f}{g})gdv} \\ \kappa(0) = 1. \end{cases} \tag{4.8}$$

and

$$C(t, v) = -\int_0^v \frac{B(t, \omega)}{D(t, \omega)} d\omega \tag{4.9}$$

$$u(t) = \frac{1}{\rho} \int_0^1 v f(t, v) dv. \tag{4.10}$$

Then the main result in [3] is stated as the following theorem.

Theorem 4.2 *Assume that $\Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$ with $\alpha \in (1, 2]$ and $\Phi(x) = x \log x$ in the limit case $\alpha = 1$. Consider a global smooth in time solution f of (4.1) with u, B and D depending nonlinearly on f according to (4.10), (4.2) and (4.3), for some smooth function m_2. Let $\rho = \int_0^1 f(t, v)dv \in (0, 1]$. Assume that $E[f|g]$ is well defined and C^1 in t, with E, g and κ defined by (4.6), (4.7) and (4.8). Assume furthermore that $\gamma < 3$ and that $\limsup_{t\to\infty} \kappa(t) < \infty$. If there exists an $\epsilon \in (0, 1/2)$ such that*

$$\epsilon < u(t) < 1 - \epsilon \qquad \forall\, t > 0,$$

then, as $t \to +\infty$, $f(t,\cdot)$ converges a.e. to a stationary solution f_∞ of (4.1) where $\rho = \int_0^1 f_\infty dv$ and $u_\infty := \frac{1}{\rho}\int_0^1 v f_\infty(v)dv$ are related as in the fundamental diagram.

Remarks:

- The assumption on u to be ϵ distance away from 0 and 1 is to ensure that no concentration occurs, i.e., not $f(t,\cdot) \to \rho\cdot\delta_{v=0}$ or $\rho\cdot\delta_{v=1}$ as $t \to \infty$. However, if we apply the same method to our model with "residual" diffusion, the trivial solution is removed and no concentration occurs.

- Another assumption that the function κ is bounded from above is also made to avoid concentrations, otherwise, with $\limsup_{t\to\infty}\kappa(t) = \infty$ the decrease of $E[f|g]$ is consistent with the formation of concentrations in f. Therefore, this assumption can be removed by adding "residual" diffusion as well.

- This theorem shows that the time-dependent kinetic density will converge to an steady solution, even if more than one exists in the considered range of parameters, provided that some entropy estimates are uniformly bounded. The asymptotic state is clearly selected by the initial conditions. Therefore, this global stability of the steady solutions implies no Stop-and-Go regimes. However, the multi-valued fundamental diagram combined with density fluctuations should produce Stop-and-Go phenomena for the full VFP equation.

4.2 Two examples

We hope to examine in more details the application of entropy methods in illustrating asymptotic behavior of the time-dependent solution of our VFP equation **with "residual" diffusion**. Then, we make some assumptions on the average speed in the example of section 4.1.1, which leads to a single-valued fundamental diagram. We try specific entropy functionals to illustrate the convergence of the kinetic densitys to the unique equilibria. Although these assumptions are not justified to describe the real cases, the method of the proof of Theorem 4.2 in [3] is very similar with those methods in these simpler cases.

Recall the example in section 4.1.1, we have the drift diffusion equation (4.5) given by

$$\partial_t f = \varepsilon \frac{\partial^2}{\partial v^2} f - K|u-v|(u-v)\frac{\partial}{\partial v} f + 2K|v-u| f \tag{4.11}$$

with $B = K|u-v|(u-v)$ and $D_\varepsilon = \varepsilon$. It is easy to check that $\frac{B}{\varepsilon}$ is monotone in u, therefore this example has unique equilibrium. We apply entropy methods in two cases: u is a given constant and $u(t)$ is a given function whose limit is a constant u_∞.

Consider a solution $f(t,v)$ of the drift diffusion equation (4.11). Define

$$\begin{aligned} C(t,v) &:= -\int_0^v \frac{B(t,\omega)}{D_\varepsilon(t,\omega)} d\omega \\ &= -\int_0^v \frac{K|u(t)-\omega|(u(t)-\omega)}{\varepsilon} d\omega \\ &= \frac{K|(u(t)-v)|^3}{3\varepsilon} - \frac{Ku(t)^3}{3\varepsilon} \end{aligned} \tag{4.12}$$

and **a local equilibrium**

$$g(t,v) = \rho \frac{e^{-C(t,v)}}{\int_0^1 e^{-C(t,v)} dv} \tag{4.13}$$

with zero flux:

$$F_g(t,v) = 0 \quad \forall\ (t,v) \in \mathcal{R}^+ \times (0,1),$$

but which is **not** *in general* a stationary solution of (4.11) since $g_t \equiv 0$ is **not** guaranteed. We may also replace $C(t,v)$ by $C(t,v) + C_0(t)$ for $C_0(t)$ an arbitrary function without changing the values of $g(t,v)$. In addition, we have

$$g' = -gC' = g\frac{B}{D_\varepsilon}.$$

We introduce a general **logarithmic entropy functional** of the solution

$$f \mapsto E[t, f] := \int_0^1 f(\log f + C(t, v))dv. \tag{4.14}$$

Two properties of the entropy functional are essential in the application: *boundedness* and *monotonicity*.

Proposition 4.3 *$g(t, v)$ given by (4.13) is a unique minimizer of the functional (4.14) on $\mathrm{L}^1_+(0, 1)$ under the constraint $\int_0^1 f dv = \rho$, where $L^1_+(0, 1)$ includes all the nonnegative functions in $L^1(0, 1)$.*

Proof.

t is just a parameter and can be omitted in the proof. Solving

$$\min_{f \in L^1_+(0,1),\ \int_0^1 f dv = \rho} \left(E[f] = \int_0^1 f(\log f + C)dv\right)$$

is equivalent to solving

$$\min_{f \in L^1_+(0,1),\ \int_0^1 f dv = \rho} E[f] + C_0\rho = \min_{f \in L^1_+(0,1),\ \int_0^1 f dv = \rho} \left\{E_1[f] = \int_0^1 f(\log f + C + C_0)dv\right\}$$

where $C_0(t)$ is an arbitrary function.

Assume that a minimizer f^* of E_1 exists. Compute the first variation of E_1 and let it to be 0 as follows. For $\forall\ \varphi \in C_+[0, 1]$ and $\varepsilon > 0$,

$$\begin{aligned}
\frac{d}{d\varepsilon}|_{\varepsilon=0} E_1[f + \varepsilon\varphi] &= \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_0^1 (f + \varepsilon\varphi)\left[\log(f + \varepsilon\varphi) + C + C_0\right] dv \\
&= \int_0^1 \varphi\left(\log f + C + C_0 + 1\right) dv = 0.
\end{aligned}$$

Therefore,

$$\log f^* + C + C_0 + 1 = 0 \tag{4.15}$$

and

$$f^*(v) = \frac{e^{-C(v)}}{e^{1+C_0}}.$$

Also we have $\int_0^1 f^*(v)dv = \rho$,

$$e^{1+C_0} = \frac{\int_0^1 e^{-C(v)}dv}{\rho},$$

and

$$f^*(v) = \rho \frac{e^{-C(v)}}{\int_0^1 e^{-C(v)}dv} = g(v).$$

To verify that $f^*(v)$ is actually a minimizer given uniquely by $\rho\frac{e^{-C(v)}}{\int_0^1 e^{-C(v)}dv}$, notice that $x \log x$ is a convex function. This implies the inequality

$$f \log f - f^* \log f^* \geq (\log f^* + 1)(f - f^*).$$

Consequently, we have

$$\begin{aligned} E_1[f] - E_1[f^*] &= \int_0^1 f \log f - f^* \log f^* + (C + C_0)(f - f^*)dv \\ &\geq \int_0^1 (\log f^* + 1 + C + C_0)(f - f^*)dv = 0 \end{aligned}$$

from (4.15).

This proves that f^* is indeed a minimizer of $E_1[f]$ and hence of $E[f]$. $\diamondsuit$

In addition, to obtain the monotonicity of the entropy in time, for the sequel we compute the derivative of the entropy with respect to t

$$\begin{aligned}
\frac{d}{dt}E[t,f] &= \int_0^1 [f_t(\log f + C + 1) + C_t f] dv \\
&= \int_0^1 [(\varepsilon f' - Bf)'(\log f + C + 1) + C_t f] dv \\
&= (\varepsilon f' - Bf)(\log f + C + 1)|_0^1 - \int_0^1 \left[(\varepsilon f' - Bf)\left(\frac{f'}{f} + C'\right) - C_t f\right] dv \\
&= F_f(t,v)(\log f + C + 1)|_0^1 - \int_0^1 \left[(\varepsilon f' - Bf)\left(\frac{f'}{f} - \frac{B}{\varepsilon}\right) - C_t f\right] dv \\
&= -\int_0^1 \left[\varepsilon f\left(\frac{f'}{f} - \frac{B}{\varepsilon}\right)^2 - C_t f\right] dv \\
&= -\int_0^1 \left[\varepsilon f\left|\frac{f'}{f} - \frac{g'}{g}\right|^2 - \frac{K}{\varepsilon}\left(|u-v|(u-v) - u^2\right) u_t' f - C_{0t}' f\right] dv.
\end{aligned}$$

We define the **entropy production** as

$$I[t,f] := -\frac{d}{dt}E[t,f]. \tag{4.16}$$

4.2.1 u is a constant

Suppose that u is NOT coupled to f by (3.23), but given and constant. Then we can choose $C_0(t)$ to be a constant, so that

$$C(v) = -\int_0^v \frac{K|u-\omega|(u-\omega)}{\varepsilon}$$

and

$$g(v) = \rho \frac{e^{-C(v)}}{\int_0^1 e^{-C(v)} dv}$$

are independent on t and

$$\begin{aligned} E[t,f] = \int_0^1 (f(\log f + C(v))\,dv \geq E[g] &= \int_0^1 g(v)(\log g(v) + C(v))dv \\ &= -\rho \log \frac{\rho}{\int_0^1 e^{-C(v)}dv} := E_g. \end{aligned}$$

Therefore, we obtain a constant lower bound E_g for the entropy functional $E[t,f]$ for $\forall\, t \geq 0, f \in \mathrm{L}^1_+(0,1)$. At the same time, the second term $C_t f$ in the integral of the entropy production

$$I[t,f] = -\int_0^1 \left[\varepsilon f \left(\frac{f'}{f} - \frac{B}{\varepsilon}\right)^2 - C_t f\right] dv$$

vanishes, so the entropy production is non positive. This implies the monotonicity of the entropy as t increases. Note that in this case $g(v)$ is actually a *general* equilibrium solution.

Theorem 4.4 *Suppose $u \in (0,1)$ in (4.11) is a constant. Let f be any solution of (4.11) in $L^1(0,1)$ such that*

$$\|f(t,\cdot)\|_{L^1(0,1)} = \int_0^1 f(t,v)dv = \rho \in (0,1).$$

Then the entropy functional $E[t, f(t,v)]$ is bounded from below by E_g, non-increasing, and converges to E_g as $t \to +\infty$. Moreover,

$$\lim_{t\to+\infty} f(t,v) = g(v)$$

in $L^1[0,1]$.

To prove this theorem, we will use the following Csiszár-Kullback inequality and the Lebesgue's Dominated Convergence Theorem.

Proposition 4.5 (Csiszár-Kullback inequality) *Assume that* Ω *is a domain in* R^N. *Let* v_1 *and* v_2 *be two nontrivial nonnegative functions in* $L^1(\Omega)$, *and consider a nonnegative strictly convex function* $\phi \in C^2(0,+\infty)$. *If*

$$A := \inf_{v>0} v\phi''(v) > 0,$$

then

$$\int_\Omega (\phi(v_1) - \phi(v_2) - \phi'(v_2)(v_1 - v_2))dx \geq \frac{A}{4}\left\{\min_{i=1,2} \|v_i\|_{L^1(\Omega)}^{-1}\right\} \|v_1 - v_2\|_{L^1(\Omega)}^2.$$

[for a more general inequality and proof, see [23]]

Lemma 4.6 (Lebesgue's Dominated Convergence Theorem) *Suppose that* $\{f_n\}_{n\in\mathcal{N}}$ *is a sequence of measurable functions, that* $f_n \to f$ *pointwise almost everywhere as* $n \to \infty$, *and that* $|f_n| \leq h$ *for all* n, *where* h *is integrable. Then* f *is integrable, and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Proof of Theorem 4.4.

From Proposition 4.3, we know $E[t,f]$ is bounded by E_g from below. The derivative of the entropy

$$\frac{d}{dt}E[t,f] = -\int_0^1 \frac{|\varepsilon f' - Bf|^2}{\varepsilon f}dv = -4\varepsilon \int_0^1 \left|(f^{\frac{1}{2}})' - \frac{1}{2}\frac{B(v)}{\varepsilon}f^{\frac{1}{2}}\right|^2 dv \le 0.$$

It follows that the entropy $E[t,f]$ is non increasing in t for f in $L^1(0,1)$.

Integrating the entropy production $I[t,f] = -\frac{d}{dt}E[t,f]$ over $t \in (0,+\infty)$, we find

$$\int_0^\infty I[t,f]dt = 4\varepsilon \int_0^\infty \int_0^1 \left|(f^{1/2})' - \frac{1}{2}\frac{B(v)}{\varepsilon}f^{1/2}\right|^2 dvdt < \infty.$$

This implies that there exist an *increasing unbounded* time sequence $(t_n)_{n\in\mathcal{N}}$ such that

$$\lim_{n\to\infty}\int_0^1 \left|(f_n^{1/2})' - \frac{1}{2}\frac{B(v)}{\varepsilon}f_n^{1/2}\right|^2 dv = 0 \tag{4.17}$$

where $(f_n(v) := f(t_n,v))_{n\in\mathcal{N}}$ for $v \in (0,1)$ is a bounded sequence in $\mathrm{L}^1(0,1;dv)$ because $\|f_n\|_{L^1} = \rho$ for any $n \in \mathcal{N}$.

We develop the square in (4.17) and use Cauchy's inequality, as follows.

$$\begin{aligned}
&\int_0^1 \left|(f_n^{1/2})' - \frac{1}{2}\frac{B(v)}{\varepsilon}f_n^{1/2}\right|^2 dv \\
= &\int_0^1 (f_n^{1/2})'^2 dv + \int_0^1 \frac{1}{4}\frac{B(v)^2}{\varepsilon^2}f_n dv - \int_0^1 \frac{B(v)}{\varepsilon}f_n^{1/2}(f_n^{1/2})'dv \\
\ge &\int_0^1 (f_n^{1/2})'^2 dv + \int_0^1 \frac{1}{4}\frac{B(v)^2}{\varepsilon^2}f_n dv - 2\left(\int_0^1 (f_n^{1/2})'^2 dv\right)^{1/2}\left(\int_0^1 \frac{1}{4}\frac{B(v)^2}{\varepsilon^2}f_n dv\right)^{1/2}
\end{aligned}$$

$$= \left[\left(\int_0^1 (f_n^{1/2})'^2 dv \right)^{1/2} - \left(\int_0^1 \frac{1}{4} \frac{B(v)^2}{\varepsilon^2} f_n dv \right)^{1/2} \right]^2$$

As $B(v) = K|u - v|(u - v)$ is continuous in v and $\int_0^1 f_n dv = \rho$, the second term $\int_0^1 \frac{1}{4} \frac{B(v)^2}{\varepsilon^2} f_n dv$ is bounded uniformly in n. It follows that there is a constant C such that $\forall\ n$

$$\int_0^1 (f_n^{1/2})'^2 dv < C. \tag{4.18}$$

and as $\forall\ n$ also

$$\int_0^1 (f_n^{1/2})^2 dv = \rho < \infty, \tag{4.19}$$

we have that the sequence $(f_n^{1/2})_{n \in \mathcal{N}}$ is bounded in $H^{1,2}(0,1)$. By the Sobolev-Rellich-Kondrachov Imbedding Theorem (see [22]), $H^{1,2}(0,1)$ is compactly imbedded in the space $L^2(0,1)$. Up to the extraction of a subsequence, we may assume that $(f_n^{1/2})_{n \in \mathcal{N}}$ strongly converges in $L^2(0,1)$, and a.e., to $f_\infty(v)^{1/2}$.

Also (4.17) is satisfied by $(f_n^{1/2})_{n \in \mathcal{N}}$. We pass to the limit as follows. For any $\varphi(v) \in C_0^\infty(0,1)$

$$\begin{aligned} & \lim_{n \to +\infty} \left| \int_0^1 \left[\left((f_n^{1/2})' - \frac{1}{2} \frac{B}{\varepsilon} f_n^{1/2} \right) - \left((f_\infty^{1/2})' - \frac{1}{2} \frac{B}{\varepsilon} f_\infty^{1/2} \right) \right] \varphi dv \right| \\ = & \lim_{n \to +\infty} \left| \int_0^1 \left((f_n^{1/2})' - (f_\infty^{1/2})' \right) \varphi dv - \frac{1}{2\varepsilon} \int_0^1 B \left(f_n^{1/2} - f_\infty^{1/2} \right) \varphi dv \right| \end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to+\infty}\left|\int_0^1 (f_n^{1/2} - f_\infty^{1/2})\left(-\varphi' - \frac{1}{2}\frac{B}{\varepsilon}\varphi\right) dv\right| \\
&\le \lim_{n\to+\infty} \|f_n^{1/2} - f_\infty^{1/2}\|_{L^2(0,1)} \left\|\varphi' + \frac{1}{2}\frac{B}{\varepsilon}\varphi\right\|_{L^2(0,1)} \\
&= 0
\end{aligned}$$

This implies

$$\lim_{n\to+\infty}\int_0^1 \left((f_n^{1/2})' - \frac{1}{2}\frac{B}{\varepsilon}f_n^{1/2}\right)\varphi dv = \int_0^1 \left((f_\infty^{1/2})' - \frac{1}{2}\frac{B}{\varepsilon}f_\infty^{1/2}\right)\varphi dv.$$

Also from (4.17), we have

$$\begin{aligned}
&\lim_{n\to+\infty}\left|\int_0^1 \left((f_n^{1/2})' - \frac{1}{2}\frac{B}{\varepsilon}f_n^{1/2}\right)\varphi dv\right| \\
\le\ &\lim_{n\to+\infty}\left[\int_0^1 \left((f_n^{1/2})' - \frac{1}{2}\frac{B}{\varepsilon}f_n^{1/2}\right)^2 dv\right]^{1/2} \left(\int_0^1 \varphi^2 dv\right)^{1/2} = 0,
\end{aligned}$$

so

$$\int_0^1 \left((f_\infty^{1/2})' - \frac{1}{2}\frac{B}{\varepsilon}f_\infty^{1/2}\right)\varphi dv = 0$$

for any $\varphi \in C_0^\infty(0,1)$. Therefore, at least in the sense of distributions, we obtain

$$(f_\infty^{1/2})' - \frac{1}{2}\frac{B(v)}{\varepsilon}f_\infty^{1/2} = 0$$

and

$$f_\infty(v) = const \cdot e^{-C(v)} \qquad v \in (0,1)$$

By the normalization of the preserved density, the constant has to be $\frac{\rho}{\int_0^1 e^{-C(\omega)}d\omega}$, so

$$f_\infty(v) = \frac{\rho e^{-C(v)}}{\int_0^1 e^{-C(\omega)}d\omega} = g(v).$$

By the convergence of $f_n^{1/2}$ to $f_\infty^{1/2}$ in $L^2(0,1)$, f_n converges to f_∞ in $L^1(0,1)$ and point wise a.e. as $t_n \to +\infty$ because

$$\begin{aligned} \|f_n - f_\infty\|_{L^1(0,1)} &\leq \left[\int_0^1 \left(f_n^{1/2} - f_\infty^{1/2}\right)^2 dv\right]^{1/2} \left[\int_0^1 \left(f_n^{1/2} + f_\infty^{1/2}\right)^2 dv\right]^{1/2} \\ &\leq \left\|f_n^{1/2} - f_\infty^{1/2}\right\|_{L^2(0,1)} \cdot 2\sqrt{\rho}. \end{aligned} \tag{4.20}$$

In addition, for any $v, w \in (0,1)$ and $n \in N$, there exists a constant $\tilde{C} > \rho$ such that

$$\begin{aligned} |f_n(v) - f_n(w)| &= \left|\int_w^v f_n'(\omega)d\omega\right| \\ &= \left|\int_w^v 2\sqrt{f_n(\omega)}\left(\sqrt{f_n(\omega)}\right)' d\omega\right| \\ &\leq \int_0^1 2\sqrt{f_n(\omega)}\left(\sqrt{f_n(\omega)}\right)' d\omega \\ &\leq 2\left(\int_0^1 f_n d\omega\right)^{1/2}\left(\int_0^1 (f_n^{1/2})'^2 d\omega\right)^{1/2} \\ &\leq 2\sqrt{\rho}\cdot\sqrt{C} \leq 2\tilde{C} \end{aligned}$$

where C is in (4.18). It follows that

$$|f_n(v)| \leq 4\tilde{C}$$

for any n and $v \in (0,1)$ because the average of f_n in $[0,1]$ is $\int_0^1 f_n(v)dv = \rho \leq \tilde{C}$. As a result, for all n,

$$\begin{aligned}
\left| f_n(v) \log \frac{f_n(v)}{f_\infty(v)} \right| &\leq |f_n(v) \log f_n(v)| + |f_n(v) \log f_\infty(v)| \\
&\leq 4\tilde{C} \log(4\tilde{C}) + 4\tilde{C} \left| \log f_\infty(v) \right| \\
&\leq C_1 + 4\tilde{C} \cdot |C(v)| \\
&= C_1 + 4\tilde{C} \left| \frac{K|u-v|^3}{3\varepsilon} - \frac{Ku^3}{3\varepsilon} \right| := h(v)
\end{aligned}$$

where $C_1 = 4\tilde{C} \left[\log(4\tilde{C}) + \left| \log \frac{\rho}{\int_0^1 e^{-C(\omega)} d\omega} \right| \right]$ and $h(v)$ is an integrable function in $L^1(0,1)$. Also,

$$f_n(v) \log \frac{f_n(v)}{f_\infty(v)} \to f_\infty(v) \log \frac{f_\infty(v)}{f_\infty(v)} = 0$$

pointwise a.e.. Therefore, from Lebesgue's Dominated Convergence Theorem stated as Lemma 4.6, we have

$$\lim_{n \to \infty} \int_0^1 f_n(v) \log \frac{f_n(v)}{f_\infty(v)} dv = 0,$$

and the entropy $E[t_n, f_n]$ converges to the constant $E[g]$ as

$$\lim_{n \to \infty} E[t_n, f_n] = \lim_{n \to \infty} \int_0^1 f_n(v) \log \frac{f_n(v)}{f_\infty(v)} dv + E[g] = E[g].$$

It is easy to see that the entropy function $E[t, f(t,v)]$ converges to $E[g]$ as well as $t \to \infty$ since it is non increasing in t.

Finally, we apply the Csiszár-Kullback inequality (Proposition 4.5) by choosing $\phi(x) = x \log x$, $v_1 = f$ and $v_2 = f_\infty$ to obtain

$$\|f(t,v) - f_\infty(v)\|^2_{L^1(0,1)} \leq 4 \int_0^1 f(t,v) \log\left(\frac{f(t,v)}{f_\infty(v)}\right) dv = 4(E[t,f] - E[g]) \to 0$$

as $t \to \infty$. This implies the convergence of the solution $f(t,v)$ of (3.21) to the stationary solution $g(v) = f_\infty(v)$ in L^1 space for large time. $\diamond$

4.2.2 Assume $u = u(t)$ (given) and $\lim_{t\to\infty} u(t) = u_\infty$

In this case, we assume the average speed is a given time-dependent function $u(t)$ which strongly converges to a constant u_∞ as $t \to \infty$. We show that the solution $f(t,v)$ of (4.11) still converges to the equilibrium as in the previous case.

When u is dependent on t,

$$C(t,v) = \frac{K|u(t) - v|^3}{3\varepsilon} - \frac{Ku(t)^3}{3\varepsilon}$$

and

$$g(t,v) = \rho \frac{e^{-C(t,v)}}{\int_0^1 e^{-C(t,v)} dv}$$

are both time-dependent functions. If we again consider the general logarithmic entropy $E[t,f] = \int_0^1 f\left(\log f + C(t,v)\right) dv$ as in (4.14), we will lose control of the sign of the entropy production $I[t,f]$ because of the existence of the term $-\frac{K}{\varepsilon}[|u(t)-v|(u(t)-v)-u(t)^2]u'(t)f - C_0'(t)f$, so the previous reasoning is not applicable here. We use instead a variation of the entropy method.

First, we define a new entropy functional of the solution $f(t,v)$ as

$$e[t,f] := \int_0^1 (f\log f - g\log g + C(t,v)(f-g))dv - \iint_{(0,1)\times(0,t)} C_t(s,v)(f-g)(s,v)dvds. \tag{4.21}$$

With zero-flux (mass preserving) boundary condition: $F_f(t,0) = F_f(t,1) = 0$ for $t \geq 0$,

$$\begin{aligned}
\frac{d}{dt}e[t,f(t,\cdot)] &= -\int_0^1 [f_t(\log f + C + 1)dv - g_t(\log g + C + 1)]dv \\
&= -\int_0^1 \varepsilon f\left|\frac{f'}{f} - \frac{g'}{g}\right|^2 dv - \int_0^1 g_t(\log g + C + 1)]dv \\
&= -\int_0^1 \varepsilon f\left|\frac{f'}{f} - \frac{g'}{g}\right|^2 dv
\end{aligned}$$

since

$$\begin{aligned}
\int_0^1 g_t(\log g + C + 1)dv &= [(\int g_t dv)(\log g + C + 1)]|_0^1 - \int_0^1 (\int g_t dv)(\log g + C + 1)'dv \\
&= (\int_0^1 g dv)_t[(\log g + C + 1)|_0^1] - \int_0^1 (\int g_t dv)(\frac{g'}{g} + C')dv \\
&= 0.
\end{aligned}$$

A non positive derivative is obtained so that $e[t,f]$ is non increasing with respect to t. Similarly to the logarithmic entropy method, we need a lower bound. As mentioned before, the function g is a unique minimizer of the entropy functional $E[t, f]$ and then the first term of $e[t, f]$, $\int_0^1 (f \log f - g \log g + C(f-g))dv$, is nonnegative. The other term, $-\iint_{(0,1)\times(0,t)} C_t(s,v)(f - g)(s,v)dvds$, is bounded below for $u(t) \in (0,1)$ as well. In fact, using

$$\int_0^1 [|u(s) - v|(u(s) - v) - u(s)^2](f-g)dv \le 2\int_0^1 |f-g|dv \le 4\rho$$

for $u(t) \in (0,1)$, we have

$$\begin{aligned}
& -\iint_{(0,1)\times(0,t)} C_t(s,v)(f-g)(s,v)dvds \\
= & -\frac{K}{\varepsilon}\iint_{(0,1)\times(0,t)} \left[|u(s)-v|(u(s)-v) - u(s)^2\right] u'(s)(f-g)dvds \\
= & -\frac{K}{\varepsilon}\int_0^t u'(s)\int_0^1 \left[|u(s)-v|(u(s)-v) - u(s)^2\right](f-g)dvds \\
\ge & -\frac{K}{\varepsilon}\cdot 4\rho \cdot 2.
\end{aligned}$$

Hence, the entropy $e[t, f]$ is bounded from below.

Consider any increasing unbounded subsequence of times $(t_n)_{n\in\mathcal{N}}$. For $(t,v) \in (0,1)^2$ denote

$$f_n(t,v) = f(t_n + t, v),$$

$$g_n(t,v) = g(t_n + t, v)$$

and

$$C_n(t,v) = C(t_n + t, v).$$

The entropy sequence

$$\begin{aligned} e[t_n, f_n, g_n] := \int_0^1 (f_n \log f_n &- g_n \log g_n + C_n(t,v)(f_n - g_n))dv \\ &- \iint_{(0,1)\times(0,t)} (C_n)_t(s,v)(f_n - g_n)(s,v)dvds, \end{aligned}$$

is bounded from below and non increasing in n. We obtain

$$\lim_{n\to\infty} \frac{2}{\varepsilon} \int_0^1 I[t_n, f_n](t)dt = \lim_{n\to\infty} \int_0^1 \int_0^1 \left| (f_n^{1/2})' - \frac{1}{2}\frac{B_n(t,v)}{\varepsilon} f_n^{1/2} \right|^2 dvdt = 0. \tag{4.22}$$

Developing the square,

$$\begin{aligned} &\int_0^1 \int_0^1 \left| (f_n^{1/2})' - \frac{1}{2}\frac{B_n(t,v)}{\varepsilon} f_n^{1/2} \right|^2 dvdt \\ \geq\ & \left[\left(\int_0^1 \int_0^1 (f_n^{1/2})'^2 dvdt \right)^{1/2} - \left(\frac{1}{4} \int_0^1 \int_0^1 \frac{B_n(t,v)^2}{\varepsilon^2} f_n dvdt \right)^{1/2} \right]^2. \end{aligned}$$

Similar to the case where $u(t)$ is a constant, $\{\int_0^1 \int_0^1 \frac{B_n(t,v)^2}{\varepsilon^2} f_n dvdt\}_{n\in\mathcal{N}}$ is bounded uniformly in n for $u(t) \in [0,1]$, and therefore,

$$\int_0^1 \int_0^1 (f_n^{1/2})'^2 dvdt \leq C$$

$$\int_0^1 \int_0^1 (f_n^{1/2})^2 dvdt \leq C$$

are satisfied for some positive constant C. We can rewrite this as

$$\int_0^1 \|f_n^{1/2}(t,\cdot)\|_{1,2}^2 dt \leq 2C$$

for any n. $\|\cdot\|_{1,2}$ is the norm of the Sobolev space $H^{1,2}(0,1;dv)$. This inequality implies that for each t_n, there exists a $\tau_n \in [0,1]$ such that

$$\|f_n^{1/2}(\tau_n, v)\|_{1,2} = \|f^{1/2}(t_n + \tau_n, v)\|_{1,2} \leq \sqrt{2C}.$$

For this new time sequence $\{t'_n := t_n + \tau_n\}$, the sequence $\{f^{1/2}(t'_n, v)\}$ is bounded in $H^{1,2}(0,1;dv)$, which is compactly imbedded in $L^2(0,1;dv)$. Then we have a subsequence of $\{f^{1/2}(t'_n, v)\}$ converging in $L^2(0,1;dv)$, still denoted as $\{f^{1/2}(t'_n, v)\}$, to $f_\infty^{1/2}(v)$. This also implies that the subsequence $\{f(t'_n, v)\}$ converges to $f_\infty(v)$ in $L^1(0,1;dv)$ space by the same argument as (4.20). As for an arbitrary sequence $\{t_n\}$ we can always find a new unbounded sequence $\{t'_n\}$ satisfying the above conditions, we simply denote, without loss of generality, $f_n(t,v) = f(t'_n + t, v)$ for $(t,v) \in [0,1]^2$.

For each n, consider now the two equations:

$$\partial_t f_n + (B_n f_n - \varepsilon f'_n)' = 0 \tag{4.23}$$

$$\partial_t \tilde{f}_n + (B_\infty \tilde{f}_n - \varepsilon \tilde{f}'_n)' = 0 \tag{4.24}$$

for $(t,v) \in [0,1]^2$ where $f_n(t,v) = f(t'_n + t, v)$ and $\tilde{f}_n(t,v) = \tilde{f}(t'_n + t, v)$. $B_\infty(v) = K|u_\infty - v|(u_\infty - v)$ is the limit of $B_n(t,v) = K|u_n(t) - v|(u_n(t) - v)$ as $n \to \infty$.

We consider both equations with the same initial values for each n,

$$f_n(0,v) = \tilde{f}_n(0,v) = f(t'_n, v).$$

Basically, equation (4.24) for each n has a solution $\tilde{f}_n(t,v)$ defined on $[0,1]^2$. As the subsequence of the initial value $f(t'_n, v) \to f_\infty(v)$ in $L^1(0,1;dv)$ as $n \to \infty$, the corresponding subsequence of the solutions, still denoted as $\tilde{f}_n(t,v)$, converges to the solution $f_\infty(t,v)$ of the IVP (Initial Value Problem):

$$\begin{cases} \partial_t f_\infty + (B_\infty f_\infty - \varepsilon f'_\infty)' = 0 & \text{for}(t,v) \in [0,1]^2; \\ f_\infty(0,v) = f_\infty(v) \overset{L^1}{=} \lim_{n\to\infty} f(t'_n, v) & \text{for } v \in [0,1] \end{cases} \tag{4.25}$$

in $L^1[0,1]^2$ because of the continuous dependence of the solution on the initial value.

This also implies the sequence $\{f_n^{1/2}(t,v)\}$ converges to $f_\infty^{1/2}(t,v)$ in space $L^2[0,1]^2$ as $n \to \infty$ because

$$\begin{aligned}
\lim_{n\to\infty} \|f_n^{1/2} - f_\infty^{1/2}\|_{L^2}^2 &= \lim_{n\to\infty} \int_0^1 \int_0^1 |f_n^{1/2} - f_\infty^{1/2}|^2 dvdt \\
&= \lim_{n\to\infty} \int_0^1 \int_0^1 \frac{|f_n^{1/2} - f_\infty^{1/2}|^2}{|f_n - f_\infty|} |f_n - f_\infty| dvdt \\
&= \lim_{n\to\infty} \int_0^1 \int_0^1 \frac{|f_n^{1/2} - f_\infty^{1/2}|}{|f_n^{1/2} + f_\infty^{1/2}|} |f_n - f_\infty| dvdt \\
&\le \lim_{n\to\infty} \int_0^1 \int_0^1 |f_n - f_\infty| dvdt \\
&= \lim_{n\to\infty} \|f_n - f_\infty\| = 0
\end{aligned}$$

when both $f_n(t,v)$ and $f_\infty(t,v)$ are positive.

Setting the difference sequence between the solutions of (4.23) and (4.24)

$$d_n(t,v) = f_n(t,v) - \tilde{f}_n(t,v),$$

we have $d_n(0,v) = 0$ and $d_n(t,v)$ satisfies

$$\partial_t d_n + (B_\infty d_n - \varepsilon d_n')' = -[(B_n(t) - B_\infty) f_n]'. \tag{4.26}$$

The right-hand side of the equation (4.26) is a small quantity for large n since B_n strongly converges to B_∞. With zero initial condition, (4.26) give a sequence of solutions $d_n(t,v)$ convergent to 0 as $n \to \infty$. This shows that as n grows, the sequences of solutions of equations (4.23) and (4.24) share the same convergence: $f_n(t,v)$ converge to $f_\infty(t,v)$ in $L^1[0,1]^2$ space (and $f_n^{1/2}(t,v)$ converge to $f_\infty^{1/2}(t,v)$ in $L^2[0,1]^2$).

With the convergence of the sequence $(f_n^{1/2})_{n\in\mathcal{N}}$ to $f_\infty^{1/2}(t,v)$ in $L^2(0,1)^2$ and $u_n(t)$ to u_∞ strongly as $n \to +\infty$, we pass to the limit. For any $\varphi(t,v) \in C_0^\infty(0,1)^2$,

$$\lim_{n\to+\infty} \left| \int_0^1 \int_0^1 \left[\left((f_n^{1/2})' - \frac{B_n}{2\varepsilon} f_n^{1/2} \right) - \left((f_\infty^{1/2})' - \frac{B_\infty}{2\varepsilon} f_\infty^{1/2} \right) \right] \varphi dv dt \right|$$

$$\begin{aligned}
&= \lim_{n\to+\infty}\left|\int_0^1\int_0^1 \left[(f_n^{1/2})' - (f_\infty^{1/2})'\right]\varphi - \frac{1}{2\varepsilon}\left(B_n f_n^{1/2} - B_\infty f_n^{1/2} + B_\infty f_n^{1/2} - B_\infty f_\infty^{1/2}\right)\varphi dvdt\right| \\
&= \lim_{n\to+\infty}\left|\int_0^1\int_0^1 \left[(f_n^{1/2}) - (f_\infty^{1/2})\right]\left(-\varphi' - \frac{B_\infty}{2\varepsilon}\varphi\right) - \frac{f_n^{1/2}}{2\varepsilon}(B_n - B_\infty)\varphi dvdt\right| \\
&\le \lim_{n\to+\infty}\left(\|f_n^{1/2} - f_\infty^{1/2}\|_{L^2(0,1)^2}\left\|\varphi' + \frac{B_\infty}{2\varepsilon}\varphi\right\|_{L^2(0,1)^2} + \frac{1}{2\varepsilon}\left|\int_0^1\int_0^1 f_n^{1/2}(B_n - B_\infty)\varphi dvdt\right|\right) \\
&= 0.
\end{aligned}$$

Also since (4.22),

$$\lim_{n\to+\infty}\left\|(f_n^{1/2})' - \frac{B_n}{2\varepsilon}f_n^{1/2}\right\|_{L^2(0,1)^2} = 0,$$

we have

$$\begin{aligned}
\left\langle (f_\infty^{1/2})' - \frac{B_\infty}{2\varepsilon}f_\infty^{1/2}, \varphi\right\rangle &= \lim_{n\to+\infty}\left\langle (f_n^{1/2})' - \frac{B_n}{2\varepsilon}f_n^{1/2}, \varphi\right\rangle \\
&\le \lim_{n\to+\infty}\left\|(f_n^{1/2})' - \frac{B_n}{2\varepsilon}f_n^{1/2}\right\|_{L^2(0,1)^2}\|\varphi\|_{L^2(0,1)^2} = 0.
\end{aligned}$$

In the sense of distribution, we obtain again that $f_\infty(t,v)$ satisfies

$$(f_\infty^{1/2})' - \frac{B_\infty(v)}{2\varepsilon}f_\infty^{1/2} = 0$$

and

$$f_\infty(t,v) = f_\infty(v) = const\cdot e^{-C_\infty(v)} \qquad \text{for} \quad v \in (0,1)$$

where

$$C_\infty(v) = -\int_0^v \frac{B_\infty(w)}{\varepsilon}dw$$

is the limit of $C_n(t,v)$ as $n \to +\infty$ and the *const* $= \frac{\rho}{\int_0^1 e^{-C_\infty(v)}dv}$.

From the proof above, we notice that the subsequence of $(f_n(t,v))_{n\in\mathcal{N}}$ for any unbounded increasing sequence $(t_n)_{n\in\mathcal{N}}$ actually converges to a global time independent equilibrium $f_\infty(v)$. From Proposition 3.2 we know that the equilibrium is unique, therefore, we state that all these sequences will converge to this unique equilibrium $\frac{\rho e^{-C_\infty(v)}}{\int_0^1 e^{-C_\infty(v)}dv}$ as time grows. We state this as a theorem.

Theorem 4.7 *Given that $u(t) \in (0,1)$ converges to u_∞ as $t \to \infty$, the solution $f(t,v)$ of (4.11) is assumed as in Theorem 4.4. Then the entropy functional* e$[t,f]$ *defined by (4.21) is bounded from below by $-\frac{8K\rho}{\varepsilon}$ and non-increasing. Moreover, for any unbounded time sequence $\{t_n\}$,*

$$\lim_{n\to\infty} f(t_n+t,v) = \frac{\rho e^{-C_\infty(v)}}{\int_0^1 e^{-C_\infty(v)}dv}$$

in $L^1[0,1]^2$.

4.2.3 The real case

In the real model we introduced, $u(t)$ is actually coupled with the solution $f(t,v)$ by

$$u(t) = \frac{1}{\rho}\int_0^1 vf(t,v)dv. \tag{4.27}$$

From the points of view showed in the above simpler cases, we know that if we can prove the convergence of $u(t)$ to a constant or that of $u(t_n + t)$ to $u_\infty(t)$ for $t \in [0,1]$ loosely enough, the solution $f(t_n + t, v)$ will converge to the equilibrium after a long period of time. In [3], it is shown that the sequence $\{u_n(t) = u(t_n + t)\}$ with increasing unbounded time sequence t_n would converge uniformly to a function $u_\infty(t)$ on $[0,1]$. This kind of argument will apply to our simpler case and with this uniform convergence, we could obtain by the method in Section 4.2.2 the convergence of $f(t,v)$.

Chapter 5

Conclusions and future work

5.1 Conclusions

In this thesis, we introduced the Vlasov-Fokker-Planck type kinetic models for multi-lane traffic flow. We paid special attention to the spatially homogeneous case of the model and convergence to equilibria. We discussed fundamental diagrams and conditions for which they are multi-valued and we discovered that the monotonicity of the $\frac{B}{D}$ term leads to a single-valued fundamental diagrams. We also examined the large time behavior of the kinetic density of our spatially homogeneous VFP type model with "residual" diffusion by the entropy methods based on the results from [3] and by the application of this model on two examples.

Kinetic models seek to describe the traffic flow in a statistical way by setting kinetic density as a distribution of the speed, hence they provide more advantages in describing multi-lane traffic situations. Our Vlasov-Fokker-Planck type kinetic models include drivers' behavior like braking/acceleration, diffusion and lane-changing with non locality and time delay. This type of model allows the four properties that a multi-lane traffic model should possess as listed in the introduction.

In the spatially homogeneous case in which the traffic patterns are identical for both lanes and independent of the position, an enforced continuous and differentiable steady solution and a trivial equilibrium, which should be allowed in proper kinetic models, are obtained, provided that the equilibrated macroscopic density and the average speed are related by the fundamental diagram $u[\rho]$. An interesting property of this fundamental diagram is that in a reasonable density region, it allows multi-valued average speed. The reason for this behavior is the non-zero passing probability in the braking scenarios. Furthermore, we explored this result by finding out that the monotonicity of a $\frac{B}{D}$ term guaranteed the unique equilibrium, stated as Theorem 3.2.

Considering drivers' inaccurate estimation of the leading traffic conditions, an "overshoot" occurs in braking or accelerating. Thus, it is reasonable to introduce a small positive value ε added to the diffusion term. With this "residual" diffusion, the original degenerate drift diffusion equation becomes nondegenerate at $v = u$, which means we obtain naturally the continuity of the steady solutions. Furthermore, the trivial equilibrium is removed.

After trying and realizing the invalidity of the commonly used Maximum Principle to show the long-time behavior of the time-dependent solution of our drift diffusion equation, we shifted our attention to the entropy methods. Compared with the results from an existing paper [3] for the VFP model without "residual" diffusion, we expect the same results based on simpler assumptions on the model with "residual" diffusion. To reveal how the entropy methods are applied in the proof of asymptotic behavior of the solutions, we examined special examples with unrealistic assumptions on average speed: u is a constant and u is a convergent function. It is believed that for the real case of our model with "residual diffusion", the convergence of the solution could be obtained once the average speed function coupled with f is uniformly convergent in some sense, and this is verified in [3].

5.2 Future work

Although we introduced the general Vlasov-Fokker-Planck type kinetic model with dependencies of the braking/acceleration force, diffusion and passing probability, incorporating non locality and time delay based on realistic guidelines, we focused on the spatially homogeneous case without time delay. Therefore, the case with time-delay is worth investigation.

Another aspect of our model worth studying is stability of the equilibria in the multi-valued fundamental diagram. In this thesis, we examined the large-time behavior of the kinetic density and showed its convergence to an equilibrium. The equilibrium will be uniquely given by the macroscopic density if the term $\frac{B}{D_\varepsilon}$ is monotone in u. However, in the case where the fundamental diagram is three-valued, to which one of the three equilibrated states will the kinetic density converge? To answer this question, we have to study their stability. Combined with the analysis of the stabilities, we could also investigate the bifurcations of the fundamental diagram to see how the three-valued density interval changes according to changes of the parameters.

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