Majorization results based upon the Bernardi integral operator

Al-Shbeil, I., Srivastava, H., Arif, M., Haq, M., Khan, N., & Khan, B.

2022

This article was originally published at:
https://doi.org/10.3390/sym14071404

symmetry

MDPI

Article

Majorization Results Based upon the Bernardi Integral Operator

Isra Al-Shbeil [1,*], Hari Mohan Srivastava [2,3,4,5], Muhammad Arif [6], Mirajul Haq [6], Nazar Khan [7] and Bilal Khan [8]

1 Department of Mathematics, Faculty of Science, the University of Jordan, Amman 11942, Jordan
2 Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
3 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
4 Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan
5 Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy
6 Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan; marifmaths@awkum.edu.pk (M.A.); merajkhan054@gmail.com (M.H.)
7 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22010, Pakistan; nazarmaths@gmail.com
8 School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200241, China; bilalmaths789@gmail.com
* Correspondence: i.shbeil@ju.edu.jo

Abstract: By making use of some families of integral and derivative operators, many distinct subclasses of analytic, starlike functions, and symmetric starlike functions have already been defined and investigated from numerous perspectives. In this article, with the help of the one-parameter Bernardi integral operator, we investigate several majorization results for the class of normalized starlike functions, which are associated with the Janowski functions. We also give some particular cases of our main results. Finally, we direct the interested readers to the possibility of examining the fundamental or quantum (or $\mathfrak{q}$-) extensions of the findings provided in this work in the concluding section. However, the $(\mathfrak{p}, \mathfrak{q})$-variations of the suggested $\mathfrak{q}$-results will provide relatively minor and inconsequential developments because the additional (rather forced-in) parameter $\mathfrak{p}$ is obviously redundant.

Keywords: analytic (holomorphic or regular) functions; univalent functions; starlike functions; majorization problems; Bernardi integral operator; carathéodory class of functions

MSC: 30C45; 30C50

check for updates

Citation: Al-Shbeil, I.; Srivastava, H.M.; Arif, M.; Haq, M.; Khan, N.; Khan, B. Majorization Results Based upon the Bernardi Integral Operator. *Symmetry* **2022**, *14*, 1404. https://doi.org/10.3390/sym14071404

Academic Editors: Sergei D. Odintsov and Ioan Rașa

Received: 31 May 2022
Accepted: 29 June 2022
Published: 8 July 2022

1. Introduction and Motivations

We denote by $\mathcal{H}(\mathbb{U})$ the class of analytic (holomorphic or regular) functions (symmetric under rotation) in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\}.$$

Suppose that $\mathcal{A}$ is the subclass of $\mathcal{H}(\mathbb{U})$ defined by

$$\mathcal{A} := \left\{ f : f \in \mathcal{H}(\mathbb{U}) \quad \text{and} \quad f(z) = \sum_{k=1}^{\infty} a_k z^k \qquad (a_1 := 1) \right\}. \tag{1}$$

The family $\mathcal{S} \subset \mathcal{A}$ contains all functions, which are also univalent in $\mathbb{U}$. We denote by $\mathcal{S}^*$ and $\mathcal{K}$ the subclasses of $\mathcal{S}$ consisting of functions f, such that the range $f(\mathbb{U})$ is, respectively, starlike with respect to the origin and convex. We also let $\mathcal{P}$ be the Carathéodory class of functions p, which are holomorphic in $\mathbb{U}$ and satisfy the following conditions:

$$p(0) = 1 \qquad \text{and} \qquad \Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

To better explain our main results in this paper, we first need to clarify some relevant ideas, so we start with the idea of quasi-subordination, which was introduced by Roberston [1] in 1970. Two analytic functions Δ_1 and Δ_2 are said to satisfy the relationship of quasi-subordination, denoted by

$$\Delta_1 \prec_q \Delta_2, \tag{2}$$

if there exist functions $\vartheta, \Theta \in \mathcal{A}$, such that the function

$$\frac{z\vartheta'(z)}{\Theta(z)}$$

is holomorphic in $\mathbb{U}$ with

$$|\Theta(z)| \leqq 1, \quad \vartheta(0) = 0 \qquad \text{and} \qquad |\vartheta(z)| \leqq |z|.$$

Completing the relationship of quasi-subordination, we have

$$\Delta_1(z) = \Theta(z)\Delta_2\big(\vartheta(z)\big) \qquad (z \in \mathbb{U}). \tag{3}$$

By choosing

$$\Theta(z) = 1 \qquad \text{and} \qquad \vartheta(z) = z,$$

we obtain the relatively more familiar concept of subordination in the geometric function theory between the functions $\Delta_1(z)$ and $\Delta_2(z)$, each of which is analytic in $\mathbb{U}$. In fact, if $\Delta_2(z) \in \mathcal{S}$, then the subordination relationship for $\Delta_1(z), \Delta_2(z) \in \mathcal{A}$ implies that

$$\Delta_1(z) \prec \Delta_2(z) \iff \Delta_1(\mathbb{U}) \subset \Delta_2(\mathbb{U}) \quad \text{and} \quad \Delta_1(0) = \Delta_2(0).$$

By taking $\vartheta(z) = z$, the above definition of quasi-subordination becomes a majorization between the analytic functions $\Delta_1(z)$ and $\Delta_2(z)$, which is written as follows:

$$\Delta_1(z) \ll \Delta_2(z) \qquad (z \in \mathbb{U}),$$

if the function $\Theta \in \mathcal{A}$ exists, with $|\Theta(z)| \leqq 1$, in such a way that

$$\Delta_1(z) = \Theta(z)\Delta_2(z) \qquad (z \in \mathbb{U}). \tag{4}$$

This idea was presented by MacGregor [2].

Definition 1. *Let the function p given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

be analytic and regular in $\mathbb{U}$ and satisfy the following subordination condition:

$$p(z) \prec \frac{1 + Az}{1 + Bz} \qquad (-1 \leqq B < A \leqq 1).$$

The function $\frac{1+Az}{1+Bz}$ is referred to as the Janowski function and the above class of functions p is represented by $\mathcal{P}(A, B)$.

Geometrically, $p(z) \in \mathcal{P}(A, B)$ if and only if $p(0) = 1$ and $p(\mathbb{U})$ lies inside the domain $\Omega(A, B)$ specified by

$$\Omega(A,B) = \left\{\omega : \left|\omega - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}\right\}$$

with its diameter end points at

$$p(-1) = \frac{1-A}{1-B} \qquad \text{and} \qquad p(1) = \frac{1+A}{1+B}.$$

Definition 2 ([3])**.** *Let $\mathcal{S}^*(A,B)$ be the class of functions $f(z) \in \mathcal{A}$, with*

$$f(0) = 0 = f'(0) - 1,$$

which satisfies the following equivalence relation:

$$f(z) \in \mathcal{S}^*(A,B) \iff \frac{zf'(z)}{f(z)} \prec \mathcal{P}(A,B),$$

in which the class $\mathcal{P}(A,B)$ is given by Definition 1.

Under the conditions $M > \frac{1}{2}$ and $0 \leqq \alpha < 1$, some special selections of the parameters A and B provide the following known function classes.

1. $S^*(-1,1) = S^*$, which is the class of starlike functions with respect to the origin.
2. $S^*(1-2\alpha,-1) = S^*(\alpha) \quad (0 \leqq \alpha < 1)$, which is the class of starlike functions of order α.
3. $S^*(1,0) = S^*(1)$, which is the function class defined by
$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad (z \in \mathbb{U}).$$
4. $S^*(\alpha,0) = S^*_*(\alpha)$, which is the function class defined by
$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1).$$
5. $S^*\left(1, \frac{1}{M} - 1\right) = S^*_*(M)$, which is the function class defined by
$$\left|\frac{zf'(z)}{f(z)} - M\right| < M \qquad \left(z \in \mathbb{U};\ M > \frac{1}{2}\right).$$
6. $S^*(\alpha,-\alpha) = S^*_{**}(\alpha)$, which is the function class defined by
$$\left|\frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1}\right| < \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1).$$

In the year 1969, Bernardi [4] defined a one-parameter integral operator $\mathfrak{J}_\gamma$ for functions $f \in \mathcal{A}$ by

$$\mathfrak{J}_\gamma\{f(z)\} := \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}\, f(t)\, dt \qquad (\gamma > -1;\ f \in \mathcal{A}). \tag{5}$$

Bernardi proved that, if $f \in \mathcal{S}^*$, then $\mathfrak{J}_\gamma f$ defined by Equation (5) is univalent and starlike in the unit disk $\mathbb{U}$. Differentiating both sides of Equation (5), we have

$$(\gamma+1)f(z) = z\mathfrak{J}'_\gamma(z) + \gamma\mathfrak{J}_\gamma(z). \tag{6}$$

Motivated by the special case of the Bernardi operator $\mathfrak{J}_\gamma$ when $\gamma = 1$, Calys [5] determined the radius of univalence and the radius of the starlikeness of functions $f \in \mathcal{A}$, which satisfy the following condition:

$$\mathfrak{F}(z) := \frac{2}{z}\int_0^z \frac{f(t)g(t)}{t}\,dt, \tag{7}$$

where $\mathfrak{F}(z) \in \mathcal{S}^*$ and (i) $g(z) \in \mathcal{K}$, (ii) $g(z) \in \mathcal{S}$ and (iii) $\frac{g(z)}{z} \in \mathcal{P}$.

In many recent articles, the majorization problems were discussed for different classes of analytic and univalent functions, which are subordinate to the Janowski function $\frac{1+Az}{1+Bz}$ (see, for example, [6–12]). In this paper, we study the problem of majorization by using the Bernardi integral operator Equation (5) in the following form:

$$\mathfrak{J}_\gamma\{f(z)g(z)\} := \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)g(t)\,dt \qquad (\gamma > -1;\ f,g \in \mathcal{A}) \tag{8}$$

and its special case $(\gamma = 1)$ given by

$$\mathfrak{J}\{f(z)g(z)\} := \frac{2}{z}\int_0^z \frac{f(t)g(t)}{t}\,dt \qquad (f,g \in \mathcal{A}), \tag{9}$$

in which the function $f(z) \in \mathcal{A}$ and (i) $g(z) \in \mathcal{K}$, (ii) $g(z) \in \mathcal{S}^*$ and (iii) $\frac{g(z)}{z} \in \mathcal{P}$.

Our investigation is organized as, in Section 1, we have given some known consequences and definitions. In Section 2, some new and known Lemmas are given, which help to prove our main results. In Section 3, with the help of the one-parameter Bernardi integral operator, we investigate several majorization results for the class of normalized starlike functions, which are associated with the Janowski functions. There, we also give some particular cases of our main results.

2. A Set of Lemmas

To prove our main results, we need the following Lemmas.

Lemma 1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be regular in $\mathfrak{U}_d$, $\mathfrak{N}$ map $\mathfrak{U}_d$ onto a many-sheeted starlike region,*

$$\mathfrak{M}(0) = \mathfrak{N}(0) = 0, \frac{\mathfrak{M}'(0)}{\mathfrak{N}'(0)} = 1,\ \frac{\mathfrak{M}'(z)}{\mathfrak{N}'(z)} \in \mathcal{S}^*(A,B).$$

Then $\frac{\mathfrak{M}(z)}{\mathfrak{N}(z)} \in \mathcal{S}^*(A,B)$.

Proof. let

$$\frac{\mathfrak{M}'(z)}{\mathfrak{N}'(z)} \in \mathcal{S}^*(A,B),$$

if and only if

$$\frac{\mathfrak{M}'(z)}{\mathfrak{N}'(z)} = \frac{1+Aw(z)}{1+Bw(z)} \prec \frac{1+Az}{1+Bz},$$

where $w(z)$ is regular in $\mathfrak{U}_d$ and meets the following requirements

$$w(0) = 0,\ |w(z)| < 1,\ z \in \mathfrak{U}_d.$$

Moreover, the function $\Xi(z) = \frac{1+Az}{1+Bz}$ maps $|z| < r$ onto the following disk

$$|\Xi(z) - \sigma(r)| < \rho(r), \tag{10}$$

where

$$\sigma(r) = \frac{1-Ar^2}{1-B^2r^2} \text{ and } \rho(r) = \frac{(B-A)r}{1-B^2r^2}.$$

We can write
$$\left|\frac{\mathfrak{M}'(z)}{\mathfrak{N}'(z)} - \sigma(r)\right| < \rho(r),\ |z| < r,\ 0 < r < 1,$$
because $\frac{\mathfrak{M}'(z)}{\mathfrak{N}'(z)}$ takes value in the disk Equation (10) . Select $\phi_a(z)$, so that
$$\mathfrak{N}'(z)\phi_a(z) = \mathfrak{M}'(z) - \sigma(r)\mathfrak{N}'(z).$$

Then $|\mathfrak{M}(z)| < \rho(r)$. Fix z_0 in $\mathfrak{U}_d$. Let the joining segment of 0 and $\mathfrak{N}'(z)$ be represented by L, which lies in one sheet of the starlike image of $\mathfrak{U}_d$ by $\mathfrak{N}$. Let L^{-1} be the inverse of L under $\mathfrak{N}$. So
$$\begin{aligned}|\mathfrak{M}(z_0) - \sigma(r)\mathfrak{N}(z_0)| &= \left|\int_0^{z_0} \mathfrak{M}'(t) - \sigma(r)\mathfrak{N}'(t)dt\right| \\ &= \left|\int_{L^{-1}} \phi_a(z)\mathfrak{N}'(t)dt\right| < \rho(r)|\mathfrak{N}(z_0)|,\end{aligned}$$
that is
$$\left|\frac{\mathfrak{M}(z)}{\mathfrak{N}(z)} - \sigma(r)\right| < \rho(r),$$
that is $\frac{\mathfrak{M}(z)}{\mathfrak{N}(z)} \prec \frac{1+Az}{1+Bz}$, or equivalently $\frac{\mathfrak{M}(z)}{\mathfrak{N}(z)} \in \mathcal{S}^*(A,B)$.
□

We obtain the following Lemma by substituting $A = -1$ and $B = 1$ in the Lemma 1.

Lemma 2 ([4]). *Let $\mathfrak{M}(z)$ and $\mathfrak{N}(z)$ be regular in the $\mathfrak{U}_d$, $\mathfrak{N}(z)$ map $\mathfrak{U}_d$ onto a many-sheeted starlike region,*
$$\begin{aligned}\mathfrak{M}(0) &= \mathfrak{N}(0) = 0,\ \frac{\mathfrak{M}'(0)}{\mathfrak{N}'(0)} = 1, \\ \Re\left(\frac{\mathfrak{M}'(z)}{\mathfrak{N}'(z)}\right) &> 0.\ \text{Then } \Re\left(\frac{\mathfrak{M}(z)}{\mathfrak{N}(z)}\right) > 0.\end{aligned}$$

Lemma 3. *Let $g \in \mathcal{S}^*(A,B)$*
$$\mathcal{J}(z) = \int_0^z t^{\gamma-1} g(t)dt. \tag{11}$$

Then $\mathcal{J}$ is $(1+\gamma)-$valent starlike for $\gamma = 1, 2, 3, \cdots$, in $\mathfrak{U}_d$, and the proof is analogous to the one given in [4].

Lemma 4. *let $g \in \mathcal{S}^*(A,B)$, and $\mathfrak{J}(z) = \frac{1+\gamma}{z^\gamma}\mathcal{J}(z)$, $\gamma = 1,2,3,\cdots$, where $\mathcal{J}(z)$ is given by Equation (11). Then $\mathfrak{J}(z) \in \mathcal{S}^*(A,B)$.*

Proof. Let
$$\mathfrak{J}'(z) = \frac{1+\gamma}{z}g(z) - \frac{\gamma}{z}\mathfrak{J}(z).$$
Then
$$\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)} = \frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)}.\frac{z^\gamma}{z^\gamma} = \frac{z^\gamma g(z) - \gamma\mathcal{J}(z)}{\mathcal{J}(z)} = \frac{\mathfrak{M}(z)}{\mathfrak{N}(z)},$$
where
$$\mathfrak{M}(z) = z^\gamma g(z) - \gamma\mathcal{J}(z) \text{ and } \mathfrak{N}(z) = \mathcal{J}(z).$$

By Lemma 3, $\mathfrak{N} = \mathcal{J}$ is $(1+\gamma)$-valent starlike for $\gamma = 1, 2, 3, \cdots$, in $\mathfrak{U}_d$,
$$\frac{\mathfrak{M}'(0)}{\mathfrak{N}'(0)} = \frac{zg'(0)}{g(0)} = 1$$

and since $g \in \mathcal{S}^*(A,B)$, we have

$$\frac{\mathfrak{M}'(z)}{\mathfrak{N}'(z)} = \frac{zg'(z)}{g(z)} \in \mathcal{P}(A,B).$$

From Lemma 1 we come to a conclusion

$$\frac{\mathfrak{M}(z)}{\mathfrak{N}(z)} = \frac{zg'(z)}{g(z)} \in \mathcal{P}(A,B),$$

that is $\mathfrak{J}(z) \in \mathcal{S}^*(A,B)$. □

3. Main Results

This section starts with the following statement of our first key finding.

Theorem 1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(A,B)$. Suppose also that $f(z) \ll g(z)$ in $\mathbb{U}$. Then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq r_1),$$

where r_1 is the smallest positive root of the following equation:

$$\begin{aligned}(1-Ar)\left(1-r^2\right)[(\gamma+1)+(A+B\gamma)r]-r(A-B)\\ -2r(1+Br)[(\gamma+1)+(A+B\gamma)r]=0.\end{aligned} \tag{12}$$

Proof. Since $g \in S^*(A,B)$, by applying Lemma 4, we have $\mathfrak{J}(z) \in S^*(A,B)$. Let

$$\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}).$$

Suppose also that there exists a function $\vartheta(z)$, analytic in $\mathbb{U}$ with

$$\vartheta(0) = 0 \qquad \text{and} \qquad |\vartheta(z)| \leqq |z|$$

in such a way that

$$\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)} = \frac{1+A\vartheta(z)}{1+B\vartheta(z)}. \tag{13}$$

Differentiating both sides of Equation (13) logarithmically, we have

$$1+\frac{z\mathfrak{J}''(z)}{\mathfrak{J}'(z)} - \frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)} = \frac{z\vartheta'(z)(A-B)}{[1+B\vartheta(z)][1+A\vartheta(z)]}.$$

On the other hand, if we differentiate both sides of Equation (6) logarithmically, we have

$$\begin{aligned}\frac{zg'(z)}{g(z)} &= \left(\frac{1+\frac{z\mathfrak{J}''(z)}{\mathfrak{J}'(z)} - \frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)}}{\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)}+\gamma}+1\right)\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)}\\ &= \frac{z\vartheta'(z)(A-B)}{[1+B\vartheta(z)]](1+B\vartheta(z))\gamma+1+A\vartheta(z)]} + \frac{1+A\vartheta(z)}{1+B\vartheta(z)}.\end{aligned} \tag{14}$$

Now, using Equation (14) and a known inequality for the Schwarz function $\vartheta(z)$, we find that

$$|\vartheta(z)| \leqq |z| \text{ and } |\vartheta'(z)| \leqq \frac{1-|\vartheta(z)|^2}{1-|z|^2} = \frac{1-R^2}{1-|z|^2} \leqq \frac{1}{1-r^2} \tag{15}$$

$$\left(R := |\vartheta(z)|;\ r := |z|\right).$$

It follows that

$$\left|\frac{zg'(z)}{g(z)}\right| \geqq \frac{1-Ar}{1+Br} - \frac{(A-B)r}{(1-r^2)(1+Br)[(\gamma+1)+(A+B\gamma)r]}, \tag{16}$$

which, after some simple calculations, yields

$$\left|\frac{zg'(z)}{g(z)}\right| \geqq \frac{(1-Ar)(1-r^2)[(\gamma+1)+(A+B\gamma)r]-(A+B)r}{(1+Br)(1-r^2)[(A+B\gamma)r+(\gamma+1)]}$$

or

$$\left|\frac{g(z)}{g'(z)}\right| \leqq \frac{r(1-r^2)(1+Br)[(\gamma+1)+(A+B\gamma)r]}{(1-Ar)(1-r^2)[(\gamma+1)+(A+B\gamma)r]-r(A-B)}. \tag{17}$$

We now write Equation (4) as follows:

$$f(z) = \Theta(z)g(z) \tag{18}$$

Differentiating both sides of Equation (18), we obtain

$$f'(z) = \Theta'(z)g(z) + \Theta(z)g'(z) = g'(z)\left(\Theta'(z)\,\frac{g(z)}{g'(z)} + \Theta(z)\right). \tag{19}$$

Moreover, the Schwarz function Θ satisfies the inequality Equation (15). Therefore, by applying Equations (15) and (17) in Equation (19), we have

$$|f'(z)| \leqq \left(|\Theta(z)| + \frac{r\left(1-|\Theta(z)|^2\right)(1+Br)[(1+Br)\gamma+1+Ar]}{(1-Ar)(1-r^2)[(1+Br)\gamma+1+Ar]-r(A-B)}\right)|g'(z)|. \tag{20}$$

If we now use

$$|\Theta'(z)| = \rho \qquad (\rho \in [0,1]) \tag{21}$$

in Equation (20), we have

$$|f'(z)| \leqq \Phi_1(r,\rho)|g'(z)|,$$

where

$$\Phi_1(r,\rho) = |\Theta(z)| + \frac{r\left(1-|\Theta(z)|^2\right)(1+Br)[(\gamma+1)+(A+B\gamma)r]}{(1-Ar)(1-r^2)[(\gamma+1)+(A+B\gamma)r]-r(A-B)}.$$

To determine r_1, it is sufficient to show that

$$r_1 = \max\{r \in [0,1) : \Phi_1(r,\rho) \leqq 1 \qquad (\forall\, \rho \in [0,1])$$

or, equivalently,

$$r_1 = \max\{r \in [0,1) : \Psi_1(r,\rho) \geqq 0 \qquad (\forall\, \rho \in [0,1]),$$

where

$$\begin{aligned}\Psi_1(r,\rho) = {} & \left[1-r^2+Ar\left(r^2-1\right)\right][(\gamma+1)+(A+B\gamma)r]-r(A-B)\\ & -r(1+\rho)(1+Br)[(\gamma+1)+(A+B\gamma)r].\end{aligned}$$

By putting $\rho = 1$, the function $\Psi_1(r,\rho)$ takes on its smallest value given by

$$\begin{aligned}\Lambda(r) :&= \min\{\Psi_1(r,1)\}\\ &= (1-Ar)\left(1-r^2\right)[(\gamma+1)+(A+B\gamma)r]-r(A-B)\\ &\quad -2r(1+Br)[(\gamma+1)+(A+B\gamma)r].\end{aligned}$$

Next, we have the following inequalities:

$$\Lambda(0) = \gamma + 1 > 0$$

and

$$\Lambda(1) = -\big((A-B) + 2(1+B)[(\gamma+1) + (A+B\gamma)]\big) < 0.$$

Hence, for all $r \in [0, r_1]$, there exits r_1, such that

$$\Lambda(r) \geqq 0,$$

where r_1 is the smallest positive root of the cubic equation Equation (12). The proof of Theorem 1 is complete. □

Each of the following corollaries and consequences of Theorem 1 is worthy of note.

I. For A =−B = 1, Theorem 1 yields the following result.

Corollary 1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$. Suppose also that $f(z) \ll g(z)$ in $\mathbb{U}$. Then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq r_2),$$

where r_2 is the smallest positive root of the following cubic equation:

$$(1+r)\left(1-r^2\right)[(\gamma+1) + (\gamma-1)r] - 2r \\ - 2r(1-r)[(\gamma+1) + (-1+\gamma)r] = 0. \tag{22}$$

II. If, in Theorem 1, we take

$$A = 1 - 2\alpha \qquad \text{and} \qquad B = -1,$$

we have the following result.

Corollary 2. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(\alpha)$. Suppose also that $f(z) \ll g(z)$ in $\mathbb{U}$. Then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq r_3),$$

where r_3 is the smallest positive root of the following cubic equation:

$$[1-(1-2\alpha)r]\left(1-r^2\right)[(\gamma+1) + ((1-2\alpha)-\gamma)r] - (2-2\alpha)r \\ - 2r(1-r)[(\gamma+1) + ((1-2\alpha)-\gamma)r] = 0. \tag{23}$$

III. For $A - 1 = B = 0$, Theorem 1 yields the following result.

Corollary 3. *Let $tf \in \mathcal{A}$ and $g \in \mathcal{S}^*(1)$. Suppose also that $f(z) \ll g(z)$ in $\mathbb{U}$. Then,*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq r_4)$$

where r_4 is the smallest positive root of the following cubic equation:

$$\left(1-r^2\right)(1-r)[(\gamma+1) + r] - r - 2r(r+\gamma+1) = 0. \tag{24}$$

IV. Putting $A = \alpha$ and $B = 0$ in Theorem 1, we have the following Corollary.

Corollary 4. *Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(\alpha)$. Suppose also that $f(z) \ll g(z)$ in $\mathbb{U}$. Then*

$$|f'(z)| \leqq |g'(z)| \qquad (|z| \leqq r_5),$$

where r_5 is the smallest positive root of the following cubic equation:

$$(1-\alpha r)\left(1-r^2\right)[(\gamma+1)+\alpha r]-\alpha r-2r(\gamma+1+\alpha r)=0. \tag{25}$$

V. If we put $A=-B=\alpha$, then Theorem 1 leads us to the following result.

Corollary 5. *Let $f\in\mathcal{A}$ and $g\in\mathcal{S}^{**}(\alpha)$. Suppose also that $f(z)\ll g(z)$ in $\mathbb{U}$. Then*

$$|f'(z)|\leqq|g'(z)| \qquad (|z|\leqq r_6),$$

where r_6 is the smallest positive root of the following cubic equation:

$$(1-\alpha r)^2\left(1-r^2\right)[(\gamma+1)+(\alpha-\alpha\gamma)r]-2\alpha r \\ -2r(1-\alpha r)[(\gamma+1)+(\alpha-\alpha\gamma)r]=0. \tag{26}$$

Our second set of main results is now presented as Theorem 2 below.

Theorem 2. *Let the functions $f(z)\in\mathcal{A}$ and $g(z)$ be regular in $\mathbb{U}$ and satisfy Equation (9). Suppose also that $f(z)\ll g(z)$ in $\mathbb{U}$. Then, each of the following assertions holds true*:

1. *If $g(z)\in\mathcal{K}$ for $|z|\leqq r_7$, then*

$$|f'(z)|\leqq|g'(z)|,$$

where r_7 is the smallest positive root of the following quartic equation:

$$r^4-r^3-5r^2-r+1=0. \tag{27}$$

2. *If $g(z)\in\mathcal{S}^*$ for $|z|\leqq r_8$, then*

$$|f'(z)|\leqq|g'(z)|,$$

where r_8 is the smallest positive root of the following cubic equation:

$$3-2r^3-6r^2-2r-2r\left(1+r^2\right)=0. \tag{28}$$

3. *If $\frac{g(z)}{z}\in\mathcal{P}$ for $|z|\leqq r_9$, then*

$$|f'(z)|\leqq|g'(z)|,$$

where r_9 is the smallest positive root of the following quartic equation:

$$r^4-2r^3-4r^2-2r+2-2r\left(1+r^2\right)=0. \tag{29}$$

Proof. We give an item-wise demonstration of Theorem 2.

1. Let $g(z)\in\mathcal{K}$. Then, since $\mathfrak{J}(z)\in\mathcal{S}^*$, there exists a function ϑ, which is regular in $\mathbb{U}$, such that

$$|\vartheta(z)|\leqq r \qquad (z\in\mathbb{U})$$

and

$$\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)}=\frac{1-z\vartheta(z)}{1+z\vartheta(z)}=\frac{\xi(z)-\int_0^z\frac{\xi(t)}{t}}{\int_0^z\frac{\xi(t)}{t}\,dt},$$

where

$$\xi(z)=f(z)g(z).$$

We thus find that

$$2(1+z\vartheta(z))^{-1}\int_0^z\frac{\xi(t)}{t}\,dt=\xi(z)$$

and

$$\frac{zg'(z)}{g(z)} = \frac{2 - z\vartheta(z) - z^2\vartheta'(z)}{1 + z\vartheta(z)} - \frac{zg'(z)}{g(z)}. \tag{30}$$

Since $g(z) \in \mathcal{K}$, we have

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{1}{1 - |z|}. \tag{31}$$

Now, by using Equation (15) and Equation (31) in Equation (30), we have

$$\left|\frac{zg'(z)}{g(z)}\right| \geqq \frac{r^4 - r^3 - 5r^2 - r + 1}{(1 - r^2)(1 + r^2)}$$

or

$$\left|\frac{g(z)}{g'(z)}\right| \leqq \frac{r(1 - r^2)(1 + r^2)}{r^4 - r^3 - 5r^2 - r + 1}. \tag{32}$$

Moreover, by using u Equations (15) and (32) in Equation (19), we easily obtain

$$|f'(z)| \leqq \left(|\Theta(z)| + \frac{r\left(1 - |\Theta(z)|^2\right)(r^2 + 1)}{(r^4 - r^3 - 5r^2 - r + 1)}\right)|g'(z)|.$$

Finally, if we make use of Theorem 1 in conjunction with Equation (21), we obtain the assertion 1 of Theorem 2.

2. Let $g(z) \in \mathcal{S}^*$. Then

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{1 + |z|}{1 - |z|}. \tag{33}$$

By using Equations (15) and (33) in Equation (30), we have

$$\left|\frac{zg'(z)}{g(z)}\right| \geqq \frac{3 - 2r^3 - 6r^2 - 2r}{(1 + r^2)(1 - r^2)}$$

or

$$\left|\frac{g(z)}{g'(z)}\right| \leqq \frac{r(1 + r^2)(1 - r^2)}{3 - 2r^3 - 6r^2 - 2r}. \tag{34}$$

Moreover, by using Equations (15) and (34) in Equation (19), we easily have

$$|f'(z)| \leqq \left(|\Theta(z)| + \frac{r(1 + r^2)\left(1 - |\Theta(z)|^2\right)}{3 - 2r^3 - 6r^2 - 2r}\right)|g'(z)|.$$

Finally, if we apply Theorem 1 in conjunction with Equation (21), we obtain the assertion 2 of Theorem 2.

3. Let $\frac{g(z)}{z} \in \mathcal{P}$. Then

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{2|z|}{1 - |z|^2}. \tag{35}$$

Now, by using Equations (15) and (35) in Equation (30), we have

$$\left|\frac{zg'(z)}{g(z)}\right| \geqq \frac{r^4 - 2r^3 - 4r^2 - 2r + 2}{(1 - r^2)(1 + r^2)}$$

or

$$\left|\frac{g(z)}{g'(z)}\right| \leqq \frac{r(1 - r^2)(1 + r^2)}{r^4 - 2r^3 - 4r^2 - 2r + 2}. \tag{36}$$

Moreover, if we make use of Equations (15) and (36) in Equation (19), we easily have

$$\left|f'(z)\right| \leqq \left(|\Theta(z)| + \frac{r\left(1-|\Theta(z)|^2\right)(1+r^2)}{r^4-2r^3-4r^2-2r+2}\right)\left|g'(z)\right|.$$

Finally, by applying Theorem 1 in conjunction with Equation (21), we obtain the assertion 3 of Theorem 2. □

4. Conclusions

In this article, with the help of the familiar Bernardi integral operator $\mathfrak{J}_\gamma \;\; (\gamma > -1)$ and its special case when $\gamma = 1$, we investigated majorization and other results for such subclasses of normalized analytic (or holomorphic or regular) functions, such as, for example, starlike and convex functions associated with the Janowski functions. We also highlighted some special cases and new consequences of our main results.

In order to conclude our current study, we attract the attention of interested readers to the potential of examining the fundamental or quantum (or $\mathfrak{q}$-) extensions of the results obtained in this work. Srivastava's newest survey-cum-expository review study [13] (see also [14,15] has impacted and driven this research area. However, it has already been highlighted by Srivastava (see [13] (p. 340) and [16] (Section 5, pp. 1511–1512)), that the $(\mathfrak{p}, \mathfrak{q})$ variations of the proposed $\mathfrak{q}$-results will lead to insignificant research because the forced-in parameter $\mathfrak{p}$ is obviously redundant. In addition, in light of Srivastava's more recent explanatory article [16], interested readers should not be misled into believing that the so-called k-Gamma function gives a "generalization" of the classical (Euler's) Gamma function. Similar observations will be made for all applications of the so-called k-Gamma function, such as the so-called (k, s) extensions of Riemann–Liouville, as well as other fractional integrals and fractional derivative operators.

Author Contributions: writing—original draft preparation, I.A.-S., H.M.S., M.A., M.H., N.K. and B.K.; writing—review and editing, I.A.-S., H.M.S., M.A., M.H., N.K. and B.K.; All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Roberston, M.S. Quasi-subordination and coefficient conjectures. *Bull. Am. Math. Soc.* **1970**, *76*, 1–9.
2. MacGregor, T.H. Majorization by univalent functions. *Duke Math. J.* **1967**, *34*, 95–102. [CrossRef]
3. Janowski, W. Some extremal problems for certain families of analytic functions. *Ann. Pol. Math.* **1973**, *28*, 297–326. [CrossRef]
4. Bernardi, S.D. Convex and starlike univalent functions. *Trans. Am. Math. Soc.* **1969**, *135*, 429–446. [CrossRef]
5. Calys, E.G. The radius of univalence and starlikeness of some classes of regular functions. *Compos. Math.* **1971**, *23*, 467–470.
6. Altintaş, O.; Özkan, Ö.; Srivastava, H.M. Majorization by starlike functions of complex order. *Complex Var. Theory Appl.* **2001**, *46*, 207–218. [CrossRef]
7. Altintaş, O.; Srivastava, H.M. Some majorization problems associated with p-valently starlike and convex functions of complex order. *East Asian Math. J.* **2001**, *17*, 175–183.
8. Kim, Y.C.; Lee, K.S.; Srivastava, H.M. Certain classes of integral operators associated with the Hardy spaces of analytic functions. *Complex Var. Theory Appl.* **1992**, *20*, 1–12. [CrossRef]
9. Tang, H.; Aouf, M.K.; Deng, G.-T. Majorization problems for certain subclasses of meromorphic multivalent functions associated with the Liu–Srivastava operator. *Filomat* **2015**, *29*, 763–772. [CrossRef]
10. Tang, H.; Deng, G.-T. Majorization problems for certain classes of multivalent analytic functions related with the Srivastava–Khairnar–More operator and exponential function. *Filomat* **2018**, *32*, 5319–5328. [CrossRef]
11. Tang, H.; Deng, G.-T.; Li, S.-H. Majorization properties for certain classes of analytic functions involving a generalized differential operator. *J. Math. Res. Appl.* **2013**, *33*, 578–586.

12. Tang, H.; Srivastava, H.M.; Li, S.-H.; Deng, G.-T. Majorization results for subclass of starlike functions based on the sine and cosine functions. *Bull. Iran. Math. Soc.* **2020**, *46*, 381–391. [CrossRef]
13. Srivastava, H.M. Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
14. Khan, B.; Liu, Z.-G.; Shaba, T.G.; Araci, S.; Khan, N.; Khan, M.G. Applications of q-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomials. *J. Math.* **2022**, *2022*, 8162182. [CrossRef]
15. Srivastava, H.M.; Khan, B.; Khan, N.; Tahir, M.; Ahmad, S.; Khan, N. Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function. *Bull. Sci. Math.* **2021**, *167*, 102942. [CrossRef]
16. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformatioons. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.