FORMAL JUSTICE AND FUNCTIONAL EQUATIONS

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Abstract. We define and classify all continuous functions $m : (\mathbb{R}_+)^N \longrightarrow \mathbb{R}_+$ which, if used as wage functions, will satisfy the first isomorphic requirement of formal proportional justice.

1. INTRODUCTON.

The requirement of formal justice in treatments of people leads to some fundamental problems in sociology. First, one has to address the question of measurement of a "compensable property." Such a property might, for example, be the (number of years of) education of an individual; the experience in a certain profession or trade; or, on the negative side, a crime that has been committed by the individual and that calls for an appropriate (and just) punishment. These examples already show that measurement of compensable properties is a difficult endeavor and may be hard (or even impossible) on a numerical scale. Nevertheless, measurement theory is all about assigning real numbers in some way to objects (or properties). We call such an assignment a

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measurement system if it is (a) reliable, *i.e* if it is invariant with respect to the user of the system, and if it is (b) accurate, *i.e.* if relations between objects are reflected by relations between their assigned numbers.

More details about measurement theory can be found in [Sołtan]. Here, we shall focus on wage systems, which are measurement systems assigning treatments (wages) to persons. We call a wage system ***formally just*** if

(i) both the concerned set of people and the set of possible wages are recognized as relational systems,

(ii) the wage system is reliable,

(iii) the wage system is accurate, and

(iv) the wage system possesses an accurate inverse in the sense that to any two different wages there are well-defined different "prototypes" of people who qualify for these wages.

Following [Sołtan], we suggest that (iii) means that the wage system is a homomorphism from the system of (prototypes of) persons to the positive real numbers (expressive homomorphic requirement); (iv) means that this homomorphism is actually an isomorphism (justificatory homomorphic requirement). Sołtan refers to both together as the ***first isomorphic requirement*** of formal justice.

2. RATIO SCALES AND A FIRST FUNCTIONAL EQUATION.

Little mathematics can be done in the generality which we have allowed so far. We will now focus on the case where both the compensable property and the wage are measured on a *ratio scale*; more precisely, we assume that the property is measured by real numbers $x \geq 0$; a prototype possessing the property (by definition) to degree 1 has been chosen, and the relation of all other prototypes to this one is defined to be the quotient $\frac{x}{1}$ $(= x)$. The relation between people measured as $x > 0$ and $y > 0$ is then the quotient $\frac{x}{y}$.

The wage system will now be a function $m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined on the (measured) values of the compensable property. If the relation between any two wages is also given by their ratios and if we assume in addition that the prototype measuring in at 1 gets a salary measured at 1, then the first isomorphic requirement entails that m must satisfy the functional equation

$$m\left[\frac{x}{y}\right] = \frac{m(x)}{m(y)} \tag{1}$$

(*i.e.*, the ratio between the measured compensable properties is mapped to the ratio between their assigned wages).

Remarks. There are several problems with the approach.

(a) The major drawback is that there is no rigorous reason why one should use ratios as the appropriate relation for either the comparison of the degree to which individuals possess a compensable property, or for the comparison of their wages. There are many possible alternatives: One could consider differences rather than ratios, or one could consider differences on one side and ratios on the other and arrive at other functional equations. The use of completely different relations is also conceivable.

One thing that points to ratio scales is tradition. The principle of proportional justice, which goes back to Aristotle, suggests that

$$\frac{m(x)}{m(y)} = \frac{x}{y}\,, \tag{2}$$

i.e. $m(x) = \text{const.}\ x$.

We shall see that (2) is a more restrictive version of justice than the generalization of (1) which we propose in Section 3.

Besides such historical reasons, there are other arguments that can be used to justify the use of ratios; we will discuss them elsewhere, because they have nothing to do with mathematics.

(b) A smaller problem is raised by the assumption that $m(1) = 1$. The assumption, as well as Equation (1), is obviously not invariant under scale changes

$$x' = \epsilon x, \qquad m' = \delta \cdot m.$$

This lack of scale invariance is a major flaw of (1), and we will address it in the next section.

(c) Third, we have apparently not taken into account market influences. This point will be addressed in more detail in Section 6.

We now settle the question which functions $m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfy (1). The result is nothing but an exercise in real analysis, but we include the proof for the sake of completeness.

Theorem 1. *The only functions* $m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *which satisfy* (1) *and are continuous*[2] *at* $x = 1$ *are of the form*

$$m(x) = x^p \qquad (p \in \mathbb{R}). \tag{3}$$

Proof. As $m\left[\frac{1}{y}\right] = \frac{1}{m(y)}$, (1) is equivalent to $m(x \cdot y) = m(x) \cdot m(y)$. Then $h := \ell n\, m$ satisfies $h(x \cdot y) = h(x) + h(y)$, and if $x = e^t$, $y = e^s$, $g(t) := h(e^t)$ satisfies

$$g(t+s) = g(t) + g(s). \tag{4}$$

As m is continuous at 1, g is continuous at 0. By (4), g must satisfy

$$g(q) = c \cdot q \tag{5}$$

for all rational q, where $c = g(1)$. By the continuity at 0, (5) must also hold for all other $q \in \mathbb{R}_+$: If $x = \lim_{n \to \infty} q_n$,

$$\begin{aligned} g(x) &= g(x-q_n) + g(q_n) \\ &= g(x-q_n) + c \cdot q_n . \end{aligned}$$

[2]The continuity condition can presumably be relaxed (to measurability), but such a generalization is, we believe, of no importance for practical applications.

The right hand side converges to $c \cdot x$ as $n \longrightarrow \infty$, and so $g(x) = c \cdot x$ for all $x \geq 0$. Finally,

$$m(x) = e^{h(x)} = e^{g(\ell n\ x)}$$
$$= e^{c\ \ell n\ x} = x^c.$$

The proof is complete. ∎

3. A SCALE INVARIANT FORMULATION OF FORMAL JUSTICE.

We now address the problem of (the lack of) scale invariance which we already pointed out in Remark (b), above. The problem has to do with the arbitrariness of the choices of scale for both the compensable property and the salary. Formal justice should in no way depend on whether we measure time in hours or minutes, or whether we measure a salary in dollars or cents. However, the equations (1) and (3) are certainly not invariant under such changes (except in the trivial and irrelevant case $p = 0$):

If $\tilde{m}(x) = C \cdot m(x)$ and m satisfies (1), $\tilde{m}$ satisfies a functional equation

$$\tilde{m}\left[\frac{x}{y}\right] = C\,\frac{\tilde{m}(x)}{\tilde{m}(y)}$$

rather than (1). And if $\bar{m}(x) = m(\epsilon x)$, we have

$$\bar{m}\left[\frac{x}{y}\right] = m(\epsilon) \cdot \frac{\bar{m}(x)}{\bar{m}(y)} .$$

Of course, this lack of invariance is also evident from the explicit form $m(x) = x^p$: $\tilde{m}(x) = C\,x^p$ and $\bar{m}(x) = \epsilon^p\,x^p$ are not of the admitted type for (1) unless $C = 1$ and $p = 0$. Conversely, it follows that whenever m is a continuous function $\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which satisfies

$$m\left[\frac{x}{y}\right] = C\,\frac{m(x)}{m(y)} ,$$

(*i.e.* $m(x) = C\,x^p$, according to Theorem 1), then the rescaled function $\tilde{m}(x) = \frac{1}{C}\,m(x)$ satisfies (1).

Inspired by these observations, we now attempt a rigorous definition of formal justice for ratio scales.

Definition and Corollary. A function $m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is suitable as a formally just wage function if there is a positive constant C such that

$$m\left[\frac{x}{y}\right] = C\,\frac{m(x)}{m(y)}$$

for all $x,y > 0$. If such a function is continuous for at least one $x > 0$, then m is necessarily of the form

$$m(x) = C \cdot x^p, \quad \text{where} \quad p \in \mathbb{R}.$$

4. TWO COMPENSABLE PROPERTIES

Things get a lot more interesting if, as in real situations, more than one compensable property is to be considered. For simplicity, we first consider the case of two such properties.

Formal justice in this case means that a wage function $m(x,y)$ must be such that all individuals which possess property x to the same degree must be treated justly with respect to y, and *vice versa*. Under the simple assumption that

$$m : \mathbb{R}_+ \times \mathbb{R}_+$$

be continuous, Theorem 1 then implies that there are functions $C(x) > 0$, $K(y) > 0$ and $p(x)$, $q(y)$ such that

$$\begin{aligned} m_x(y) &:= m(x,y) = C(x) \cdot y^{p(x)} \\ \text{and} \quad m_y(x) &:= m(x,y) = K(y) \cdot x^{q(y)} \end{aligned} \tag{6}$$

We next investigate which functions $m(x,y)$ satisfy equations (6).

From (6),

$$\ell n\ C(x) + p(x) \cdot \ell n\ y = \ell n\ K(y) + q(y) \cdot \ell n\ x\ . \tag{7}$$

Let $c(x) = \ell n\ C(x)$, $k(y) = \ell n\ K(y)$ and put $x = 1$ in (7). It follows that

$$c(1) + p(1) \ell n\ y = k(y),$$

i.e.

$$K(y) = e^{c(1)} \cdot y^{p(1)},$$

and

$$C(x) = e^{k(1)} \cdot x^{q(1)}.$$

So

$$m(x,y) = K(1) \cdot x^{q(1)} \cdot y^{p(x)}$$
$$= C(1) \cdot y^{p(1)} \cdot x^{q(y)}.$$

For $x = y = 1$ we get $K(1) = C(1) = :C$, so

$$x^{q(1)} \cdot y^{p(x)} = y^{p(1)} \cdot x^{q(y)}$$

or

$$y^{p(x)-p(1)} = x^{q(y)-q(1)}.$$

Taking logarithms and separating variables, we find

$$(p(x)-p(1)) \ \ell n \ y = (q(y)-q(1)) \ \ell n \ x,$$

i.e.

$$\frac{\ell n \ x}{p(x) - p(1)} = \text{const.} = \frac{\ell n \ y}{q(y) - q(1)} ,$$

(the cases $p(x) \equiv p(1)$ or $q(y) = q(1)$ are trivial), or

$$p(x) = p(1) + \text{const.} \ \ell n \ x$$
$$q(y) = q(1) + \text{const.} \ \ell n \ y.$$

Summarizing, we find

$$m(x,y) = C \ x^{q(1)} \cdot y^{p(1)} \cdot y^{\text{const.} \ \ell n \ x}$$

or

$$m(x,y) = C \ y^{p(1)} \cdot x^{q(1)} \cdot x^{\text{const.} \ \ell n \ y}.$$

Let $p := p(1)$, $q := q(1)$, $\alpha :=$ const., then

$$m(x,y) = C\, x^q\, y^p\, e^{\alpha\, \ell n\, x \cdot \ell n\, y}. \tag{8}$$

We have proved

THEOREM 2. *The only continuous functions* $m : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *which permit simultaneous representations*

$$m(x,y) = C(x) \cdot y^{p(x)}$$
$$m(x,y) = K(y) \cdot x^{q(y)}$$

are given by (8).

Remark. The continuity assumption can clearly be weakened — it is enough to require that for every $y > 0$, $m_y(x) = m(x,y)$ is formally just and continuous for <u>one</u> x, and that for every $x > 0$, $m_x(y) = m(x,y)$ is formally just and continuous for <u>one</u> y; our discussion shows that m must then be continuous everywhere.

5. N COMPENSABLE PROPERTIES

Before we consider the general case, we review $N = 1$ and $N = 2$. For $N = 1$, possible functions were $m(x) = C\, x^p$ $(C > 0,\ p \in \mathbb{R})$; for $N = 2$, we found

$$m(x,y) = C\, x^p\, y^q\, e^{\alpha\, \ell n\, x\, \ell n\, y} \qquad (C > 0).$$

We observe that a formally just wage (or treatment) function can depend on two parameters (C and p) for $N = 1$ and on 4 parameters (C, q, p and α) for $N = 2$. The obvious guess is that there are 2^N parameters for general N, and we verify this now.

First, it is useful to rewrite $C = e^{p_0}$, $p = p_1$, $q = p_2$, $\alpha = p_{1,2}$, and $x^{p_1} = e^{p_1 \ell n\, x}$ *etc.* We find

$$m(x) = e^{p_0} \cdot e^{p_1 \ell n\, x} \quad \text{for} \quad N = 1,$$

$$m(x,y) = e^{(p_0 + p_1 \ell n\, x + p_2 \ell n\, y + p_{1,2} \ell n\, x \, \ell n\, y)} \quad \text{for} \quad N = 2.$$

It is now easy to guess the general case. It must be

$$m(x_1,\cdots,x_N) = \exp\left[\sum_{M\subset\{1,\cdots,N\}} p_M \prod_{i\in M} \ell n\, x_i\right]. \tag{9}$$

Here $\sum\limits_{M\subset\{1,\cdots,N\}}$ denotes a summation over all the 2^N subsets of $\{1,\cdots,N\}$ and the p_M are real parameters indexed by M.

If $|M| = N - 1$ and i is the only index <u>not</u> in M, let $X_M = (x_1,\cdots,x_{i-1}, x_{i+1},\cdots,x_N) \in \mathbb{R}^{N-1}$ and $m_{x_M}(x_i) = m(x_1,\cdots,x_i,\cdots,x_N)$. It is straightforward to check that all the functions given by (9) satisfy equations

$$m_{x_M}(x_i) = C_i(x_M)\, e^{q_i(x_M)\cdot \ell n\, x_i}, \tag{10}$$

$$i = 1,\cdots,N, \quad M = \{1,\cdots,N\}\backslash\{i\}, \quad C_i(x_M) > 0.$$

THEOREM 3. *Every continuous function*

$$m : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+$$

which satisfies (10) *is of the form* (9).

Proof. We prove this by induction over N. By Theorems 1 and 2, the assertion holds for $N = 1$ and 2. Suppose it holds for all $L < N$. If the variable x_N is the new one, then, by (10),

$$m(x_1,\cdots,x_N) = C_N(x_1,\cdots,x_{N-1})\ e^{q_N(x_1,\cdots,x_{N-1})\cdot \ell n\ x_N}. \tag{11}$$

If we consider x_N as a parameter, then, on the other hand, by the inductive assumption,

$$m(x_1,\cdots,x_N) = \exp\left[\sum_{M\subset\{1,\cdots,N-1\}} p_M(x_N) \prod_{i\in M} \ell n\ x_i\right]. \tag{12}$$

Our job will be done if we can identify the functions C_N, q_N and all p_M. This is again done inductively. Let $c_N = \ell n\ C_N$ for convenience. The exponents in the right hand sides of (11) and (12) must be equal. We equate them and set first $x_N = 1$. The result is

$$c_N(x_1,\cdots,x_{N-1}) = \sum_{M\subset\{1,\cdots,N-1\}} p_M(1) \prod_{i\in M} \ell n\ x_i\ ,$$

i.e. we have (from (11) and (12)) the identity

$$q_N(x_1,\cdots,x_{N-1})\cdot \ell n\ x_N = \sum_{M\subset\{1,\cdots,N-1\}} (p_M(x_N)-p_M(1)) \prod_{i\in M} \ell n\ x_i . \qquad (13)$$

Let $x_1 = x_2 = \cdots = x_{N-1} = 1$ in (13). We get

$$p_\emptyset(x_N) = p_\emptyset(1) + q_N(1,\cdots,1)\ell n\ x_N \qquad (14)$$

($\emptyset$ denotes the empty set).

We have to introduce some notation before we can proceed. For $M \subset \{1,\cdots,N\}$ and $x \in (\mathbb{R}_+)^N$, let $x|_M \in (\mathbb{R}_+)^N$ be defined by

$$(x|_M)_i = \begin{cases} x_i & \text{if } i \in M \\ 1 & \text{if } i \notin M \end{cases} .$$

We claim that there are real constants K_M and $C(M)$ (indexed by subsets of $\{1,\cdots,N-1\}$) such that

$$q_N(X|_M) = \sum_{\tilde{M}\subset M} K_{\tilde{M}} \cdot \prod_{i\in\tilde{M}} \ell n\ x_i \qquad (15)$$

and that

$$p_M(x_N) - p_M(1) = C(M)\ \ell n\ x_N \quad \text{for each } M \subset \{1,\cdots,N-1\}. \qquad (16)$$

The assertion is true (by (14)) for $M = \emptyset$. We prove it by induction over the cardinality of M. Suppose (15) and (16) hold for all M such that $|M| \le \ell < N$. To make the induction step, choose an M such that $|M| = \ell$, a $k \notin M$, and consider

$$q_N(x|_{M\cup\{k\}})\ \ell n\ x_N = \sum_{\tilde{M}\subset M\cup\{k\}} (p_{\tilde{M}}(x_N) - p_{\tilde{M}}(1)) \prod_{i\in\tilde{M}} \ell n\ x_i . \tag{17}$$

(This follows from (13)).

The right hand side of (17) decomposes into

$$\sum_{\tilde{M}\subsetneq M\cup\{k\}} (p_{\tilde{M}}(x_N) - p_{\tilde{M}}(1)) \prod_{i\in\tilde{M}} \ell n\ x_i$$

$$+ (p_{M\cup\{k\}}(x_N) - p_{M\cup\{k\}}(1)) \prod_{i\in M\cup\{k\}} \ell n\ x_i .$$

The induction hypothesis applies to the first term and yields

$$\sum_{\tilde{M}\subsetneq M\cup\{k\}} C(\tilde{M})\ \ell n\ x_N \prod_{i\in\tilde{M}} \ell n\ x_i .$$

We insert this into (17) and separate x_N from the other variables: The equation

$$\ell n\ x_N \left[q_N(X|_{M\cup\{k\}}) - \sum_{\tilde{M}\subsetneq M\cup\{k\}} C(\tilde{M}) \prod_{i\in\tilde{M}} \ell n\ x_i \right]$$

$$= (p_{M\cup\{k\}}(x_N) - p_{M\cup\{k\}}(1)) \prod_{i\in M\cup\{k\}} \ell n\ x_i$$

implies that

$$\frac{\ell n\, x_N}{(p_{M\cup\{k\}}(x_N) - p_{M\cup\{k\}}(1))} = \frac{\prod_{i\in M\cup\{k\}} \ell n\, x_i}{q_N(x|_{M\cup\{k\}}) - \sum_{\tilde{M}\subsetneq M\cup\{k\}} C(\tilde{M}) \prod_{i\in\tilde{M}} \ell n\, x_i}. \qquad (18)$$

Note that the case where $p_{M\cup\{k\}}$ is constant is trivial. In (18), the variable x_N occurs only on the left, all others only on the right, so both sides must be equal to a constant. Equations (15) and (16) now follow with little effort, and so does the assertion of the theorem. ■

Corollary. *Let* $m : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+$ *be such that for every* $i \in \{1,\cdots,N\}$, $M = \{1,\cdots,N\}\setminus\{i\}$ *and* $x_M \in \mathbb{R}_+^{N-1}$, *the function* $m_{x_M}(x_i)$ *(see* (10) *for the notation) is continuous for at least one* $x_i \in \mathbb{R}_+$ *and satisfies a functional equation*

$$m_{x_M}\left[\frac{x}{y}\right] = C(x_M)\,\frac{m_{x_M}(x)}{m_{x_M}(y)} .$$

Then m *is of the form* (9).

6. CONCLUSIONS

We have discovered that in an investigation where a wage function $m : (\mathbb{R}_+)^N \longrightarrow \mathbb{R}_+$ (which is observed in reality) satisfies our requirement of formal justice, 2^N parameters have to be identified (one for each subset of $\{1,\cdots,N\}$. Because of the exponential growth of 2^N with N, it seems hopeless to test formal justice when numerous compensable properties are considered simultaneously. For $N = 2$ or 3, however, it is a feasible task and well worth a sociological study.

We also suggest that market influences may actually be included in the functions permitted. Higher demand for a certain compensable property must not necessarily violate the first isomorphic principle; it can simply lead to a change in the parameters which determine m.

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