On the Cyclic Structure of the Peripheral Point Spectrum of Perron-Frobenius Operators.

by

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BSc, University of Victoria, 2005

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

Master of Science

in the Department of Mathematics and Statistics

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Abstract

The Frobenius-Perron operator $P_T : L^1[0,1] \to L^1[0,1]$ and the Koopman operator $K_T : L^\infty[0,1] \to L^\infty[0,1]$ for a given nonsingular transformation $T : [0,1] \to [0,1]$ can be shown to have cyclic spectrum by referring to the theory of lattice homomorphisms on a Banach lattice. In this paper, it is verified directly that the peripheral point spectrum of P_T and the point spectrum of K_T are fully cyclic. Under some restrictions on T, P_T is known to be a well defined linear operator on the Banach space $BV[0,1]$. It is also shown that the peripheral point spectrum of $P_T : BV[0,1] \to BV[0,1]$ is fully cyclic.

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Acknowledgements

I wish to thank my supervisor for showing me guidance throughout my studies at UVic, without which I surely would have faltered long ago.

I wish to thank Professor Anthony Quas for helpful discussions streamlining the proof of Theorem 4.1.5.

I wish to thank my supervisory committee for having the patience to allow me to finish at my own pace.

I wish to thank the Department of Mathematics and Statistics at UVic for providing a large portion of the funding necessary for me to be able to complete this thesis.

Chapter 1

Introduction

1.1 Frobenius-Perron Operators

Consider the finite measure space $([0,1], \mathcal{B}, m)$, where $[0,1]$ is the unit interval on the real line, $\mathcal{B}$ is the set of all Borel sets on $[0,1]$, and m is the Lebesgue measure on $\mathcal{B}$. If T is a measurable transformation of the interval such that $m(T^{-1}A) = 0$ whenever $m(A) = 0$, then there exists a linear operator $P_T : L^1[0,1] \to L^1[0,1]$, called the Frobenius-Perron operator, defined implicitly by

$$\int_A P_T f\, dm = \int_{T^{-1}A} f\, dm \tag{1.1}$$

for any measurable set A in $\mathcal{B}$ and any integrable function f in $L^1[0,1]$.

P_T is an important operator because of its uses in finding absolutely continuous invariant measures (acim's). If $f \in L^1[0,1]$ is a positive fixed point of P_T, ie. a positive eigenfunction corresponding to the eigenvalue 1, then $d\mu = f\, dm$ is an acim since, for any measurable set A,

$$\mu(A) = \int_A f\, dm = \int_A P_T f\, dm = \int_{T^{-1}A} f\, dm = \mu(T^{-1}A).$$

Thus knowledge of the spectrum of P_T may help in finding invariant densities for T.

Since P_T has operator norm equal to 1 on $L^1[0,1]$, if 1 is in the spectrum of P_T it is in the peripheral spectrum. Schaefer in [24] has shown that, for certian conditions on a positive operator on a space of functions, the peripheral point spectrum is fully cyclic. Here, it is verified that $P_T : L^1[0,1] \to L^1[0,1]$ satisfies these conditions. Theorem 4.1.5 shows that if P_T has point spectrum on the unit circle S^1 in the complex plane, then 1 is guaranteed to be in the point spectrum and if f is an eigenfunction for some eigenvalue α with modulus equal to 1, then $|f|$ is guaranteed to be a (positive) fixed point of P_T.

1.2 Outline

In this thesis results regarding the structure of the spectrum of the Frobenius-Perron and Koopman operators for various interval maps are discussed. The purpose of this is to write a paper that is a short, self-contained collection of these results presented in [24]. In order to establish the necessary background, Chapter 2 recovers some basic facts for positive operators (and specifically Frobenius-Perron and Koopman operators) on complex Banach lattices.

Section 2.1 focuses on positive operators on Banach lattices. The primary spaces considered throughout this work are introduced and shown to be complex Banach lattices, and a few simple characterizations for positive operators and lattice homomorphisms are given. This section follows closely the work of Schaefer in [24], and proofs are given here for the sake of completeness.

In Section 2.2, the Frobenius-Perron and Koopman operators are introduced. The Frobenius-Perron operator is shown to be a positive operator according to the definition given in Section 2.1, and likewise the Koopman operator is shown to be a lattice homomorphism. Some other well-known facts about the Frobenius-Perron and Koopman operators are gathered and verified. Also, when acting on the space of functions of bounded variation, the Frobenius-Perron operator was shown in [17] to satisfy the Lasota-Yorke inequality, which is stated here. Excellent sources on the Frobenius-Perron and Koopman operators include [8] and [16].

In Section 2.3, a collection of results in spectral theory are given. These results are applied in the latter chapters to positive matrices and the Frobenius-Perron and Koopman operators. Definitions and results in this section are drawn from [3], [7] and [9], again with proofs for completeness.

Chapter 3 develops the spectral theory of $n \times n$ matrices over $\mathbb{C}$. In Section 3.1, positive matrices are considered. The two main results, Theorem 3.1.6 and Theorem 3.1.11, are due to Schaefer in [24]. These two theorems state that the peripheral spectrum of a positive matrix is cyclic, and under certain conditions, it is fully cyclic. Next a simple proof is given to show that the peripheral spectrum of a column-stochastic matrix is fully cyclic. Excellent books on the subject of positive matrices are [1], [12] and [25].

Section 3.2 focuses on the spectral theory of lattice homomorphisms on $\mathbb{C}^n$, which turn out to have very simple matrices in the standard basis. The main result is Theorem 3.2.2, which states that the spectrum of an $n \times n$ lattice homomorphism is fully cyclic. This section is a direct application of the results of Schaefer in [24].

Chapter 4 turns to the spectral theory of the Frobenius-Perron and Koopman operators. In Section 4.1, the Frobenius-Perron operator on L^1 and Koopman operator on L^∞ are studied. The main results in this section are analogous to the main results in Chapter 3. Theorem 4.1.5 states that the peripheral point spectrum of the Frobenius-Perron operator is fully cyclic, which is the analogue of Theorem 3.1.17 for the Frobenius-Perron operator on L^1. This theorem is a direct application of the theory of Schaefer in [24], as was noted by Ding *et al.* in [5]. Theorem 4.1.10 states that the point spectrum of the Koopman operator is fully cyclic, which is the analogue of Theorem 3.2.2 for the Koopman operator on L^∞.

Section 4.2 presents simple cases for which the spectrum of the Frobenius-Perron operator is either the entire unit disk $\mathbb{D}$ or a subset of the boundary of the unit disk $\delta\mathbb{D}$. This result is due to Ding *et al.* in their work [5]. This illustrates the importance of considering the Frobenius-Perron operator as acting on the space of functions of bounded variation, on which the Frobenius-Perron operator is a quasi-compact operator, in order to find isolated spectral points.

In Section 4.3, the Frobenius-Perron operator is considered to be acting on the space of functions of bounded variation. The main result is Theorem 4.3.2, which is a restatement of Theorem 4.1.5 for the space of functions of bounded variation. This Theorem is part of a larger result by Rychlik in [23], which is due again to results in [24]. Other excellent sources of information on the spectral structure of Frobenius-Perron and Koopman operators include [4], [6], [18] and [19].

Chapter 2

Preliminaries

2.1 Banach Lattices and Lattice Homomorphisms

2.1.1 Real Banach Lattices

Definition 2.1.1. A *partially ordered set* (or *poset* for short) is an ordered pair $(X, \preceq)$ where X is a set and $\preceq$ is a relation on X such that, for any x, y, z in X:

- $x \preceq x$ (reflexive);
- if $x \preceq y$ and $y \preceq x$, then $x = y$ (antisymmetric);
- if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (transitive).

$x \prec y$ will mean that $x \preceq y$ and $x \neq y$, $x \succeq y$ will mean that $y \preceq x$ and $x \succ y$ will mean $y \prec x$. The $\preceq$ in $(X, \preceq)$ will usually be suppressed when it is understood, so that the poset $(X, \preceq)$ is simply referred to as the poset X.

Definition 2.1.2. For any subset $B \subset X$, the set B is called *bounded above* if there exists c in X such that $b \preceq c$ for all b in B, and c is called an *upper bound* for B. Similarly, a set $B \subset X$ is *bounded below* if there exists c in X such that $b \succeq c$ for all b in B, and c is called a *lower bound* for B. If there exists an upper bound c_0 on the set B such that c_0 is also a lower bound for the set of upper bounds of B, then by the antisymmetric property it is the unique upper bound to do this, called the *supremum* of B, and is denoted $c_0 = \sup B$. Similarly, if there exists a lower bound such that it is a supremum for all lower bounds of B, it is unique, called the *infimum* of B, and is denoted $\inf B$.

Definition 2.1.3. A *lattice* is a poset $(X, \preceq)$ in which there exists a supremum and infimum for every pair of elements a, b from X. The supremum of a and b is called their *join*, and written $\sup\{a, b\} = a \vee b$, and the infimum is called their *meet*, and written $\inf\{a, b\} = a \wedge b$.

Definition 2.1.4. An *ordered vector space* is a poset $(X, \preceq)$ where X is a vector space over $\mathbb{R}$ such that, for any x, y, z in X:

- if $x \preceq y$ then $x + z \preceq y + z$;
- if $x \preceq y$ then $tx \preceq ty$ for all $t \geq 0$.

A *vector lattice* is an ordered vector space $(X, \preceq)$ that is a lattice with respect to the order $\preceq$ on X.

If $x \preceq y$ for x, y in X, then for $t \geq 0$ in $\mathbb{R}$, $tx \preceq ty$ by the second item in Definition 2.1.4. Then by two applications of the first item in Definition 2.1.4,

$$tx \preceq ty \Rightarrow 0 \preceq ty - tx \Rightarrow -ty \preceq -tx.$$

Then given $x \preceq y$ for x, y in X and $s < 0$, there exists $t > 0$ such that $s = -t$, and thus $sy \preceq sx$.

In an ordered vector space, a vector x is called nonnegative if $0 \preceq x$, and positive if $0 \prec x$. For any x in a vector lattice X, there exist the nonnegative vectors $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := (-x) \vee x$. They are called the positive component, negative component and modulus, respectively.

Proposition 2.1.5. *Let X be a vector lattice. For any x in X, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Moreover, for y, z in X and $t \geq 0$ in $\mathbb{R}$:*

1. $x + (y \vee z) = (x + y) \vee (x + z)$;
2. $x + (y \wedge z) = (x + y) \wedge (x + z)$;
3. $t(x \vee y) = (tx) \vee (ty)$;
4. $t(x \wedge y) = (tx) \wedge (ty)$.

Proof. Let x, y, z be vectors in X. Since

$$z \preceq y \vee z \text{ and } y \preceq y \vee z,$$

by Definition 2.1.4 it follows that

$$x + z \preceq x + (y \vee z) \text{ and } x + y \preceq x + (y \vee z).$$

Suppose that $x + z \preceq v$ and $x + y \preceq v$ for some v in X. Then, again from 2.1.4

$$z \preceq v - x \text{ and } y \preceq v - x,$$

so that

$$y \vee z \preceq v - x$$

or

$$x + (y \vee z) \preceq v.$$

Therefore, $x + (y \vee z) = (x + y) \vee (x + z)$, which gives item *1*. The equalities *2*, *3* and *4* can be shown similarly.

By use of *1*,

$$x + x^- = x + \big((-x) \vee 0\big) = (x - x) \vee (x + 0) = 0 \vee x = x^+,$$

or $x = x^+ - x^-$, giving the first assertion, and by use of *1*, *3* and the fact that $x = x^+ - x^-$,

$$x^+ + x^- = x + 2x^- = x + (-2x \vee 0) = (-x) \vee x = |x|,$$

which finishes the proof. $\square$

Remark. The proof that $x + (y \vee z) = (x + y) \vee (x + z)$ in Proposition 2.1.5 does not depend on the fact that $(y \vee z)$ is the supremum of finitely many elements of X, so that when $\sup A$ exists, $x + \sup A = \sup(x + A)$ by the same method of proof ($x + A = \{x + a : a \in A\}$). Similarly, for x in X and $t \geq 0$, $x + \inf A = \inf(x + A)$, $t \sup A = \sup(tA)$ and $t \inf A = \inf(tA)$ (where $tA = \{ta : a \in A\}$).

Proposition 2.1.6. *Let X be a vector lattice. $|\cdot|$ as defined above satisfies:*

1. $|x| \succeq 0$ *for all* $x \in X$*;*
2. $|x| = 0$ *if and only if* $x = 0$*;*
3. $|tx| = |t||x|$ *for all* $t \in \mathbb{R}$ *and* $x \in X$*;*
4. $|x + y| \preceq |x| + |y|$ *for all* $x, y \in X$*.*

Proof. 1. For any x in X, $x^+ \succeq 0$ and $x^- \succeq 0$ implies that

$$0 \preceq x^+ \preceq x^+ + x^- = |x|.$$

2. $|x| = 0$ if and only if both $x^+ = 0$ and $x^- = 0$. But $x = 0$ if and only if both $x^+ = 0$ and $x^- = 0$.

3. Let x be in X. If $t \geq 0$, by Proposition 2.1.5

$$|tx| = (tx)^+ + (tx)^- = t(x^+) + t(x^-) = |t||x|.$$

If $t < 0$

$$|tx| = (tx)^+ + (tx)^- = (-t)((-x)^+) + (-t)((-x)^-) = |t|x^- + |t|x^+ = |t||x|.$$

4. For x, y in X, $x \preceq x^+$ and $y \preceq y^+$, so that

$$(x + y)^+ = (x + y) \vee 0 \preceq (x^+ + y) \vee 0 \preceq (x^+ + y^+) \vee 0 = x^+ + y^+.$$

Similarly, $(x + y)^- \preceq x^- + y^-$, so that

$$|x + y| = (x + y)^+ + (x + y)^- \preceq x^+ + x^- + y^+ + y^- = |x| + |y|.$$

$\square$

Definition 2.1.7. Let X be a vector lattice. Suppose that X is a normed vector space with the norm $\|\cdot\|$ satisfying $\|x\| \leq \|y\|$ whenever $|x| \preceq |y|$. Then the vector lattice X is called a *normed vector lattice* and the norm $\|\cdot\|$ is called a *lattice norm*. If X is complete in the norm, it is called a *Banach lattice*.

Example 2.1.8. The vector space $\mathbb{R}$ together with the order $\leq$ is an ordered vector space, since $\leq$ satisfies the two conditions in Definition 2.1.4. This is the only order on $\mathbb{R}$ that will be considered throughout. For any two numbers $x, y \in \mathbb{R}$, their max and min both exist, so $(\mathbb{R}, \leq)$ is a vector lattice.

$|\cdot|$ on $\mathbb{R}$ is called the Euclidean norm, and $\mathbb{R}$ is complete in this norm. For any norm $\|\cdot\|$ on $\mathbb{R}$, $\|1\| = a > 0$ so that for any x in $\mathbb{R}$

$$\|x\| = \|1 \cdot x\| = \|1\||x| = a|x|.$$

Therefore every norm on $\mathbb{R}$ may be obtained as a positive multiple of $|\cdot|$. Thus $\mathbb{R}$ is complete in any norm since $a|\cdot|$ is equivalent to $|\cdot|$ whenever $a > 0$. Then any norm $a|\cdot|$ applied to $\mathbb{R}$ is a lattice norm since $a|x| \leq a|y|$ whenever $|x| \leq |y|$, so that $\mathbb{R}$ with any norm is a Banach lattice.

Also, it is noted here that the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \bigcup \{\infty, -\infty\}$ form a lattice with the extensions

$$\begin{aligned} -\infty &< x, \\ x &< \infty, \\ -\infty &< \infty, \end{aligned}$$

for all real numbers x. It is usual for the operations of multiplication and addition to be extended in a commutative and associative way to $\overline{\mathbb{R}}$ in an attempt to create a vector space. For x in $\mathbb{R}$,

- $\infty + x = \infty$ and $-\infty + x = -\infty$,
- $\infty x = \infty$ if $x > 0$ and $\infty x = -\infty$ if $x < 0$,
- $-\infty x = -\infty$ if $x > 0$ and $-\infty x = \infty$ if $x < 0$,
- $\infty + \infty = \infty$ and $-\infty - \infty = -\infty$,
- $\infty\infty = (-\infty)(-\infty) = \infty$ and $(-\infty)\infty = -\infty$.

However, it is not so simple when dealing with the terms $\infty 0$ and $\infty - \infty$. By convention, $0(\pm\infty) = 0$ unless stated otherwise. The expression $\infty - \infty$ will remain undefined. Thus $\overline{\mathbb{R}}$ is not a vector space and hence not a vector lattice.

The reason for discussing this extension is that $\overline{\mathbb{R}}$ is complete in the sense of suprema and infima of subsets: $\sup A$ and $\inf A$ exist in $\overline{\mathbb{R}}$ for any subset A of $\overline{\mathbb{R}}$. Thus given a sequence x_n in $\overline{\mathbb{R}}$ (and more specifically $\mathbb{R}$) the limit superior and limit inferior exist in $\overline{\mathbb{R}}$, where

$$\limsup x_n = \inf_{k \geq 1} \big(\sup_{n \geq k} x_n \big) \text{ and } \liminf x_n = \sup_{k \geq 1} \big(\inf_{n \geq k} x_n \big).$$

Example 2.1.9. The vector space $\mathbb{R}^n$ has the order $\leq$ defined for any x, y in $\mathbb{R}^n$ by $x \leq y$ if and only if $x_i \leq y_i$ for all i in $\{1, \ldots, n\}$. Then, for any x, y, z in $\mathbb{R}^n$ and

$t \geq 0$,

$$\begin{aligned} x \leq y &\Leftrightarrow x_i \leq y_i \ \forall i \in \{1, \ldots, n\} \\ &\Leftrightarrow x_i + z_i \leq y_i + z_i \ \forall i \in \{1, \ldots, n\} \\ &\Leftrightarrow x + z \leq y + z, \end{aligned}$$

and

$$\begin{aligned} x \leq y &\Leftrightarrow x_i \leq y_i \ \forall i \in \{1, \ldots, n\} \\ &\Leftrightarrow tx_i \leq ty_i \ \forall i \in \{1, \ldots, n\} \\ &\Leftrightarrow tx \leq ty, \end{aligned}$$

so that $\mathbb{R}^n$ is an ordered vector space with this order. For any $x, y \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$, $(x \vee y)_i = x_i \vee y_i$ and $(x \wedge y)_i = x_i \wedge y_i$ both exist, so that $(\mathbb{R}^n, \leq)$ is a vector lattice. Hence $\mathbb{R}^n$ with $\leq$ has an absolute value given by $|x|_i = |x_i|$ for a vector x in the ith coordinate.

The Euclidian norm on $\mathbb{R}^n$ is defined by

$$||x|| = \sqrt{\sum_{i=1}^{n} (x_i)^2},$$

which is a lattice norm since $|x| \leq |y|$ implies that $\sum_{i=1}^{n} (x_i)^2 \leq \sum_{i=1}^{n} (y_i)^2$. $\mathbb{R}^n$ is complete in this norm, so that $\mathbb{R}^n$ is a Banach lattice.

Example 2.1.10. Let $[0, 1]$ be the set of real numbers x satisfying $0 \leq x \leq 1$, $\mathcal{B}$ be the Borel sets on $[0, 1]$ and m be the Lebesgue measure on $\mathcal{B}$. Let M be the set of all real-valued $\mathcal{B}$-measurable functions on $[0, 1]$. As is shown in [22], M is a vector space under pointwise addition and scalar multiplication from $\mathbb{R}$.

Similar to Example 2.1.9, M has the ordering $f \leq g$ if $f(x) \leq g(x)$ for all x in $[0, 1]$ and f, g in M, and under this ordering M is an ordered vector space. For any f_1, f_2 in M, let

$$g(x) = \sup(f_1(x), f_2(x)) = f_1(x) \vee f_2(x)$$

for all x in $[0, 1]$. Then for any α in $\mathbb{R}$,

$$\{x \in [0, 1] : g(x) < \alpha\} = \{x \in [0, 1] : f_1(x) < \alpha\} \bigcap \{x \in [0, 1] : f_2(x) < \alpha\},$$

which is a measurable set since f_1 and f_2 are measurable functions. Therefore g is measurable. Similarly, $h(x) = f_1(x) \wedge f_2(x)$ is measurable, and so $f_1 \vee f_2$ and $f_1 \wedge f_2$ exist as measurable functions. Thus M with this order is a vector lattice. This order then gives the absolute value of a measurable function f as

$$|f|(x) = |f(x)|$$

for all x in $[0, 1]$.

As discussed in [11], M can be extended to the set of all $\overline{\mathbb{R}}$-valued $\mathcal{B}$-measurable functions, where a set B of $\overline{\mathbb{R}}$ is measurable if $B \bigcap \mathbb{R}$ is a Borel set. Care must be taken to avoid the term $\infty - \infty$, so the functions are only allowed to take the value ∞ on a set of measure zero, ie. they are finite almost everywhere. In this case, scalar multiplication from $\mathbb{R}$ is well-defined following the rules from Example 2.1.8. Further, if $h(x) = 0$ whenever $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise, then h is a measurable function and M has been extended to a new real vector space.

Example 2.1.11. The vector space $L^1_{\mathbb{R}}([0,1], \mathcal{B}, m)$, or briefly $L^1_{\mathbb{R}}$, is the space of equivalence classes of real-valued (hence the subscript $\mathbb{R}$) $\mathcal{B}$-measurable functions on $[0,1]$ satisfying $\int_0^1 |f(x)|\, dm(x) < \infty$. Two such functions f, g are equivalent if $f = g$ almost everywhere (ie. equal except on a set of measure zero). Given $f \in L^1$, members $\hat{f}$ of the equivalence class of f are called versions (or representatives) of f. Define an order on L^1 as follows: for any f, g in $L^1_{\mathbb{R}}$, $f \leq g$ if and only if there exists versions $\hat{f}$ and $\hat{g}$ of f and g, respectively, satisfying $\hat{f}(x) \leq \hat{g}(x)$ for every x in $[0,1]$. Similar to the case $\mathbb{R}^n$, $L^1_{\mathbb{R}}$ with this order is an ordered vector space. In fact, these equivalence classes are of extended real-valued measurable functions, which are finite almost everywhere.

For a full development of the Banach space $L^1_{\mathbb{R}}$ for a measure space $(X, \mathcal{A}, \mu)$ and some standard results, see [22]. From this point on, elements of the ordered vector space $L^1_{\mathbb{R}}$ will be referred to simply as functions, and it will be specifically mentioned when a version of an equivalence class is needed.

For any f, g in $L^1_{\mathbb{R}}$, let $\hat{f}, \hat{g}$ be versions and define $\widehat{f \vee g} = \hat{f} \vee \hat{g}$ in accordance with Example 2.1.10. Then the versions $\widehat{f \vee g}$ are equal almost everywhere no matter the choice of versions $\hat{f}, \hat{g}$, since changing $\hat{f}$ and $\hat{g}$ on sets of measure zero changes $\widehat{f \vee g}$ on a subset of the union of these two sets, which has measure zero. Thus the versions $\widehat{f \vee g}$ determine a class h. Since $\hat{f} \leq \widehat{f \vee g}$ and $\hat{g} \leq \widehat{f \vee g}$, $f \leq h$ and $g \leq h$. If $f \leq h_0$ and $g \leq h_0$ for some function h_0, then there exists versions $\hat{f}, \hat{g}, \hat{h_0}$ such that $\hat{f} \leq \hat{h_0}$ and $\hat{g} \leq \hat{h_0}$, so that $\widehat{f \vee g} \leq \hat{h_0}$ and therefore $h \leq h_0$. Thus h can be properly relabelled as $f \vee g$. $f \wedge g$ is similarly defined.

Since $|(\hat{f} \vee \hat{g})(x)| \leq |\hat{f}(x)| + |\hat{g}(x)|$ for all x in $[0,1]$ and versions $\hat{f}, \hat{g}$ of f, g,

$$\int_0^1 |(\hat{f} \vee \hat{g})(x)|\, dx \leq \int_0^1 (|\hat{f}(x)| + |\hat{g}(x)|)\, dx = \int_0^1 |\hat{f}(x)|\, dx + \int_0^1 |\hat{g}(x)|\, dx < \infty,$$

so that versions of $f \vee g$ are integrable and thus the join is in $L^1_{\mathbb{R}}$. Similarly, the meet satisfies $|(\hat{f} \wedge \hat{g})(x)| \leq |\hat{f}(x)| + |\hat{g}(x)|$ for all x in $[0,1]$, so that $f \wedge g$ is in $L^1_{\mathbb{R}}$, and $L^1_{\mathbb{R}}$ is then a vector lattice. This order gives the absolute value $|f|$ for a function f of $L^1_{\mathbb{R}}$ to be the equivalence class of all versions of the form $|\hat{f}(x)|$ for every x in $[0,1]$, where $\hat{f}$ is a version of f. The norm on $L^1_{\mathbb{R}}$ is given by $\|f\|_1 = \int_0^1 |f|\, dm$ and if $|f| \leq |g|$ then $\|f\|_1 \leq \|g\|_1$, so that $\|\cdot\|_1$ is a lattice norm. $L^1_{\mathbb{R}}$ with this norm is known to be a Banach space, so that $L^1_{\mathbb{R}}$ is now a Banach lattice.

Example 2.1.12. The vector space $L^\infty_{\mathbb{R}}([0,1], \mathcal{B}, m)$, or briefly $L^\infty_{\mathbb{R}}$, of equivalence classes of real-valued $\mathcal{B}$-measurable functions f on $[0,1]$ satisfying $|\hat{f}(x)| \leq M$ for all

x in $[0,1]$ for some $M>0$ and some version $\hat{f}$ of f, has the order $\leq$ defined for any f,g in $L^\infty_{\mathbb{R}}$ by $f\leq g$ if and only if $\hat{f}(x)\leq\hat{g}(x)$ for every x in $[0,1]$ for some versions $\hat{f}$ and $\hat{g}$ of f and g, respectively. Similar to $L^1_{\mathbb{R}}$, $L^\infty_{\mathbb{R}}$ is an ordered vector space. Again, elements of $L^\infty_{\mathbb{R}}$ will be referred to as functions, and versions will be specified when needed.

For functions f,g in $L^\infty_{\mathbb{R}}$ the functions $f\vee g$ and $f\wedge g$ are defined as in Example 2.1.11. Again, for any two functions f,g in $L^\infty_{\mathbb{R}}$ with versions $\hat{f},\hat{g}$, $|\hat{f}(x)\vee\hat{g}(x)|\leq|\hat{f}(x)|+|\hat{g}(x)|$ so that, if M is an essential bound for f and M' is an essential bound for g, then

$$|\hat{f}(x)\vee\hat{g}(x)|\leq|\hat{f}(x)|+|\hat{g}(x)|\leq M+M'$$

for almost every x in $[0,1]$. Therefore $f\vee g$ is in $L^\infty_{\mathbb{R}}$. Similarly, $|\hat{f}\wedge\hat{g}|\leq|\hat{f}|+|\hat{g}|$, so that $f\wedge g$ is in $L^\infty_{\mathbb{R}}$ and $L^\infty_{\mathbb{R}}$ is a vector lattice. The order gives the absolute value $|f|$ for a function f of $L^\infty_{\mathbb{R}}$ to be the equivalence class of all versions of the form $|\hat{f}(x)|$ for every x in $[0,1]$, where $\hat{f}$ is a version of f. The norm on $L^\infty_{\mathbb{R}}$ is given by $\|f\|_\infty=\inf\{M:|f|\leq M\mathbf{1}\}$ and if $|f|\leq|g|$, $\|f\|_\infty\leq\|g\|_\infty$, so that $\|\cdot\|_\infty$ is a lattice norm. $L^\infty_{\mathbb{R}}$ with this norm is known to be a Banach space, so that $L^\infty_{\mathbb{R}}$ is now a Banach lattice.

Example 2.1.13. The vector space $BV_{\mathbb{R}}$ of equivalence classes of real-valued measurable functions on $[0,1]$ satisfying $\bigvee\hat{f}=\bigvee_0^1\hat{f}<\infty$ for some version $\hat{f}$ of f is a subspace of $L^1_{\mathbb{R}}$ and has the order $\leq$ defined for any f,g in $BV_{\mathbb{R}}$ by $f\leq g$ if and only if $\hat{f}(x)\leq\hat{g}(x)$ for every x in $[0,1]$ for some versions $\hat{f}$ and $\hat{g}$ of f and g, respectively. $BV_{\mathbb{R}}$ is in fact a subspace of both $L^1_{\mathbb{R}}$ and $L^\infty_{\mathbb{R}}$. Similar to $L^1_{\mathbb{R}}$ and $L^\infty_{\mathbb{R}}$, $BV_{\mathbb{R}}$ is an ordered vector space.

For any function f in $BV_{\mathbb{R}}$, the function $f\vee 0$ is measurable, as was previously noted in Example 2.1.11. Let $0=b_1<b_2<\ldots<b_k=1$ be any finite partition of the interval. For i in $\{1,\ldots,k\}$, consider $|(\hat{f}\vee 0)(b_i)-(\hat{f}\vee 0)(b_{i-1})|$ for some version $\hat{f}$ of f. If $\hat{f}(b_i)\geq 0$ and $\hat{f}(b_{i-1})\geq 0$, then

$$|(\hat{f}\vee 0)(b_i)-(\hat{f}\vee 0)(b_{i-1})|=|\hat{f}(b_i)-\hat{f}(b_{i-1})|.$$

If $\hat{f}(b_i)\geq 0$ and $\hat{f}(b_{i-1})<0$, then

$$|(\hat{f}\vee 0)(b_i)-(\hat{f}\vee 0)(b_{i-1})|=\hat{f}(b_i)-0<\hat{f}(b_i)+(-\hat{f}(b_{i-1}))=|\hat{f}(b_i)-\hat{f}(b_{i-1})|,$$

and similarly for $\hat{f}(b_i)<0$ and $\hat{f}(b_{i-1)}\geq 0$. If $\hat{f}(b_i)<0$ and $\hat{f}(b_{i-1})<0$, then

$$|\hat{f}\vee 0(b_i)-\hat{f}\vee 0(b_{i-1})|=0+0\leq|\hat{f}(b_i)-\hat{f}(b_{i-1})|.$$

Therefore, $\sum_{i=1}^k|\hat{f}\vee 0(b_i)-\hat{f}\vee 0(b_{i-1})|\leq\sum_{i=1}^k|\hat{f}(b_i)-\hat{f}(b_{i-1})|$, so that $\bigvee_0^1(\hat{f}\vee 0)\leq\bigvee_0^1\hat{f}$, so that this particular version of $f\vee 0$ has bounded variation. Thus $f\vee 0$ is in $BV_{\mathbb{R}}$. Then for any g,h in $BV_{\mathbb{R}}$, the function $g+\big((h-g)\vee 0\big)$ is in $BV_{\mathbb{R}}$. But if $\hat{g}(x)\geq\hat{h}(x)$ for versions $\hat{g},\hat{h}$ of g,h respectively,

$$\Big(\hat{g}+\big((\hat{h}-\hat{g})\vee 0\big)\Big)(x)=\hat{g}(x)+0=\hat{g}(x),$$

and if $\hat{g}(x) < \hat{h}(x)$,

$$\Big(\hat{g} + \big((\hat{h} - \hat{g}) \vee 0\big)\Big)(x) = \hat{g}(x) + (\hat{h} - \hat{g})(x) = \hat{g}(x) - \hat{g}(x) + \hat{h}(x) = \hat{h}(x),$$

so that $g \vee h = g + \big((h-g) \vee 0\big)$ is in $BV_{\mathbb{R}}$. Similarly $g \wedge h$ is in $BV_{\mathbb{R}}$ and $BV_{\mathbb{R}}$ is a vector lattice. This gives the absolute value $|f|$ for a function f of $BV_{\mathbb{R}}$ to be the equivalence class of all versions of the form $|\hat{f}(x)|$ for every x in $[0,1]$, where $\hat{f}$ is a version of f.

For a function f in $BV_{\mathbb{R}}$, as is shown in [23], there exists a version f_0 of f with minimal variation, and in writing $\bigvee f := \bigvee f_0$ unless explicitly stated otherwise. The norm on $BV_{\mathbb{R}}$ is given by $\|f\|_{BV_{\mathbb{R}}} = \bigvee f + \|f\|_1$. From [23] $BV_{\mathbb{R}}$ is a Banach space with this norm. However, $\|\cdot\|_{BV_{\mathbb{R}}}$ is not a lattice norm, since

$$\hat{f}(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2 - 2x & \text{if } 1/2 < x \leq 1, \end{cases}$$

is bounded above by the function $\mathbf{1}$, but

$$\|\mathbf{1}\|_{BV_{\mathbb{R}}} = \bigvee \mathbf{1} + \|\mathbf{1}\|_1 = 0 + 1 = 1$$

and

$$\|\hat{f}\|_{BV_{\mathbb{R}}} = \bigvee \hat{f} + \|\hat{f}\|_1 = 2 + 1/2 = 5/2 > 1.$$

Therefore $BV_{\mathbb{R}}$ with $\|\cdot\|_{BV_{\mathbb{R}}}$ is a Banach space and a vector lattice, but not a Banach lattice.

These orderings of $\mathbb{R}^n$, $L^1_{\mathbb{R}}$, $L^\infty_{\mathbb{R}}$ and $BV_{\mathbb{R}}$ are called canonical orderings, and are the only orderings of these vector lattices that will be considered throughout the remainder of this work.

Example 2.1.14. Consider the set F of all real-valued continuous piecewise linear functions on the interval $[0,1]$ with finitely many pieces. This set forms a real vector space under pointwise addition of functions and pointwise multiplication by scalars. Then, the canonical order given again by $f \leq g$ if and only if $f(x) \leq g(x)$ for all x in $[0,1]$ makes F an ordered vector space. Also, for any two functions f and g in F, $f \vee g$ and $f \wedge g$ exist as elements of F, so that F is a vector lattice. Similar to the previous examples, the absolute value of a function f in F is given by $|f|(x) = |f(x)|$ for all x in $[0,1]$.

Definition 2.1.15. Let X, Y be ordered vector spaces and let $S : X \to Y$ be a linear map. Then S is called positive (written $S \geq 0$) if $Sx \succeq 0$ for all $x \succeq 0$.

Proposition 2.1.16. *Let X, Y be vector lattices and let $S : X \to Y$ be a linear map. S is positive if and only if $|Sx| \preceq S|x|$ for all x in X.*

Proof. Suppose S is positive. Let x be any vector in X. Then

$$0 \preceq |x| - x \text{ and } 0 \preceq |x| + x,$$

so that

$$0 \preceq S(|x| - x) \text{ and } 0 \preceq S(|x| + x),$$

or

$$Sx \preceq S|x| \text{ and } -Sx \preceq S|x|,$$

since S is positive and linear. Since $0 \preceq |x|$, $0 \preceq S|x|$, so that

$$0 \preceq S|x| \text{ and } -Sx \preceq S|x|,$$

which implies $(Sx)^- = -Sx \vee 0 \preceq S|x|$. Similarly, $(Sx)^+ = Sx \vee 0 \preceq S|x|$, so that

$$|Sx| = (Sx)^+ \vee (Sx)^- \preceq S|x|,$$

and the condition is met.

Conversely, suppose that $|Sx| \preceq S|x|$ for all x in X. Then, for any $x \succeq 0$, $|x| = x$ and

$$0 \preceq |Sx| \preceq S|x| = Sx,$$

so that S is positive. □

Definition 2.1.17. Let X, Y be vector lattices and let $S : X \to Y$ be a linear map. Then, S is called a lattice homomorphism if $S(x_1 \vee x_2) = (Sx_1) \vee (Sx_2)$ and $S(x_1 \wedge x_2) = (Sx_1) \wedge (Sx_2)$ for all x_1, x_2 in X.

Proposition 2.1.18. *Let X, Y be vector lattices and let $S : X \to Y$ be a linear map. If S is a lattice homomorphism, then S is positive.*

Proof. Suppose S is a lattice homomorphism. For any $x \succeq 0$, $x = x \vee 0$ so that

$$Sx = S(x \vee 0) = (Sx) \vee (S0) = (Sx) \vee 0 \succeq 0.$$

Therefore $Sx \succeq 0$ and S is positive. □

Proposition 2.1.19. *Let X, Y be vector lattices and let $S : X \to Y$ be a linear map. Then, S is a lattice homomorphism if and only if $|Sx| = S|x|$ for all x in X.*

Proof. Suppose S is a lattice homomorphism. Then, for any x in X,

$$\begin{aligned} S(x^+) &= S(x \vee 0) \\ &= S(x) \vee S(0) \\ &= S(x) \vee 0 \\ &= (Sx)^+, \end{aligned}$$

and similarly $S(x^-) = (Sx)^-$, so that

$$|Sx| = (Sx)^+ + (Sx)^- = S(x^+) + S(x^-) = S(x^+ + x^-) = S|x|.$$

Conversely, assume that $|Sx| = S|x|$ for all x in X. By Proposition 2.1.16, S is positive. Since $x \preceq x^+$ for all x in X, $Sx \preceq S(x^+)$ so that $(Sx)^+ \preceq S(x^+)$. Similarly

$(Sx)^- \preceq S(x^-)$, but by hypothesis $|Sx| = S|x|$, or $(Sx)^+ + (Sx)^- = S(x^+) + S(x^-)$, so that $(Sx)^+ = S(x^+)$ and $(Sx)^- = S(x^-)$ for any x in X. Therefore, for x, y in X,

$$\begin{aligned} S(x \vee y) &= S\big(x + \big((y-x) \vee 0\big)\big) \\ &= Sx + S\big((y-x)^+\big) \\ &= Sx + \big(S(y-x)\big)^+ \\ &= Sx + (Sy - Sx)^+ \\ &= (Sx) \vee (Sy). \end{aligned}$$

Similarly $S(x \wedge y) = (Sx) \wedge (Sy)$, and S is a lattice homomorphism. □

Example 2.1.20. Consider the linear map $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$S(x, y) = (x + y, x + y).$$

Then S is positive since for x and y both nonnegative, $x + y$ is nonnegative. But S is not a lattice homomorphism since

$$|S(-1, 1)| = |(1 - 1, 1 - 1)| = (0, 0)$$

and

$$S|(-1, 1)| = S(1, 1) = (2, 2) \neq (0, 0).$$

Linear transformations of this type will be discussed in detail in Chapter 3.

2.1.2 Complexification of Real Banach Lattices

For a real vector space V, the complexification $V_{\mathbb{C}}$ of V is the additive group $V \oplus \mathrm{i}V$ together with scalar multiplication from $\mathbb{C}$ defined by

$$(a + \mathrm{i}b)(x + \mathrm{i}y) = ax - by + \mathrm{i}(ay + bx).$$

For a (real) vector lattice X, which has the modulus function $|x| = x^+ + x^-$, the function $|\cdot| : X_{\mathbb{C}} \to X$ is defined by

$$|z| = |x + \mathrm{i}y| := \sup_{0 \le \theta < 2\pi} |\cos\theta x + \sin\theta y| \tag{2.1}$$

if the supremum exists in X. Note that $z \in X_{\mathbb{C}}$ implies that $|z| \in X$.

Remark. The supremum in (2.1) may not always exist in X. An example of when existence of the modulus may fail is given in Example 2.1.29.

Definition 2.1.21. A *complex vector lattice* is the complexification $X_{\mathbb{C}}$ of a real vector lattice X such that the modulus (2.1) exists for all $z = x + \mathrm{i}y$ in $X_{\mathbb{C}}$.

Proposition 2.1.22. *Let $X_{\mathbb{C}}$ be a complex vector lattice. Then $|\cdot|$ as defined in* (2.1) *satisfies:*

1. $|z| \succeq 0$ *for all* $z \in X_{\mathbb{C}}$;

2. $|z| = 0$ *if and only if* $z = 0 + i0$*;*

3. $|\alpha z| = |\alpha||z|$ *for all* $\alpha \in \mathbb{C}$ *and* $z \in X_{\mathbb{C}}$*;*

4. $|z_1 + z_2| \preceq |z_1| + |z_2|$ *for all* $z_1, z_2 \in X_{\mathbb{C}}$*.*

Proof. 1. For any $z = x + \mathrm{i}y$ in $X_{\mathbb{C}}$, $|\cos\theta x + \sin\theta y| \succeq 0$ for all θ by Proposition 2.1.6, so that

$$|z| = \sup_{0 \le \theta < 2\pi} |\cos\theta x + \sin\theta y| \succeq 0.$$

2. For $z = x + \mathrm{i}y$,

$$\begin{aligned} |z| = 0 \quad &\Rightarrow \quad \sup_{0 \le \theta < 2\pi} |\cos\theta x + \sin\theta y| = 0 \\ &\Rightarrow \quad |\cos 0x + \sin 0y| \preceq 0 \text{ and } |\cos\frac{\pi}{2}x + \sin\frac{\pi}{2}y| \preceq 0 \\ &\Rightarrow \quad x = 0 \text{ and } y = 0 \\ &\Rightarrow \quad z = 0 + \mathrm{i}0, \end{aligned}$$

where Proposition 2.1.6 was used. For $z = 0 + \mathrm{i}0$,

$$\begin{aligned} |z| &= \sup_{0 \le \theta < 2\pi} |\cos\theta 0 + \sin\theta 0| \\ &= \sup_{0 \le \theta < 2\pi} |0| \\ &= 0, \end{aligned}$$

again, by using Proposition 2.1.6.

3. For any $z = x + \mathrm{i}y$ in $X_{\mathbb{C}}$ and any $\alpha = |\alpha|(\cos\phi + \mathrm{i}\sin\phi)$ in $\mathbb{C}$,

$$\alpha z = |\alpha|(\cos\phi x - \sin\phi y + \mathrm{i}\cos\phi y + \mathrm{i}\sin\phi x),$$

so that

$$\begin{aligned} |\alpha z| &= \sup_{0 \le \theta < 2\pi} \Big| |\alpha| \big(\cos\theta(\cos\phi x - \sin\phi y) + \sin\theta(\cos\phi y + \sin\phi x) \big) \Big| \\ &= \sup_{0 \le \theta < 2\pi} |\alpha| \big| (\cos\theta\cos\phi + \sin\theta\sin\phi)x + (\sin\theta\cos\phi - \cos\theta\sin\phi)y \big| \\ &= |\alpha| \sup_{0 \le \theta < 2\pi} |\cos(\theta - \phi)x + \sin(\theta - \phi)y| \\ &= |\alpha| \sup_{0 \le \theta < 2\pi} |\cos\theta x + \sin\theta y| \\ &= |\alpha||z|, \end{aligned}$$

where Proposition 2.1.6 and the remarks following Proposition 2.1.5 were used.

4. For $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ in $X_{\mathbb{C}}$,

$$\begin{aligned} |z_1 + z_2| &= \sup_{0\le\theta<2\pi} |\cos\theta(x_1+x_2) + \sin\theta(y_1+y_2)| \\ &\preceq \sup_{0\le\theta<2\pi} \big(|\cos\theta x_1 + \sin\theta y_1| + |\cos\theta x_2 + \sin\theta y_2|\big) \\ &\preceq \sup_{0\le\theta_1<2\pi} |\cos\theta_1 x_1 + \sin\theta_1 y_1| + \sup_{0\le\theta_2<2\pi} |\cos\theta_2 x_2 + \sin\theta_2 y_2| \\ &= |z_1| + |z_2|, \end{aligned}$$

where Proposition 2.1.6 and the remarks following Proposition 2.1.5 were used. $\square$

Definition 2.1.23. Let $X_{\mathbb{C}}$ be a complex vector lattice such that the vector lattice X has a lattice norm $\|\cdot\|$. Then a norm $\|\cdot\|_{\mathbb{C}}$ on $X_{\mathbb{C}}$ satisfying $\|z\|_{\mathbb{C}} = \big\||z|\big\|$ for all z in $X_{\mathbb{C}}$ is called a *lattice norm.* A *complex Banach lattice* $X_{\mathbb{C}}$ is the complexification of a Banach lattice X endowed with the lattice norm $\|\cdot\|_{\mathbb{C}}$.

Let $X_{\mathbb{C}}$ be a complex Banach lattice with lattice norm $\|\cdot\|_{\mathbb{C}}$. If $|z_1| \preceq |z_2|$, then

$$\|z_1\|_{\mathbb{C}} = \big\||z_1|\big\| \le \big\||z_2|\big\| = \|z_2\|_{\mathbb{C}},$$

since $\|\cdot\|$ is a lattice norm. Therefore, $|z_1| \preceq |z_2| \Rightarrow \|z_1\|_{\mathbb{C}} \le \|z_2\|_{\mathbb{C}}$ for any lattice norm $\|\cdot\|_{\mathbb{C}}$.

Remark. As in the case of $\mathbb{C}$ over $\mathbb{R}$, no attempt is made to extend the order $\preceq$ on X to $X_{\mathbb{C}}$. Therefore it is important to note that the term complex vector lattice does not imply directly a lattice structure, but rather that the complex vector space was derived from a real vector lattice.

Example 2.1.24. As in Example 2.1.8, $\mathbb{R}$ with the order $\le$ and the Euclidean norm is a Banach lattice. The complexification of $\mathbb{R}$ is the vector space $\mathbb{C}$, and for any $z = x + \mathrm{i}y$ in $\mathbb{C}$, (2.1) gives

$$|z| = \sup_{0\le\theta<2\pi} |\cos\theta x + \sin\theta y|.$$

This supremum exists in $\mathbb{R}$ as it is the supremum of a set of positive numbers bounded above by $|x| + |y|$. In fact, if (x, y) is viewed as a vector in $\mathbb{R}^2$, then $\cos\theta x + \sin\theta y$ is the dot product of the unit vector $(\cos\theta, \sin\theta)$ and (x, y). Thus, $|\cos\theta x + \sin\theta y|$ is maximized when $(\cos\theta, \sin\theta)$ either points in the same direction or in the opposite direction of (x, y). The unique θ_0 for which $(\cos\theta_0, \sin\theta_0)$ points in the same direction as (x, y) is called the argument of z, and the modulus of z will sometimes be labelled as r. Since $(\cos\theta_0, \sin\theta_0)$ is a unit vector and (x, y) has modulus r,

$$(r\cos\theta_0, r\sin\theta_0) = (x, y),$$

so that $r\cos\theta_0 = x$ and $r\sin\theta_0 = y$, giving the identity $r^2 = x^2 + y^2$.

The absolute value $|\cdot|$ on $\mathbb{R}$ is a lattice norm that makes it a Banach lattice, and the modulus $|\cdot|$ on $\mathbb{C}$ is a lattice norm since, for any $z = x + \mathrm{i}y$ in $\mathbb{C}$,

$$\begin{aligned} |z| &= \sup_{0\le\theta<2\pi} |\cos\theta x + \sin\theta y| \\ &= \big|\sup_{0\le\theta<2\pi} |\cos\theta x + \sin\theta y|\big| \\ &= \big||z|\big|. \end{aligned}$$

Therefore, $\mathbb{C}$ with the modulus $|\cdot|$ is a complex Banach lattice.

Example 2.1.25. As in Example 2.1.9, $\mathbb{R}^n$ for a positive integer n with the canonical ordering and the Euclidean norm is a Banach lattice. The complexification of $\mathbb{R}^n$ is the vector space $\mathbb{C}^n$, and for any $x = u + \mathrm{i}v$ in $\mathbb{C}^n$, (2.1) yields

$$\begin{aligned} |x| &= \sup_{0\le\theta<2\pi} |\cos(\theta)u + \sin(\theta)v| \\ &= \sup_{0\le\theta<2\pi} |\cos(\theta)(u_1,\ldots,u_n) + \sin(\theta)(v_1,\ldots,v_n)| \\ &= \sup_{0\le\theta<2\pi} |(\cos(\theta)u_1 + \sin(\theta)v_1,\ldots,\cos(\theta)u_n + \sin(\theta)v_n)| \\ &= \sup_{0\le\theta<2\pi} (|\cos(\theta)u_1 + \sin(\theta)v_1|,\ldots,|\cos(\theta)u_n + \sin(\theta)v_n|). \end{aligned}$$

For each coordinate i, $u_i = r_i\cos(\theta_i)$ and $v_i = r_i\sin(\theta_i)$, so that

$$|\cos(\theta)r_i\cos(\theta_i) + \sin(\theta)r_i\sin(\theta_i)| = r_i|(\cos(\theta),\sin(\theta))\cdot(\cos(\theta_i),\sin(\theta_i))|,$$

the absolute value of the dot product of the unit directional vector $(\cos(\theta),\sin(\theta))$ and the vector $(\cos(\theta_i),\sin(\theta_i))$ in the direction of x_i in the complex plane, for which the supremum exists since it is bounded above by $1 + 1 = 2$, and it is attained for $0 \le \theta < 2\pi$ when $\theta = \theta_i$. Then,

$$r_i|(\cos(\theta),\sin(\theta))\cdot(\cos(\theta_i),\sin(\theta_i))| \le r_i(\cos^2(\theta_i) + \sin^2(\theta_i)) = r_i.$$

Then $|x| = (r_1,\ldots,r_n) = (|x_1|,\ldots,|x_n|)$.

To place a lattice norm $\|\cdot\|_{\mathbb{C}}$ on $\mathbb{C}^n$ that agrees with the Euclidean norm $\|\cdot\|$ on the subset $\mathbb{R}^n + \mathrm{i}0$ of $\mathbb{C}^n$, it is required that

$$\begin{aligned} \|x\|_{\mathbb{C}} &= \big\||x|\big\| \\ &= \|(|x_1|,\ldots,|x_n|)\| \\ &= \sqrt{\sum_{i=1}^{n} |x_i|^2}. \end{aligned}$$

Since $\mathbb{R}^n$ is a Banach lattice with the Euclidean norm and the modulus (2.1) exists for all x in the complexification $\mathbb{C}^n$, then together with the Euclidean norm (from here written as $\|\cdot\|$), $\mathbb{C}^n$ is a complex Banach lattice.

Example 2.1.26. As in Example 2.1.11, $L^1_{\mathbb{R}}$ with the canonical ordering and the lattice norm $\|\cdot\|_1$ is a Banach lattice. The complexification of $L^1_{\mathbb{R}}$ is the vector space $L^1_{\mathbb{C}} = L^1_{\mathbb{R}} \oplus \mathrm{i}L^1_{\mathbb{R}}$, so that for f in $L^1_{\mathbb{C}}$, $f = f_1 + \mathrm{i}f_2$ for real-valued integrable functions f_1, f_2. Then, for $f = f_1 + \mathrm{i}f_2$ and almost every x in $[0,1]$, define

$$\begin{aligned} |f|(x) &= \sup_{0\le\theta<2\pi} |\cos(\theta)f_1 + \sin(\theta)f_2|(x) \\ &= \sup_{0\le\theta<2\pi} |\cos(\theta)f_1(x) + \sin(\theta)f_2(x)|, \end{aligned}$$

since the order on $L^1_{\mathbb{R}}$ is defined pointwise. Similar to Example 2.1.25, if $f_1(x) = r(x)\cos(\theta_0(x))$ and $f_2(x) = r(x)\sin(\theta_0(x))$, then for almost every x in $[0,1]$,

$$\begin{aligned} |f|(x) &= \sup_{0\le\theta<2\pi} \Big|r(x)\Big(\big(\cos(\theta)\cos(\theta_0(x))\big) + \big(\sin(\theta)\sin(\theta_0(x))\big)\Big)\Big| \\ &= |r(x)||\cos^2(\theta_0(x)) + \sin^2(\theta_0(x))| \\ &= |r(x)|. \end{aligned}$$

Thus, $|f|(x) = |f(x)|$ for almost every x in $[0,1]$. This supremum exists in $L^1_{\mathbb{R}}$ since, for almost every x in $[0,1]$,

$$\begin{aligned} |f|(x) &= \sup_{0\le\theta<2\pi} |\cos(\theta)f_1 + \sin(\theta)f_2|(x) \\ &\le \sup_{0\le\theta<2\pi} \Big(|\cos(\theta)f_1|(x) + |\sin(\theta)f_2|(x)\Big) \\ &\le \sup_{0\le\theta_1<2\pi} |\cos(\theta_1)f_1|(x) + \sup_{0\le\theta_2<2\pi} |\sin(\theta_2)f_2|(x) \\ &= |f_1|(x) + |f_2|(x) \\ &= (|f_1| + |f_2|)(x), \end{aligned}$$

so that $|f| \le |f_1| + |f_2|$, and thus $\int |f| \le \int |f_1| + \int |f_2| < \infty$.

The norm on $L^1_{\mathbb{R}}$ is given by $\|f_1\|_1 = \int |f_1|$ for all f_1 in $L^1_{\mathbb{R}}$. Then extending the norm to any f in $L^1_{\mathbb{C}}$ gives

$$\|f\|_{\mathbb{C}} = \big\||f|\big\|_1,$$

which is well defined since $|f|$ exists and is integrable. Since $L^1_{\mathbb{R}}$ is a Banach lattice with the norm $\|\cdot\|_1$ and the modulus (2.1) exists for all f in the complexification $L^1_{\mathbb{C}}$, $L^1_{\mathbb{C}}$ (from here written as L^1) together with the norm $\|\cdot\|_{\mathbb{C}}$ (from here written as $\|\cdot\|_1$) is a complex Banach lattice.

Example 2.1.27. As in Example 2.1.12, $L^\infty_{\mathbb{R}}$ with the canonical ordering and the lattice norm $\|\cdot\|_\infty$ is a Banach lattice. The complexification of $L^\infty_{\mathbb{R}}$ is the vector space $L^\infty_{\mathbb{C}} = L^\infty_{\mathbb{R}} \oplus \mathrm{i}L^\infty_{\mathbb{R}}$, so that for f in $L^\infty_{\mathbb{C}}$, $f = f_1 + \mathrm{i}f_2$ for real-valued essentially bounded functions f_1, f_2.

As in Example 2.1.26, for f in $L^\infty_{\mathbb{C}}$ and for almost every x in $[0,1]$,

$$|f|(x) = |f(x)|.$$

Also, as in Example 2.1.26, if $f = f_1 + \mathrm{i}f_2$ is in $L^\infty_{\mathbb{C}}$, then

$$|f| \le |f_1| + |f_2|,$$

so that if $\|f_1\|_\infty = M$ and $\|f_2\|_\infty = M'$, then for almost every x in $[0,1]$,

$$|f|(x) \le |f_1|(x) + |f_2|(x) \le M + M',$$

which implies that $|f|$ is in $L^\infty_{\mathbb{R}}$.

The norm on $L^\infty_{\mathbb{R}}$ is given by $\|f_1\|_\infty = \inf\{M \in \mathbb{R} : |f_1| \le M \text{ a.e.}\}$ for all f_1 in $L^\infty_{\mathbb{R}}$. Then extending the norm to any f in $L^\infty_{\mathbb{C}}$ gives

$$\|f\|_{\mathbb{C}} = \big\| |f| \big\|_\infty,$$

which is well defined since $|f|$ exists and is essentially bounded. Since $L^\infty_{\mathbb{R}}$ is a Banach lattice with the norm $\|\cdot\|_\infty$ and the modulus (2.1) exists for all f in the complexification $L^\infty_{\mathbb{C}}$, $L^\infty_{\mathbb{C}}$ (from here written as L^∞) together with the norm $\|\cdot\|_{\mathbb{C}}$ (from here written as $\|\cdot\|_\infty$) is a complex Banach lattice.

Example 2.1.28. As in Example 2.1.13, $BV_{\mathbb{R}}$ with the canonical ordering and the norm $\|\cdot\|_{BV}$ is a vector lattice. The complexification of $BV_{\mathbb{R}}$ is the vector space $(BV)_{\mathbb{C}} = BV_{\mathbb{R}} \oplus \mathrm{i}BV_{\mathbb{R}}$, so that for f in $(BV)_{\mathbb{C}}$, $f = f_1 + \mathrm{i}f_2$ for real-valued functions of bounded variation f_1, f_2.

As in Example 2.1.26, for f in $(BV)_{\mathbb{C}}$ and for almost every x in $[0,1]$,

$$|f|(x) = |f(x)|.$$

Let $f = f_1 + \mathrm{i}f_2$ be in $(BV)_{\mathbb{C}}$ and suppose that f_1, f_2 are versions with minimal variation. Let $0 = b_0 < b_1 < \ldots < b_k = 1$ be any partition of the interval, then

$$\begin{aligned}
\sum_{i=1}^{k} |f(b_i) - f(b_{i-1})| &= \sum_{i=1}^{k} |f_1(b_i) + \mathrm{i}f_2(b_i) - f_1(b_{i-1}) - \mathrm{i}f_2(b_{i-1})| \\
&= \sum_{i=1}^{k} \big|\big(f_1(b_i) - f_1(b_{i-1})\big) + \mathrm{i}\big(f_2(b_i) - f_2(b_{i-1})\big)\big| \\
&\le \sum_{i=1}^{k} |f_1(b_i) - f_1(b_{i-1})| + \sum_{i=1}^{k} |f_2(b_i) - f_2(b_{i-1})| \\
&\le \bigvee f_1 + \bigvee f_2 \\
&< \infty,
\end{aligned}$$

which implies that the variation of f is bounded in the sense of a complex valued function (ie. $f \in BV_{\mathbb{C}}$). This shows that $(BV)_{\mathbb{C}} \hookrightarrow BV_{\mathbb{C}}$ in a natural way. It is an easy exercise to establish a similar embedding $BV_{\mathbb{C}} \hookrightarrow (BV)_{\mathbb{C}}$. Thus, from here $(BV)_{\mathbb{C}} = BV_{\mathbb{C}}$ will be written as BV.

Define the norm $\|\cdot\|_{\mathbb{C}}$ on BV by

$$\|f\|_{\mathbb{C}} = \bigvee f + \big\||f|\big\|_1 = \bigvee f + \|f\|_1 < \infty,$$

as $f = f_1 + \mathrm{i}f_2$ in $BV_{\mathbb{R}} \oplus \mathrm{i}BV_{\mathbb{R}}$ implies that f_1 and f_2 are both integrable, so that f is integrable as in Example 2.1.26.

Since BV is a vector lattice and the modulus (2.1) exists for all f in the complexification BV, BV together with the norm $\|\cdot\|_{\mathbb{C}}$ (from here written as $\|\cdot\|_{BV}$) is a complex vector lattice and a Banach space, however it is not a complex Banach lattice in the sense of Definition 2.1.21 since $\|\cdot\|_{BV}$ is not extended from a lattice norm.

This next example shows that the complexification of a real vector lattice is not always a complex vector lattice.

Example 2.1.29. The vector space F of continuous piecewise linear real-valued functions on $[0,1]$ with finitely many pieces as described in Example 2.1.14 is a real vector lattice when given the canonical ordering. Let $F_{\mathbb{C}}$ be the complexification of F. Then, considering $f = f_1 + \mathrm{i}f_2$ in $F_{\mathbb{C}}$ where $f_1(x) = 1 - x$ and $f_2(x) = x$, the set of functions

$$\{|\cos(\theta)f_1 + \sin(\theta)f_2|\}$$

is bounded above by the constant function $\mathbf{2}$ (which is in F) since f_1 and f_2 are both bounded above by the function $\mathbf{1}$.

For any x in $[0,1]$,

$$\begin{aligned}
|f(x)| &= \sup_{0\le\theta<2\pi} |\cos(\theta)f_1(x) + \sin(\theta)f_2(x)| \\
&= \sup_{0\le\theta<2\pi} |\cos(\theta)(1-x) + \sin(\theta)x| \\
&= \sqrt{(1-x)^2 + x^2} \\
&= \sqrt{2x^2 - 2x + 1},
\end{aligned}$$

so that for any upperbound f_0 on the set $\{|\cos(\theta)f_1 + \sin(\theta)f_2|\}$ must satisfy

$$f_0(x) \geq \sqrt{2x^2 - 2x + 1}$$

for all x in $[0,1]$.

A plot of the function $\tilde{f} = \sqrt{2x^2 - 2x + 1}$ shows that $\tilde{f}$ is strictly convex. Then for any upperbound f_0 in F of the set $\{|\cos(\theta)f_1 + \sin(\theta)f_2|\}$, by the strict convex nature of $\tilde{f}$, it is possible to construct a new upperbound h in F such that $h < f_0$. Thus the set of upperbounds is not uniquely bounded below in F, and therefore the supremum $|f|$ does not exist in F.

Definition 2.1.30. Let $X_{\mathbb{C}}, Y_{\mathbb{C}}$ be complex vector lattices and let $S : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ be a linear map. S is called real if $SX \subset Y$ (where X is viewed as the subset of $X_{\mathbb{C}}$ of all $x + \mathrm{i}0$ for x in X, and similarly for Y). S is called positive (written $S \geq 0$) if it

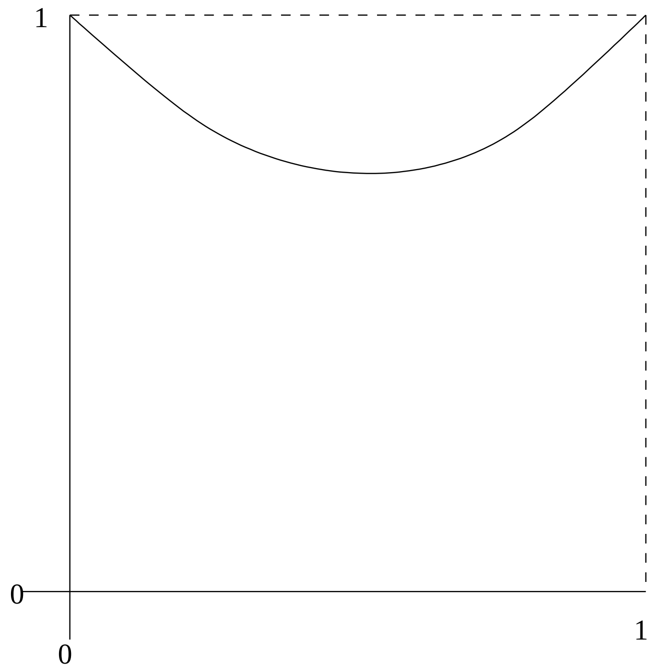

Figure 2.1: $\tilde{f}(x) = \sqrt{2x^2 - 2x + 1}$

is real and its restriction to X is positive. S is called a lattice homomorphism if it is real and its restriction to X is a lattice homomorphism.

Remark. Thus an operator $S : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ is positive if the positive cone of X is mapped into the positive cone of Y, and is a lattice homomorphism if it is a lattice homomorphism from X to Y. Since there is no lattice structure defined on the larger space $X_{\mathbb{C}}$, this is the only requirement for S to be a lattice homomorphism.

Theorem 2.1.31. *Let $X_{\mathbb{C}}, Y_{\mathbb{C}}$ be complex vector lattices and let $S : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ be a linear map. S is positive if and only if $|Sz| \preceq S|z|$ for all z in $X_{\mathbb{C}}$.*

Proof. Suppose that S is positive. Then, for any $z = x + \mathrm{i}y$ in $X_{\mathbb{C}}$, $Sz = Sx + \mathrm{i}Sy$ where Sx and Sy are in X (since S is real) and

$$\begin{aligned} |Sz| &= \sup_{0 \le \theta < 2\pi} |\cos(\theta)Sx + \sin(\theta)Sy| \\ &= \sup_{0 \le \theta < 2\pi} |S\big(\cos(\theta)x + \sin(\theta)y\big)|. \end{aligned}$$

For any θ_0,

$$|\cos(\theta_0)x + \sin(\theta_0)y| \preceq \sup_{0 \le \theta < 2\pi} |\cos(\theta)x + \sin(\theta)y|,$$

so that

$$S|\cos(\theta_0)x + \sin(\theta_0)y| \preceq S \sup_{0 \le \theta < 2\pi} |\cos(\theta)x + \sin(\theta)y|$$

for any θ_0. But

$$|S\big(\cos(\theta_0)x + \sin(\theta_0)y\big)| \preceq S|\cos(\theta_0)x + \sin(\theta_0)|,$$

since S is positive. This implies that

$$\sup_{0\le\theta<2\pi} |S\big(\cos(\theta)x + \sin(\theta)y\big)| \preceq S \sup_{0\le\theta<2\pi} |\cos(\theta)x + \sin(\theta)y| = S|z|,$$

and hence $|Sz| \preceq S|z|$.

Conversely, suppose that $|Sz| \preceq S|z|$ for all z in $X_{\mathbb{C}}$. For any $x_0 \succeq 0$ in X,

$$0 \preceq |Sx_0| \preceq S|x_0| = Sx_0.$$

Then for any x in X, $Sx^+ \succeq 0$ and $Sx^- \succeq 0$ and in particular

$$Sx^+ - Sx^- = S(x^+ - x^-) = Sx$$

is in Y, so that S is real. But since $|Sz| \preceq S|z|$ for all z in $X_{\mathbb{C}}$, $|Sx| \preceq S|x|$ for all x in X, and by Theorem 2.1.16, S is positive on X. Thus S is positive on $X_{\mathbb{C}}$. □

In Section 11 of Chapter II in [24], Schaefer states the following result. Since it is not used in this work, it is given without proof.

Theorem 2.1.32. *Let $X_{\mathbb{C}}, Y_{\mathbb{C}}$ be complex vector lattices and let $S : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ be a linear map. S is a lattice homomorphism if and only if $|Sz| = S|z|$ for all z in $X_{\mathbb{C}}$.*

2.2 Frobenius-Perron and Koopman Operators

Let T be a measurable transformation on $([0,1], \mathcal{B}, m)$ such that, for measurable sets $A \in \mathcal{B}$, $m(T^{-1}A) = 0$ whenever $m(A) = 0$. Such a measurable transformation is called nonsingular. Then, T has associated with it an operator K_T on $L^\infty([0,1], \mathcal{B}, m)$, called the Koopman operator for T, that is defined by the relation

$$K_T g = g \circ T \tag{2.2}$$

for any function g in L^∞. The nonsingularity of T here is necessary since for versions $\hat{g}_1, \hat{g}_2$ of $g_1, g_2 \in L^\infty$, $\hat{g}_1(x) = \hat{g}_2(x)$ almost everywhere implies that $\hat{g}_1(Tx) = \hat{g}_2(Tx)$ almost everywhere.

Theorem 2.2.1. *Let K_T be the Koopman operator on L^∞ for a nonsingular transformation T on $[0,1]$. Then*

1. *K_T is a lattice homomorphism;*

2. *T^N is measurable and nonsingular for any positive integer N, and $K_{T^N} = K_T^N$.*

Proof. *1.* For any g_1, g_2 in L^∞,

$$K_T(\hat{g}_1 \vee \hat{g}_2)(x) = (\hat{g}_1 \vee \hat{g}_2)(Tx) = \hat{g}_1(Tx) \vee \hat{g}_2(Tx) = K_T\hat{g}_1 \vee K_T\hat{g}_2(x)$$

almost everywhere, so that $K_T(g_1 \vee g_2) = K_T g_1 \vee K_T g_2$. Similarly, $K_T(g_1 \wedge g_2) = K_T g_1 \wedge K_T g_2$ and thus K_T is a lattice homomorphism.

2. For $N = 1$, $T^1 = T$ is given to be measurable and nonsingular. Suppose for some $N \geq 1$, T^N is measurable and nonsingular. Then $T^{N+1} = T \circ T^N$ and $(T^{N+1})^{-1} = T^{-N} \circ T^{-1}$. For any measurable set A, $T^{-1}A$ is measurable, since T is measurable, which implies that $T^{-N}T^{-1}A = (T^{N+1})^{-1}A$ is measurable, since T^N is measurable, so that T^{N+1} is measurable. If $m(A) = 0$, then $m(T^{-1}A) = 0$, since T is nonsingular, which implies that $m(T^{-N}T^{-1}A) = m((T^{N+1})^{-1}A) = 0$, since T^N is nonsingular, so that T^{N+1} is nonsingular.

By repeated applications of (2.2), for any g in L^∞

$$K_T^N g = K_T^{N-1} g \circ T = K_T^{N-2} g \circ T^2 = \ldots = g \circ T^N = K_{T^N} g.$$

Since g was arbitrarily chosen, $K_T^N = K_{T^N}$. □

T also induces an operator P_T on the space $L^1([0,1], \mathcal{B}, m)$. Suppose that f is in L^1, and consider the finite signed measure $d\mu = f\, dm$. Composing μ with T^{-1} gives a new finite signed measure $\mu \circ T^{-1} \ll m$ as follows:

- $(\mu \circ T^{-1})(A) = \int_{T^{-1}A} f\, dm$, for all $A \in \mathcal{B}$;
- if $m(A) = 0$, then $m(T^{-1}A) = 0$ so that $(\mu \circ T^{-1})(A) = \int_{T^{-1}A} f\, dm = 0$.

By the Radon-Nikodym Theorem [11], there exists $\tilde{f} \in L^1$ such that

$$\int_{T^{-1}A} f\, dm = (\mu \circ T^{-1})(A) = \int_A \tilde{f}\, dm.$$

Then define $P_T f := \tilde{f}$. This gives the equation

$$\int_A P_T f\, dm = \int_{T^{-1}A} f\, dm, \tag{2.3}$$

which is Equation (1.1), or equivalently

$$\int (P_T f) \mathbf{1}_A\, dm = \int f (K_T \mathbf{1}_A)\, dm. \tag{2.4}$$

Then for any simple function $\phi = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}$,

$$\begin{aligned}\int (P_T f)\phi \, dm &= \sum_{i=1}^{n} a_i \int (P_T f)\mathbf{1}_{A_i} \, dm \\ &= \sum_{i=1}^{n} a_i \int f(K_T \mathbf{1}_{A_i}) \, dm \\ &= \int f(K_T \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}) \, dm \\ &= \int f(K_T \phi) \, dm,\end{aligned}$$

where the linearity of both the integral and K_T was used. For any g in L^∞, there exists a sequence ϕ_n of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |g|$, $\phi_n \to g$ uniformly and versions of ϕ_n converge pointwise almost everywhere to the versions of g ([11]). Then by twice applying the Dominated Convergence Theorem from [11],

$$\begin{aligned}\int (P_T f) g \, dm &= \lim_{n\to\infty} \int (P_T f)\phi_n \, dm \\ &= \lim_{n\to\infty} \int f(K_T \phi) \, dm \\ &= \int f(K_T g) \, dm,\end{aligned}$$

since versions of $K_T \phi_n$ converge pointwise almost everywhere to versions of $K_T g$. Thus, the Frobenius-Perron operator P_T and the Koopman operator K_T satisfy the relation

$$\int (P_T f) g \, dm = \int f(K_T g) \, dm \tag{2.5}$$

for any f in L^1 and any g in L^∞. The act of integrating integrable functions against a bounded function, as in Equation (2.5), gives a bounded linear functional on L^1 (ie. an element of $(L^1)^*$), and as is shown at the begining of Section 2.3 K_T is thus the dual of P_T. More basic properties of $P_T : L^1 \to L^1$ are collected below.

Definition 2.2.2. A measure μ that satisfies $\mu = \mu \circ T^{-1}$ is called *invariant* (with respect to T). An invariant measure that is absolutly continuous (with respect to m) is then called an *absolutely continuous invariant measure*, or briefly *acim*.

Theorem 2.2.3. *Let P_T be the Frobenius-Perron operator on L^1 for a nonsingular measurable transformation T on $[0, 1]$. Then*

1. *P_T is linear and positive;*

2. *$\int P_T f = \int f$ for every f in L^1;*

3. *$\|P_T f\|_1 = \|f\|_1$ for $f \geq 0$ in L^1;*

4. *T^N is measurable and nonsingular for any positive integer N, and $P_{T^N} = P_T^N$;*

5. *For f in L^1, $d\mu = f\,dm$ is an acim if and only if $P_T f = f$.*

Proof. *1.* By the definition of P_T in (2.3), for f_1, f_2 in L^1, a, b in $\mathbb{C}$ and any A in $\mathcal{B}$,

$$\begin{aligned}
\int_A P_T(af_1 + bf_2) &= \int_{T^{-1}A} af_1 + bf_2 \\
&= a\int_{T^{-1}A} f_1 + b\int_{T^{-1}A} f_2 \\
&= a\int_A P_T(f_1) + b\int_A P_T(f_2) \\
&= \int_A aP_T(f_1) + bP_T(f_2),
\end{aligned}$$

where the linearity of the integral was used. Since A was an arbitrarily chosen measurable set, $P_T(af_1 + bf_2) = aP_T f_1 + bP_T f_2$, and hence P_T is linear.

Let A be any measurable set and suppose $f \geq 0$. Then, by (2.3)

$$\int_A P_T f = \int_{T^{-1}A} f \geq 0.$$

Since $\int_A P_T f \geq 0$ for any measurable set A, $P_T f \geq 0$. Therefore, by Definition 2.1.30, P_T is positive.

2. Let f be any function in L^1. Since $T^{-1}[0,1] = [0,1]$, by (2.3)

$$\int P_T f = \int_0^1 P_T f = \int_{T^{-1}[0,1]} f = \int_0^1 f = \int f.$$

3. Let $f \geq 0$, so that $P_T f \geq 0$ and

$$\|P_T f\|_1 = \int |P_T f| = \int P_T f = \int f = \int |f| = \|f\|_1.$$

4. As in Theorem 2.2.1, T^N is measurable and nonsingular. For any f in L^1 and any measurable set A, by repeated applications of (2.3)

$$\int_A P_T^N f = \int_{T^{-1}A} P_T^{N-1} f = \int_{T^{-2}A} P_T^{N-2} = \ldots = \int_{T^{-N}A} f = \int_A P_{T^N} f.$$

Since the functions $P_T^N f$ and $P_{T^N} f$ integrate to the same number over any measurable set A, they are in fact equal. Since this is true for any f in L^1, $P_T^N = P_{T^N}$.

5. Suppose that $d\mu = f\,dm$ is an absolutely continuous invariant measure. Then, for any measurable set A,

$$\int_A P_T f = \int_{T^{-1}A} f = \int_A f$$

by (2.3), so that $P_T f$ and f integrate to the same number. Since A was chosen arbitrarily, $P_T f = f$.

Conversely, suppose $P_T f = f$. Then $d\mu = f\,dm$ is absolutely continuous, and for any measurable set A,

$$\mu(T^{-1}A) = \int_{T^{-1}A} f = \int_A P_T f = \int_A f = \mu(A)$$

by (2.3). Therefore, μ is an acim. □

From Theorem 2.2.3 P_T is a positive operator, and from (2.2) $K_T\mathbf{1} = \mathbf{1}$, so that there exists a strictly positive linear functional ϕ on L^1, namely $\phi(f) = \int f\,dm$, satisfying $K_T\phi \leq \phi$. This will be seen (proof of Theorem 4.1.5) to be crucial in showing that the peripheral point spectrum of P_T is fully cyclic.

A transformation S on $([0,1], \mathcal{B}, m)$ will be called piecewise C^2 if there exists a partition $0 = a_0 < a_1 < \ldots < a_n = 1$ of the unit interval such that the restrictions S_i of S to the subintervals (a_{i-1}, a_i) are C^2 functions which can be extended continuously to functions on the intervals $[a_{i-1}, a_i]$. A transformation S will be called piecewise C^2 and monotone if it is piecewise C^2 and it is monotone on each of its branches. As is shown in [17], for a piecewise C^2 and monotone transformation S on $[0,1]$, and for a version $\hat{f}$ of a function f in L^1,

$$\widehat{P_S f}(x) = \sum_{i=1}^{n} \frac{\hat{f}(S_i^{-1}(x))}{|S_i'(x)|}\mathbf{1}_i(x), \tag{2.6}$$

where S_i is the branch of the transformation S on the interval $[a_{i-1}, a_i]$, and $\mathbf{1}_i(x)$ is the characteristic function of the interval $S_i[a_{i-1}, a_i]$.

A piecewise C^2 transformation S will be called expanding if $\inf(|S'|(x)) > 1$, where the infimum is taken over all x such that S is differentiable at x (note that this implies S is monotone on each branch). Suppose now that T is required to be piecewise C^2 and expanding on $([0,1], \mathcal{B}, m)$ (which implies nonsingularity). Then, for any positive integer N, T^N is piecewise C^2 on intervals of the form $I_{i_1} \bigcap T^{-1}I_{i_2} \bigcap \ldots \bigcap T^{-(N-1)}I_{i_{(N-1)}}$, and by the chain rule, $|(T^N)'(x)| > \inf(|T'|)^N > 1$ for all x such that T^N is differentiable at x, so that T^N is piecewise C^2 and expanding.

The space $BV = BV_{\mathbb{C}}$ as discussed in Example 2.1.28 is a vector subspace of L^1. Then P_T is a transformation from BV to L^1. In [17] Lasota and Yorke derive the inequality

$$\bigvee_0^1 P_{T^N} f \leq \alpha\|f\|_1 + \frac{2}{\inf(|T'|)^N}\bigvee_0^1 f, \tag{2.7}$$

for any positive integer N. $T^N = \tau$ has partition $0 = b_0 < b_1 < \ldots < b_q = 1$, so that if τ_i is the corresponding branch of τ on the interval $[b_{i-1}, b_i]$,

$$\alpha = \frac{\sup\left|\frac{d}{dx}|\tau_i^{-1}|\right|}{\inf|\tau_i^{-1}|} + \frac{2}{\min_i(b_i - b_{i-1})}$$

is a positive number. Then, for the case $N = 1$, P_T maps BV back into BV.

Theorem 2.2.4. *Let P_T be the Frobenius-Perron operator on BV for a nonsingular transformation T on $[0,1]$. Then*

1. *P_T is positive;*
2. *T^N is measurable and nonsingular for any positive integer N, and $P_{T^N} = P_T^N$;*
3. *For f in BV, $d\mu = f\,dm$ is an acim if and only if $P_T f = f$.*

Proof. All results follow directly from Theorem 2.2.3 since BV is an invariant subspace of L^1. $\square$

2.3 Spectral Theory

Let $X \neq \{0\}$ be a complex Banach space. Denote by $\mathscr{L}(X)$ the space of **bounded** linear operators on X. The space of bounded linear forms (ie. complex-valued bounded linear maps) on X is then a complex Banach space with the operator norm and will be denoted by X^*. This is called the dual space, or simply dual, of X. For $x \in X$ and $y \in X^*$, the value of y at x will be written

$$y(x) = \langle x, y \rangle.$$

This is a bilinear functional $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{C}$. That is, the following properties are satisfied:

- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for all x in X, y in X^* and α in $\mathbb{C}$;
- $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$ for all x_1, x_2 in X and y in X^*;
- $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$ for all x in X, y in X^* and α in $\mathbb{C}$;
- $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$ for all x in X and y_1, y_2 in X^*.

For any $B \in \mathscr{L}(X)$, there exists a unique $B^* \in \mathscr{L}(X^*)$, called the adjoint of B, such that

$$\langle Bx, y \rangle = \langle x, B^* y \rangle$$

for all $x \in X$ and $y \in X^*$. (For proofs of the above statements see [11])

Remark. It is possible that a dual space be constructed with the bounded semilinear forms on X, so that $y(x) = \langle x, y \rangle$ is a sesquilinear functional on $X \times X^*$ (ie. $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$). This is natural in the finite dimensional case, as the norm is directly described by this inner-product. However the bilinear functionald is desired for the results for the Frobenius-Perron and Koopman operators, so it is used throughout.

Definition 2.3.1. A bounded linear operator B in $\mathscr{L}(X)$ is called an *isometry* if $\|Bx\| = \|x\|$ for all x in X.

Definition 2.3.2. The *resolvent set* $\rho(B)$ of a bounded linear operator B is the set of complex numbers λ such that $\lambda I - B$ has bounded linear inverse. For $\lambda \in \rho(B)$ the *resolvent operator* (or simply *resolvent*) of B is the bounded linear operator $R(\lambda, B) := (\lambda I - B)^{-1}$. The *spectrum* $\sigma(B)$ is the complement of the resolvent set, ie. $\sigma(B) = \mathbb{C} \setminus \rho(B)$.

Theorem 2.3.3. *The resolvent set $\rho(B)$ for $B \in \mathscr{L}(X)$ is open. The function $R(\lambda, B)$ is analytic for λ in $\rho(B)$.*

Proof. Let λ be a fixed point in $\rho(B)$, and let μ satisfy $|\mu| < \|R(\lambda, B)\|^{-1}$. Then, it will be shown that $\mu + \lambda \in \rho(B)$. Since $\|\mu R(\lambda, B)\| < 1$, the series

$$S(\mu) = \sum_{k=0}^{\infty}(-\mu)^k R(\lambda, B)^{k+1} = \sum_{k=0}^{\infty}(-1)^k(\mu R(\lambda, B))^k R(\lambda, B) \tag{2.8}$$

converges in the uniform topology of $\mathscr{L}(X)$. But then

$$\begin{aligned}
((\lambda+\mu)I - B)S(\mu) &= (\lambda I - B)S(\mu) + \mu S(\mu) \\
&= \sum_{k=0}^{\infty}(-\mu)^k(\lambda I - B)R(\lambda, B)^{k+1} \\
&\quad + \sum_{k=0}^{\infty}\mu(-\mu)^k R(\lambda, B)^{k+1} \\
&= \sum_{k=0}^{\infty}(-\mu)^k R(\lambda, B)^k \\
&\quad + \sum_{k=0}^{\infty}(-1)(-\mu)^{k+1} R(\lambda, B)^{k+1} \\
&= \sum_{k=0}^{\infty}(-\mu R(\lambda, B))^k - (-\mu R(\lambda, B))^{k+1} \\
&= (-\mu R(\lambda, B))^0 \\
&= I.
\end{aligned}$$

Therefore $\lambda + \mu \in \rho(B)$ and $R(\lambda + \mu, B)$ admits the series expansion $S(\mu)$ given by (2.8), so that $R(\lambda + \mu, B)$ is analytic at the point $\mu = 0$. $\square$

Corollary 2.3.4. *For any $\lambda \in \rho(B)$, if $d(\lambda)$ is the distance from λ to the spectrum $\sigma(B)$, then*

$$\|R(\lambda, B)\| \geq \frac{1}{d(\lambda)}.$$

Proof. From the proof of Theorem 2.3.3, if $|\mu| < \|R(\lambda, B)\|^{-1}$, then $\lambda + \mu \in \rho(B)$. Thus, $d(\lambda) \geq \|R(\lambda, B)\|^{-1}$, so that the statement follows. $\square$

Remark. From Corollary 2.3.4 it is immediate that $\|R(\lambda, B)\|$ becomes unbounded as λ approaches the spectrum.

Theorem 2.3.5. *The spectrum $\sigma(B)$ for $B \in \mathscr{L}(X)$ is compact and nonempty.*

Proof. For $|\lambda| > \|B\|$, $\|\frac{B}{\lambda}\| < 1$ so that the series $\hat{R}(\lambda) = \sum_{k=0}^{\infty} \frac{B^k}{\lambda^{k+1}}$ converges in the norm of $\mathscr{L}(X)$. Then

$$\begin{aligned}(\lambda I - B)\hat{R}(\lambda) &= \sum_{k=0}^{\infty} \frac{B^k}{\lambda^k} - \sum_{k=0}^{\infty} \frac{B^{k+1}}{\lambda^{k+1}} \\ &= \frac{B^0}{\lambda^0} \\ &= I.\end{aligned}$$

Similarly $\hat{R}(\lambda)(\lambda I - B) = I$, and $\lambda \in \rho(B)$ with $R(\lambda, B) = \hat{R}(\lambda)$, so that $\sigma(B)$ is bounded. From Theorem 2.3.3 $\rho(B)$ is open, so that $\sigma(B)$ is closed. Hence $\sigma(B)$ is compact.

Suppose that $\sigma(B)$ is empty. Then by Theorem 2.3.3, $R(\lambda, B)$ is an entire function. Since

$$\hat{R}(1/\lambda) = \lambda \sum_{k=0}^{\infty} \lambda^k B^k,$$

which is equal to 0 when $\lambda = 0$, $R(\lambda, B)$ is analytic at infinity. Thus, by Liouville's Theorem, $R(\lambda, B)$ is constant. Since $R(\lambda, B) = \sum_{k=0}^{\infty} \frac{B^k}{\lambda^{k+1}} \to 0$ as $\lambda \to \infty$, $R(\lambda, B) = 0$ for all λ, which is contradictory to the assumption that $R(\lambda, B)$ is invertible at λ, since $X \neq \{0\}$. Therefore $\sigma(B)$ is not empty. □

Since the spectrum of a bounded linear operator is a bounded set in the complex plane, there exists a least upper bound on the set $\{|\lambda| : \lambda \in \sigma(B)\}$.

Definition 2.3.6. Let $B \in \mathscr{L}(X)$. The *spectral radius* $r(B)$ of the bounded set $\sigma(B)$ is

$$r(B) = \sup\{|\lambda| : \lambda \in \sigma(B)\}.$$

Remark. Since all of the singularities of the analytic function $R(\lambda, B)$ lie in the disk $\{\lambda \in \mathbb{C} : |\lambda| \leq r(B)\}$, the Laurent series about infinity $\sum_{k=0}^{\infty} B^k/\lambda^{k+1}$ from the proof of Theorem 2.3.5 converges for $|\lambda| > r(B)$ to the resolvent of B at λ.

The Principle of Uniform Boundedness is an important tool in the study of bounded linear operators of a Banach space. It is stated here and the reader is referred to II.3.20-21 of [9] for the proof.

Theorem 2.3.7. *Let B_n be bounded linear operators on a Banach space X. Then the following statements are equivalent:*

- $\sup_n \|B_n\| < \infty$;
- $\sup_n \|B_n x\| < \infty$, $\forall x \in X$;
- $\sup_n |\langle B_n x, y\rangle| < \infty$, $\forall x \in X$, $\forall y \in X^*$.

Lemma 2.3.8. *For $B_1, B_2 \in \mathscr{L}(X)$, $\|B_1 B_2\| \leq \|B_1\|\|B_2\|$.*

Proof. Let x be any element of X. Then

$$\|B_1B_2x\| \leq \|B_1\|\|B_2x\| \leq \|B_1\|\|B_2\|\|x\|.$$

Thus, $\|B_1B_2\| \leq \|B_1\|\|B_2\|$. $\square$

Remark. From this simple lemma, it can be shown that $\|B^k\| \leq \|B\|^k$ for $k \geq 2$ by induction. But the case when $k = 1$ is trivially true, since $\|B^1\| = \|B\| = \|B\|^1$, and for $\|B\| \neq 0$ the case when $k = 0$ is true, since $\|B^0\| = \|I\| = 1 = \|B\|^0$.

The following Theorem depends on a standard result on subadditive sequences, namely that for a subadditive sequence b_k, $\lim_{k\to\infty} \frac{b_k}{k}$ exists and is equal to $\inf_k \frac{b_k}{k}$ (possibly taking the value $-\infty$). The reader is referred to [10] or [26] for a proof of this result.

Theorem 2.3.9. *Let $B \in \mathscr{L}(X)$. Then*

$$r(B) = \lim_{k\to\infty} \|B^k\|^{1/k} = \inf_k \|B^k\|^{1/k} \leq \|B\|.$$

Proof. Suppose that $\|B\| \neq 0$. Since $\|B^k\|^{1/k} \geq 0$ for all positive integers k,

$$\inf_k \|B^k\|^{1/k} \geq 0.$$

Since $\|B\| \in \{\|B^k\|^{1/k} : k = 1, 2, \ldots\}$, it is immediate that

$$\inf_k \|B^k\|^{1/k} \leq \|B\|.$$

Let $b_k = \log(\|B^k\|)$. Then, since $\|B^{n+m}\| \leq \|B^n\|\|B^m\|$ by Lemma 2.3.8,

$$b_{n+m} \leq b_n + b_m.$$

Then b_k is a subadditive sequence, and by the standard Theorem 4.9 in [26] (originally from [10]), $\lim_k \frac{b_k}{k}$ exists and is equal to $\inf_k \frac{b_k}{k}$ (possibly being equal to $-\infty$).

Denote for the rest of the proof $\lim_k \|B^k\|^{1/k}$ by l. Given $\epsilon > 0$, if $|\lambda| \geq l + \epsilon$, there exists n such that

$$|\lambda|^{-1}\|B^k\|^{1/k} \leq (l+\epsilon)^{-1}(l+\epsilon/2),$$

for all $k > n$, so that

$$\left\|\frac{B^k}{\lambda^k}\right\| \leq \left(\frac{l+\epsilon/2}{l+\epsilon}\right)^k.$$

Therefore, since ϵ was arbitrary, the series $\sum_{k=0}^{\infty} \lambda^{-(k+1)}B^k$ is now shown to converge for $|\lambda| > l$. Again, the simple calculation from Theorem 2.3.5 that

$$(\lambda I - B)\sum_{k=0}^{\infty} \frac{B^k}{\lambda^{k+1}} = \sum_{k=0}^{\infty} \frac{B^k}{\lambda^{k+1}}(\lambda I - B) = I$$

shows that this series is the resolvent for λ. Thus $r(B) \leq l$.

Let $x \in X$ and $y \in X^*$ be given. Then the function

$$\lambda \to \langle R(\lambda, B)x, y\rangle$$

is analytic for $|\lambda| > r(B)$, since the resolvent is analytic in $\rho(B)$. This implies that the series

$$\sum_{k=0}^{\infty} \left\langle \frac{B^k}{\lambda^{k+1}} x, y \right\rangle$$

converges for $|\lambda| > r(B)$. Thus

$$\sup_k \left| \left\langle \frac{B^k}{\lambda^{k+1}} x, y \right\rangle \right| < \infty,$$

so by the Principle of Uniform Boundedness

$$\sup_k \left\| \frac{B^k}{\lambda^{k+1}} \right\| \leq M_\lambda < \infty,$$

and hence $\|B^k\|^{1/k} \leq |\lambda|(M_\lambda|\lambda|)^{1/k}$, which shows that $l \leq |\lambda|$. But λ was chosen arbitrarily to satisfy $|\lambda| > r(B)$, so that $l \leq r(B)$.

Suppose now that $\|B\| = 0$, which implies that $B = 0$. Then it is immediate that $\|B^k\|^{1/k} = 0$ for any positive integer k. Thus $\inf_k \|B^k\|^{1/k} = 0$ and $\lim_k \|B^k\|^{1/k}$ exists and is equal to 0. Then it only remains to show that $r(B) = 0$. But for any $\lambda \neq 0$, $\lambda I - B = \lambda I$, which is bounded and has an inverse (namely $\lambda^{-1}I$). Thus $\sigma(B) = \{0\}$ which implies that $r(B) = 0$, and the proof is complete. $\square$

Another tool that is important in the study of bounded linear operators on a Banach space that will be needed here is the Bounded Inverse Theorem. It is stated here and the reader is referred to [9] for the proof (II.2.2).

Theorem 2.3.10. *(Bounded Inverse Theorem) If B is a bounded linear bijective operator on X, then B^{-1} is also a bounded linear (bijective) operator on X.*

Thus, in order to show that $R(\lambda, B)$ does not exist as a bounded linear operator, it suffices to show that $\lambda I - B$ is not bijective. Then the spectrum may be broken down into smaller sets where bijectivity fails in different ways.

Definition 2.3.11. The spectrum $\sigma(B)$ for B in $\mathscr{L}(X)$ is divided into three mutually disjoint parts:

- $\sigma_p(B) := \{\lambda \in \mathbb{C} : \lambda I - B \text{ is not 1-1}\}$;
- $\sigma_c(B) := \{\lambda \in \mathbb{C} : \lambda I - B \text{ is 1-1, and } \overline{(\lambda I - B)X} = X, \text{ but } (\lambda I - B)X \neq X\}$;
- $\sigma_r(B) := \{\lambda \in \mathbb{C} : \lambda I - B \text{ is 1-1, but } \overline{(\lambda I - B)X} \neq X\}$.

The sets $\sigma_p(B)$, $\sigma_c(B)$ and $\sigma_r(B)$ are called the *point spectrum*, *continuous spectrum* and *residual spectrum*, respectively. $\sigma(B) = \sigma_p(B) \cup \sigma_c(B) \cup \sigma_r(B)$.

Definition 2.3.12. Let $B \in \mathscr{L}(X)$. The set

$$\begin{aligned} \sigma_a(B) := \{\lambda \in \mathbb{C} \quad : \quad & \text{there exists a sequence } x_n \text{ in } X \text{ with} \\ & \|x_n\| = 1 \text{ and } \lim_{n\to\infty} \|(\lambda I - B)x_n\| = 0\} \end{aligned}$$

is called the *approximate point spectrum* of B.

It is immediate that the point spectrum is a subset of the approximate point spectrum. It can also be shown that the continuous spectrum is a subset of the approximate point spectrum (see [7]). Another useful fact of $\sigma_a(B)$ is that it contains the boundary of the spectrum.

Theorem 2.3.13. *Let $B \in \mathscr{L}(X)$. Then $\sigma_a(B)$ is a subset of $\sigma(B)$. The boundary of the spectrum $\partial\sigma(B)$ is entirely contained in the approximate point spectrum $\sigma_a(B)$.*

Proof. First, suppose that $\lambda \in \rho(B)$. Then for any $x \in X$

$$\begin{aligned} \|x\| &= \|R(\lambda, B)(\lambda I - B)x\| \\ &\leq \|R(\lambda, B)\|\|(\lambda I - B)x\|, \end{aligned}$$

so that $\|(\lambda I - B)x\| \geq \|R(\lambda, B)\|^{-1} > 0$ when $\|x\| = 1$. Thus any sequence $\|(\lambda I - B)x_n\|$ is bounded away from 0 when $\|x_n\| = 1$, so $\sigma_a(B) \subset \sigma(B)$.

Now, suppose that $\lambda_0 \in \partial\sigma(B)$. Let $\epsilon > 0$. Since λ_0 is in the boundary of $\sigma(B)$, there exists $\lambda \in \rho(B)$ such that $|\lambda - \lambda_0| < \epsilon/2$. By Corollary 2.3.4

$$\|R(\lambda, B)\| \geq \frac{1}{d(\lambda)} \geq \frac{2}{\epsilon},$$

where $d(\lambda)$ is the distance from the spectrum. Then x can be chosen such that $\|x\| = 1$ and

$$\|R(\lambda, B)x\| \geq \frac{1}{\epsilon}.$$

Define $x_0 = \frac{R(\lambda,B)x}{\|R(\lambda,B)x\|}$, so that $\|x_0\| = 1$. Then

$$\begin{aligned} \|(\lambda_0 I - B)x_0\| &\leq |(\lambda_0 - \lambda)|\|x_0\| + \|(\lambda I - B)x_0\| \\ &< \frac{\epsilon}{2} + \frac{\|x\|}{\|R(\lambda, B)x\|} \\ &\leq \frac{\epsilon}{2} + \epsilon = \frac{3\epsilon}{2}. \end{aligned}$$

Thus given a sequence ϵ_n decreasing to 0, there exists a sequence x_n such that $\|x_n\| = 1$ but $\|(\lambda_0 I - B)x_n\| < \epsilon_n$, so that $\lambda_0 \in \sigma_a(B)$ and the proof is complete. □

The following Lemma is result VI.2.7 from [9]. The reader is referred there for the proof.

Lemma 2.3.14. *Suppose B is a bounded linear operator on X. Then B has bounded linear inverse B^{-1} if and only if B^* has bounded linear inverse $(B^*)^{-1}$. Moreover, $(B^*)^{-1} = (B^{-1})^*$.*

Proof. VI.2.7 from [9]. □

Theorem 2.3.15. *Let $B \in \mathscr{L}(X)$. Then $\sigma(B) = \sigma(B^*)$ and for $\lambda \in \rho(B) = \rho(B^*)$, $R(\lambda, B)^* = R(\lambda, B^*)$.*

Proof. Let $\lambda \in \rho(B)$. Then $(\lambda I - B)$ is invertible. Therefore, by Lemma 2.3.14, so is $(\lambda I - B)^* = (\lambda I^* - B^*)$ on X^* and so $\lambda \in \rho(B^*)$.

Now suppose $\lambda \in \rho(B^*)$. Then, similar to the above, $\lambda \in \rho(B^{**})$. Consider for arbitrary $x \in X$ and $y \in X^*$

$$\begin{aligned}
\langle(\lambda I - B)(\lambda I^{**} - B^{**})^{-1}x - x, y\rangle &= \langle((\lambda I^* - B^*)^{-1})^* x, (\lambda I^* - B^*)y\rangle - \langle x, y\rangle \\
&= \langle x, (\lambda I^* - B^*)^{-1}(\lambda I^* - B^*)y\rangle - \langle x, y\rangle \\
&= 0.
\end{aligned}$$

Therefore, by the Hahn Banach Theorem $(\lambda I - B)(\lambda I^{**} - B^{**})^{-1}x = x$. A similar argument shows $(\lambda I^{**} - B^{**})^{-1}(\lambda I - B)x = x$, so $\lambda I - B$ is invertible. Therefore, $\lambda \in \rho(B)$. Thus, $\rho(B) = \rho(B^*)$ and so $\sigma(B) = \sigma(B^*)$.

By Lemma 2.3.14, for $\lambda \in \rho(B) = \rho(B^*)$,

$$\begin{aligned}
R(\lambda, B^*) &= (\lambda I^* - B^*)^{-1} \\
&= ((\lambda I - B)^{-1})^* \\
&= R(\lambda, B)^*,
\end{aligned}$$

and the proof is complete. □

Definition 2.3.16. The *peripheral spectrum* of $B \in \mathscr{L}(X)$ is the set $\{\lambda \in \sigma(B) : |\lambda| = r(B)\}$. The *peripheral point spectrum* of $B \in \mathscr{L}(X)$ is the set $\{\lambda \in \sigma_p(B) : |\lambda| = r(B)\}$.

Definition 2.3.17. Let $B \in \mathscr{L}(X)$ be a bounded linear operator such that $r(B) > 0$. Then, a subset S of the spectrum is called *cyclic* if, for all nonzero points $\lambda \in S$, setting $\lambda = |\lambda|\omega$ one has, for every $k \in \mathbb{Z}$, $|\lambda|\omega^k \in S$.

The notions of cyclic point spectrum and cyclic peripheral point spectrum are the central focus for the next two chapters. The main results will be that the peripheral point spectrum for positive $n \times n$ matrices and Frobenius-Perron operators is (fully) cyclic. The definition of fully cyclic will follow in the next chapters.

Chapter 3

Spectrum of Positive $n \times n$ Matrices

In this chapter the theory of positive square matrices over $\mathbb{C}$ is developed. Throughout the entire chapter, all matrices will be square $n \times n$ matrices for some given dimension n.

3.1 Positive Matrices

Recall from the Preliminaries that $\mathbb{C}^n$ is a Banach lattice when equipped with the Euclidean norm:

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

For x in $\mathbb{C}^n$, $x \geq 0$ will mean that x_i is real and greater than or equal to 0 for all $i \in \{1, \ldots, n\}$, while $x > 0$ will mean that $x \geq 0$ and $x \neq 0$. $x \gg 0$ will mean that $x_i > 0$ for all $i \in \{1, \ldots, n\}$.

The space of bounded linear operators on $\mathbb{C}^n$ is denoted $\mathscr{L}(\mathbb{C}^n)$, and each bounded linear operator B can be represented by a matrix $A = \{a_{ij}\}$ under some given basis (for the domain and range). Thus, when the basis is chosen, a matrix may be interpreted as a bounded linear operator and the results on bounded linear operators from the Preliminaries apply. $\mathbb{C}^{n\times n}$, the space of $n \times n$ matrices over $\mathbb{C}$, inherits its norm from $\mathbb{C}^n$:

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

From Definition 2.1.30, an $n \times n$ matrix A over $\mathbb{C}$ is positive (written $A \geq 0$) if

$$x \geq 0 \quad \Rightarrow \quad Ax \geq 0.$$

Remark. If A is a positive matrix, then given a positive integer k,

$$x \geq 0 \Rightarrow Ax \geq 0 \Rightarrow \ldots \Rightarrow A^k x \geq 0,$$

so that $A^k \geq 0$.

Proposition 3.1.1. *An $n \times n$ matrix $A = \{a_{ij}\}$ over $\mathbb{C}$, where the coefficients a_{ij} are taken from the standard basis representation, is positive if and only if $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$.*

Proof. Suppose A is positive. Fix l in $\{1, \ldots, n\}$. Then the standard basis vector $e_l = (\delta_{1l}, \ldots, \delta_{nl})$ where

$$\delta_{il} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{if } i \neq l, \end{cases}$$

is positive, so that

$$Ae_l = \Big(\sum_{j=1}^{n} a_{1j}\delta_{jl}, \ldots, \sum_{j=1}^{n} a_{nj}\delta_{jl} \Big) = (a_{1l}, \ldots, a_{nl}),$$

which is the lth column of A, is positive. This means that the entries in the lth column of A must all be greater than or equal to 0. Since l was arbitrarily chosen, $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$.

Conversely, suppose $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$. Let $x \geq 0$. Then $x_j \geq 0$ for all $j \in \{1, \ldots, n\}$, and

$$Ax = \Big(\sum_{j=1}^{n} a_{1j}x_j, \ldots, \sum_{j=1}^{n} a_{nj}x_j \Big).$$

Since $a_{ij} \geq 0$ and $x_j \geq 0$, $\sum_{j=1}^{n} a_{ij}x_j \geq 0$ for all i. Therefore, $Ax \geq 0$ and hence A is positive. $\square$

Example 3.1.2. The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

under the standard basis is a positive matrix. By performing a simple change of basis, the matrix can be given a different form:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

After the change to the basis $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$, the matrix becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which has -1 as an entry. Thus there may exist representations of a positive matrix that do not have all nonnegative entries.

It is due to the simple characterization of positive matrices in Proposition 3.1.1 that the only form of matrices that will be considered throughout this work are standard basis representations (so that the basis for both the domain and codomain is chosen to be the standard basis).

From Definition 2.3.2, the spectrum $\sigma(A)$ of an $n \times n$ matrix A is the set of $\lambda \in \mathbb{C}$ for which the matrix $(\lambda I - A)$ does not have an inverse. Since the dimension of $\mathbb{C}^n$ is finite, this can only happen if $(\lambda I - A)$ is not injective, and therefore λ is in the point spectrum of A. Thus, if λ is in the spectrum of A, there exists a nonzero solution x to the equation $(\lambda I - A)x = 0$, or $Ax = \lambda x$. These λ are called eigenvalues of A, and the nonzero solutions x are called eigenvectors for λ. Since an $n \times n$ matrix can have at most n eigenvalues, the spectrum of A is a finite set in the complex plane. $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator and is defined on the resolvent set

$$\rho(A) = \mathbb{C}^n \setminus \sigma(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is invertible}\}.$$

From Definition 2.3.6, the spectral radius of A is $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

Recall (Theorem 2.3.3) that, for $|\lambda| > r(A)$, $R(\lambda, A)$ exists, is analytic and admits the representation $R(\lambda, A) = \sum_{k=1}^{\infty} \frac{A^k}{\lambda^{k+1}}$. Then, if $A \geq 0$ and $\lambda > r(A)$,

$$\begin{aligned} 0 \leq x \;\; &\Rightarrow \;\; 0 \leq A^k x \;\; \forall k \geq 0 \\ &\Rightarrow \;\; 0 \leq \sum_{k=0}^{\infty} \frac{A^k x}{\lambda^{k+1}} \\ &\Rightarrow \;\; 0 \leq R(\lambda, A)x. \end{aligned}$$

Therefore, A positive and $\lambda > r(A)$ imply that $R(\lambda, A)$ is a positive matrix.

The following is the Frobenius-Perron Theorem for positive matrices, and can be found in [1]. The proof is included to motivate the calculations in the infinite dimensional cases. Recall from Example 2.1.25 that $x > 0$ means that $x \geq 0$ but $x \neq 0$, and from Equation (2.1) $|x| = (|x_1|, \ldots, |x_n|)$.

Proposition 3.1.3. *If $A \geq 0$, then $r = r(A)$ is an eigenvalue of A with at least one eigenvector $x > 0$.*

Proof. Let λ be an eigenvalue of A such that $|\lambda| = r$ and choose any sequence $\{\lambda_i\}$ of complex numbers with $|\lambda_i| > r$ such that $\{\lambda_i\}$ converges to λ. Then there exists an eigenvector $x_0 \neq 0$ associated with λ, so that

$$\begin{aligned} |R(\lambda_i, A)x_0| &= |\sum_{k=0}^{\infty} \lambda_i^{-(k+1)} A^k x_0| \\ &\leq \sum_{k=0}^{\infty} |\lambda_i^{-(k+1)} A^k x_0| \\ &= \sum_{k=0}^{\infty} |\lambda_i^{-(k+1)}||A^k x_0| \\ &\leq \sum_{k=0}^{\infty} |\lambda_i|^{-(k+1)} A^k |x_0| \\ &= R(|\lambda_i|, A)|x_0|, \end{aligned}$$

where Theorem 2.1.31 was used in the second inequality since $A \geq 0$. Therefore $|R(\lambda_i, A)x_0| \leq R(|\lambda_i|, A)|x_0|$.

Consider

$$\begin{aligned}(\lambda_i I - A)x_0 &= \lambda_i x_0 - Ax_0\\ &= \lambda_i x_0 - \lambda x_0\\ &= (\lambda_i - \lambda)x_0.\end{aligned}$$

Then $R(\lambda_i, A)x_0 = \frac{x_0}{(\lambda_i-\lambda)}$ and $|R(\lambda_i, A)x_0| = \left|\frac{x_0}{(\lambda_i-\lambda)}\right|$, which is not bounded as λ_i approaches λ, so that $R(|\lambda_i|, A)|x_0| \geq |R(\lambda_i, A)x_0|$ is also not bounded as λ_i approaches λ.

Set $z_i = R(|\lambda_i|, A)|x_0|/\|R(|\lambda_i|, A)|x_0|\|$. The set $B = \{x \in \mathbb{C}^n : \|x\| = 1\}$ is compact and $z_i \in B$ for all $i \geq 1$. Thus there exists a subsequence $\{z_{i_l}\}$of $\{z_i\}$ converging to a point x in B. Since $z_{i_l} \geq 0$, $x \geq 1$. Then,

$$\begin{aligned}(rI - A)z_{i_l} &= \big((r - |\lambda_{i_l}| + |\lambda_{i_l}|)I - A\big)z_{i_l}\\ &= \big((r - |\lambda_{i_l}|)I + (|\lambda_{i_l}|I - A)\big)z_{i_l}\\ &= (r - |\lambda_{i_l}|)z_{i_l} + (|\lambda_{i_l}|I - A)\frac{R(|\lambda_{i_l}|, A)|x_0|}{\|R(|\lambda_{i_l}|, A)|x_0|\|}\\ &= (r - |\lambda_{i_l}|)z_{i_l} + \frac{|x_0|}{\|R(|\lambda_{i_l}|, A)|x_0|\|}.\end{aligned}$$

Thus,

$$\begin{aligned}\|(rI - A)x\| &= \|r(x - z_{i_l}) + (rI - A)z_{i_l} + A(z_{i_l} - x)\|\\ &\leq \|r(x - z_{i_l})\| + \|(rI - A)z_{i_l}\| + \|A(x - z_{i_l})\|\\ &\leq (r + \|A\|)\|x - z_{i_l}\| + \|(rI - A)z_{i_l}\|\\ &= (r + \|A\|)\|x - z_{i_l}\| + \left\|(r - |\lambda_{i_l}|)z_{i_l} + \frac{|x_0|}{\|R(|\lambda_{i_l}|, A)|x_0|\|}\right\|,\end{aligned}$$

for all i. But this goes to zero as $l \to \infty$ if the final term does. The final term gives

$$\begin{aligned}\left\|(r - |\lambda_{i_l}|)z_{i_l} + \frac{|x_0|}{\|R(|\lambda_{i_l}|, A)|x_0|\|}\right\| &\leq \left\|(r - |\lambda_{i_l}|)z_{i_l}\right\| + \left\|\frac{|x_0|}{\|R(|\lambda_{i_l}|, A)|x_0|\|}\right\|\\ &= |r - |\lambda_{i_l}||\|z_{i_l}\| + \frac{\|x_0\|}{\|R(|\lambda_{i_l}|, A)|x_0|\|}\\ &= |r - |\lambda_{i_l}|| + \frac{\|x_0\|}{\|R(|\lambda_{i_l}|, A)|x_0|\|},\end{aligned}$$

which goes to zero as $l \to \infty$. Therefore, $\|(rI - A)x\| = 0$ and hence $Ax = rx$. $\square$

For $x \in \mathbb{C}$, define

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0,\\ 0 & \text{if } x = 0.\end{cases}$$

For $x \in \mathbb{C}^n$, define $\operatorname{sgn}(x)$ by $\operatorname{sgn}(x)_i = \operatorname{sgn}(x_i)$ for $1 \leq i \leq n$. For two vectors x, y in $\mathbb{C}^n$, define the product xy by $(xy)_i = x_i y_i$. $\mathbb{C}^n$ equipped with this product is an algebra. In addition, for a vector x and a nonpositive integer k, define x^k by

$$(x^k)_i = \begin{cases} x_i^k & \text{if } x_i \neq 0, \\ 0 & \text{if } x_i = 0. \end{cases}$$

Recall from Definition 2.3.16 that the peripheral spectrum of an $n \times n$ matrix A is defined to be the set $\{\lambda \in \sigma(A) : |\lambda| = r\}$, where $r = r(A)$ is the spectral radius of A. Also recall from Definition 2.3.17 that if $r > 0$ the peripheral spectrum is cyclic when $|\omega| = 1$ and $r\omega \in \sigma(A)$ implies that $r\omega^k \in \sigma(A)$ for all integers k, where $|\omega| = 1$. A partial result in this direction is the following.

Lemma 3.1.4. *Let $A \geq 0$. If $Ax = \omega x$ for some $x \neq 0$, $|\omega| = 1$, and $A|x| = |x|$, then $A(|x|\mathrm{sgn}(x)^k) = \omega^k(|x|\mathrm{sgn}(x)^k)$ for all integers k.*

Proof. Since $Ax = \omega x$, for $1 \leq i \leq n$,

$$\sum_{j=1}^{n} a_{ij}x_j = \omega x_i \Rightarrow \sum_{j=1}^{n} a_{ij}|x_j|\mathrm{sgn}(x_j) = \omega |x_i| \mathrm{sgn}(x_i).$$

Since $A|x| = |x|$, for $1 \leq i \leq n$,

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}|x_j| = |x_i| \quad &\Rightarrow \quad \sum_{j=1}^{n} a_{ij}|x_j|\omega\mathrm{sgn}(x_i) = \omega|x_i|\mathrm{sgn}(x_i), \quad 1 \leq i \leq n \\ &\Rightarrow \quad \sum_{j=1}^{n} a_{ij}|x_j|\omega\mathrm{sgn}(x_i) = \sum_{j=1}^{n} a_{ij}|x_j|\mathrm{sgn}(x_j), \quad 1 \leq i \leq n \\ &\Rightarrow \quad \sum_{j=1}^{n} a_{ij}|x_j|\left(1 - \frac{\mathrm{sgn}(x_j)}{\omega\mathrm{sgn}(x_i)}\right) = 0 \end{aligned}$$

for any i such that $x_i \neq 0$, so that $\omega\mathrm{sgn}(x_i) \neq 0$.

Fix $i \in \{1, \ldots, n\}$ such that $x_i \neq 0$. Then, for all $j \in \{1, \ldots, n\}$ such that $a_{ij}|x_j| > 0$,

$$\sum_{j} a_{ij}|x_j|\left(1 - \frac{\mathrm{sgn}(x_j)}{\omega\mathrm{sgn}(x_i)}\right) = 0. \tag{3.1}$$

Let $\mathfrak{R}(y)$ denote the real part of the complex number y. Since $|\frac{\mathrm{sgn}(x_j)}{\omega\mathrm{sgn}(x_i)}| = 1$,

$$\mathfrak{R}\left(1 - \frac{\mathrm{sgn}(x_j)}{\omega\mathrm{sgn}(x_i)}\right) \geq 0,$$

and because of (3.1), whenever $a_{ij}|x_j| > 0$,

$$\mathfrak{R}\left(1 - \frac{\mathrm{sgn}(x_j)}{\omega\mathrm{sgn}(x_i)}\right) = 0,$$

so that

$$\frac{\text{sgn}(x_j)}{\omega\text{sgn}(x_i)} = 1,$$

meaning $\omega\text{sgn}(x_i) = \text{sgn}(x_j)$.

If $x_i = 0$, then $\sum_{j=1}^{n} a_{ij}|x_j| = 0$ since $A|x| = |x|$, and since $a_{ij}|x_j| \geq 0$ for all j, it can be concluded that $a_{ij}|x_j| = 0$ for all j. Therefore, in either case, for any integer k,

$$\omega^k \text{sgn}(x_i)^k = \text{sgn}(x_j)^k$$

whenever $a_{ij}|x_j| > 0$. So

$$\begin{aligned}
\sum_{j=1}^{n} a_{ij}|x_j|\text{sgn}(x_j)^k &= \sum_{j:\ a_{ij}|x_j|>0} a_{ij}|x_j|\text{sgn}(x_j)^k \\
&= \sum_{j:\ a_{ij}|x_j|>0} a_{ij}|x_j|\omega^k\text{sgn}(x_i)^k \\
&= \omega^k\text{sgn}(x_i)^k \sum_{j=1}^{n} a_{ij}|x_j| \\
&= \omega^k|x_i|\text{sgn}(x_i)^k.
\end{aligned}$$

Therefore $A(|x|\text{sgn}(x)^k) = \omega^k(|x|\text{sgn}(x)^k)$ for all integers k. □

Remark. Lemma 3.1.4 shows that, for a positive matrix A, in addition to the spectrum on the unit circle being cyclic, the corresponding eigenvectors rotate along with the eigenvalues. This behaviour is called fully cyclic, and will be formally defined shortly (Definition 3.1.10). The term fully cyclic will refer to a cyclic spectral decomposition.

Definition 3.1.5. Let H be a subset of $\{1, .., n\}$. Then the ideal J_H of $\mathbb{C}^n$ will be defined as the subspace

$$J_H = \{x \in \mathbb{C}^n : x_i = 0\ \forall i \in H\}.$$

These are ideals in the algebra $\mathbb{C}^n$ closed under pointwise multiplication (ie. $y \in \mathbb{C}^n$ and $x \in J_H$ implies that $xy \in J_H$).

Remark. Given a vector x in $\mathbb{C}^n$, let J be the smallest subspace containing all vectors y_0 such that $|y_0| \leq |x|$, and let $H_x = \{i : x_i = 0\}$. Then given any y in J, there exists $b > 0$ such that $b|y| \leq |x|$, and in particular $(b|y|)_i = 0$ for i in H_x. Thus $J = J_{H_x}$ and the ideals in Definition 3.1.5 can be described in terms of the lattice structure on $\mathbb{C}^n$. J_{H_x} is called the ideal generated by x.

Let J_H be an ideal of $\mathbb{C}^n$, and suppose that J_H is invariant under A, that is $A(J_H) \subset J_H$. Then there exist linear maps $A_{|J_H} : J_H \to J_H$, induced by A on the subspace J_H, and $A_{J_H} : \mathbb{C}^n/J_H \to \mathbb{C}^n/J_H$, induced by A on the quotient space $\mathbb{C}^n/J_H$. The action of A_{J_H} on elements $x + J_H = \hat{x}$ of $\mathbb{C}^n/J_H$ is given by

$$A_{J_H}(x + J_H) = (Ax) + J_H.$$

As is shown in [24],

$$\sigma(A) = \sigma(A_{|J_H}) \bigcup \sigma(A_{J_H}). \tag{3.2}$$

If A has the matrix representation (a_{ij}) under the standard basis, $A_{|J_H}$ has matrix equal to the submatrix (a_{kl}) where $(k,l) \in H^c \times H^c$, and A_{J_H} has matrix equal to the submatrix (a_{rs}) where $(r,s) \in H \times H$.

The space $\mathbb{C}^n/J_H$ inherits the complex vector lattice structure from $\mathbb{C}^n$ pointwise, as $\hat{x} \geq 0$ if and only if $x_i \geq 0$ for all i in H, and $|\hat{x}|$ is the element with representative $|x|$, which has $|x|_i = |x_i|$ for all i in H. Thus $|\hat{x}| = \widehat{|x|}$.

Recall from Section 2.3 that the dual space of $\mathbb{C}^n$ is the space of all linear functionals on $\mathbb{C}^n$, which can be identified with $\mathbb{C}^n$ under the inner product

$$y(x) = \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

The Hermitian form $\sum_{i=1}^{n} x_i \bar{y}_i$ is the natural choice to construct the norm on $\mathbb{C}^n$, however it is not used here as the regular dot product is consistent with the dual used in the next chapter. From the dot product the adjoint of a matrix A is then the transpose A^t, and not the conjugate transpose $\bar{A}^t$ that would be associated with the Hermitian form.

Theorem 3.1.6. *The peripheral spectrum of every positive matrix is cyclic.*

Proof. Let A be a positive matrix. If $r(A) = 0$, then $\sigma(A) = \{0\}$, which is trivially cyclic. If $r(A) > 0$, it may be assumed without loss of generality that $r(A) = 1$, since from the Spectral Mapping Theorem in [9], $\sigma(\beta A) = \beta\sigma(A)$ with $\beta = \frac{1}{r}$, so that scalar factors will not change the cyclic nature of the peripheral spectrum. Let $Ax = \alpha x$ where $|\alpha| = 1$ and $x \neq 0$. Then

$$|x| = |\alpha||x| = |\alpha x| = |Ax| \leq A|x|$$

since $|Ax| \leq A|x|$ by Theorem 2.1.31. Therefore $\big|A|x| - |x|\big| = A|x| - |x|$.

Denote by J the smallest ideal containing $\{A^m|x| : m \in \mathbb{Z}, m \geq 0\}$. That is, J_H where for each i in H^c there exists a nonnegative integer m such that $A^m|x|$ is strictly positive in the ith coordinate, however for all i in H, $A^m|x|$ is zero in the ith coordinate for all nonnegative integers m. For any y in J, there exists a finite linear combination $c_1 A^{m_1}|x| + \ldots + c_j A^{m_j}|x|$ such that $|y| \leq c_1 A^{m_1}|x| + \ldots + c_j A^{m_j}|x|$. Then

$$|Ay| \leq A|y| \leq c_1 A^{m_1+1}|x| + \ldots + c_j A^{m_j+1}|x|,$$

so that Ay is also in J. Thus J is an A-invariant ideal.

Since A is positive with spectral radius equal to 1, so is $A^* = A^t$, so Proposition 3.1.3 may be applied to obtain $y_0 > 0$ such that $A^t y_0 = y_0$. Define K to be the subspace $\{z : \langle |z|, y_0 \rangle = 0\}$. This is an ideal since it consists of all vectors in $\mathbb{C}^n$ that are zero where y_0 is strictly positive. This is also an invariant ideal under A, since for any z in K,

$$\begin{aligned}\langle |Az|, y_0\rangle &\le \langle A|z|, y_0\rangle \\ &= \langle |z|, A^t y_0\rangle \\ &= \langle |z|, y_0\rangle = 0,\end{aligned}$$

so that Az is also in K.

It will now be shown that $\alpha^k \in \sigma(A)$ for all integers k. First assume that $J = \mathbb{C}^n$. Then if x is in K,

$$0 = \langle |x|, y_0\rangle = \langle |x|, (A^t)^m y_0\rangle = \langle A^m|x|, y_0\rangle$$

for all m, which implies that y_0 is everywhere 0 since $J = \mathbb{C}^n$, which is a contradiction. Therefore $\langle |x|, y_0\rangle > 0$ and x is not in K. However $A|x| - |x|$ is in K since

$$\begin{aligned}\langle \big|A|x| - |x|\big|, y_0\rangle &= \langle A|x| - |x|, y_0\rangle \\ &= \langle A|x|, y_0\rangle - \langle |x|, y_0\rangle \\ &= \langle |x|, A^t y_0\rangle - \langle |x|, y_0\rangle \\ &= \langle |x|, y_0\rangle - \langle |x|, y_0\rangle = 0.\end{aligned}$$

Thus, if $\hat{x}$ is the canonical representation of x in $\mathbb{C}^n/K$, then

$$\begin{aligned}A_K\hat{x} &= A_K(x + K) \\ &= (Ax) + K \\ &= (\alpha x) + K \\ &= \alpha\hat{x},\end{aligned}$$

and

$$\begin{aligned}A_K|\hat{x}| - |\hat{x}| &= A_K(|x| + K) - (|x| + K) \\ &= (A|x| + K) - (|x| + K) \\ &= (A|x| - |x|) + K \\ &= 0 + K \\ &= \hat{0},\end{aligned}$$

or $A_K|\hat{x}| = |\hat{x}|$. Then from Lemma 3.1.4 $\alpha^k \in \sigma(A_K)$ for all integers k, and hence $\alpha^k \in \sigma(A)$ for all integers k by virtue of Equation (3.2).

Now assume that $J \neq \mathbb{C}^n$. Then the same argument may be applied to the positive matrix $A_{|J}$, which is the restriction of A to J, with $A_{|J}x' = \alpha x'$, where $x' \neq 0$ is the canonical image of x in J. $\square$

Example 3.1.7. The $n \times n$ zero matrix

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

and the $n \times n$ identity matrix

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

are both positive matrices for all positive integers n. The only eigenvalue of the zero matrix is 0 and the only eigenvalue of I is 1. Hence the peripheral spectrum (and indeed the entire spectrum) of both is cyclic.

Example 3.1.8. The 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$$

has the characteristic polynomial

$$\begin{aligned} p_A(\lambda) &= (\lambda)(\lambda - 2) - (-3)(-1) \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda + 1)(\lambda - 3), \end{aligned}$$

so that the spectrum of A is equal to the set $\{-1, 3\}$. Thus the peripheral spectrum is cyclic, however the entire spectrum is not cyclic since $1 = (-1)^2|-1|$ is not in the spectrum.

Following the methods of the proof of Theorem 3.1.6, $A' = \frac{1}{3}A$ has spectral radius equal to one. Then the vector $x = (1, 3)$ is a fixed point for A', so that $J = \mathbb{C}^2$. The choice vector $y_0 = (1, 1)$ as a fixed point for $(A')^t$ gives $K = \{0\}$. Thus $A'_K = A'$.

Example 3.1.9. The 6×6 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{7}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}$$

has the characteristic polynomial

$$\begin{aligned} p_A(\lambda) &= \Big(\lambda^3 - \frac{7}{4}\lambda^2 - 1\Big)(\lambda^3 - 8) \\ &= (\lambda-2)\Big(\lambda^2 + \frac{1}{4}\lambda + \frac{1}{2}\Big)(\lambda-2)(\lambda+1-\mathrm{i}\sqrt{3})(\lambda+1+\mathrm{i}\sqrt{3}) \\ &= (\lambda-2)\Big(\lambda + \frac{1}{8} + \mathrm{i}\frac{\sqrt{31}}{8}\Big)\Big(\lambda + \frac{1}{8} - \mathrm{i}\frac{\sqrt{31}}{8}\Big) \cdot \\ &\quad \cdot (\lambda-2)(\lambda-1-\mathrm{i}\sqrt{3})(\lambda-1+\mathrm{i}\sqrt{3}), \end{aligned}$$

so that the spectrum of A is equal to the set

$$\{2, -1+\mathrm{i}\sqrt{3}, -1-\mathrm{i}\sqrt{3}, -\frac{1}{8} + \mathrm{i}\frac{\sqrt{31}}{8}, -\frac{1}{8} - \mathrm{i}\frac{\sqrt{31}}{8}\}.$$

The peripheral spectrum consits of the three points

$$\{2, -1+\mathrm{i}\sqrt{3}, -1-\mathrm{i}\sqrt{3}\},$$

and thus is cyclic, however the entire spectrum is not cyclic since

$$\Big| -\frac{1}{8} + \mathrm{i}\frac{\sqrt{31}}{8}\Big| = \Big| -\frac{1}{8} - \mathrm{i}\frac{\sqrt{31}}{8}\Big| = \sqrt{\frac{1+31}{64}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

is not in the spectrum.

$A' = \frac{1}{2}A$ has spectral radius equal to 1. Then the vector

$$x = \Big(0, 0, 0, 1, -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}, -1 + \mathrm{i}\sqrt{3}\Big)$$

satisfies $A'x = \big(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\big)x$, so that $J = \{y : y_1 = y_2 = y_3 = 0\} \neq \mathbb{C}^n$. Thus

$$A'_{|J} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

The vector

$$(2, 2, 1)$$

is a fixed point for the matrix $(A'_{|J})^t$, so that $K = \{0\}$. Thus $(A'_{|J})_K = A'_{|J}$.

Definition 3.1.10. Let A be an $n \times n$ matrix over $\mathbb{C}$ and let S be a subset of the spectrum of A. S is called *fully cyclic* if, for all $\lambda \neq 0$ in S and $x \neq 0$ in $\mathbb{C}^n$ such that $Ax = \lambda x$ with $\lambda = |\lambda|\omega$ and $x = |x|\mathrm{sgn}(x)$, then for all integers k, $|\lambda|\omega^k$ is in S and $A|x|\mathrm{sgn}(x)^k = |\lambda|\omega^k|x|\mathrm{sgn}(x)^k$.

Informally, Definition 3.1.10 differs from Definition 2.3.17 in that, in addition to the spectral points rotating in a cyclic manner, the corresponding eigenvectors rotate as well for a fully cyclic subset of the spectrum.

Theorem 3.1.11. *Let $A \geq 0$. If there exists $y_0 \gg 0$ such that $A^t y_0 \leq r(A) y_0$, then the peripheral spectrum of A is fully cyclic.*

Proof. As in Theorem 3.1.6, it may be assumed without loss of generality that the spectral radius of A is equal to 1. Let $Ax = \alpha x$ where $|\alpha| = 1$ and $x \neq 0$. The ideal K, defined with respect to y_0 in the hypothesis, in the proof of Theorem 3.1.6 is then equal to $\{0\}$. Then the equations $A_K \hat{x} = \alpha \hat{x}$ and $A_K |\hat{x}| = |\hat{x}|$ become $Ax = \alpha x$ and $A|x| = |x|$, respectively. Now when the proof of Lemma 3.1.4 is applied, it is shown that $A|x|\mathrm{sgn}(x)^k = \alpha^k |x| \mathrm{sgn}(x)^k$ for all integers k. Thus, the peripheral spectrum is fully cyclic. $\square$

Example 3.1.12. Since $I(1, \ldots, 1) = 1(1, \ldots, 1)$ (and $I^t = I$), I has fully cyclic peripheral spectrum. This is immediate, since every vector in $\mathbb{C}^n$ is a fixed point of I.

Example 3.1.13. Consider again the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{7}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

Then $A^t(2,1,4,2,2,1) = 2(2,1,4,2,2,1)$, so by Theorem 3.1.11 the peripheral spectrum is fully cyclic. If $\alpha = \frac{-1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$, the eigenvectors x for the peripheral spectral points λ are:

λ	x
2	$(1,2,4,0,0,0)$
2	$(0,0,0,1,1,2)$
2α	$\big(0,0,0,1,\alpha^2,2\alpha\big)$
$2\alpha^2$	$\big(0,0,0,1,\alpha,2\alpha^2\big)$

It is observed that for any pair λ and x of an eigenvalue in the peripheral spectrum and one of its corresponding eigenvectors, $A(|x|\mathrm{sgn}(x)^k) = \lambda^k |x| \mathrm{sgn}(x)^k$ for all integers k, and it is confirmed that A has fully cyclic peripheral spectrum.

Example 3.1.14. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then the characteristic polynomial is

$$p_A(\lambda) = (\lambda - 1)(\lambda^2 - 1) = (\lambda - 1)^2(\lambda + 1).$$

Thus, the eigenvalues are ± 1 and the spectral radius is 1. The eigenvector for 1 is $(0,0,1)$, while the eigenvector for -1 is $x_{-1} = (1,-1,0)$. But $|x_{-1}| = (1,1,0)$ is not a fixed point for A (note that it is a generalized eigenvector for the eigenvalue 1). Thus, the peripheral spectrum of A is not fully cyclic.

For any $y \gg 0$,

$$A^t y = (y_2 + y_3, y_1 + y_3, y_3).$$

Thus $A^t y \not\leq y$, since $y_1 \leq y_2$ implies that $y_1 < y_2 + y_3$, whereas $y_2 \leq y_1$ implies that $y_2 < y_1 + y_3$.

Definition 3.1.15. Let $A = (a_{ij})$ be a positive $n \times n$ matrix. A is called *row-stochastic* (or *Markov*) if $\sum_{j=1}^n a_{ij} = 1$ for all $1 \leq i \leq n$. A is called *column-stochastic* if $\sum_{i=1}^n a_{ij} = 1$ for all $1 \leq j \leq n$. A is called *doubly-stochastic* if it is both row- and column-stochastic.

Lemma 3.1.16. *The spectral radius of a row- or column-stochastic matrix is equal to* 1.

Proof. Since the transpose of a row-stochastic matrix is column-stochastic and the transpose of a column-stochastic matrix is row-stochastic, by Theorem 2.3.15 it suffices to show only that the spectral radius of a row-stochastic matrix is equal to 1.

Let A be row-stochastic, and suppose that $Ax = \lambda x$ for $x \neq 0$. Then

$$\lambda x_i = \sum_{j=1}^n a_{ij} x_j$$

for all i in $\{1, \ldots, n\}$. Taking the modulus of both sides gives

$$|\lambda x_i| = |\sum_{j=1}^n a_{ij} x_j|,$$

and by the triangle inequality

$$|\lambda x_i| \leq \sum_{j=1}^n a_{ij} |x_j|.$$

Summing both sides over i then gives

$$\begin{aligned} |\lambda| \sum_{i=1}^n |x_i| &\leq \sum_{i=1}^n \sum_{j=1}^n a_{ij} |x_j| \\ &= \sum_{j=1}^n |x_j| \sum_{i=1}^n a_{ij} \\ &= \sum_{j=1}^n |x_j| = \sum_{i=1}^n |x_i|, \end{aligned}$$

where the fact that $\sum_{i=1}^{n} a_{ij} = 1$ for all i for a row-stochastic matrix was used. Since $x \neq 0$, $\sum_{i=1}^{n} |x_i| > 0$ and thus $|\lambda| \leq 1$. Therefore $r(A) \leq 1$.

Since A is row-stochastic, $A\mathbf{1} = \mathbf{1}$, so that $1 \in \sigma(A)$ and $r(A) \geq 1$. Thus $r(a) = 1$. □

Theorem 3.1.17. *The peripheral spectrum of a column-stochastic matrix is fully cyclic.*

Proof. Let A be an $n \times n$ column-stochastic matrix. Then A is positive with spectral radius equal to 1, and

$$A^t(1, \ldots, 1) = \big(\sum_{i=1}^{n} a_{i1}, \ldots, \sum_{i=1}^{n} a_{in}\big) = (1, \ldots, 1).$$

Thus, by Theorem 3.1.11, the peripheral spectrum of A is fully cyclic. □

The same cannot be said in general about Markov matrices.

Example 3.1.18. Following the lead of Example 3.1.14, consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Then A is a Markov matrix such that

$$A(1, 1, 1) = (1, 1, 1),$$

$$A(1, -1, 0) = (-1, 1, 0) = -1(1, -1, 0)$$

and

$$A(0, 0, 1) = (0, 0, \frac{1}{3}) = \frac{1}{3}(0, 0, 1).$$

Thus the spectrum, which is equal to $\{1, -1, \frac{1}{3}\}$, is cyclic, however the peripheral spectrum, which is equal to $\{1, -1\}$, is not fully cyclic since

$$A|(1, -1, 0)| = A(1, 1, 0) = (1, 1, \frac{2}{3}) \neq |(1, -1, 0)|.$$

3.2 Lattice Homomorphisms

Recall that all matrices considered here are assumed to be given with respect to the standard basis of $\mathbb{C}^n$. As is defined in the Preliminaries, a linear operator and hence a matrix A is a lattice homomorphism if and only if $A(x \vee y) = Ax \vee Ay$ and $A(x \wedge y) = Ax \wedge Ay$ for all x, y in $\mathbb{C}^n$. Since lattice homomorphisms are positive, the results of the previous section apply to lattice homomorphisms. First, a characterization of matrices that are lattice homomorphisms.

Proposition 3.2.1. *$A = (a_{ij})$ is a lattice homomorphism if and only if A is a positive matrix with at most one nonzero entry per row of A.*

Proof. Suppose that A is a lattice homomorphism. It has already been shown that A is positive, so it remains to show that there exists at most one nonzero entry per row. Fix $i \in \{1, \dots, n\}$. Then the ith component of $A(x \vee y)$ is $\sum_{j=1}^{n} a_{ij}(x_j \vee y_j)$ and the ith component of $Ax \vee Ay$ is $\sum_{j=1}^{n} a_{ij}x_j \vee \sum_{j=1}^{n} a_{ij}y_j$, where x and y can be chosen to be any vectors in $\mathbb{C}^n$. If two entries a_{il_1} and a_{il_2} in the ith row are nonzero, then let x be the vector with $x_{l_1} = 1$, $x_{l_2} = -1$ and $x_j = 0$ for $j \neq l_1, l_2$, and let y be the vector with $y_{l_1} = -1$, $y_{l_2} = 1$ and $y_j = 0$ for $j \neq l_1, l_2$, so that

$$\begin{aligned}
\sum_{j=1}^{n} a_{ij}(x_j \vee y_j) &= a_{il_1}(1 \vee -1) + a_{il_2}((-1) \vee 1) \\
&= a_{il_1} + a_{il_2} \\
&\neq (a_{il_1} - a_{il_2}) \vee (a_{il_2} - a_{il_1}) \\
&= \sum_{j=1}^{n} a_{ij}x_j \vee \sum_{j=1}^{n} a_{ij}y_j.
\end{aligned}$$

Therefore, $A(x \vee y) \neq Ax \vee Ay$ and by contradiction, A being a lattice homomorphism implies that A is positive with at most one nonzero entry per row.

Conversely, suppose that A is positive with at most one nonzero entry per row. Fix $i \in \{1, \dots, n\}$, and if row i has a nonzero entry a_{ij_i}, then given some x, y in $\mathbb{C}^n$, the ith component of $A(x \vee y)$ is $a_{ij_i}(x_{j_i} \vee y_{j_i}) = a_{ij_i}x_{j_i} \vee a_{ij_i}y_{j_i}$, which is the ith component of $Ax \vee Ay$. Similarly, the ith component of $A(x \wedge y)$ is equal to the ith component of $Ax \wedge Ay$. Therefore, when A is positive with at most one nonzero entry per row, $A(x \vee y) = Ax \vee Ay$ and $A(x \wedge y) = Ax \wedge Ay$, and hence, A is a lattice homomorphism. □

Remark. If T is a transformation on the indices $\{1, \dots, n\}$ (ie. any function from $\{1, \dots, n\}$ back into itself), then the Markov transition matrix associated with T is the matrix $A = (a_{ij})$ where $a_{i(Ti)} = 1$ and $a_{ij} = 0$ if $j \neq Ti$. This matrix is identical to the Koopman operator for T on $\mathbb{C}^n$, and by the above Lemma, is a lattice homomorphism.

In the previous section it was shown that certain positive matrices have fully cyclic peripheral spectrum. For lattice homomorphisms, it is true that the entire spectrum is fully cyclic.

Theorem 3.2.2. *The spectrum of an $n \times n$ lattice homomorphism A over $\mathbb{C}$ is fully cyclic.*

Proof. Suppose that $Ax = \lambda x$, where $x \neq 0$ and $\lambda = \rho\omega$ where $\rho > 0$ and $|\omega| = 1$. For $i \in \{1, \dots, n\}$, since A is a lattice homomorphism,

$$(A|x|)_i = a_{ij_i}|x_{j_i}| = |a_{ij_i}x_{j_i}| = |(Ax)_i| = |\lambda x_i| = \rho|x|_i.$$

Since i was arbitrary, this shows that $A|x| = \rho|x|$.

Fix i in $\{1, \dots, n\}$. Suppose that row i of A has a nonzero entry a_{ij_i} and that x_{j_i} is nonzero. Then, for l in $\{1, \dots, n\}$, let $y_l = \operatorname{sgn}(x_l)$ so that $a_{ij_i}|x_{j_i}|y_{j_i} = \rho\omega|x_i|y_i$

and $a_{ij_i}|x_{j_i}| = \rho|x_i|$. Since $a_{ij_i}|x_{j_i}| \neq 0$, substitution yeilds

$$y_{j_i} = \omega y_i,$$

where y_{j_i}, ω and y_i are all complex numbers with modulus 1. Letting k be any integer then gives

$$y_{j_i}^k = \omega^k y_i^k,$$

so that

$$\begin{aligned} a_{ij_i}|x_{j_i}|y_{j_i}^k &= a_{ij_i}|x_{j_i}|\omega^k y_i^k \\ &= \rho|x_i|\omega^k y_i^k \\ &= \rho\omega^k|x_i|y_i^k. \end{aligned}$$

Now suppose that $a_{ij_i}|x_{j_i}| = 0$. Then, for any integer k, the above equalities all hold, with everything being equal to zero. Letting $y = (y_i)$, it has been shown that $A(|x|y^k) = \rho\omega^k|x|y^k$, whence the spectrum of A is fully cyclic. □

If A is an $n \times n$ lattice homomorphism, it may also occur that A^t, the transpose of A, is a lattice homomorphism. This means that each column of A has at most one nonzero entry. Suppose that A is a lattice homomorphism, whose transpose is also a lattice homomorphism, having m nonzero entries. Then there are $n-m$ rows with all zeros and $n-m$ columns with all zeros. For each row i with all zeros, let j_i be a uniquely chosen column with all zeros. For each row i with one (unique) nonzero entry, let j_i be the unique column for which a_{ij_i} is nonzero. Then, given any i in $\{1,\ldots,n\}$, $a_{ij} = 0$ for all $j \neq j_i$.

Define T from $\{1,\ldots,n\}$ to $\{1,\ldots,n\}$ by $Ti = j_i$, so that $Ti \neq Tl$ for $i \neq l$ by construction. Then T is a permutation on the indices $\{1,\ldots,n\}$ and A is the transition matrix for T with each position i scaled by the factor $a_{i(Ti)}$. Since T is a permutation on a finite set, it can be represented uniquely as a product of disjoint cycles.

Fix i in $\{1,\ldots,n\}$. Then there exists a positive integer p such that $T^p i = i$ and $T^q i \neq i$ for any positive integer $q < p$. Solving $Ax = \lambda x$ then yields

$$\begin{aligned} a_{i(Ti)}x_{Ti} &= \lambda x_i \\ a_{(Ti)(T^2i)}x_{T^2i} &= \lambda x_{Ti} \\ &\vdots \\ a_{(T^{p-1}i)i}x_i &= \lambda x_{T^{p-1}i}. \end{aligned}$$

If $a_{(T^qi)(T^{q+1}i)} \neq 0$ for all q in $\{0,\ldots,p-1\}$, then

$$\frac{\lambda}{a_{(T^qi)(T^{q+1}i)}} x_{T^qi} = x_{T^{q+1}i} \tag{3.3}$$

for all q. Recurrent application of (3.3) gives $\lambda^p x_i = (a_{i(Ti)} \cdot a_{(Ti)(T^2i)} \cdot \ldots \cdot a_{(T^{p-1}i)i})x_i$.

Then, if $x_i \neq 0$, $\lambda = \sqrt[p]{a_{i(Ti)} \cdot a_{(Ti)(T^2i)} \cdot \ldots \cdot a_{(T^{p-1}i)i}}$, which has p solutions arranged cyclically on the circle with radius equal to the geometric mean of the nonzero entries $a_{(T^qi)(T^{q+1}i)}$. The corresponding eigenvector is then defined by (3.3) at the indices of the cycle and zero off the cycle, since there is no communication between the disjoint cycles. If $x_i = 0$, then, by (3.3), $x_{T^qi} = 0$ for all q, and there is no contribution to the eigenvector.

If $a_{(T^qi)(T^{q+1}i)} = 0$ for some q in $\{0, \ldots, p-1\}$, then the vector with $x_{T^{q+1}i} \neq 0$ and $x_j = 0$ for $j \neq T^{q+1}i$ will be an eigenvector corresponding to the eigenvalue $\lambda = 0$. Vectors with $x_{T^{q_0+1}i} \neq 0$ and $x_j = 0$ for $j \neq T^{q_0+1}i$ where $a_{(T^{q_0}i)(T^{q_0+1}i)} \neq 0$ will be generalized eigenvectors corresponding to the eigenvalue $\lambda = 0$.

In either case, p eigenvectors and generalized eigenvectors have been found for a cycle of length p. This then gives a description for all of the eigenvalues and the corresponding eigenspaces for the matrix A.

Example 3.2.3. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then a_{15}, a_{53} and a_{31} are nonzero, and a_{24}, a_{46} and a_{62} are nonzero, so that $1 \to 5 \to 3 \to 1$ and $2 \to 4 \to 6 \to 2$ are the disjoint cycles in the permutation T. The eigenvalues then satisfy

$$\lambda = \sqrt[3]{a_{15}a_{53}a_{31}} = \sqrt[3]{1}$$

and

$$\lambda = \sqrt[3]{a_{24}a_{46}a_{62}} = \sqrt[3]{8}.$$

Thus, setting $\alpha = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$, the eigenvectors x for each corresponding eigenvalue λ can be given as follows:

λ	x
1	$(1, 0, 1, 0, 1, 0)$
α	$(1, 0, \alpha^2, 0, \alpha, 0)$
α^2	$(1, 0, \alpha, 0, \alpha^2, 0)$
2	$(0, 1, 0, 2, 0, 2)$
2α	$(0, 1, 0, 2\alpha, 0, 2\alpha^2)$
$2\alpha^2$	$(0, 1, 0, 2\alpha^2, 0, 2\alpha)$

Example 3.2.4. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the permutation T on the indices is the cycle $1 \to 2 \to 4 \to 3 \to 1$. Since $Ae_1 = 0$, e_1 is the eigenvector for the eigenvalue 0, and since

$$Ae_2 = A^2 e_4 = A^3 e_3 = e_1,$$

e_2, e_4 and e_3 are generalized eigenvectors for the eigenvalue 0.

Remark. If the nonzero entries $a_{i(Ti)}$ are allowed to be any complex numbers, then the matrix A may no longer be positive, but the above method is still applicable to find the eigenvalues and eigenvectors. The cyclic structure is still present, although the eigenvalues may no longer rotate about the origin with reference to the real axis in the complex plane (ie. each cycle may be shifted by some positive angle).

Chapter 4

Spectrum of Frobenius-Perron Operators

4.1 Spectrum of P_T on L^1

Consider the measure space $([0,1], \mathcal{B}, m)$, where $[0,1]$ is the unit interval on the real line, $\mathcal{B}$ is the σ-algebra of Borel sets on $[0,1]$ and m is the Lebesgue measure on $[0,1]$. Recall from Section 2.2 that a measurable transformation T on $[0,1]$ is called nonsingular if $m(T^{-1}A) = 0$ whenever $m(A) = 0$. Also, recall that the Koopman operator K_T defined by Equation (2.2) in Section 2.2 for T on L^∞ is given by the relation $K_T g = g \circ T$ for all g in L^∞. The basic properties of the operator K_T are collected in Theorem 2.2.1.

The Frobenius-Perron operator P_T for T on L^1 is defined by Equation (2.3) in Section 2.2 to be the operator that satisfies

$$\int_A P_T f = \int_{T^{-1}A} f \tag{4.1}$$

for all f in L^1 and A in $\mathcal{B}$. More generally, as is given by Equation (2.5) in Section 2.2, P_T is the operator that has K_T as its dual, or

$$\int (P_T f) g = \int f(K_T g) \tag{4.2}$$

for all f in L^1 and g in L^∞.

For any f in L^1, $\|P_T f\|_1 = \int |P_T f| \leq \int P_T |f| = \int |f| = \|f\|_1$, where equality holds when f is positive. Therefore, P_T has norm 1. Similarly, P_T^N for any positive integer N is a positive operator with norm 1, so that by Theorem 2.3.9 $r(P_T) = 1$. More basic properties of the operator P_T are collected in Theorem 2.2.3.

As was shown in the previous section (Proposition 3.1.3) the spectral radius of a positive matrix is an eigenvalue with a positive eigenvector. Similarly, it will now be shown that, for the positive operator P_T on the complex Banach lattice L^1, the spectral radius is in the spectrum. Since there may not always be peripheral point spectrum, a slightly modified version of the proof of Proposition 3.1.3 is given. However, if there is peripheral point spectrum, the same proof as Proposition 3.1.3 may be used, *mutatis mutandis*, to show that 1 is an eigenvalue with a corresponding positive eigenvector.

Proposition 4.1.1. *For a nonsingular transformation T on $[0,1]$, 1 is in the spectrum of P_T.*

Proof. Recall that by Definition 2.3.6, the set $\{\lambda : |\lambda| > 1\}$ is entirely contained in the resolvent set $\rho(P_T)$, and so that by Definition 2.3.2 the Resolvent $R(\lambda, P_T)$ is defined for all λ such that $|\lambda| > 1$. Suppose that 1 is not in the spectrum of P_T. Then the set of positive real numbers $\alpha > 1$ has minimum distance to the spectrum greater than some positive real number ϵ. Then, by Lemma VII.6.11 of [9], there exists a positive real number K such that $\frac{K}{\epsilon}$ is a uniform bound on $\|R(\alpha, P_T)\|$ for $\alpha > 1$.

For any complex number λ with $|\lambda| > 1$,

$$R(\lambda, P_T) = \sum_{k=0}^{\infty} \lambda^{-(k+1)} P_T^k,$$

as is shown in the proof of Theorem 2.3.5, so that for any f in L^1,

$$\begin{aligned}
|R(\lambda, P_T)f| &= |\sum_{k=0}^{\infty} \lambda^{-(k+1)} P_T^k f| \\
&\leq \sum_{k=0}^{\infty} |\lambda^{-(k+1)} P_T^k f| \\
&= \sum_{k=0}^{\infty} |\lambda^{-(k+1)}||P_T^k f| \\
&\leq \sum_{k=0}^{\infty} |\lambda|^{-(k+1)} P_T^k |f| \\
&= R(|\lambda|, P_T)|f|,
\end{aligned}$$

where Theorem 2.1.31 was used in the second inequality since $P_T \geq 0$. Therefore

$$\begin{aligned}
\big\|R(\lambda, P_T)f\big\| &= \big\||R(\lambda, P_T)f|\big\| \\
&\leq \big\|R(|\lambda|, P_T)|f|\big\| \\
&\leq \big\|R(|\lambda|, P_T)\big\|\big\||f|\big\| \\
&= \big\|R(|\lambda|, P_T)\big\|\big\|f\big\|
\end{aligned}$$

for all f in L^1 and all complex numbers λ with $|\lambda| > 1$. But $\|R(|\lambda|, P_T)\|$ has the uniform bound $\frac{K}{\epsilon}$. This implies that $\|R(\lambda, P_T)\|$ has a uniform bound for all λ such that $|\lambda| > 1$, which is contrary to Corollary 2.3.4 since, by definition of spectral radius, there exists a spectral point λ with $|\lambda| = 1$ and hence a sequence $\{\lambda_i\}$ such that $|\lambda_i| > 1$ and $\lambda_i \to \lambda$. Thus, a contradiction has been met, and 1 must then be in the spectrum. $\square$

Thus it may be concluded that 1 is always in the spectrum for the Frobenius-Perron operator P_T for a nonsingular transformation T. However it may not be concluded that 1 is always in the point spectrum, and hence there may exist nonsingular transformations T with no invariant densities in L^1.

Example 4.1.2. Let $Tx = \frac{x}{2}$, which is measurable and nonsingular. Then, for any

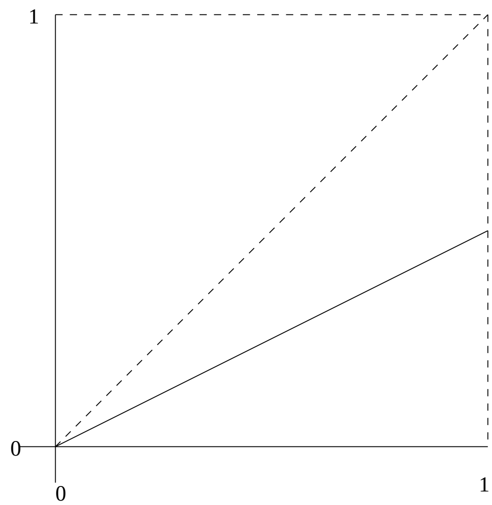

Figure 4.1: $Tx = \frac{x}{2}$

version $\hat{f}$ of any function f in L^1, the Frobenius-Perron operator is given by

$$P_T\hat{f}(x) = \begin{cases} 2\hat{f}(2x) & \text{if } 0 \le x \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x \le 1. \end{cases} \tag{4.3}$$

This comes from Equation (2.6) in Section 2.2, where there is only one branch (as is shown in Figure 4.1), $T^{-1}(x) = 2x$ everywhere, $|T'(x)| = \frac{1}{2}$ everywhere, and $T[0,1] = [0, \frac{1}{2}]$. Therefore, if f is to be a fixed point of P_T, it must be equal to $\mathbf{0}$ on the interval $(\frac{1}{2}, 1]$, which by Equation (4.3) in turn implies that it must be equal to $\mathbf{0}$ on the interval $(\frac{1}{4}, \frac{1}{2}]$ (since $\hat{f}(x) = 2\hat{f}(2x) = 2 \cdot 0$), and so on, so that the only fixed point of P_T is $\mathbf{0}$. Thus 1 is not in the point spectrum of P_T.

While Proposition 4.1.1 ensures that $1 \in \sigma(P_T)$, this fact can be verified directly as follows: consider the function $\mathbf{1} = \mathbf{1}_{[0,1]}$. Then there does not exist f in L^1 such that $(I - P_T)f = \mathbf{1}$, since attempting to solve $f = \mathbf{1} + P_T f$ implies that f must take the value 1 on the interval $(\frac{1}{2}, 1]$, which in turn implies that it must take the value 3 on the interval $(\frac{1}{4}, \frac{1}{2}]$, and 7 on the interval $(\frac{1}{8}, \frac{1}{4}]$, and so on, so that

$$\int_0^1 |f| = \sum_{n=1}^{\infty} \frac{2^n - 1}{2^n} = \infty.$$

Thus $(I - P_T)L^1 \neq L^1$ and 1 must be in the spectrum.

For a function f in L^1 or L^∞, define the argument of f as the function g in L^∞

such that, for a version $\hat{f}$ of f, g has the version $\hat{g}$ where

$$\hat{g}(x) = \begin{cases} \frac{\hat{f}(x)}{|\hat{f}|(x)} & \text{if } \hat{f}(x) \neq 0, \\ 0 & \text{if } \hat{f}(x) = 0. \end{cases}$$

For g_1, g_2 in L^∞, define $g_1 g_2$ as the function with version $\hat{g}_1 \hat{g}_2$ obtained by pointwise multiplication for versions $\hat{g}_1$ and $\hat{g}_2$ of g_1 and g_2, respectively. In addition, for g in L^∞ and a nonpositive integer k, define g^k as the function with version

$$\widehat{g^k}(x) = \begin{cases} \hat{g}(x)^k & \text{if } \hat{g}(x) \neq 0, \\ 0 & \text{if } \hat{g}(x) = 0. \end{cases}$$

for any version $\hat{g}$ of g.

Definition 4.1.3. Let P be a bounded linear operator on L^1 and let S be a subset of the point spectrum of P. S is called *fully cyclic* if, for all $\alpha \neq 0$ in S and $f \neq 0$ in L^1 such that $Pf = \alpha f$ with $\alpha = |\alpha|\omega$ and any g such that $f = |f|g$, then $|\alpha|\omega^k$ is in S and $P(|f|g^k) = |\alpha|\omega^k|f|g^k$ for all integers k.

The goal is to show that the peripheral point spectrum of P_T, which lies on the unit circle S^1, is fully cyclic. This will be an extension of Theorem 3.1.17, where the Markov operator P_T has the dual K_T (as is described in Section 2.2), and K_T satisfies $K_T\mathbf{1} \leq \mathbf{1}$. First, a simple lemma.

Lemma 4.1.4. *For any f in L^1, $P_T\bar{f} = \overline{P_T f}$.*

Proof. Given f in L^1, let g be any function in L^∞. Then

$$\begin{aligned} \int (P_T\bar{f} - \overline{P_T f})g &= \int P_T\bar{f}g - \int \overline{P_T f\bar{g}} \\ &= \int P_T\bar{f}g - \overline{\int P_T f\bar{g}} \\ &= \int \bar{f}g \circ T - \overline{\int f\bar{g} \circ T} \\ &= \int \bar{f}g \circ T - \int \bar{f}g \circ T \\ &= 0. \end{aligned}$$

Therefore $P_T\bar{f} - \overline{P_T f} = 0$, and the result follows. □

Theorem 4.1.5. *Let P_T be the Frobenius-Perron operator on L^1. The peripheral point spectrum of P_T is fully cyclic.*

Proof. Suppose that $P_T f = \alpha f$ where $|\alpha| = 1$ and $f \neq 0$. Then $|f| = |P_T f| \leq P_T|f|$ since P_T is a positive operator, and so

$$0 < \int |f|\mathbf{1} \leq \int P_T|f|\mathbf{1} = \int |f|\mathbf{1} \circ T = \int |f|\mathbf{1}.$$

Therefore $\int(P_T|f|-|f|)=0$, and since $P_T|f|-|f|\geq 0$, then $P_T|f|=|f|$.

Let $\hat{f}$ be some version of f. Define the sets $S_0=\{x\in[0,1]:\hat{f}(x)\neq 0\}$, $S_k=T^{-k}S_0\setminus\dot{\bigcup}_{i=0}^{k-1}S_i$ for[1] $k>0$ and $S=\dot{\bigcup}_{k=0}^{\infty}S_k$. Equivalently,

$$S=\{x\in[0,1]:\exists k>0,\ T^kx\in S_0\}$$

and

$$S_k=\{x\in S:T^kx\in S_0,\ T^ix\notin S_0,\ i=0\ldots k-1\}.$$

If a different version of f is used, these sets will only differ by a set of measure zero, since S_0 will only differ by a set of measure zero and T is nonsingular. Then $\mathbf{1}_{S_0}-\mathbf{1}_{T^{-1}S_0}\geq 0$ on S_0, so that $(\mathbf{1}_{S_0}-\mathbf{1}_{T^{-1}S_0})|f|\geq 0$ and

$$\begin{aligned}\int(\mathbf{1}_{S_0}-\mathbf{1}_{T^{-1}S_0})|f| &= \int|f|\mathbf{1}_{S_0}-|f|\mathbf{1}_{S_0}\circ T\\ &= \int|f|\mathbf{1}_{S_0}-P_T|f|\mathbf{1}_{S_0}\\ &= \int|f|\mathbf{1}_{S_0}-|f|\mathbf{1}_{S_0}\\ &= 0.\end{aligned}$$

This implies that $\mathbf{1}_{S_0}-\mathbf{1}_{T^{-1}S_0}=0$ on S_0 as L^∞ functions, or that $S_0\subset T^{-1}S_0$ except for possibly a set of measure zero. This in turn implies that $S=T^{-1}S$ and $S^c=T^{-1}S^c$, except possibly for sets of measure zero.

The function g_0 is defined to be the argument of $\hat{f}$, that is,

$$g_0(x)=\begin{cases}\frac{\hat{f}(x)}{|\hat{f}|(x)} & \text{if } x\in S_0,\\ 0 & \text{if } x\in S_0^c,\end{cases}$$

and is extended to the function g on $[0,1]$, where

$$g(x)=\begin{cases}\alpha^k(g_0\circ T^k)(x) & \text{if } x\in S_k,\\ 0 & \text{if } x\in S^c.\end{cases}$$

Since g_0 and α have modulus at most 1, g is in L^∞. Since $g=g_0$ on S_0, $|f|g=f$ as L^1 functions.

[1] The dot means disjoint union.

Consider

$$\begin{aligned}
\int |g_0\circ T-\bar\alpha g_0|^2|f| &= \int (g_0\circ T-\bar\alpha g_0)\overline{(g_0\circ T-\bar\alpha g_0)}|f| \\
&= \int (g_0\circ T-\bar\alpha g_0)(\bar g_0\circ T-\alpha\bar g_0)|f| \\
&= \int \big((g_0\bar g_0)\circ T+\alpha\bar\alpha g_0\bar g_0\big)|f| \\
&\quad -\int \big(\bar\alpha g_0(\bar g_0\circ T)+\alpha\bar g_0(g_0\circ T)\big)|f| \\
&= \int (\mathbf{1}_{S_0}\circ T+\mathbf{1}_{S_0})|f| \\
&\quad -\int \bar\alpha f(\bar g_0\circ T)+\alpha\bar f(g_0\circ T) \\
&= \int |f|\mathbf{1}_{S_0}\circ T+|f|\mathbf{1}_{S_0} \\
&\quad -\int \bar\alpha(P_T f)\bar g_0+\alpha(P_T\bar f)g_0 \\
&= \int P_T|f|\mathbf{1}_{S_0}+|f|\mathbf{1}_{S_0}-\int \alpha\bar\alpha f\bar g_0+\alpha\bar\alpha\bar f g_0 \\
&= \int |f|\mathbf{1}_{S_0}+|f|\mathbf{1}_{S_0}-\int |f|g_0\bar g_0+|f|\bar g_0 g_0 \\
&= 2\|f\|_1-2\|f\|_1 \\
&= 0.
\end{aligned}$$

This implies that $g_0\circ T-\bar\alpha g_0=0$ on S_0, or that $g\circ T=\bar\alpha g$ almost everywhere on S_0.

For $k>0$, $x\in S_k$ implies that $Tx\in S_{k-1}$, and

$$\begin{aligned}
g(Tx) &= \alpha^{k-1}g_0\big(T^{k-1}(Tx)\big) \\
&= \bar\alpha\alpha^k g_0(T^k x) \\
&= \bar\alpha g(x),
\end{aligned}$$

so that $g\circ T=\bar\alpha g$ almost everywhere on S_k. For almost every $x\in S^c$, both $g(Tx)$ and $g(x)$ are equal to zero. Therefore, $g\circ T=\bar\alpha g$ as L^∞ functions.

For any integer n, consider g^n. Then $g^n\circ T=(g\circ T)^n=\bar\alpha^n g^n$ for any integer n

and $0 \neq |f|g^n \in L^1$. Now, let h be any function in L^∞. Then

$$\begin{aligned}
\int \big(P_T(|f|g^n) - \alpha^n|f|g^n\big)h &= \int P_T(|f|g^n)h - \alpha^n|f|g^n h \\
&= \int |f|g^n(h \circ T) - \alpha^n|f|g^n h \\
&= \int |f|\alpha^n(g^n h) \circ T - \alpha^n|f|g^n h \\
&= \alpha^n \int P_T(|f|)g^n h - |f|g^n h \\
&= \alpha^n \int |f|g^n h - |f|g^n h \\
&= 0.
\end{aligned}$$

This shows that $P_T|f|g^n = \alpha^n|f|g^n$ for any integer n, and hence the peripheral point spectrum of P_T is fully cyclic. $\square$

Example 4.1.6. Consider $T = I$ the identity map on $[0, 1]$.

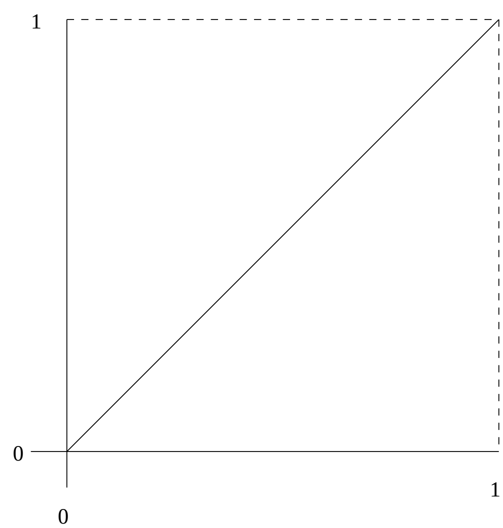

Figure 4.2: $Tx = x$

Then P_T is simply the identity on L^1, since

$$\int_A P_T f = \int_{T^{-1}A} f = \int_A f$$

for any measurable set A and integrable function f. Thus the spectrum of P_T is $\sigma(P_T) = \{1\}$, so that the peripheral point spectrum is trivially fully cyclic.

Example 4.1.7. Let $Tx = x + a - \lfloor x + a \rfloor$ be the rotation map on $[0, 1]$, where a will be chosen to be in $[0, 1)$. Then T is a nonsingular measure-preserving transformation.

This implies that $P_T\mathbf{1} = \mathbf{1}$. If $a = 0$, then $T = I$. Now suppose $a \in (0,1)$. Then T has two branches (Figure 4.3) and the Frobenius-Perron operator is described a.e. for versions (Equation (2.6)) by

$$\widehat{P_T f}(x) = \begin{cases} \hat{f}(x+1-a) & \text{if } 0 \le x \le a, \\ \hat{f}(x-a) & \text{if } a < x \le 1. \end{cases}$$

Thus it is verified that $\mathbf{1}$ is a fixed point for P_T, so that T is Lebesgue measure preserving.

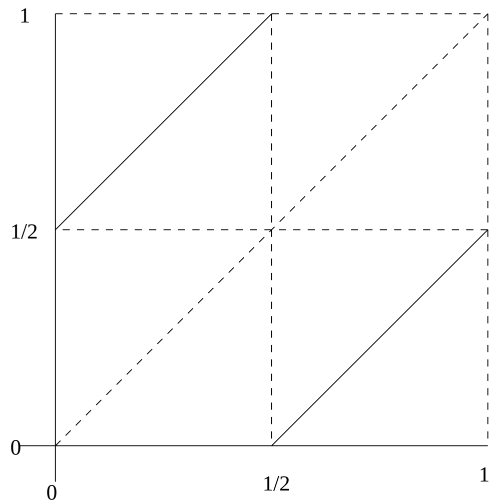

Figure 4.3: $Tx = x + \frac{1}{2} - \lfloor x + \frac{1}{2} \rfloor$

Let $a = \frac{1}{2}$. Suppose that the function f satisfies $P_T f = \alpha f$ for some complex number α with modulus at most 1. Then for almost every x in the interval $\left[0, \frac{1}{2}\right]$, $f\left(x + \frac{1}{2}\right) = \alpha f(x)$, and for almost every x in the interval $\left[\frac{1}{2}, 1\right)$, $f\left(x - \frac{1}{2}\right) = \alpha f(x)$, so that for almost every x in $[0,1]$, $f(x) = \alpha^2 f(x)$. This can only be satisfied if $f = 0$ or $\alpha^2 = 1$, so that the point spectrum of P_T is given by $\sigma(P_T) = \{1, -1\}$.

From the equation for P_T it is immediate that $P_T f = f$ if and only if $\hat{f}\left(x + \frac{1}{2}\right) = \hat{f}(x)$ a.e. in $\left[0, \frac{1}{2}\right)$ for all versions $\hat{f}$ of f, so that f is composed of two copies of the same function on the intervals $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right]$, and $P_T f = -f$ if and only if $\hat{f}\left(x + \frac{1}{2}\right) = -\hat{f}(x)$ a.e. for all versions $\hat{f}$ of f, so that f is composed of two copies of the same function, and then the second copy is reflected through the x-axis. Thus if $\hat{f}(x) = e^{4\pi i x}$, then $P_T f = f$, but if $\hat{f}(x) = e^{2\pi i x}$, then $P_T f = -f$.

If $\hat{f}\left(x + \frac{1}{2}\right) = \hat{f}(x)$ almost everywhere, then

$$|\hat{f}\big(x + \frac{1}{2}\big)|\text{sgn}(\hat{f}\big(x + \frac{1}{2}\big))^n = 1^n|\hat{f}(x)|\text{sgn}(\hat{f}(x))^n$$

almost everywhere for any integer n, and if $\hat{f}\left(x + \frac{1}{2}\right) = -\hat{f}(x)$ almost everywhere, then

$$|\hat{f}(x+\frac{1}{2})|\text{sgn}(\hat{f}(x+\frac{1}{2}))^n = (-1)^n|\hat{f}(x)|\text{sgn}(\hat{f}(x))$$

almost everywhere for any integer n. Thus, if f is an eigenfunction for the eigenvalue $\lambda = \pm 1$ with $f = |f|g$, $P_T(|f|g^n) = \lambda^n|f|g^n$ for any integer n and the peripheral point spectrum is fully cyclic.

Example 4.1.8. Consider $Tx = \beta x - \lfloor \beta x \rfloor$ the β-transformation on $[0,1]$. For $\beta > 0$, T is a nonsingular transformation, and as is discussed in [20] P_T has a unique fixed point when $\beta > 1$. Similar to Example 4.1.2, when $0 < \beta < 1$, P_T will have no fixed points, and for $\beta = 1$ every point is a fixed point, as in Example 4.1.6. Thus P_T will have interesting peripheral point spectrum only when $\beta > 1$.

For $\beta = 2$, T is the called the doubling map.

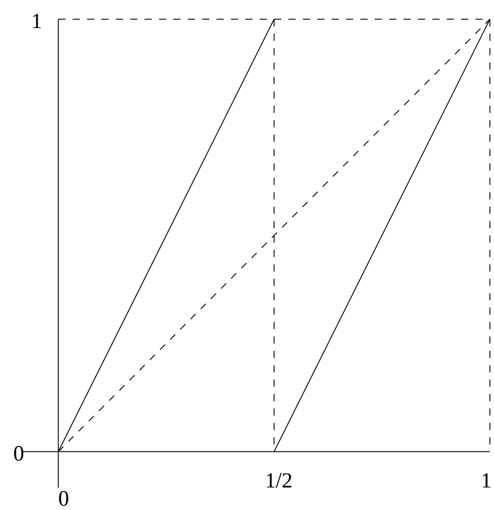

Figure 4.4: $Tx = 2x - \lfloor 2x \rfloor$

From Equation (2.6), for any version $\hat{f}$ of some function f,

$$\widehat{P_T f}(x) = \frac{1}{2}\Big(\hat{f}(\frac{x}{2}) + \hat{f}(\frac{x+1}{2})\Big)$$

for a.e. x in $[0,1]$. Then it can be shown ([21]) that $\mathbf{1}$ is the unique fixed density for P_T. Since $\mathbf{1}$ is the unique fixed denisty for P_T, then by Theorem 4.1.5, for $|\alpha| = 1$ and $f \neq 0$, if $P_T f = \alpha f$ it must be that $|f| = c\mathbf{1}$ for some constant $c \in \mathbb{C}$.

Similarly, if $Tx = nx - \lfloor nx \rfloor$ for any integer $n > 2$,

$$\widehat{P_T f}(x) = \frac{1}{n}\sum_{k=0}^{n-1} \hat{f}(\frac{x+k}{n}),$$

so that $\mathbf{1}$ is again the unique fixed density, and hence $P_T f = \alpha f$ for $|\alpha| = 1$ and $f \neq 0$ can only happen if $|f| = c\mathbf{1}$ for some constant $c \in \mathbb{C}$.

Suppose that $P_T f = \alpha f$ for some $|\alpha| = 1$, so that $|f| = c\mathbf{1}$ for some constant $c \in \mathbb{C}$. Then f is in L^2, so that it has a Fourier series expansion $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$. Computing the coefficients c_n gives

$$\begin{aligned} \alpha c_0 &= \int \alpha f \\ &= \int P_T f\, \mathbf{1} \\ &= \int f\, K_T \mathbf{1} \\ &= \int f = c_0 \end{aligned}$$

for $n = 0$, so that $\alpha = 1$ or $c_0 = 0$, and for $n \neq 0$,

$$\begin{aligned} \alpha c_n &= \int \alpha f\, e^{2\pi i n x} \\ &= \int P_T f\, e^{2\pi i n x} \\ &= \int f\, K_T e^{2\pi i n x} \\ &= \int f\, e^{2\pi i 2 n x} \\ &= c_{2n}. \end{aligned}$$

Since $|\alpha| = 1$, this is contrary to the fact that c_n converges to 0, unless $c_n = 0$ for all $n \neq 0$, in which case $f = 0$ if $\alpha \neq 1$, and $f = c_0\mathbf{1}$ if $\alpha = 1$. Thus 1 is the only point in the peripheral point spectrum of P_T.

As is shown in Section 3.3 of [4], for $|z| < 1$, z is in the point spectrum of P_T acting on L^2, so that P_T has square-integrable eigenfunctions f corresponding to the eigenvalue z. By Proposition 6.12 of [11], f is integrable since $[0, 1]$ is a finite measure space, and so $P_T f = zf$ in L^1. Since the spectrum is a closed subset of the unit disc $\mathbb{D}$, this implies that $\sigma(P_T) = \mathbb{D}$, so that all points on the unit circle $\partial\mathbb{D}$ are in the spectrum, but all unimodular points not equal to 1 are outside of the point spectrum.

Remark. Since T in Example 4.1.8 has two branches and $\inf |T'(x)| = 2 > \sqrt{2}$, Bowen showed in [2] that T is weakly mixing for the Lebesgue measure. Thus, T has no unimodular eigenvalues other than 1. Details of this are given in the example.

Example 4.1.9. Let $Tx = rx(1 - x)$ be the logistic map on $[0, 1]$ with parameter r satisfying $0 < r \leq 4$. Then, T is a nonsingular transformation, and as is shown by Theorem 6.5.1 and Example 6.5.1 following it in [16], for $r = 4$, T has unique invariant density f_0 such that

$$\hat{f}_0(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

for versions $\hat{f}_0$ of f_0. Thus if $P_T f = \alpha f$ for some $|\alpha| = 1$ and $f \neq 0$, then by

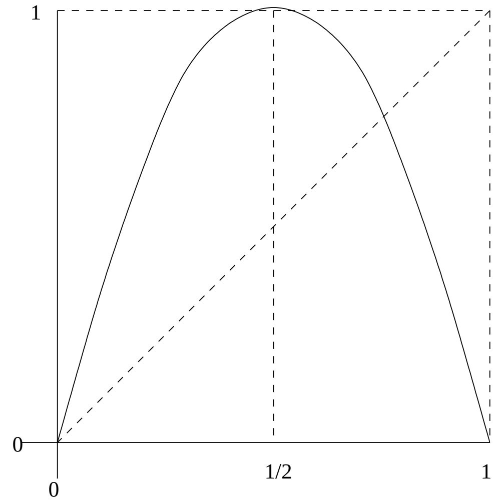

Figure 4.5: $Tx = 4x(1-x)$

Theorem 4.1.5 it must be that $|f| = f_0$.

As is shown in Section 4.4 of [8], the logistic map has the same spectral decomposition as the slope 2 tent map, since they are topologically conjugate by a smooth conjugacy. Similarly to Example 4.1.8, the slope 2 tent map has peripheral spectrum on the entire unit circle $\partial\mathbb{D}$, but peripheral point spectrum only at 1, and hence so does the logistic map.

P_T is a positive operator such that its dual operator K_T satisfies $K_T\phi \leq \phi$ for some strictly positive linear form ϕ, namely $\phi = \mathbf{1}$. Theorem 4.1.5 is analogous then to Theorem 3.1.17. Similarly, an analogue for 3.2.2 may be proven for the lattice homomorphism K_T.

Theorem 4.1.10. *Let K_T be the Koopman operator on L^∞. The point spectrum of K_T is fully cyclic.*

Proof. Suppose that $K_T g = \lambda g$ for some $g \neq 0$ and some $\lambda \neq 0$, so that $g = |g|h$ where h is the argument of g and $\lambda = r\omega$ where $|\omega| = 1$ and $r = |\lambda| > 0$. Then

$$K_T|g| = |K_T g| = |\lambda g| = |\lambda||g| = r|g|.$$

Let $S_0 = \{x \in [0,1] : \hat{g}(x) \neq 0\}$ for some version $\hat{g}$ of g. Then, on S_0,

$$g \circ T = K_T g = \lambda g,$$

where $\lambda \neq 0$ and $g \neq 0$, so that $S_0 \subset T^{-1}S_0$. On S_0^c, the same equality shows that $g \circ T = 0$, so that $S_0^c \subset T^{-1}S_0^c$.

Let k be an integer. On S_0,

$$\begin{aligned} K_T(|g|h^k) &= K_T|g|K_Th^k \\ &= r|g|\frac{K_Tg^k}{K_T|g|^k} \\ &= r|g|\frac{\lambda^k g^k}{r^k|g|^k} \\ &= r|g|\frac{r^k\omega^k|g|^k h^k}{r^k|g|^k} \\ &= r|g|\omega^k h^k \\ &= r\omega^k|g|h^k, \end{aligned}$$

where the division by $K_T|g|^k$ is justified since $K_T|g|^k \neq 0$ on S_0. On S_0^c,

$$K_T(|g|h^k) = 0 = r\omega^k|g|h^k.$$

Therefore, $K_T(|g|h^k) = r\omega^k|g|h^k$, and hence, the point spectrum of K_T is fully cyclic. □

4.2 A Spectral Dichotomy

For this section, T is again a measurable nonsingular transformation on the space $([0,1], \mathcal{B}, m)$. The main result of this section, Theorem 4.2.7, is due to Ding *et al.* in [5], and it is given here to show what the entire spectrum may look like under some simple assumptions on T.

Definition 4.2.1. T is called *onto* (with respect to m) if there does not exist any A in $\mathcal{B}$ with $m(A) > 0$ such that $T([0,1])$ is contained in the set $[0,1]\backslash A$. m is called *regular* (with respect to T) if A is measurable and $m(A) > 0$ implies that $m(T^{-1}A) > 0$. m is called *normal* (with respect to T) if A is measurable and $m(A) = 0$ implies that TA is measurable and $m(TA) = 0$. If T is invertible, this says that T^{-1} is nonsingular.

Definition 4.2.2. A measure μ is called *absolutely continuous with respect to* a measure ν (written $\mu \ll \nu$) if, for A in $\mathcal{B}$, $\nu(A) = 0$ implies that $\mu(A) = 0$. Two measures μ and ν on $([0,1], \mathcal{B})$ are called *equivalent* if they are absolutely continuous with respect to each other, or $\mu \ll \nu$ and $\nu \ll \mu$.

Proposition 4.2.3. *m is regular if and only if m is absolutely continuous with respect to $m \circ T^{-1}$.*

Proof. Suppose that m is regular. Let A be a measurable set such that $m(T^{-1}A) = 0$. Then $m(A) = 0$ since m is regular. Thus m and $m \circ T^{-1}$ are absolutely continuous with respect to each other.

Conversely, suppose that $m \ll m \circ T^{-1}$. Let A be a measurable set such that $m(A) > 0$. Since m is absolutely continuous with respect to $m \circ T^{-1}$, this implies that $m(T^{-1}A) > 0$, so that m is regular and the proof is complete. □

Corollary 4.2.4. *m is regular and nonsingular with respect to T if and only if m and $m \circ T^{-1}$ are equivalent.*

Proof. Suppose that m is regular and nonsingular with respect to T. Since m is regular, $m \ll m \circ T^{-1}$ by Proposition 4.2.3. Since m is nonsingular, $m \circ T^{-1} \ll m$ by definition. Thus, m and $m \circ T^{-1}$ are equivalent.

Conversely, suppose that m and $m \circ T^{-1}$ are equivalent. By Proposition 4.2.3, m is regular with respect to T. Since $m \circ T^{-1} \ll m$, m is nonsingular with respect to T. Thus, m is regular and nonsingular, and the proof is complete. □

Proposition 4.2.5. *If m is regular, then T is onto. If T is onto and m is normal, then m is regular.*

Proof. Suppose first that T is not onto, so that there exists a measurable set A with positive measure such that $T([0,1])$ is contained in the set $[0,1] \setminus A$. Then $m(T^{-1}A) = 0$ since $T^{-1}A$ is a nullset. Therefore m is not regular. Thus if m is regular then T is onto.

Suppose now that m is normal but not regular. Then there exists a measurable set A such that $m(A) > 0$ but $m(T^{-1}A) = 0$. Consider the L^1 function $\mathbf{1} = \mathbf{1}_{[0,1]}$. This is a positive function, so that $P_T\mathbf{1}$ is positive by Theorem 2.2.3. By the definition of P_T,

$$\int_A P_T\mathbf{1}\, dm = \int_{T^{-1}A} \mathbf{1}\, dm = 0,$$

since $T^{-1}A$ is a nullset. Now, $P_T\mathbf{1}$ is positive and integrates to 0 over A, so that if S is the support of some version of $P_T\mathbf{1}$, $m(A \bigcap S) = 0$ or for almost every x in S, x is in $[0,1] \setminus A$. Then

$$\begin{aligned} \int_0^1 \mathbf{1}\, dm &= \int_0^1 P_T\mathbf{1}\, dm \\ &= \int_S P_T\mathbf{1}\, dm \\ &= \int_{T^{-1}S} \mathbf{1}\, dm, \end{aligned}$$

which implies that $[0,1] = B \bigcup B'$, where $B \bigcap B' = \emptyset$, $B \subset T^{-1}S$ and $m(B') = 0$. Since m is normal, TB' is measurable with measure zero. Let $S' = S \cup TB'$ and $A' = A \setminus TB'$, so that $m(S') = m(S)$ and $m(A') = m(A)$. Further, $T[0,1] \subset S'$ since $TB \subset S'$ and $TB' \subset S'$. Since $S' \subset [0,1] \setminus A'$, this implies that $T[0,1] \subset [0,1] \setminus A'$, or that T is not onto. Hence, T onto implies that m is regular when m is normal, and the proof is complete. □

Recall at this point that K_T is the Koopman operator on the space L^∞ associated with the transformation T on $[0,1]$.

Lemma 4.2.6. *m is regular if and only if K_T is an isometry.*

Proof. Suppose first that m is regular. Let g be any nonzero function in L^∞. Then for any $\epsilon > 0$ and version $\hat{g}$ of g, let

$$A = \{x \in X : |\hat{g}(x)| > \|g\|_\infty - \epsilon\}.$$

Then $m(A) > 0$ by the definition of the essential supremum norm. So $m(T^{-1}A) > 0$ since m is regular, where

$$T^{-1}A = \{x \in X : |\hat{g} \circ T(x)| > \|g\|_\infty - \epsilon\}.$$

This implies that $\|K_T g\|_\infty > \|g\|_\infty - \epsilon$ for any positive ϵ, so that $\|K_T g\|_\infty \geq \|g\|_\infty$. Since $\|K_T\| = 1$, then $\|K_T g\|_\infty \leq \|g\|_\infty$ for any nonzero g in L^∞. Hence, $\|K_T g\| = \|g\|$ for any nonzero g in L^∞, so that by Definition 2.3.1 K_T is an isometry.

Conversely, suppose now that K_T is an isometry. Let A be measurable such that $m(A) > 0$. Then $\mathbf{1}_A$ is nonzero and hence $K_T \mathbf{1}_A = \mathbf{1}_{T^{-1}A}$ is nonzero, since K_T is an isometry. Thus $m(T^{-1}A) > 0$ and m is regular. □

Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ be the closed unit disk in $\mathbb{C}$, and $\partial\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ be the boundary of $\mathbb{D}$, the unit circle.

Theorem 4.2.7. *Let m be regular with respect to T. If $0 \in \sigma(P_T)$, then $\sigma(P_T) = \mathbb{D}$ and if $0 \notin \sigma(P_T)$, then $\sigma(P_T) \subset \delta\mathbb{D}$.*

Proof. Suppose that m is regular, so that by Lemma 4.2.6 K_T is an isometry, and let λ be given such that $|\lambda| < 1$. If there exists a g_0 in L^∞ with $(\lambda I - K_T)g_0 = 0$, then

$$\|g_0\|_\infty = \|K_T g_0\|_\infty = |\lambda| \|g_0\|_\infty,$$

so that $\|g_0\|_\infty = 0$. Thus $\lambda I - K_T$ is injective. Then, for any g in L^∞,

$$\|(\lambda I - K_T)g\|_\infty \geq \|K_T g\|_\infty - \|\lambda g\|_\infty = (1 - |\lambda|)\|g\|_\infty,$$

so that $\lambda I - K_T$ is bounded below and hence λ is not in the approximate point spectrum. Therefore, according to Theorem 2.3.13, $\lambda \notin \delta\sigma(K_T)$ for $|\lambda| < 1$, and in particular $0 \notin \delta\sigma(K_T)$.

Now suppose that $0 \in \sigma(K_T)$. If there exists $\lambda \in \rho(K_T)$ such that $|\lambda| < 1$, then it is immediate that there must exist $\lambda_0 \in \delta\sigma(K_T)$ such that $|\lambda_0| < 1$, which is contrary to what was just shown. Thus if $|\lambda| < 1$, then $\lambda \in \sigma(K_T)$. Since the spectrum of K_T is closed and a subset of $\mathbb{D}$, $\sigma(K_T) = \mathbb{D}$, and by Theorem 2.3.15 $\sigma(P_T) = \mathbb{D}$.

Suppose now that $0 \in \rho(K_T)$. If there exists $\lambda \in \sigma(K_T)$ such that $|\lambda| < 1$, then it is immediate that there must exist $\lambda_0 \in \delta\sigma(K_T)$ such that $|\lambda_0| < 1$, which again is contrary to what has already been shown. Thus if $|\lambda| < 1$, then $\lambda \in \rho(K_T)$. Since the spectrum of K_T is a subset of $\mathbb{D}$, $\sigma(K_T) \subset \delta\mathbb{D}$, whence $\sigma(P_T) \subset \delta\mathbb{D}$ again by Theorem 2.3.15. □

Remark. By Theorem 4.1.5, if there exists peripheral point spectrum λ for P_T that is an irrational root of unity, then all rotations λ^k of λ are in $\sigma(P_T)$. This forms a dense set in the unit circle. Thus, if m is regular with respect to T, the only way to possibly have any isolated spectral points is if $0 \notin \sigma(P_T)$ and the peripheral point spectrum is a finite union of finite cyclic groups in $\partial\mathbb{D}$.

In fact, results from Schaefer in [24] imply, in this case, that the entire spectrum of P_T is cyclic, so that the only way to have any isolated spectral points if m is regular is if $0 \notin \sigma(P_T)$ and none of the spectral points are irrational roots of unity.

Example 4.2.8. Recall the β-transformation from Example 4.1.8. For $0 < \beta < 1$, T is not onto, and hence m is not regular by Proposition 4.2.5. For $\beta = 1$, $T = I$ so that $T^{-1}A = A$ for any measurable set A, which implies that m is regular. Then $P_T = I$ is invertible and hence $0 \notin \sigma(P_T)$, so that the spectrum is entirely contained in $\partial\mathbb{D}$. This agrees with the fact that the spectrum of I is equal to $\{1\}$.

For $\beta > 1$ and $A \in \mathcal{B}$ with positive measure, $T^{-1}A$ consits of a finite union of measurable sets, one being $\frac{A}{\beta} = \{y : y = \frac{x}{\beta} \text{ for some } x \in A\}$, where $m\big(\frac{A}{\beta}\big) = \frac{m(A)}{\beta} > 0$ so that m is regular with respect to T. For $\beta = 2$, Equation (2.6) implies that

$$\widehat{P_T f} = \frac{1}{2}\Big(\hat{f}\big(\frac{x}{2}\big) + \hat{f}\big(\frac{x+1}{2}\big)\Big),$$

so that for

$$\hat{f}(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} < x \le 1, \end{cases}$$

$P_T f = 0$, and hence $\sigma(P_T) = \mathbb{D}$. Similarly, since $\hat{P_T} f(x) = \frac{1}{n}\sum_{i=0}^{n-1} \hat{f}\big(\frac{x+1}{n}\big)$ for $\beta = n$ when $n > 2$, it can be shown that $0 \in \sigma(P_T)$ by letting f be the function that is equal to 1 on $\big[0, \frac{1}{n}\big)$ and -1 on $\big[\frac{1}{n}, \frac{2}{n}\big)$.

For β not equal to an integer, Equation (2.6) implies that

$$\widehat{P_T f}(x) = \begin{cases} \frac{1}{\beta}\sum_{i=0}^{\lfloor\beta\rfloor} f\big(\frac{x+i}{\beta}\big) & \text{if } 0 \le x \le T1, \\ \frac{1}{\beta}\sum_{i=0}^{\lfloor\beta\rfloor-1} f\big(\frac{x+i}{\beta}\big) & \text{if } T1 < x \le 1, \end{cases}$$

For $\beta > 2$, if f is taken to be the function that is equal to 1 on $\big[0, \frac{1}{\beta}\big)$ and -1 on $\big[\frac{1}{\beta}, \frac{2}{\beta}\big)$, then $P_T f = 0$ and $0 \in \sigma(P_T)$. For $1 < \beta < 2$, if f is taken to be the function that is equal to 1 on $\big[0, 1 - \frac{1}{\beta}\big)$ and -1 on $\big[\frac{1}{\beta}, 1\big)$, then $P_T f = 0$ and $0 \in \sigma(P_T)$. Hence, $\sigma(P_T) = \mathbb{D}$ for all values of $\beta > 1$.

Example 4.2.9. For the shift map $Tx = x + a - \lfloor x + a \rfloor$ described in Example 4.1.7, given any measurable set A with $m(A) > 0$, $m(T^{-1}A) = m(A) > 0$ since T was shown to be m preserving in Example 4.1.7. Thus m is regular with respect to T. P_T is given by

$$\widehat{P_T f}(x) = \begin{cases} \hat{f}(x + 1 - a) & \text{if } 0 \le x \le a, \\ \hat{f}(x - a) & \text{if } a < x \le 1. \end{cases}$$

Then P_T is invertible having inverse Q given by

$$\widehat{Qf}(x) = \begin{cases} \hat{f}(x + a) & \text{if } 0 \le x \le 1 - a, \\ \hat{f}(x - 1 + a) & \text{if } 1 - a < x \le 1. \end{cases}$$

Thus Q is the resolvent of P_T at 0, so that $0 \notin \sigma(P_T)$ and hence $\sigma(P_T)$ is entirely contained in $\partial\mathbb{D}$.

Remark. If T is any measurable, nonsingular and invertible transformation on the interval $[0,1]$ such that T^{-1} is measurable and nonsingular, then

$$P_{T^{-1}} = P_T^{-1},$$

so that if λ is an eigenvalue of P_T, then $\lambda \neq 0$ since P_T has bounded linear inverse, and therefore there exists $f \neq 0$ such that

$$P_T f = \lambda f \Rightarrow P_{T^{-1}} f = \frac{1}{\lambda} f.$$

Thus λ and $\frac{1}{\lambda}$ both have modulus less than or equal to 1 (since they are eigenvalues of Frobenius-Perron operators), and hence both lie on the unit cirlce. This simple proof then can be used to show that, for measurable, nonsingular and invertible transformations with measurable and nonsingular inverse, all eigenvalues lie on $\partial\mathbb{D}$, the unit circle. This is slightly weaker than Theorem 4.2.7, which shows that the entire spectrum is a subset of the unit circle.

Example 4.2.10. For the logistic map $Tx = rx(1-x)$ described in Example 4.1.9, T is not onto if $r < 4$ and hence m is not regular. Therefore let $r = 4$. In Example 4.1.9, the logistic map was shown to have an invariant density f_0 with the version

$$\hat{f}_0 = \frac{1}{\pi\sqrt{x(1-x)}},$$

which is strictly positive for all x in $(0,1)$. Thus $m(A) > 0$ for some measurable set A implies that

$$\mu(A) = \int_A f_0 \, dm > 0.$$

Since μ is an acim (Definition 2.2.2), this then implies that $\mu(T^{-1}A) > 0$ which in turn implies that $m(T^{-1}A) > 0$, since μ is absolutely continuous with respect to m. This shows that m is regular with respect to T.

By Equation (2.6)

$$\widehat{P_T f}(x) = \frac{1}{4\sqrt{1-x}}\Big[\hat{f}\Big(\frac{1-\sqrt{1-x}}{2}\Big) + \hat{f}\Big(\frac{1+\sqrt{1-x}}{2}\Big)\Big]$$

almost everywhere, so that for any f satisfying

$$\hat{f}\Big(\frac{1-\sqrt{1-x}}{2}\Big) = -\hat{f}\Big(\frac{1+\sqrt{1-x}}{2}\Big)$$

almost everywhere, $P_T f = 0$ so that $0 \in \sigma(P_T)$, and hence $\sigma(P_T) = \mathbb{D}$.

Remark. In fact, the Frobenius-Perron operator for the logistic map for $0 < r < 4$ is given by

$$\widehat{P_T f}(x) = \frac{1}{r\sqrt{1-\frac{4x}{r}}}\Big[\hat{f}\Big(\frac{1-\sqrt{1-\frac{4x}{r}}}{2}\Big) + \hat{f}\Big(\frac{1+\sqrt{1-\frac{4x}{r}}}{2}\Big)\Big]$$

almost everywhere, so that for any f satisfying

$$\hat{f}\Big(\frac{1-\sqrt{1-\frac{4x}{r}}}{2}\Big) = -\hat{f}\Big(\frac{1+\sqrt{1-\frac{4x}{r}}}{2}\Big)$$

almost everywhere, $P_T f = 0$ and $0 \in \sigma(P_T)$. However, it may not be concluded from Theorem 4.2.7 that $\sigma(P_T) = \mathbb{D}$, since T in this case is not regular.

Corollary 4.2.11. *Let $m \circ T^{-1}$ and m be equivalent. If $0 \in \sigma(P_T)$, then $\sigma(P_T) = \mathbb{D}$ and if $0 \notin \sigma(P_T)$, then $\sigma(P_T) \subset \delta\mathbb{D}$.*

Proof. This follows immediately from Theorem 4.2.7 and Proposition 4.2.4. □

Corollary 4.2.12. *Let T be onto and m normal. If $0 \in \sigma(P_T)$, then $\sigma(P_T) = \mathbb{D}$ and if $0 \notin \sigma(P_T)$, then $\sigma(P_T) \subset \delta\mathbb{D}$.*

Proof. This follows immediately from Theorem 4.2.7 and Proposition 4.2.5. □

4.3 Spectrum of P_T on BV

In addition to being nonsingular, T is now required to be piecewise C^2 and expanding on $[0, 1]$.[2] Then, for any positive integer N, T^N is piecewise C^2 on intervals of the form $I_{i_1} \bigcap T^{-1} I_{i_2} \bigcap \ldots \bigcap T^{-(N-1)} I_{i_{(N-1)}}$, and by the chain rule, $|(T^N)'(x)| > \inf(|T'|)^N > 1$ for all x such that T^N is differentiable at x, so that T^N is expanding.

For any positive integer n, consider $P_T^n = P_{T^n}$. Since P_T preserves integrals, $\|\mathbf{1}\|_{BV} \le \|P_{T^n}\mathbf{1}\|_{BV}$, as $\bigvee_0^1 \mathbf{1} = 0$, so that $\|P_{T^n}\|_{BV} \ge 1$ and the spectral radius of P_T is at least 1. Now, from the Lasota-Yorke inequality (2.7), for any f in BV and any

[2] As is defined in Section 2.2, T is expanding if $\inf |T'|(x) > 1$.

positive integer k,

$$\begin{aligned}
\|P_{T^{kN}} f\|_{BV} &= \bigvee_0^1 P_{T^{kN}} f + \|P_{T^{kN}} f\|_1 \\
&\leq \bigvee_0^1 P_{T^{kN}} f + \|f\|_1 \\
&\leq \alpha \|f\|_1 (1 + \beta + \ldots + \beta^{k-1}) + \beta^k \bigvee_0^1 f + \|f\|_1 \\
&\leq \left(\frac{\alpha}{1-\beta}\right) \|f\|_1 + \bigvee_0^1 f + \|f\|_1 \\
&\leq \left(\frac{\alpha}{1-\beta}\right) (\bigvee_0^1 f + \|f\|_1) + \bigvee_0^1 f + \|f\|_1 \\
&= \left(\frac{\alpha}{1-\beta} + 1\right) \|f\|_{BV}.
\end{aligned}$$

Therefore, $\|P_{T^{kN}}\| \leq \frac{\alpha}{1-\beta} + 1$ for any positive integer k. Since the spectral radius is at least one, and the sequence $(\frac{\alpha}{1-\beta} + 1)^{1/kN}$ decreases to one as k increases, the spectral radius is equal to one.

Given a function $f \in BV$, one may define the set

$$B_f = \{g \in L^\infty : |f| g \in BV\}.$$

This is a vector subspace of L^∞. The functions $\mathbf{1}_{S_0}$, where S_0 is the support of some version of f, and g_0, the argument of f, are in B_f.

Lemma 4.3.1. *For any nonzero $k \in \mathbb{Z}$ and any $z_1, z_2 \in \mathbb{C}$, $\big||z_1|\omega_1^k - |z_2|\omega_2^k\big| \leq |k||z_1 - z_2|$, where $\omega_i = \operatorname{sgn}(z_i)$ for $i = 1, 2$.*

Proof. Suppose without loss of generality that $|z_1| \leq |z_2|$. If $z_1 = 0$, the result is trivial. Suppose also that $k > 1$ (since the inequality is trivial when $k = 1$). Then

$$\begin{aligned}
\big||z_1|\omega_1^k - |z_2|\omega_2^k\big| &= \big||z_1|\omega_1^k - |z_1|\omega_1^{k-1}\omega_2 + |z_1|\omega_1^{k-1}\omega_2 - |z_1|\omega_1^{k-2}\omega_2^2 + \\
&\quad \ldots + |z_1|\omega_1\omega_2^{k-1} - |z_2|\omega_2^k\big| \\
&\leq |z_1| \sum_{j=0}^{k-2} |(\omega_1^{k-j}\omega_2^j - \omega_1^{k-j-1}\omega_2^{j+1})| + \\
&\quad + \big||z_1|\omega_1\omega_2^{k-1} - |z_2|\omega_2^k\big| \\
&= (k-1)\big||z_1|\omega_1 - |z_1|\omega_2\big| + \big||z_1|\omega_1 - |z_2|\omega_2\big| \\
&= (k-1)\big|z_1 - |z_1|\omega_2\big| + |z_1 - z_2|.
\end{aligned}$$

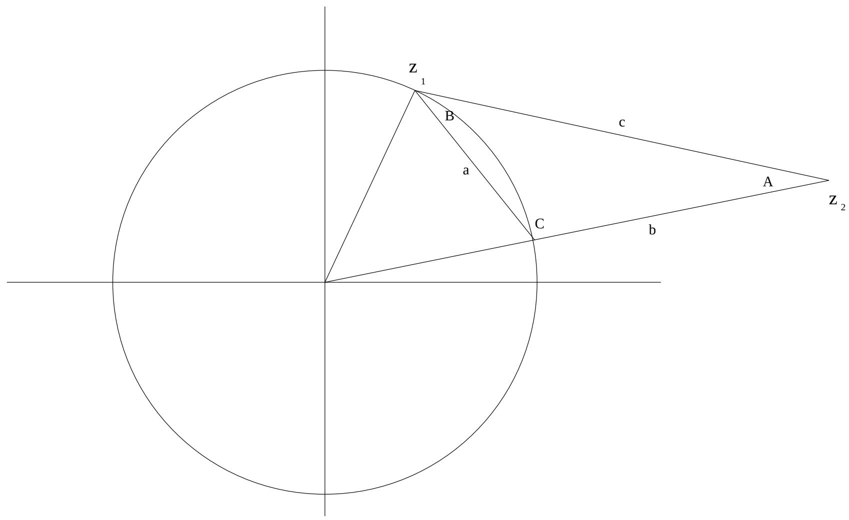

Figure 4.6

As in the Figure, the quantities $\big|z_1 - |z_1|\omega_2\big|$, $\big||z_1|\omega_2 - z_2\big|$ and $|z_1 - z_2|$ will be relabeled as a, b and c, respectively. By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos(C)$. Since the angle C will always be between $\pi/2$ and π, $-\cos(C) \geq 0$, so that $a^2 \leq c^2$. But $a = \big|z_1 - |z_1|\omega_2\big|$ and $c = |z_1 - z_2|$, so $\big|z_1 - |z_1|\omega_2\big| \leq |z_1 - z_2|$. Therefore, $\big||z_1|\omega_1^k - |z_2|\omega_2^k\big| \leq k|z_1 - z_2|$.

Suppose now that $k < 0$. Then let $u_1 = \bar{z}_1$ and $u_2 = \bar{z}_2$, and apply the same argument with $-k > 0$, so that

$$\big||z_1|\omega_1^k - |z_2|\omega_2^k\big| = \big||z_1|\bar{\omega}_1^{-k} - |z_2|\bar{\omega}_2^{-k}\big| \leq (-k)|u_1 - u_2|.$$

But for any $z \in \mathbb{C}$, $|z| = |\bar{z}|$, so that $|u_1 - u_2| = |\overline{u_1 - u_2}| = |\bar{u}_1 - \bar{u}_2|$, and the result follows. □

Theorem 4.3.2. *Let T be piecewise C^2 and expanding on $[0,1]$. Let P_T be the Frobenius-Perron operator on BV. The peripheral point spectrum of P_T is fully cyclic.*

Proof. Suppose that $P_T f = \alpha f$ where $|\alpha| = 1$ and $f \neq 0$. It is shown that $P_T|f| = |f|$ in L^1 in Theorem 4.1.5, and it will be shown now that $|f|$ is in BV. Let $0 = b_0 < b_1 < \ldots < b_n = 1$ be any partition of the interval and let $\hat{f}$ be a version of f with minimal variation. For any $i = 1, \ldots, n$,

$$|\hat{f}|(b_i) - |\hat{f}|(b_{i-1}) \leq |\hat{f}(b_i) - \hat{f}(b_{i-1})|$$

and

$$|\hat{f}|(b_{i-1}) - |\hat{f}|(b_i) \leq |\hat{f}(b_i) - \hat{f}(b_{i-1})|$$

by the triangle inequality, so that

$$\big||\hat{f}|(b_i) - |\hat{f}|(b_{i-1})\big| \leq |\hat{f}(b_i) - \hat{f}(b_{i-1})|.$$

Therefore
$$\sum_{i=1}^{n} \big||\hat f|(b_i) - |\hat f|(b_{i-1})\big| \le \sum_{i=1}^{n} |\hat f(b_i) - \hat f(b_{i-1})|,$$
which implies that $\bigvee_0^1 |\hat f| \le \bigvee_0^1 \hat f$, so that $|f|$ has a version with bounded variation. Thus, $|f|$ is in BV, and $P_T|f| = |f|$ in BV.

Let g_0, g, S and S_k, $k = 0, 1, \ldots$, be as defined in Theorem 4.1.5. Then the proof is the same as the proof of Theorem 4.1.5, *mutatis mutandis*, where the only precaution needed is to make sure that g^n is in B_f. Let $0 = b_1 < b_2 < \ldots < b_j = 1$ be any partition of the interval. Then from Lemma 4.3.1, for $n \neq 0$ and any version $\hat g$ of g such that $\hat f(x) = |\hat f|(x)\hat g(x)$ for all x in $[0, 1]$,
$$\begin{aligned}
\sum_{i=0}^{j-1} \big||\hat f|\hat g^n(b_{i+1}) - |\hat f|\hat g^n(b_i)\big| &\le \sum_{i=0}^{j-1} |n|\big||\hat f|\hat g(b_{i+1}) - |\hat f|\hat g(b_i)\big| \\
&= |n| \sum_{i=0}^{j-1} \big|\hat f(b_{i+1}) - \hat f(b_i)\big| \\
&\le |n| \bigvee_0^1 \hat f.
\end{aligned}$$
Therefore, for $n \neq 0$, $|\hat f|\hat g^n$ has bounded variation, so that g^n is in B_f. Since $n = 0$ is the case $P_T|f| = |f|$, which was verified, the proof is complete. □

Example 4.3.3. For $\beta > 1$, the β-transformation $Tx = \beta x - \lfloor \beta x \rfloor$ is piecewise C^2 with $\lceil \beta \rceil$ linear branches. Also, $T'(x) = \beta$ at every x for which T is differentiable, so that $\inf(|T'|) = \beta > 1$. Thus the peripheral point spectrum of P_T acting on BV is fully cyclic by Theorem 4.3.2.

Example 4.3.4. For any $a \in [0, 1)$, the shift map $Tx = x + a - \lfloor x + a \rfloor$ is piecewise C^2 with 2 linear branches. However, $T'(x) = 1$ for every x at which T is differentiable. Thus Theorem 4.3.2 cannot be applied.

Example 4.3.5. For any $r \in (0, 4]$, the logistic map $Tx = rx(1 - x)$ is C^2 on the entire interval $(0, 1)$. However, $T'\left(\frac{1}{2}\right) = 0$ for all r, so that Theorem 4.3.2 cannot be applied.

Returning to the case of piecewise C^2 and expanding maps T on $[0, 1]$, it has been shown that P_T satisfies the Lasota Yorke inequality (2.7). As was shown by Hofbauer and Keller [13], the conditions for the Ionescu Tulcea-Marinescu Theorem ([14] or [15]) are then met. Thus, it may be applied to show that the peripheral spectrum of P_T is made up of finitely many points, all of which are point spectrum, and the rest of the spectrum is contained in a disk centred at the origin with radius strictly smaller than 1.

Theorem 4.3.2 then implies that the peripheral spectrum is a finite union of cyclic groups of eigenvalues. The eigenfunctions corresponding to the eigenvalues are also cyclic, as has been shown.

Remark. Alternatively, following the work of Rychlik in [23], it can be shown without the Ionescu Tulcea-Marinescu Theorem that P_T is quasi-compact. By the Uniform Ergodic Theory of [9], the same result may then be obtained.

Chapter 5

Conclusions

This thesis has gathered some results concerning the spectrum of Frobenius-Perron and Koopman operators acting on spaces of functions. The purpose of this work has been to write a short, self-contained paper describing the fully cyclic nature of the peripheral point specturm of the Frobenius-Perron operator and of the entire point spectrum of the Koopman operator.

In Chapter 3, results about the peripheral spectrum were given for positive matrices acting on the finite dimensional space $\mathbb{C}^n$. Theorem 3.1.11 shows that the peripheral spectrum of certain positive matrices is fully cyclic, and Theorem 3.2.2 shows that the entire spectrum of a lattice homomorphism is fully cyclic.

In Chapter 4, corresponding results for the Frobenius-Perron and Koopman operators were given. Theorem 4.1.5 shows that the peripheral point spectrum of the Frobenius-Perron operator acting on integrable functions is fully cyclic, and Theorem 4.1.10 shows that the entire point spectrum of the Koopman operator is fully cyclic.

Theorem 4.2.7 shows that there are some cases when the spectrum is either the entire unit disk $\mathbb{D}$ or a subset of the unit circle $\delta\mathbb{D}$. Thus the possibility exists that there may be no isolated spectral points. Since it is conventional to sometimes consider the Frobenius-Perron operator as acting on the space of functions of bounded variation to overcome this difficulty, Theorem 4.3.2 verifies that the peripheral point spectrum of the Frobenius-Perron operator is still fully cyclic in this case.

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