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Research Article

Third-Order Differential Subordination and Superordination Results for Meromorphically Multivalent Functions Associated with the Liu-Srivastava Operator

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There are many articles in the literature dealing with the first-order and the second-order differential subordination and superordination problems for analytic functions in the unit disk, but only a few articles are dealing with the above problems in the third-order case (see, e.g., Antonino and Miller (2011) and Ponnusamy et al. (1992)). The concept of the third-order differential subordination in the unit disk was introduced by Antonino and Miller in (2011). Let Ω be a set in the complex plane $\mathbb{C}$. Also let $\mathfrak{p}$ be analytic in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$ and suppose that $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$. In this paper, we investigate the problem of determining properties of functions $\mathfrak{p}(z)$ that satisfy the following third-order differential superordination: $\Omega \subset \{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\}$. As applications, we derive some third-order differential subordination and superordination results for meromorphically multivalent functions, which are defined by a family of convolution operators involving the Liu-Srivastava operator. The results are obtained by considering suitable classes of admissible functions.

1. Introduction, Definitions, and Preliminaries

Let $\mathscr{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}. \tag{1}$$

For $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $a \in \mathbb{C}$, let

$$\mathscr{H}[a, n] = \Big\{f : f \in \mathscr{H}(\mathbb{U}),\quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\Big\} \tag{2}$$

and suppose that $\mathscr{H} = \mathscr{H}[1, 1]$.

Let f and F be members of the analytic function class $\mathscr{H}(\mathbb{U})$. The function f is said to be subordinate to F, or F is superordinate to f, if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0) = 0, \quad |\mathfrak{w}(z)| < 1 \quad (z \in \mathbb{U}), \tag{3}$$

such that

$$f(z) = F(\mathfrak{w}(z)). \tag{4}$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z). \tag{5}$$

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence (see, for details, [1]):

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = F(0), \quad f(\mathbb{U}) \subset F(\mathbb{U}). \tag{6}$$

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N}) \tag{7}$$

which are analytic and multivalent in the punctured unit disk:

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}. \tag{8}$$

For the function f given by (7) and the function g given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (p \in \mathbb{N};\ z \in \mathbb{U}^*), \tag{9}$$

the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{10}$$

For parameters $\alpha_i \in \mathbb{C}$ $(i = 1, 2, \ldots, q)$ and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\mathbb{Z}_0^- = 0, -1, -2, \ldots;\ j = 1, 2, \ldots, s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is defined by (see, for example, [2, 3])

$$\begin{gathered} {}_qF_s\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!} \\ \left(q \le s+1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}\right), \end{gathered} \tag{11}$$

where $(\nu)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$\begin{aligned} (\nu)_k &= \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \\ &= \begin{cases} 1 & (k = 0;\ \nu \in \mathbb{C} \setminus \{0\}), \\ \nu(\nu+1)\cdots(\nu+k-1) & (k \in \mathbb{N};\ \nu \in \mathbb{C}). \end{cases} \end{aligned} \tag{12}$$

Recently, Tang et al. [4] introduced a function $h_p^{\lambda,\mu}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$\begin{aligned} & h_p^{\lambda,\mu}\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) \\ &\quad = (1 - \lambda + \mu) z_q^{-p} F_s\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) \\ &\qquad + (\lambda - \mu) z\left[z_q^{-p} F_s\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right)\right]' \\ &\qquad + \lambda\mu z^2\left[z_q^{-p} F_s\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right)\right]'' \\ &\qquad\qquad (p \in \mathbb{N};\ \lambda, \mu \ge 0;\ z \in \mathbb{U}). \end{aligned} \tag{13}$$

In particular, when $\lambda = \mu = 0$, we obtain

$$h_p^{0,0}\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) = h_p\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right), \tag{14}$$

which was introduced and studied by Liu and Srivastava [5].

Corresponding to the function $h_p^{\lambda,\mu}(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ given by (13), we consider a convolution operator

$$H_p^{\lambda,\mu}\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s\right) : \Sigma_p \longrightarrow \Sigma_p \tag{15}$$

defined by the following Hadamard product (or convolution):

$$\begin{aligned} & H_p^{\lambda,\mu}\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s\right) f(z) \\ &\qquad = h_p^{\lambda,\mu}\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) * f(z). \end{aligned} \tag{16}$$

For the sake of convenience, we write

$$H_{p,q,s}^{\lambda,\mu}(\beta_1) = H_{p,q,s}^{\lambda,\mu}(\alpha_1) = H_p^{\lambda,\mu}\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s\right). \tag{17}$$

It is easily verified from definition (16) that

$$\begin{aligned} z\left(H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z)\right)' &= \beta_1 H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) \\ &\quad - (\beta_1 + p) H_{p,q,s}^{\lambda,\mu}(\beta_1 + 1) f(z), \end{aligned} \tag{18}$$

$$\begin{aligned} z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1) f(z)\right)' &= \alpha_1 H_{p,q,s}^{\lambda,\mu}(\alpha_1 + 1) f(z) \\ &\quad - (\alpha_1 + p) H_{p,q,s}^{\lambda,\mu}(\alpha_1) f(z). \end{aligned} \tag{19}$$

We note that, for $\lambda = \mu = 0$, the operator $H_{p,q,s}^{0,0}(\alpha_1)$ reduces to the Liu-Srivastava operator $H_{p,q,s}(\alpha_1)$ (see [5, 6]; see also [7]), while the Liu-Srivastava operator is the meromorphic analogous of the Dziok-Srivastava operator (see [8–10]; see also [11, 12]), which includes (as its special cases) the meromorphic analogous of the Carlson-Shaffer convolution operator $L_p(a, c) = H_{p,2,1}^{0,0}(1, a; c)$ (see [13, 14]), the meromorphic analogous of the Ruscheweyh derivative operator $D^{n+1} = L_p(n + p, 1)$ (see [15]), and the operator

$$J_{\delta,p} = \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t)\, dt = L_p(\delta, \delta + 1) \quad (\delta > 0) \tag{20}$$

studied by Uralegaddi and Somanatha [16].

Let Ω be any set in $\mathbb{C}$. Also let $\mathfrak{p}$ be analytic in $\mathbb{U}$ and suppose that $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$. Recently, Antonino and Miller [17] have extended the theory of second-order differential subordinations in $\mathbb{U}$ introduced by Miller and Mocanu [1] to the third-order case. They determined properties of functions $\mathfrak{p}(z)$ that satisfy the following third-order differential subordination:

$$\left\{\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega. \tag{21}$$

We will now recall some definitions and a theorem due to Antonino and Miller [17], which are required in our next investigations.

Definition 1 (see [17], p. 440, Definition 1). Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the following third-order differential subordination:

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \prec h(z), \tag{22}$$

then $\mathfrak{p}(z)$ is called a solution of the differential subordination. A univalent function $\mathfrak{q}(z)$ is called a dominant of the solutions of the differential subordination or, more simply, a dominant if $\mathfrak{p}(z) \prec \mathfrak{q}(z)$ for all $\mathfrak{p}(z)$ satisfying (22). A dominant $\widetilde{\mathfrak{q}}(z)$ that satisfies $\widetilde{\mathfrak{q}}(z) \prec \mathfrak{q}(z)$ for all dominants $\mathfrak{q}(z)$ of (22) is said to be the best dominant.

Definition 2 (see [17], p. 441, Definition 2). Let $\mathcal{Q}$ denote the set of functions $\mathfrak{q}$ that are analytic and univalent on the set $\overline{\mathbb{U}} \setminus E(\mathfrak{q})$, where

$$E(\mathfrak{q}) = \left\{\xi : \xi \in \partial\mathbb{U} \text{ and } \lim_{z \to \xi} \mathfrak{q}(z) = \infty\right\}, \tag{23}$$

is such that

$$\min \left|\mathfrak{q}'(\xi)\right| = \rho > 0 \tag{24}$$

for $\xi \in \partial\mathbb{U} \setminus E(\mathfrak{q})$. Further, let the subclass of $\mathcal{Q}$ for which $\mathfrak{q}(0) = a$ be denoted by $\mathcal{Q}(a)$ and

$$\mathcal{Q}(1) = \mathcal{Q}_1. \tag{25}$$

Definition 3 (see [17], p. 449, Definition 3). Let Ω be a set in $\mathbb{C}$, $\mathfrak{q} \in \mathcal{Q}$, and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi_n[\Omega, \mathfrak{q}]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; z) \notin \Omega \tag{26}$$

whenever

$$\begin{gathered} r = \mathfrak{q}(\xi), \quad s = k\xi\mathfrak{q}'(\xi), \\ \Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)} + 1\right), \\ \Re\left(\frac{u}{s}\right) \geqq k^2\Re\left(\frac{\xi^2\mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right), \end{gathered} \tag{27}$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(\mathfrak{q})$, and $k \geqq n$.

Theorem 4 (see [17], p. 449, Theorem 1). *Let $\mathfrak{p} \in \mathcal{H}[a, n]$ with $n \in \mathbb{N} \setminus \{1\}$. Also let $\mathfrak{q} \in \mathcal{Q}(a)$ and satisfy the following conditions*:

$$\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right) \geqq 0, \qquad \left|\frac{z\mathfrak{p}'(z)}{\mathfrak{q}'(\xi)}\right| \leqq k, \tag{28}$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(\mathfrak{q})$, and $k \geqq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, \mathfrak{q}]$ and

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \in \Omega, \tag{29}$$

then

$$\mathfrak{p}(z) \prec \mathfrak{q}(z). \tag{30}$$

In this paper, following the theory of second-order differential superordinations in the unit disk introduced by Miller and Mocanu [18], we consider the dual problem of determining properties of functions $\mathfrak{p}(z)$ that satisfy the following third-order differential superordination:

$$\Omega \subset \left\{\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) : z \in \mathbb{U}\right\}. \tag{31}$$

In other words, we determine the conditions on Ω, Δ, and ψ for which the following implication holds true:

$$\begin{aligned} &\Omega \subset \left\{\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) : z \in \mathbb{U}\right\} \\ &\quad \Longrightarrow \Delta \subset \mathfrak{p}(\mathbb{U}), \end{aligned} \tag{32}$$

where Δ is any set in $\mathbb{C}$.

If either Ω or Δ is a simply connected domain, then (32) can be rephrased in terms of superordination. If $\mathfrak{p}(z)$ is univalent in $\mathbb{U}$, and if Δ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is a conformal mapping $\mathfrak{q}$ of $\mathbb{U}$ onto Δ such that $\mathfrak{q}(0) = \mathfrak{p}(0)$. In this case, (32) can be rewritten as follows:

$$\begin{aligned} &\Omega \subset \left\{\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) : z \in \mathbb{U}\right\} \\ &\quad \Longrightarrow \mathfrak{q}(z) \prec \mathfrak{p}(z). \end{aligned} \tag{33}$$

If Ω is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of $\mathbb{U}$ onto Ω such that $h(0) = \psi(\mathfrak{p}(0), 0, 0, 0; 0)$. In addition, if the function

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \tag{34}$$

is univalent in $\mathbb{U}$, then (33) can be rewritten as

$$\begin{aligned} &h(z) \prec \psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \\ &\quad \Longrightarrow \mathfrak{q}(z) \prec \mathfrak{p}(z). \end{aligned} \tag{35}$$

There are three key ingredients in the implication relationship (33): the differential operator ψ, the set Ω, and the "dominating" function $\mathfrak{q}$. If two of these entities were given, one would hope to find conditions on the third entity so that (33) would be satisfied. In this paper, we start with a given set Ω and a given function $\mathfrak{q}$, and we then determine a set of "admissible" operators ψ so that (33) holds true.

We first introduce the following definition.

Definition 5. Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and the function $h(z)$ be analytic in $\mathbb{U}$. If the functions $\mathfrak{p}(z)$ and

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \tag{36}$$

are univalent in $\mathbb{U}$ and satisfy the following third-order differential superordination:

$$h(z) \prec \psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right), \tag{37}$$

then $\mathfrak{p}(z)$ is called a solution of the differential superordination. An analytic function $\mathfrak{q}(z)$ is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $\mathfrak{q}(z) \prec \mathfrak{p}(z)$ for $\mathfrak{p}(z)$ satisfying (37). A univalent subordinant $\tilde{\mathfrak{q}}(z)$ that satisfies the condition

$$\mathfrak{q}(z) \prec \tilde{\mathfrak{q}}(z) \tag{38}$$

for all subordinants $\mathfrak{q}(z)$ of (37) is said to be the best subordinant. We note that the best subordinant is unique up to a rotation of $\mathbb{U}$.

For Ω a set in $\mathbb{C}$, with ψ and $\mathfrak{p}$ as given in Definition 5, we suppose that (37) is replaced by

$$\Omega \subset \left\{\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) : z \in \mathbb{U}\right\}. \tag{39}$$

Although this more general situation is a "differential containment," yet we also refer to it as a differential superordination, and the definitions of solution, subordinant, and best subordinant as given above can be extended to this more general case.

We will use the following lemma [[17], p. 445, Lemma D] from the theory of third-order differential subordinations in $\mathbb{U}$ to determine subordinants of the third-order differential superordinations.

Lemma 6 (see [17]). *Let $\mathfrak{p} \in \mathcal{Q}(a)$, and let $\mathfrak{q}(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $\mathfrak{q}(z) \neq a$ and $n \in \mathbb{N} \setminus \{1\}$. If $\mathfrak{q}$ is not subordinate to $\mathfrak{p}$, then there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(\mathfrak{p})$, and an $m \geqq n$ for which $\mathfrak{q}(\mathbb{U}_{r_0}) \subset \mathfrak{p}(\mathbb{U})$,*

(i) $\mathfrak{q}(z_0) = \mathfrak{p}(\xi_0)$,
(ii) $\Re(\xi_0\mathfrak{p}''(\xi_0)/\mathfrak{p}'(\xi_0)) \geqq 0$ *and* $|z\mathfrak{q}'(z)/\mathfrak{p}'(\xi_0)| \leqq m$,
(iii) $z_0\mathfrak{q}'(z_0) = m\xi_0\mathfrak{p}'(\xi_0)$,
(iv) $\Re(1 + z_0\mathfrak{q}''(z_0)/\mathfrak{q}'(z_0)) \geqq m\Re(1 + \xi_0\mathfrak{p}''(\xi_0)/\mathfrak{p}'(\xi_0))$,
(v) $\Re(z_0^2\mathfrak{q}'''(z_0)/\mathfrak{q}'(z_0)) \geqq m^2\Re(\xi_0^2\mathfrak{p}'''(\xi_0)/\mathfrak{p}'(\xi_0))$.

2. Admissible Functions and a Fundamental Result

We next define the class of admissible functions referred to in the preceding section.

Definition 7. Let Ω be a set in $\mathbb{C}$, $\mathfrak{q} \in \mathscr{H}[a, n]$ and $\mathfrak{q}'(z) \neq 0$. The class of admissible functions $\Psi_n'[\Omega, \mathfrak{q}]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; \xi) \in \Omega \tag{40}$$

whenever

$$r = \mathfrak{q}(z), \quad s = \frac{z\mathfrak{q}'(z)}{m},$$
$$\Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m}\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right), \tag{41}$$

$$\Re\left(\frac{u}{s}\right) \leqq \frac{1}{m^2}\Re\left(\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right), \tag{42}$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U}$, and $m \geqq n \geqq 2$.

If $\psi : \mathbb{C}^2 \times \overline{\mathbb{U}} \to \mathbb{C}$ and $\mathfrak{q} \in \mathscr{H}[a, n]$, then the admissibility condition (41) reduces to the following form:

$$\psi\left(\mathfrak{q}(z), \frac{z\mathfrak{q}'(z)}{m}; \xi\right) \in \Omega \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ m \geqq n \geqq 2). \tag{43}$$

If $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ and $\mathfrak{q} \in \mathscr{H}[a, n]$ with $\mathfrak{q}'(z) \neq 0$, then the admissibility condition (41) reduces to the following form:

$$\psi(r, s, t; \xi) \in \Omega \tag{44}$$

whenever $r = \mathfrak{q}(z)$, $s = z\mathfrak{q}'(z)/m$, and

$$\Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m}\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right) \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ m \geqq n \geqq 2). \tag{45}$$

The next theorem is a foundation result in the theory of the third-order differential superordinations in $\mathbb{U}$.

Theorem 8. *Let $\mathfrak{q} \in \mathscr{H}[a, n]$ and $\psi \in \Psi_n'[\Omega, \mathfrak{q}]$. If*

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \tag{46}$$

is univalent in $\mathbb{U}$ and $\mathfrak{p} \in \mathcal{Q}(a)$ satisfy the following conditions:

$$\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right) \geqq 0, \qquad \left|\frac{z\mathfrak{p}'(z)}{\mathfrak{q}'(z)}\right| \leqq m \quad (z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ m \geqq n \geqq 2), \tag{47}$$

then

$$\Omega \subset \left\{\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) : z \in \mathbb{U}\right\} \tag{48}$$

implies that

$$\mathfrak{q}(z) \prec \mathfrak{p}(z). \tag{49}$$

Proof. Suppose that

$$\mathfrak{q}(z) \not\prec \mathfrak{p}(z). \tag{50}$$

Then, by the above lemma, there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(\mathfrak{p})$, and an $m \geqq n \geqq 2$ that satisfy conditions (i)–(v) of the above lemma. Using these conditions with $r = \mathfrak{p}(\xi_0)$, $s = \xi_0\mathfrak{p}'(\xi_0)$, $t = \xi_0^2\mathfrak{p}''(\xi_0)$, $u = \xi_0^3\mathfrak{p}'''(\xi_0)$, and $\xi = \xi_0$ in Definition 7, we obtain

$$\psi\left(\mathfrak{p}(\xi_0), \xi_0\mathfrak{p}'(\xi_0), \xi_0^2\mathfrak{p}''(\xi_0), \xi_0^3\mathfrak{p}'''(\xi_0); \xi_0\right) \in \Omega, \tag{51}$$

which contradicts (48), so we have

$$\mathfrak{q}(z) \prec \mathfrak{p}(z). \tag{52}$$

□

In the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of $\mathbb{U}$ onto Ω, we denote this class $\Psi_n'[h(\mathbb{U}), \mathfrak{q}]$ by $\Psi_n'[h, \mathfrak{q}]$. The following result is an immediate consequence of Theorem 8.

Theorem 9. *Let $\mathfrak{q} \in \mathscr{H}[a, n]$. Also let the function h be analytic in $\mathbb{U}$ and suppose that $\psi \in \Psi_n'[h, \mathfrak{q}]$. If $\mathfrak{p} \in \mathcal{Q}(a)$ satisfies condition (47) and*

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \tag{53}$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \tag{54}$$

implies that

$$\mathfrak{q}(z) \prec \mathfrak{p}(z). \tag{55}$$

Theorems 8 and 9 can only be used to obtain subordinants of the third-order differential superordination of the forms (48) or (54).

Theorem 10. *Let the function h be analytic in $\mathbb{U}$ and let $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation*

$$\psi\left(\mathfrak{q}(z), z\mathfrak{q}'(z), z^2\mathfrak{q}''(z), z^3\mathfrak{q}'''(z); z\right) = h(z) \tag{56}$$

has a solution $\mathfrak{q} \in \mathcal{Q}(a)$. If $\psi \in \Psi_n'[h, \mathfrak{q}]$, $\mathfrak{p} \in \mathcal{Q}(a)$, and

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \tag{57}$$

is univalent in $\mathbb{U}$, then (54) implies that

$$\mathfrak{q}(z) \prec \mathfrak{p}(z) \tag{58}$$

and $\mathfrak{q}(z)$ is the best subordinant.

Proof. Since $\psi \in \Psi_n'[h, \mathfrak{q}]$, by applying Theorem 9, we deduce that $\mathfrak{q}$ is a subordinant of (54). Since $\mathfrak{q}$ satisfies (56), it is also a solution of the differential superordination (54). Therefore, all subordinants of (54) will be subordinate to $\mathfrak{q}$. It follows that $\mathfrak{q}(z)$ will be the best subordinant of (54). □

In the next two sections, by making use of the third-order differential subordination results of Antonino and Miller [17] in the unit disk $\mathbb{U}$ and the third-order differential superordination results in $\mathbb{U}$ obtained in Section 2 (see, for details, Theorems 8, 9, and 10), we determine certain appropriate classes of admissible functions and investigate some third-order differential subordination and differential superordination properties of meromorphically multivalent functions associated with the operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$ defined by (16). It should be remarked in passing that, in recent years, several authors obtained many interesting results involving various linear and nonlinear convolution operators associated with (second-order) differential subordination and superordination, and the interested reader may refer to several earlier works including (for example) [19] to [20–23].

3. Third-Order Differential Subordination of the Operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$

We first define the following class of admissible functions, which are required in proving the differential subordination theorem involving the operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$ defined by (16).

Definition 11. Let Ω be a set in $\mathbb{C}$ and $\mathfrak{q} \in \mathcal{Q}_1 \cap \mathscr{H}$. The class of admissible functions $\Phi_H[\Omega, \mathfrak{q}]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(a, b, c, d; z) \notin \Omega \tag{59}$$

whenever

$$\begin{gathered} a = \mathfrak{q}(\xi), \qquad b = \frac{k\xi\mathfrak{q}'(\xi) + \beta_1\mathfrak{q}(\xi)}{\beta_1}, \\ \Re\left(\frac{(\beta_1+1)(c-a)}{b-a} - (2\beta_1+1)\right) \geqq k\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)} + 1\right), \\ \Re\left(\frac{(\beta_1+1)(\beta_1+2)(d-3c+3b-a)}{b-a}\right) \\ \geqq k^2\Re\left\{\frac{\xi^2\mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right\}, \end{gathered} \tag{60}$$

where $z \in \mathbb{U}$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $\xi \in \partial\mathbb{U} \setminus E(\mathfrak{q})$, and $k \in \mathbb{N} \setminus \{1\}$.

Theorem 12. *Let $\phi \in \Phi_H[\Omega, \mathfrak{q}]$. If the functions $f \in \Sigma_p$ and $\mathfrak{q} \in \mathcal{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right) \geqq 0, \qquad \left|\frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)}{\mathfrak{q}'(\xi)}\right| \leqq k, \tag{61}$$

$$\begin{gathered} \left\{\phi\left(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z),\right.\right. \\ z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1) f(z), \\ \left.\left. z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2) f(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega, \end{gathered} \tag{62}$$

then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) \prec \mathfrak{q}(z). \tag{63}$$

Proof. Define the analytic function $\mathfrak{p}(z)$ in $\mathbb{U}$ by

$$\mathfrak{p}(z) = z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z). \tag{64}$$

Then, differentiating (64) with respect to z and using (18), we have

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) = \frac{z\mathfrak{p}'(z) + \beta_1\mathfrak{p}(z)}{\beta_1}. \tag{65}$$

Further computations show that

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1)f(z) = \frac{z^2\mathfrak{p}''(z)+2(\beta_1+1)z\mathfrak{p}'(z)+\beta_1(\beta_1+1)\mathfrak{p}(z)}{\beta_1(\beta_1+1)}, \tag{66}$$

$$\begin{aligned} z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2)f(z) &= \Big(z^3\mathfrak{p}'''(z)+3(\beta_1+2)z^2\mathfrak{p}''(z) \\ &\quad +3(\beta_1+1)(\beta_1+2)z\mathfrak{p}'(z) \\ &\quad +\beta_1(\beta_1+1)(\beta_1+2)\mathfrak{p}(z)\Big) \\ &\quad \times(\beta_1(\beta_1+1)(\beta_1+2))^{-1}. \end{aligned} \tag{67}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\begin{aligned} a(r,s,t,u) &= r, \qquad b(r,s,t,u) = \frac{s+\beta_1 r}{\beta_1}, \\ c(r,s,t,u) &= \frac{t+2(\beta_1+1)s+\beta_1(\beta_1+1)r}{\beta_1(\beta_1+1)}, \end{aligned} \tag{68}$$

$$\begin{aligned} d&(r,s,t,u) \\ &= (u+3(\beta_1+2)t+3(\beta_1+1)(\beta_1+2)s \\ &\quad +\beta_1(\beta_1+1)(\beta_1+2)r) \\ &\quad \times(\beta_1(\beta_1+1)(\beta_1+2))^{-1}. \end{aligned} \tag{69}$$

Let

$$\begin{aligned} &\psi(r,s,t,u;z) \\ &= \phi(a,b,c,d;z) \\ &= \phi\Bigg(r, \frac{s+\beta_1 r}{\beta_1}, \frac{t+2(\beta_1+1)s+\beta_1(\beta_1+1)r}{\beta_1(\beta_1+1)}, \\ &\qquad \frac{u+3(\beta_1+2)t+3(\beta_1+1)(\beta_1+2)s+\beta_1(\beta_1+1)(\beta_1+2)r}{\beta_1(\beta_1+1)(\beta_1+2)};z\Bigg). \end{aligned} \tag{70}$$

The proof will make use of Theorem 4. Using (64) to (67), we find from (70) that

$$\begin{aligned} &\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z);z\right) \\ &\quad = \phi\Big(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z), \\ &\qquad z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1)f(z), \\ &\qquad z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2)f(z);z\Big). \end{aligned} \tag{71}$$

Hence, clearly, (62) becomes

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z);z\right) \in \Omega. \tag{72}$$

We note that

$$\begin{aligned} \frac{t}{s}+1 &= \frac{(\beta_1+1)(c-a)}{b-a} - (2\beta_1+1), \\ \frac{u}{s} &= \frac{(\beta_1+1)(\beta_1+2)(d-3c+3b-a)}{b-a}. \end{aligned} \tag{73}$$

Thus, the admissibility condition for $\phi \in \Phi_H[\Omega,\mathfrak{q}]$ in Definition 11 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega,\mathfrak{q}]$ as given in Definition 3 with $n = 2$. Therefore, by using (61) and Theorem 4, we have

$$\mathfrak{p}(z) \prec \mathfrak{q}(z) \tag{74}$$

or, equivalently,

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z) \prec \mathfrak{q}(z), \tag{75}$$

which evidently completes the proof of Theorem 12. □

Our next result is an extension of Theorem 12 to the case where the behavior of $\mathfrak{q}(z)$ on $\partial\mathbb{U}$ is not known.

Corollary 13. *Let $\Omega \subset \mathbb{C}$ and let the function $\mathfrak{q}$ be univalent in $\mathbb{U}$ with $\mathfrak{q}(0) = 1$. Suppose also that $\phi \in \Phi_H[\Omega,\mathfrak{q}_\rho]$ for some $\rho \in (0,1)$, where $\mathfrak{q}_\rho(z) = \mathfrak{q}(\rho z)$. If the functions $f \in \Sigma_p$ and $\mathfrak{q}_\rho$ satisfy the following conditions:*

$$\begin{aligned} &\Re\left(\frac{\xi\mathfrak{q}_\rho''(\xi)}{\mathfrak{q}_\rho'(\xi)}\right) \geqq 0, \\ &\left|\frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z)}{\mathfrak{q}_\rho'(\xi)}\right| \leqq k \quad \left(z \in \mathbb{U};\ \xi \in \partial\mathbb{U}\setminus E(\mathfrak{q}_\rho)\right), \\ &\phi\Big(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1)f(z), \\ &\qquad z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1)f(z), \\ &\qquad z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2)f(z);z\Big) \in \Omega, \end{aligned} \tag{76}$$

then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z) \prec \mathfrak{q}(z). \tag{77}$$

Proof. We note from Theorem 12 that

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1)f(z) \prec \mathfrak{q}_\rho(z). \tag{78}$$

The result asserted by Corollary 13 is now deduced from the following subordination property:

$$\mathfrak{q}_\rho(z) \prec \mathfrak{q}(z). \tag{79}$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_H[h(\mathbb{U}), \mathbf{q}]$ is written as $\Phi_H[h, \mathbf{q}]$. The following two results are immediate consequences of Theorem 12 and Corollary 13.

Theorem 14. *Let $\phi \in \Phi_H[h, \mathbf{q}]$. If the functions $f \in \Sigma_p$ and $\mathbf{q} \in \mathcal{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{\xi \mathbf{q}''(\xi)}{\mathbf{q}'(\xi)}\right) \geqq 0, \qquad \left|\frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)}{\mathbf{q}'(\xi)}\right| \leqq k, \tag{80}$$

$$\begin{aligned}\phi\Big(&z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z),\\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1) f(z),\\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2) f(z); z\Big) \prec h(z),\end{aligned} \tag{81}$$

then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) \prec \mathbf{q}(z). \tag{82}$$

Corollary 15. *Let $\Omega \subset \mathbb{C}$ and let the function $\mathbf{q}$ be univalent in $\mathbb{U}$ with $\mathbf{q}(0) = 1$. Suppose also that $\phi \in \Phi_H[h, \mathbf{q}_\rho]$ for some $\rho \in (0,1)$, where $\mathbf{q}_\rho(z) = \mathbf{q}(\rho z)$. If the functions $f \in \Sigma_p$ and $\mathbf{q}_\rho$ satisfy the following conditions:*

$$\Re\left(\frac{\xi \mathbf{q}_\rho''(\xi)}{\mathbf{q}_\rho'(\xi)}\right) \geqq 0,$$

$$\left|\frac{z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)}{\mathbf{q}_\rho'(\xi)}\right| \leqq k \quad \left(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E\left(\mathbf{q}_\rho\right)\right), \tag{83}$$

$$\begin{aligned}\phi\Big(&z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z),\\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1) f(z),\\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2) f(z); z\Big) \prec h(z),\end{aligned}$$

then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) \prec \mathbf{q}(z). \tag{84}$$

Our next theorem yields the best dominant of the differential subordination (70).

Theorem 16. *Let the function h be univalent in $\mathbb{U}$. Also let $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and ψ be given by* (70). *Suppose that the differential equation*

$$\psi\left(\mathbf{q}(z), z\mathbf{q}'(z), z^2\mathbf{q}''(z), z^3\mathbf{q}'''(z); z\right) = h(z) \tag{85}$$

has a solution $\mathbf{q}(z)$ with $\mathbf{q}(0) = 1$, which satisfies condition (61). *If the function $f \in \Sigma_p$ satisfies condition* (81) *and the function*

$$\begin{aligned}\phi\Big(&z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z),\\ &z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2) f(z); z\Big)\end{aligned} \tag{86}$$

is analytic in $\mathbb{U}$, then

$$z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) \prec \mathbf{q}(z) \tag{87}$$

and $\mathbf{q}(z)$ is the best dominant.

Proof. By applying Theorem 12, we deduce that $\mathbf{q}$ is a dominant of (81). Since $\mathbf{q}$ satisfies (85), it is also a solution of (81). Therefore, $\mathbf{q}$ will be dominated by all dominants. Hence $\mathbf{q}$ is the best dominant. □

In view of Definition 11, in the particular case when $\mathbf{q}(z) = 1 + Mz$ $(M > 0)$, the class $\Phi_H[\Omega, \mathbf{q}]$ of admissible functions, denoted simply by $\Phi_H[\Omega, M]$, is described below.

Definition 17. Let Ω be a set in $\mathbb{C}$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, and $M > 0$. The class $\Phi_H[\Omega, M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ such that

$$\begin{aligned}\phi\Bigg(&1 + Me^{i\theta}, 1 + \frac{k+\beta_1}{\beta_1} Me^{i\theta},\\ &1 + \frac{L + (2k+\beta_1)(\beta_1+1) Me^{i\theta}}{\beta_1(\beta_1+1)},\\ &1 + \frac{N + 3(\beta_1+2) L + (3k+\beta_1)(\beta_1+1)(\beta_1+2) Me^{i\theta}}{\beta_1(\beta_1+1)(\beta_1+2)}; z\Bigg)\\ &\notin \Omega\end{aligned} \tag{88}$$

whenever $z \in \mathbb{U}$, $\Re(Le^{-i\theta}) \geqq (k-1)kM$ and $\Re(Ne^{-i\theta}) \geqq 0$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$.

Corollary 18. *Let $\phi \in \Phi_H[\Omega, M]$. If the function $f \in \Sigma_p$ satisfies the following conditions:*

$$\begin{aligned}&\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)\right| \leqq kM \quad (k \in \mathbb{N} \setminus \{1\}; M > 0),\\ &\phi\Big(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z),\\ &\quad z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1) f(z),\\ &\quad z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2) f(z); z\Big) \in \Omega,\end{aligned} \tag{89}$$

then

$$\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) - 1\right| < M \quad (M > 0). \tag{90}$$

In the special case when

$$\Omega = \mathbf{q}(\mathbb{U}) = \{\omega : |\omega - 1| < M\ (M > 0)\}, \tag{91}$$

the class $\Phi_H[\Omega, M]$ is denoted, for brevity, by $\Phi_H[M]$. Corollary 18 can now be rewritten in the following form.

Corollary 19. *Let $\phi \in \Phi_H[M]$. If the function $f \in \Sigma_p$ satisfies the following conditions:*

$$\begin{aligned}
&\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)\right| \leqq kM \quad (k \in \mathbb{N} \setminus \{1\}; M > 0), \\
&\left|\phi\left(z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z), z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z),\right.\right. \\
&\quad z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1) f(z), \\
&\quad \left.\left. z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2) f(z); z\right) - 1\right| < M,
\end{aligned} \tag{92}$$

then

$$\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) - 1\right| < M \quad (M > 0). \tag{93}$$

Corollary 20. *Let $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ with $\Re(\beta_1) \geqq -1/2$ and $M > 0$. If the function $f \in \Sigma_p$ satisfies the following conditions:*

$$\begin{aligned}
&\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)\right| \leqq kM \quad (k \in \mathbb{N} \setminus \{1\}), \\
&\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z) - 1\right| < M,
\end{aligned} \tag{94}$$

then

$$\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) - 1\right| < M. \tag{95}$$

Proof. Corollary 20 follows from Corollary 19 by setting

$$\phi(a, b, c, d; z) = b = 1 + \frac{k + \beta_1}{\beta_1} M e^{i\theta}. \tag{96}$$

□

Corollary 21. *Let $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, $k \in \mathbb{N} \setminus \{1\}$, and $M > 0$. If the function $f \in \Sigma_p$ satisfies the following conditions*:

$$\begin{aligned}
&\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1) f(z)\right| \leqq kM, \\
&\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-2) f(z) - z^p H_{p,q,s}^{\lambda,\mu}(\beta_1-1) f(z)\right| < \frac{M}{|\beta_1|},
\end{aligned} \tag{97}$$

then

$$\left|z^p H_{p,q,s}^{\lambda,\mu}(\beta_1+1) f(z) - 1\right| < M. \tag{98}$$

Proof. Let

$$\phi(a, b, c, d; z) = d - c, \quad \Omega = h(\mathbb{U}), \tag{99}$$

where

$$h(z) = \frac{Mz}{|\beta_1|} \quad (M > 0). \tag{100}$$

In order to use Corollary 18, we need to show that $\phi \in \Phi_H[\Omega, M]$; that is, the admissibility condition (88) is satisfied. This follows easily, since

$$\begin{aligned}
&\left|\phi\left(1 + Me^{i\theta}, 1 + \frac{k+\beta_1}{\beta_1} Me^{i\theta},\right.\right. \\
&\quad 1 + \frac{L + (2k+\beta_1)(\beta_1+1) Me^{i\theta}}{\beta_1(\beta_1+1)}, \\
&\quad \left.\left. 1 + \frac{N + 3(\beta_1+2)L + (3k+\beta_1)(\beta_1+1)(\beta_1+2) Me^{i\theta}}{\beta_1(\beta_1+1)(\beta_1+2)}; z\right)\right| \\
&= \left|\frac{N + 3(\beta_1+2)L + (3k+\beta_1)(\beta_1+1)(\beta_1+2) Me^{i\theta}}{\beta_1(\beta_1+1)(\beta_1+2)}\right. \\
&\quad \left. - \frac{L + (2k+\beta_1)(\beta_1+1) Me^{i\theta}}{\beta_1(\beta_1+1)}\right| \\
&= \left|\frac{Ne^{-i\theta} + 2(\beta_1+2) Le^{-i\theta} + k(\beta_1+1)(\beta_1+2) M}{\beta_1(\beta_1+1)(\beta_1+2) e^{-i\theta}}\right| \\
&\geqq \frac{\Re\left(Ne^{-i\theta}\right) + 2|\beta_1+2| \Re\left(Le^{-i\theta}\right) + k|\beta_1+1||\beta_1+2| M}{|\beta_1(\beta_1+1)(\beta_1+2)|} \\
&\geqq \frac{k(2k - 2 + |\beta_1+1|) M}{|\beta_1||\beta_1+1|} \\
&\geqq \frac{M}{|\beta_1|}
\end{aligned} \tag{101}$$

whenever $z \in \mathbb{U}$, $\Re(Le^{-i\theta}) \geqq (k-1)kM$, and $\Re(Ne^{-i\theta}) \geqq 0$ for all $\theta \in \mathbb{R}$ and $k \in \mathbb{N} \setminus \{1\}$. The required result now follows from Corollary 18. □

4. Third-Order Differential Superordination of the Operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$

In this section, we obtain the third-order differential superordination results for meromorphically multivalent functions associated with the operator $H_{p,q,s}^{\lambda,\mu}(\beta_1)$ defined by (16). Because of this, the class of admissible functions is given in the following definition.

Definition 22. Let Ω be a set in $\mathbb{C}$ and $\mathfrak{q} \in \mathscr{H}$ with $\mathfrak{q}'(z) \neq 0$. The class of admissible functions $\Phi_H'[\Omega, \mathfrak{q}]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(a, b, c, d; \xi) \in \Omega \tag{102}$$

whenever

$$a = \mathfrak{q}(z), \qquad b = \frac{z\mathfrak{q}'(z) + m\beta_1 \mathfrak{q}(z)}{m\beta_1},$$

$$\Re\left(\frac{(\beta_1+1)(c-a)}{b-a} - (2\beta_1+1)\right) \leqq \frac{1}{m}\Re\left\{\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right\}, \tag{103}$$

$$\Re\left(\frac{(\beta_1+1)(\beta_1+2)(d-3c+3b-a)}{b-a}\right) \leqq \frac{1}{m^2}\Re\left\{\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right\},$$

where $z \in \mathbb{U}$, $\beta_1 \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$, $\xi \in \partial\mathbb{U}$, and $m \in \mathbb{N}\setminus\{1\}$.

Theorem 23. *Let $\phi \in \Phi'_H[\Omega, \mathfrak{q}]$. If the functions $f \in \Sigma_p$ and $z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1)f(z) \in \mathcal{Q}_1$ satisfy the following conditions:*

$$\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right) \geqq 0, \qquad \left|\frac{z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z)}{\mathfrak{q}'(z)}\right| \leqq m, \tag{104}$$

$$\phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right) \tag{105}$$

is univalent

$$\Omega \subset \left\{\phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right) : z \in \mathbb{U}\right\} \tag{106}$$

implies that

$$\mathfrak{q}(z) \prec z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z). \tag{107}$$

Proof. Let the function $\mathfrak{p}(z)$ be defined by (64) and ψ by (70). Since $\phi \in \Phi'_H[\Omega, \mathfrak{q}]$, (71) and (106) yield

$$\Omega \subset \left\{\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) : z \in \mathbb{U}\right\}. \tag{108}$$

We see from (68) and (69) that the admissible condition for $\phi \in \Phi'_H[\Omega, \mathfrak{q}]$ in Definition 22 is equivalent to the admissible condition for ψ as given in Definition 7 with $n = 2$. Hence $\psi \in \Psi'_2[\Omega, \mathfrak{q}]$, and by using (104) and Theorem 8, we have

$$\mathfrak{q}(z) \prec \mathfrak{p}(z) \tag{109}$$

or, equivalently,

$$\mathfrak{q}(z) \prec z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), \tag{110}$$

which evidently completes the proof of Theorem 23. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, then the class $\Phi'_H[h(\mathbb{U}), \mathfrak{q}]$ is written simply as $\Phi'_H[h, \mathfrak{q}]$. With proceedings similar as in the preceding section, the following result is an immediate consequence of Theorem 23.

Theorem 24. *Let $\phi \in \Phi'_H[h, \mathfrak{q}]$. Also let the function h be analytic in $\mathbb{U}$. If the functions $f \in \Sigma_p$ and $z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z) \in \mathcal{Q}_1$ satisfy condition (104) and*

$$\phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right) \tag{111}$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right) \tag{112}$$

implies that

$$\mathfrak{q}(z) \prec z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z). \tag{113}$$

Theorems 23 and 24 can only be used to obtain subordinations involving the third-order differential superordination of the forms (106) or (112). The following theorem proves the existence of the best subordinant of (112) for a suitable chosen ϕ.

Theorem 25. *Let the function h be analytic in $\mathbb{U}$, and let $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ and ψ be given by (70). Suppose that the differential equation*

$$\psi\left(\mathfrak{q}(z), z\mathfrak{q}'(z), z^2\mathfrak{q}''(z), z^3\mathfrak{q}'''(z); z\right) = h(z) \tag{114}$$

has a solution $\mathfrak{q}(z) \in \mathcal{Q}_1$. If the functions $f \in \Sigma_p$ and $z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z) \in \mathcal{Q}_1$ satisfy condition (104) and

$$\phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right), \tag{115}$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1-2) f(z); z\right) \tag{116}$$

implies that

$$\mathfrak{q}(z) \prec z^p H^{\lambda,\mu}_{p,q,s}(\beta_1+1) f(z) \tag{117}$$

and $\mathfrak{q}$ is the best subordinant.

Proof. The proof of Theorem 25 is similar to that of Theorem 16 and it is being omitted here. □

By combining Theorems 14 and 24, we obtain the following sandwich-type result.

Corollary 26. *Let the functions h_1 and $\mathfrak{q}_1$ be analytic functions in $\mathbb{U}$. Also let the function h_2 be univalent in $\mathbb{U}$, $\mathfrak{q}_2 \in \mathcal{Q}_1$ with $\mathfrak{q}_1(0) = \mathfrak{q}_2(0) = 1$ and $\phi \in \Phi_H[h_2, \mathfrak{q}_2] \cap \Phi'_H[h_1, \mathfrak{q}_1]$. If the function $f \in \Sigma_p$, $z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z) \in \mathcal{Q}_1 \cap \mathcal{H}$ and*

$$\phi\Big(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 2) f(z); z\Big) \tag{118}$$

is univalent in $\mathbb{U}$, and the conditions (61) *and* (104) *are satisfied, then*

$$h_1(z) \prec \phi\Big(z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 1) f(z), z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 - 2) f(z); z\Big) \prec h_2(z) \tag{119}$$

implies that

$$\mathfrak{q}_1(z) \prec z^p H^{\lambda,\mu}_{p,q,s}(\beta_1 + 1) f(z) \prec \mathfrak{q}_2(z). \tag{120}$$

Remark 27. By setting $\lambda = \mu = 0$ in all results of this paper, we can obtain the corresponding results for the well-known Liu-Srivastava operator $H_{p,q,s}(\beta_1)$.

5. Concluding Remarks and Observations

In our present investigation, we have derived several third-order differential subordination and superordination results for meromorphically multivalent functions in the punctured unit disk involving the operator $H^{\lambda,\mu}_{p,q,s}(\beta_1)$ defined by (16) with respect to the parameter $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, which is associated with the Liu-Srivastava operator $H_{p,q,s}(\beta_1)$. Our results have been obtained by considering suitable classes of admissible functions. Furthermore, if we use relation (19), we can obtain the corresponding third-order differential subordination and superordination results for the operator $H^{\lambda,\mu}_{p,q,s}(\alpha_1)$ with respect to the parameter $\alpha_1 \in \mathbb{C}$ and here we choose to omit the details involved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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