SOME ASYMPTOTIC FORMULAS EXHIBITING THE BEHAVIORS OF THE TRIPLE HYPERGEOMETRIC SERIES F_S AND F_T NEAR THE BOUNDARIES OF THEIR CONVERGENCE REGIONS

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ABSTRACT

The authors prove a number of asymptotic formulas which depict the behaviors of the triple hypergeometric series F_S and F_T near the boundaries of their regions of convergence. Results of this type are potentially useful in several seemingly diverse areas of applications utilizing these classes of multiple hypergeometric functions. Some indication of applicability is also provided.

1. INTRODUCTION

In many areas of applications using various classes of hypergeometric functions of one, two, and more variables, one finds the need for the asymptotic behaviors of these functions near the boundaries of the regions of convergence of the series defining them. The asymptotic behavior of the Gaussian hypergeometric function ${}_2F_1(z)$ near $z = 1$ is given by a well-known analytic continuation formula. An asymptotic formula exhibiting the behavior of a certain Clausenian hypergeometric function ${}_3F_2(z)$ near $z = 1$ was stated (without proof) by Srinivasa Ramanujan (1887–1920); indeed, it has motivated a number of recent works which are already reported in our earlier paper [13] dealing similarly with the generalized hypergeometric function ${}_pF_{p-1}(z)$ near $z = 1$. Several further asymptotic formulas of this type, involving more general families of hypergeometric functions of one, two, and more variables, have appeared in the literature (*cf.*, *e.g.*, [7] to [14]); some of these results were, in fact, applied in order to solve various boundary value problems involving the Euler–Darboux equation, and also in the study of certain operators of fractional calculus (*cf.* [6] and [22]).

The object of the present paper is to prove a number of asymptotic formulas which exhibit the behaviors of the triple hypergeometric functions F_S and F_T near the boundaries of the unit cube:

$$\Omega = \{(x,y,z) : |x| < 1, \ |y| < 1, \ |z| < 1\},$$

which is, in fact, the region of convergence of the triple hypergeometric series defining each of these functions.

2. DEFINITIONS, NOTATIONS, AND PRELIMINARY RESULTS

We begin by recalling the definitions and properties of various hypergeometric functions relevant to our present investigation.

The generalized hypergeometric series in one variable is defined by (see, *e.g.*, [17] and [20])

$$(2.1) \qquad {}_pF_q[(\alpha_p);(\beta_q);z] = {}_pF_q\left[\begin{matrix}(\alpha_p); \\ \\ (\beta_q);\end{matrix}\ z\right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_m}{\prod_{j=1}^{q} (\beta_j)_m} \frac{z^m}{m!}$$

$$(p \leqq q \text{ and } |z| < \infty; \ p = q+1 \text{ and } |z| < 1)$$

in terms of the Pochhammer symbol $(\lambda)_m$ given by

$$(2.2)\qquad (\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } m=0 \\ \lambda(\lambda+1)\cdots(\lambda+m-1), & \text{if } m \in \mathbb{N} = \{1,2,3,\cdots\}. \end{cases}$$

Here, and in what follows, (α_p) abbreviates the array of p parameters $\alpha_1,\cdots,\alpha_p$, with similar interpretations for (β_q), *etc.*

For the generalized hypergeometric series, we readily obtain the simple identity:

$$(2.3)\qquad {}_pF_q[(\alpha_{p-1}),1;\ (\beta_{q-1}),2;\ z]$$

$$= \frac{\prod_{j=1}^{q-1} (\beta_j-1)}{\prod_{j=1}^{p-1} (\alpha_j-1)}\ \frac{1}{z}\left\{{}_{p-1}F_{q-1}[(\alpha_{p-1}-1);\ (\beta_{q-1}-1);\ z] - 1\right\}.$$

The most useful special case of the series in (2.1) is the Gauss hypergeometric function ${}_2F_1$ which has the open unit disk:

$$\mathcal{U} = \{z : |z| < 1\}$$

for the region of convergence of the series defining it (see [24, Chapter 14]). Furthermore, we have (*cf.* [5] and [7])

$$(2.4)\qquad {}_2F_1(\alpha,\beta;\ \alpha+\beta;\ z) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n!)^2}$$

$$\cdot\ [2\psi(n+1) - \psi(\alpha+n) - \psi(\beta+n) - \log(1-z)]\,(1-z)^n,$$

$$(|\arg(1-z)| < \pi;\quad |1-z| < 1);$$

(2.5)
$$ {}_2F_1(\alpha,\beta;\ \alpha+\beta+m;\ z) = \frac{\Gamma(m)\Gamma(\alpha+\beta+m)}{\Gamma(\alpha+m)\Gamma(\beta+m)} \sum_{n=0}^{m-1} \frac{(\alpha)_n(\beta)_n}{n!\,(1-m)_n} (1-z)^n $$

$$ + \frac{\Gamma(\alpha+\beta+m)}{\Gamma(\alpha)\Gamma(\beta)} (z-1)^m \sum_{n=0}^{\infty} \frac{(\alpha+m)_n(\beta+m)_n}{n!\,(n+m)!} (1-z)^n $$

$$ \cdot \left[\psi(n+1) + \psi(m+n+1) - \psi(\alpha+m+n) - \psi(\beta+m+n) - \log(1-z)\right], $$

$$(|\arg(1-z)| < \pi;\quad |1-z| < 1;\quad m \in \mathbb{N});$$

(2.6)
$$ {}_2F_1(\alpha,\beta;\ \alpha+\beta-m;\ z) = \frac{\Gamma(m)\Gamma(\alpha+\beta-m)}{\Gamma(\alpha)\Gamma(\beta)} (1-z)^{-m} \sum_{n=0}^{m-1} \frac{(\alpha-m)_n(\beta-m)_n}{n!\,(1-m)_n} (1-z)^n $$

$$ + (-1)^m \frac{\Gamma(\alpha+\beta-m)}{\Gamma(\alpha-m)\Gamma(\beta-m)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(n+m)!} (1-z)^n $$

$$ \cdot \left[\psi(n+1) + \psi(m+n+1) - \psi(\alpha+n) - \psi(\beta+n) - \log(1-z)\right], $$

$$(|\arg(1-z)| < \pi;\quad |1-z| < 1;\quad m \in \mathbb{N});$$

(2.7)
$$ {}_2F_1(\alpha,\beta;\ \gamma;\ 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \qquad (\mathrm{Re}(\gamma-\alpha-\beta) > 0); $$

(2.8)
$$ {}_2F_1(\alpha,\beta;\ \alpha+\beta;\ 1-\rho) = -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[2\gamma + \psi(\alpha) + \psi(\beta) + \log\rho\right] $$

$$ + o(1), \qquad (\rho \rightarrow 0+), $$

where $\psi(z)$ denotes the Psi function defined by

(2.9)
$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

which possess the properties (see [5])

(2.10)
$$\psi(m+1) = -\gamma + \sum_{n=1}^{m} \frac{1}{n} = -\gamma + \sum_{n=0}^{m-1} \frac{(1)_n}{(2)_n}, \qquad (m \in \mathbb{N})$$

and

(2.11)
$$\psi(z+m) = \psi(z) + \sum_{n=0}^{m-1} \frac{1}{z+n} = \psi(z) + \frac{1}{z} \sum_{n=0}^{m-1} \frac{(z)_n}{(z+1)_n}, \qquad (m \in \mathbb{N}),$$

γ being the Euler–Mascheroni constant:

(2.12)
$$\gamma = -\psi(1) \cong 0.5772156649\cdots.$$

For the higher-order hypergeometric series ${}_3F_2$ and ${}_4F_3$, it is known that (see [8] and [13])

(2.13)
$${}_3F_2\left[\begin{matrix} \alpha_1, \alpha_2, \alpha_3; \\ \\ \beta_1, \beta_2; \end{matrix} \; 1-\rho\right] = -\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} [2\gamma + \psi(\alpha_1) + \psi(\alpha_2) + \log \rho]$$

$$+\frac{(\beta_1-\alpha_3)(\beta_2-\alpha_3)\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3)}\ {}_4F_3\left[\begin{matrix}\beta_1-\alpha_3+1,\ \beta_2-\alpha_3+1,\ 1,\ 1;\\ \\ \alpha_1+1,\ \alpha_2+1,\ 2;\end{matrix}\ 1\right]$$

$$+\ o(1),\quad (\rho\longrightarrow 0+)\qquad [\alpha_1+\alpha_2+\alpha_3=\beta_1+\beta_2;\ \operatorname{Re}(\alpha_3)>0];$$

$$(2.14)\quad {}_4F_3\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3,\alpha_4;\\ \\ \beta_1,\beta_2,\beta_3;\end{matrix}\ 1-\rho\right]$$

$$=-\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}\ [2\gamma+\psi(\alpha_1)+\psi(\alpha_2)+\log\rho]$$

$$+\frac{(\beta_1-\alpha_3)(\alpha_1+\alpha_2-\beta_1)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3)\Gamma(\alpha_4)}\ {}_4F_3\left[\begin{matrix}\beta_1-\alpha_3+1,\alpha_1+\alpha_2-\beta_1+1,1,1;\\ \\ \alpha_1+1,\alpha_2+1,2;\end{matrix}\ 1\right]$$

$$+\frac{(\beta_2-\alpha_4)(\beta_3-\alpha_4)\Gamma(\beta_2)\Gamma(\beta_3)}{\Gamma(\alpha_4)\Gamma(\beta_2+\beta_3-\alpha_4+1)}\ F^{0:3;3}_{1:1;1}\left[\begin{matrix}\text{———}: & \alpha_1,\alpha_2,\alpha_3; & \beta_2-\alpha_4+1,\beta_3-\alpha_4+1,1;\\ & & \\ \beta_2+\beta_3-\alpha_4+1: & \beta_1; & 2;\end{matrix}\ 1,1\right]$$

$$+\ o(1),\quad (\rho\longrightarrow 0+)$$

$$[\alpha_1+\alpha_2+\alpha_3+\alpha_4=\beta_1+\beta_2+\beta_3;\quad \operatorname{Re}(\alpha_3)>0;\quad \operatorname{Re}(\alpha_4)>0],$$

where $F^{p:r;u}_{q:s;v}$ denotes a general double hypergeometric series defined by (*cf.* [1, p. 150] and [20, p. 27])

$$(2.15)\quad F^{p:r;u}_{q:s;v}\left[\begin{matrix}(a_p):(\alpha_r);(\lambda_u); \\ \\ (b_q):(\beta_s);(\mu_v);\end{matrix}\; x,\, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{m+n}\prod_{j=1}^{r}(\alpha_j)_m\prod_{j=1}^{u}(\lambda_j)_n}{\prod_{j=1}^{q}(b_j)_{m+n}\prod_{j=1}^{s}(\beta_j)_m\prod_{j=1}^{v}(\mu_j)_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

We also have the following expansion:

$$(2.16)\quad F^{p:r;u}_{q:s;v}\left[\begin{matrix}(a_p):(\alpha_r);(\lambda_u); \\ \\ (b_q):(\beta_s);(\mu_v);\end{matrix}\; x,\, y\right]$$

$$= \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_m\prod_{j=1}^{r}(\alpha_j)_m}{\prod_{j=1}^{q}(b_j)_m\prod_{j=1}^{s}(\beta_j)_m}\;{}_{p+u}F_{q+v}\left[\begin{matrix}(a_p+m),(\lambda_u); \\ \\ (b_q+m),(\mu_v);\end{matrix}\; y\right]\frac{x^m}{m!}.$$

Some special cases of $F^{p:r;u}_{q:s;v}$, relevant to our present investigation, are the Appell series F_1 and F_3 defined by (see [1] and [20])

$$(2.17)\qquad F_1(\alpha,\beta_1,\beta_2;\gamma;x,y) = F^{1:1;1}_{1:0;0}\left[\begin{matrix}\alpha:\beta_1;\beta_2; \\ \\ \gamma: —; —;\end{matrix}\; x,y\right]$$

$$= \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}$$

and

$$(2.18) \qquad F_3(\alpha_1,\alpha_2,\beta_1,\beta_2;\ \gamma;\ x,y) = F^{0:2;2}_{1:0;0}\left[\begin{array}{l} -:\ \alpha_1,\beta_1;\ \alpha_2,\beta_2; \\ \qquad\qquad\qquad\qquad\quad x,y \\ \gamma:\ \text{——};\ \text{——}; \end{array}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_m(\beta_1)_n(\beta_2)_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!},$$

respectively. In particular, for the Appell function F_1 it is known that [1, p. 34]

$$(2.19) \qquad F_1(\alpha,\beta_1,\beta_2;\ \gamma;\ x,y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta_2)_m}{(\gamma)_m} F(\alpha+m,\beta_1+\beta_2+m;\ \gamma+m;\ x)\frac{(y-x)^m}{m!}.$$

Finally, the triple hypergeometric functions F_S and F_T, which are the subject of our present investigation, are defined by (see [4, p. 114], [3, pp. 67–68], and [20, p. 43]).

$$(2.20) \qquad F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\ \gamma,\gamma,\gamma;\ x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$

and

$$(2.21) \qquad F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma,\gamma,\gamma;\ x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.$$

Each of these triple hypergeometric functions may be expressed in terms of the Gauss series ${}_2F_1$, or the Appell series F_1 and F_3, as follows:

(2.22) $F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}}\ {}_2F_1(\alpha_2+n,\beta_3;\ \gamma+m+n;\ z)\ \frac{x^m}{m!}\ \frac{y^n}{n!}\ ;$$

(2.23) $F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_m(\beta_3)_n}{(\gamma)_{m+n}}\ {}_2F_1(\alpha_2+n,\beta_2;\ \gamma+m+n;\ y)\ \frac{x^m}{m!}\ \frac{z^n}{n!}\ ;$$

(2.24) $F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_2)_{m+n}(\beta_2)_m(\beta_3)_n}{(\gamma)_{m+n}}\ {}_2F_1(\alpha_1,\beta_1;\ \gamma+m+n;\ x)\ \frac{y^m}{m!}\ \frac{z^n}{n!}\ ;$$

(2.25) $F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_1)_m(\beta_1)_m}{(\gamma)_m}\ F_1(\alpha_2,\beta_2,\beta_3;\ \gamma+m;\ y,z)\ \frac{x^m}{m!}\ ;$$

(2.26) $F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_2)_m(\beta_2)_m}{(\gamma)_m} F_3(\alpha_1,\alpha_2+m,\beta_1,\beta_3;\ \gamma+m;\ x,z)\ \frac{y^m}{m!}\ ;$$

(2.27) $F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_2)_m(\beta_3)_m}{(\gamma)_m} F_3(\alpha_1,\alpha_2+m,\beta_1,\beta_2;\ \gamma+m;\ x,y)\ \frac{z^m}{m!}\ .$$

(2.28) $F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}}\ {}_2F_1(\alpha_2+n,\beta_1+m;\ \gamma+m+n;\ z)\ \frac{x^m}{m!}\ \frac{y^n}{n!}\ ;$$

(2.29) $F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_{m+n}}{(\gamma)_{m+n}}\ {}_2F_1(\alpha_2+n,\beta_2;\ \gamma+m+n;\ y)\ \frac{x^m}{m!}\ \frac{z^n}{n!}\ ;$$

(2.30) $F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_2)_{m+n}(\beta_2)_m(\beta_1)_n}{(\gamma)_{m+n}}\ {}_2F_1(\alpha_1,\beta_1+n;\ \gamma+m+n;\ x)\ \frac{y^m}{m!}\ \frac{z^n}{n!}\ ;$$

(2.31) $F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_1)_m(\beta_1)_m}{(\gamma)_m} F_1(\alpha_2,\beta_2,\beta_1+m;\ \gamma+m;\ y,z)\ \frac{x^m}{m!}\ ;$$

(2.32) $F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_2)_m(\beta_2)_m}{(\gamma)_m}\ F_1(\beta_1,\alpha_1,\alpha_2+m;\ \gamma+m;\ x,z)\ \frac{y^m}{m!}\ ;$$

(2.33) $F_T(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \gamma,\gamma,\gamma;\ x,y,z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_2)_m(\beta_1)_m}{(\gamma)_m}\ F_3(\alpha_1,\alpha_2+m,\beta_1+m,\beta_2;\ \gamma+m;\ x,y)\ \frac{z^m}{m!}\ .$$

3. ASYMPTOTIC BEHAVIORS OF THE TRIPLE HYPERGEOMETRIC SERIES F_S AND F_T

In this section we shall prove several asymptotic formulas which exhibit the behaviors of the triple hypergeometric series F_S and F_T near one or the other of the following boundary points of the unit cube Ω:

(i) $x = 1$; (ii) $y = 1$; (iii) $z = 1$; (iv) $x = y = 1$;

(v) $x = z = 1$; (vi) $y = z = 1$; (vii) $x = y = z = 1$.

In view of expansions like (2.22) and (2.23), or (2.26) and (2.27), we need not consider the cases (ii) and (v) separately. Consequently, we give the asymptotic behaviors of F_S in the remaining five cases only. The asymptotic behaviors of F_T, on the other hand, are also given in all these five cases except the case (vii) in which the result becomes overly involved.

(S1) $F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_2+b_3;\ x,y,1-\rho)$

$$= \frac{\Gamma(a_2+b_3)}{\Gamma(a_2+1)\Gamma(b_3+1)}\, a_1 b_1 x\ F^{0:4;2}_{1:2;0}\left[\begin{matrix} \text{—}: & a_1+1, b_1+1, 1, 1; & a_2, b_2; \\ a_2+1: & b_3+1, 2; & \text{—}; \end{matrix}\ x,y\right]$$

$$+ \frac{\Gamma(a_2+b_3)}{\Gamma(a_2+1)\Gamma(b_3)}\, b_2 y\ F^{1:2;1}_{1:1;0}\left[\begin{matrix} b_2+1: & a_2, 1; & 1; \\ 2: & a_2+1; & -; \end{matrix}\ y,y\right]$$

$$- \frac{\Gamma(a_2+b_3)}{\Gamma(a_2)\Gamma(b_3)}\,(1-y)^{-b_2}\,[2\gamma + \psi(a_2) + \psi(b_3) + \log\rho] + o(1), \quad (\rho \to 0+).$$

(S2) $F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_1+b_1;\ 1-\rho,y,z)$

$$= \frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\, a_2 b_2 y\ {}_4F_3\left[\begin{matrix} a_2+1,\ b_2+1,\ 1,\ 1; \\ a_1+1,\ b_1+1,\ 2; \end{matrix}\ y\right]$$

$$+ \frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\, a_2 b_3 z\ F^{2:1;2}_{2:0;1}\left[\begin{matrix} a_2+1,\ 1: & b_2; & b_3+1,\ 1; \\ a_1+1,\ b_1+1: & \text{—}; & 2; \end{matrix}\ y,z\right]$$

$$- \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\,[2\gamma + \psi(a_1) + \psi(b_1) + \log\rho] + o(1), \quad (\rho \to 0+).$$

(S3) $F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_2+b_2+b_3;\ x,1-\rho y,1-\rho z)$

$$= \frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2+1)\Gamma(b_2+b_3+1)}\, a_1 b_1 x \;\; {}_4F_3\left[\begin{matrix} a_1+1,\; b_1+1,\; 1,\; 1; \\ \\ a_2+1,\; b_2+b_3+1,\; 2; \end{matrix}\; x\right]$$

$$+ \frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3+1)}\, b_3\left[1-\frac{z}{y}\right] \;\; {}_3F_2\left[\begin{matrix} b_3+1,\; 1,\; 1; \\ \\ b_2+b_3+1,\; 2; \end{matrix}\; 1-\frac{z}{y}\right]$$

$$- \frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3)}\, [2\gamma + \psi(a_2) + \psi(b_2+b_3) + \log \rho y] + o(1), \quad (\rho \longrightarrow 0+).$$

(S4) $F_S(a_1,a_2,a_2,b_1,b_2,b_3;\, a_1+b_1;\, 1-\rho x, 1-\rho y, z)$

$$= \frac{\Gamma(a_2+b_2)}{\Gamma(a_2+2)\Gamma(b_2+2)}\, a_1 a_2 b_2 (b_1+1) \;\; {}_4F_3\left[\begin{matrix} a_1+1,\; b_1+2,\; 1,\; 1; \\ \\ a_2+2,\; b_2+2,\; \quad 2; \end{matrix}\; 1\right]$$

$$+ \frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+2)}\, b_1 a_2 b_2 \;\; {}_4F_3\left[\begin{matrix} a_2+1,\; b_2+1,\; 1,\; 1; \\ \\ a_1+1,\; b_1+2,\; 2; \end{matrix}\; 1\right]$$

$$- \frac{\Gamma(a_2+b_2)}{\Gamma(a_2+1)\Gamma(b_2)}\, b_3 z \;\; F^{1:2;1}_{1:1;0}\left[\begin{matrix} b_3+1:\; a_2,\; 1;\; 1; \\ \\ 2:\;\; a_2+1;\; -; \end{matrix}\; z, z\right]$$

$$+ \frac{\Gamma(a_2+b_2)}{\Gamma(a_2+2)\Gamma(b_2+1)}\, a_1 b_1 a_2 b_3 z$$

$$\cdot\, F^{0:4;3}_{1:2;1}\left[\begin{matrix} \text{---}:\; a_1+1,\; b_1+1,\; 1,\; 1;\; a_2+1,\; b_3+1,\; 1; \\ \\ a_2+2: \qquad b_2+1, \quad 2; \qquad\qquad 2; \end{matrix}\; 1, z\right]$$

$$-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\,[2\gamma+\psi(a_1)+\psi(b_1)+\log\rho x]$$

$$-\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}\,(1-z)^{-b_3}\,[2\gamma+\psi(a_2)+\psi(b_2)+\log\rho y]$$

$$-\frac{\Gamma(a_2+b_2+1)}{\Gamma(a_2+1)\Gamma(b_2+1)}+o(1), \qquad (\rho\longrightarrow 0+;\ \ a_1+b_1=a_2+b_2).$$

(S5) $$F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_1+b_1;\ 1-\rho x,1-\rho y,1-\rho z)$$

$$=\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\,\frac{a_2(b_2+b_3+1)}{(a_1+1)(b_1+1)}\ {}_4F_3\left[\begin{matrix}a_2+1,\ b_2+b_3+2,\ 1,\ 1;\\ a_1+2,\ b_1+2,\ 2;\end{matrix}\ 1\right]$$

$$+\frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2+1)\Gamma(b_2+b_3)}\,\frac{a_1b_1}{b_2+b_3+1}\ {}_4F_3\left[\begin{matrix}a_1+1,\ b_1+1,\ 1,\ 1;\\ a_2+1,\ b_2+b_3+2,\ 2;\end{matrix}\ 1\right]$$

$$+\frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3+1)}\,b_3\left[1-\frac{z}{y}\right]\ {}_3F_2\left[\begin{matrix}b_3+1,\ 1,\ 1;\\ b_2+b_3+1,\ 2;\end{matrix}\ 1-\frac{z}{y}\right]$$

$$-\frac{\Gamma(a_1+b_1+1)}{\Gamma(a_1+1)\Gamma(b_1+1)}$$

$$-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\,[2\gamma+\psi(a_1)+\psi(b_1)+\log\rho x]$$

$$-\frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3)}\,[2\gamma+\psi(a_2)+\psi(b_2+b_3)+\log\rho y]+o(1),$$

$$(\rho\longrightarrow 0+;\ \ a_1+b_1=a_2+b_2+b_3).$$

(T1) $F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_2+b_1;\ x,y,1-\rho)$

$$=-\frac{\Gamma(a_2+b_1)}{\Gamma(a_2+1)\Gamma(b_1)}\,b_2y(1-x)^{-a_1}\ F^{1:2;1}_{1:1;0}\left[\begin{array}{l} b_2+1:\ a_2,\ 1;\ 1; \\ \qquad\qquad\qquad\qquad\qquad y,y \\ \quad 2:\ \ a_2+1;\ -; \end{array}\right]$$

$$-\frac{\Gamma(a_2+b_1)}{\Gamma(a_2)\Gamma(b_1+1)}\,a_1x(1-y)^{-b_2}\ F^{1:2;1}_{1:1;0}\left[\begin{array}{l} a_1+1:\ b_1,\ 1;\ 1; \\ \qquad\qquad\qquad\qquad\qquad x,x \\ \quad 2:\ \ b_1+1;\ -; \end{array}\right]$$

$$-\frac{\Gamma(a_2+b_1)}{\Gamma(a_2)\Gamma(b_1)}\,(1-x)^{-a_1}(1-y)^{-b_2}\,[2\gamma+\psi(a_2)+\psi(b_1)+\log\rho]+o(1),$$

$$(\rho\longrightarrow 0+).$$

(T2) $F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_1+b_1;\ 1-\rho,y,z)$

$$=\frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\,a_2b_2y\ F^{1:3;1}_{1:2;0}\left[\begin{array}{l} a_2+1:\ b_2+1,\ 1,\ 1;\ b_1; \\ \qquad\qquad\qquad\qquad\qquad\qquad y,z \\ b_1+1:\qquad a_1+1,\ 2;\ —; \end{array}\right]$$

$$-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1+1)}\,a_2\ F^{1:2;1}_{1:1;0}\left[\begin{array}{l} a_2+1:\ b_1,\ 1;\ 1; \\ \qquad\qquad\qquad\qquad\qquad z,z \\ \quad 2:\ \ b_1+1;\ -; \end{array}\right]$$

$$-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}(1-z)^{-a_2}\,[2\gamma+\psi(a_1)+\psi(b_1)+\log\rho]+o(1),\quad(\rho\longrightarrow 0+).$$

(T3) $F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_2+b_1+b_2;\ x,1-\rho y,1-\rho z)$

$$=\frac{\Gamma(a_2+b_1+b_2)}{\Gamma(a_2)\Gamma(b_1+b_2+1)}\ \frac{a_1b_1}{b_1+b_2}\,x\ F^{2:2;1}_{2:1;0}\left[\begin{array}{l} a_1+1,\ b_1+1:\ b_1+b_2,\ 1;\ 1; \\ \\ b_1+b_2+1,\ 2:\ \ b_1+b_2+1;\ -; \end{array}\ x,x\right]$$

$$+\frac{\Gamma(a_2+b_1+b_2)}{\Gamma(a_2)\Gamma(b_1+b_2+1)}\,b_1\left[1-\frac{z}{y}\right]\ F^{1:1;2}_{1:0;1}\left[\begin{array}{l} \quad b_1+1:\ a_1;\ 1,\ 1; \\ \\ b_1+b_2+1:\ —;\quad 2; \end{array}\ x,\ 1-\frac{z}{y}\right]$$

$$-\frac{\Gamma(a_2+b_1+b_2)}{\Gamma(a_2)\Gamma(b_1+b_2)}\ {}_2F_1(a_1,b_1;\ b_1+b_2;\ x)\,[2\gamma+\psi(a_2)+\psi(b_1+b_2)+\log\rho y]$$

$$+\,o(1),\qquad(\rho\longrightarrow 0+).$$

(T4) $F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_1+b_1;\ 1-\rho x,1-\rho y,z)$

$$=\frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1)}\ \frac{a_2b_2}{b_1\ +\ 1}\ F^{1:3;1}_{1:2;0}\left[\begin{array}{l} a_2+1:\ b_2+1,\ 1,\ 1;\ b_1+1; \\ \\ b_1+2:\quad a_1+1,\ 2;\ —; \end{array}\ 1,z\right]$$

$$+\frac{\Gamma(a_2+b_2)}{\Gamma(a_2+2)\Gamma(b_2+2)}\,a_1(b_1+1)a_2b_2$$

$$\cdot\ F^{1:3;2}_{1:2;1}\left[\begin{array}{l} b_1+2:\ a_1+1,\ 1,\ 1;\ a_2+1,\ b_1; \\ \\ a_2+2:\quad b_2+2,\ \ 2;\qquad b_1+1; \end{array}\ 1,z\right]$$

$$-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1+1)}\, a_2\; F^{1:2;1}_{1:1;0}\left[\begin{matrix} a_2+1:\ b_1,\ 1;\ 1; \\ \\ 2:\ \ b_1+1;\ -; \end{matrix}\ z,z\right]$$

$$-\frac{\Gamma(a_2+b_2)}{\Gamma(a_2+1)\Gamma(b_2)}\; F^{1:2;1}_{1:1;0}\left[\begin{matrix} b_1:\ a_2,\ 1;\ 1; \\ \\ 1:\ \ a_2+1;\ -; \end{matrix}\ z,z\right]$$

$$-\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2+1)}\,(1-z)^{-b_1}$$

$$-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\,(1-z)^{-a_2}\,[2\gamma+\psi(a_1)+\psi(b_1)+\log\rho x]$$

$$-\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}\,(1-z)^{-b_1}\,[2\gamma+\psi(a_2)+\psi(b_2)+\log\rho y]+o(1),$$

$$(\rho\longrightarrow 0+;\quad a_2+b_2=a_1+b_1).$$

Remark. In each of the above asymptotic formulas, and in what follows, we find it to be convenient to write the denominator parameters of F_S and F_T only once.

Proof of (S1). If we substitute the relation (2.5) into the series (2.24), we obtain

(3.1) $F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_2+b_3;\ x,y,1-\rho)$

$$= \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(a_2+b_3)_{m+n}} \; {}_2F_1(a_2+n, b_3;\, a_2+b_3+m+n;\, 1-\rho) \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(a_2+b_3)_{m+n}} \left\{ \frac{\Gamma(m)\Gamma(a_2+b_3+m+n)}{\Gamma(a_2+m+n)\Gamma(b_3+m)} \sum_{k=0}^{m-1} \frac{(a_2+n)_k (b_3)_k}{(1-m)_k} \frac{\rho^k}{k!} \right.$$

$$+ \frac{\Gamma(a_2+b_3+m+n)}{\Gamma(a_2+n)\Gamma(b_3)} (-\rho)^m \sum_{k=0}^{\infty} \frac{(a_2+m+n)_k (b_3+m)_k}{k!\,(m+k)!} \rho^k$$

$$\left. \cdot \left[\psi(k+1) + \psi(m+k+1) - \psi(a_2+m+n+k) - \psi(b_3+m+k) - \log \rho\right] \right\} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= \frac{\Gamma(a_2+b_3)}{\Gamma(a_2)\Gamma(b_3)} \left\{ S_1 + S_2 + S_3 - S_4 - S_5 - S_6 \right\}.$$

By rounding the term multiplied by ρ into the order $\mathcal{O}(\rho)$, the series S_1 is calculated as follows:

$$(3.2) \qquad S_1 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{m-1} \frac{(a_1)_m (b_1)_m (b_2)_n (a_2)_{n+k} (b_3)_k \; (m-k-1)!}{(a_2)_{m+n} (b_3)_m \; m! \; n! \; k!} x^m y^n (-\rho)^k$$

$$= \frac{a_1 b_1}{a_2 b_3} x \sum_{p,n,k=0}^{\infty} \frac{(a_1+1)_{p+k} (b_1+1)_{p+k} (b_2)_n (a_2)_{n+k} (b_3)_k (1)_p}{(a_2+1)_{p+n+k} (b_3+1)_{p+k} (2)_{p+k} \; n! \; k!} x^{p+k} y^n (-\rho)^k$$

$$= \frac{a_1 b_1}{a_2 b_3} x \sum_{p,n=0}^{\infty} \frac{(a_1+1)_p (b_1+1)_p (b_2)_n (a_2)_n (1)_p}{(a_2+1)_{p+n} (b_3+1)_p (2)_p \; n!} x^p y^n + \mathcal{O}(\rho)$$

$$= \frac{a_1 b_1}{a_2 b_3} \times F^{0:4;2}_{1:2;0}\left[\begin{matrix} \text{——}: a_1+1,\ b_1+1,\ 1,\ 1;\ a_2,\ b_2; \\ a_2+1: \quad b_3+1,\ 2;\ \text{——}; \end{matrix}\ x, y\right]$$

$$+ O(\rho), \qquad (\rho \to 0+).$$

The series S_2 is written in the form:

$$(3.3) \quad S_2 = \sum_{m,n,k=0}^{\infty} \frac{(a_1)_m (b_1)_m (b_2)_n (a_2)_{m+n+k} (b_3)_{m+k}}{(a_2)_{m+n} (b_3)_m (1)_{m+k}\ m!\ n!\ k!} (-\rho x)^m y^n \rho^k \left[-\gamma + \sum_{\ell=0}^{k-1} \frac{(1)_\ell}{(2)_\ell}\right],$$

which, upon rounding the terms involving ρ into $O(\rho)$, becomes

$$(3.4) \qquad S_2 = -\gamma(1-y)^{-b_2} + O(\rho), \qquad (\rho \to 0+),$$

where the formula (2.10) is applied.

In a similar manner, we have

$$(3.5) \qquad S_6 = \log \rho \sum_{m,n,k=0}^{\infty} \frac{(a_1)_m (b_1)_m (b_2)_n (a_2)_{m+n+k} (b_3)_{m+k}}{(a_2)_{m+n} (b_3)_m (1)_{m+k}\ m!\ n!\ k!} (-\rho x)^m y^n \rho^k$$

$$= (1-y)^{-b_2} \log \rho + O(\rho \log \rho), \qquad (\rho \to 0+);$$

$$(3.6) \qquad S_4 = \sum_{m,n,k=0}^{\infty} \frac{(a_1)_m (b_1)_m (b_2)_n (a_2)_{m+n+k} (b_3)_{m+k}}{(a_2)_{m+n} (b_3)_m (1)_{m+k}\ m!\ n!\ k!} (-\rho x)^m y^n \rho^k$$

$$\cdot \left[\psi(a_2) + \frac{1}{a_2} \sum_{\ell=0}^{m+n+k-1} \frac{(a_2)_\ell}{(a_2+1)_\ell} \right]$$

$$= \psi(a_2) \sum_{n=0}^{\infty} \frac{(b_2)_n}{n!} y^n + \frac{1}{a_2} \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \frac{(b_2)_n (a_2)_\ell}{(a_2+1)_\ell \, n!} y^n + O(\rho)$$

$$= \psi(a_2)(1-y)^{-b_2} + \frac{b_2}{a_2} y \, F^{1:2;1}_{1:1;0} \left[\begin{matrix} b_2+1: a_2, \ 1; \ 1; \\ \\ 2: \ a_2+1; \ -; \end{matrix} \ y, y \right] + O(\rho), \quad (\rho \to 0+),$$

by virtue of (2.11).

Finally, the series S_3 and S_5 are calculated in the forms:

$$S_3 = -\gamma(1-y)^{-b_2} + O(\rho), \quad (\rho \to 0+); \tag{3.7}$$

$$S_5 = \psi(b_3)(1-y)^{-b_2} + O(\rho), \quad (\rho \to 0+). \tag{3.8}$$

Thus we obtain the result (S1).

Proof of (S2). By using the relation (2.24), we have

$$F_S(a_1, a_2, a_2, b_1, b_2, b_3; \ a_1+b_1; \ 1-\rho, y, z) \tag{3.9}$$

$$= \sum_{m,n=0}^{\infty} \frac{(a_2)_{m+n} (b_2)_m (b_3)_n}{(a_1+b_1)_{m+n} \, m! \, n!} \, {}_2F_1(a_1, b_1; \ a_1+b_1+m+n; \ 1-\rho) \, y^m z^n$$

$$= {}_2F_1(a_1, b_1; \ a_1+b_1; \ 1-\rho)$$

$$+\frac{a_2b_2}{a_1+b_1}\,y\sum_{m=0}^{\infty}\frac{(a_2+1)_m(b_2+1)_m}{(a_1+b_1+1)_m(2)_m}\,F(a_1,b_1;\,a_1+b_1+1+m;\,1-\rho)\,y^m$$

$$+\frac{a_2b_3}{a_1+b_1}\,z\sum_{m,n=0}^{\infty}\frac{(a_2+1)_{m+n}(b_2)_m(b_3+1)_n}{(a_1+b_1+1)_{m+n}\,m!\,(2)_n}$$

$${}_2F_1(a_1,b_1;\,a_1+b_1+1+m+n;\,1-\rho)\,y^mz^n.$$

Now apply the formulas (2.7) and (2.8), and we obtain

$$(3.10)\quad F_S(a_1,a_2,a_2,b_1,b_2,b_3;\,a_1+b_1;\,1-\rho,y,z)$$

$$=-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\,[2\gamma+\psi(a_1)+\psi(b_1)+\log\rho]$$

$$+\frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\,a_2b_2\,y\sum_{m=0}^{\infty}\frac{(a_2+1)_m(b_2+1)_m(1)_m}{(a_1+1)_m(b_1+1)_m(2)_m}\,y^m$$

$$+\frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\,a_2b_3\,z\sum_{m,n=0}^{\infty}\frac{(a_2+1)_{m+n}(b_2)_m(b_3+1)_n(1)_{m+n}}{(a_1+1)_{m+n}(b_1+1)_{m+n}(2)_n\,m!}\,y^mz^n$$

$$+\,o(1),\qquad(\rho\longrightarrow 0+),$$

which is just the relation (S2).

Proof of (S3). If we apply (2.25) and (2.19), we find that

$$(3.11)\quad F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_2+b_2+b_3;\ x,\ 1-\rho y,\ 1-\rho z)$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(b_1)_m}{(a_2+b_2+b_3)_m\ m!}\, x^m\ F_1(a_2,b_2,b_3;\ a_2+b_2+b_3+m;\ 1-\rho y,\ 1-\rho z)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a_1)_m(a_2)_n(b_1)_m(b_3)_n}{(a_2+b_2+b_3)_{m+n}\ m!\ n!}\ {}_2F_1(a_2+n,b_2+b_3+n;\ a_2+b_2+b_3+m+n;\ 1-\rho y)$$

$$\cdot\ x^m[\rho(y-z)]^n$$

$$= \left\{ \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} + \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \right\} \frac{(a_1)_m(a_2)_n(b_1)_m(b_3)_n}{(a_2+b_2+b_3)_{m+n}\ m!\ n!}$$

$$\cdot\ {}_2F_1(a_2+n,b_2+b_3+n;\ a_2+b_2+b_3+m+n;\ 1-\rho y)\ x^m[\rho(y-z)]^n$$

$$= S_1 + S_2.$$

Using the relations (2.5) and (2.6), the series S_1 and S_2 are represented as follows:

$$(3.12)\qquad S_1 = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{(a_1)_m(a_2)_n(b_1)_m(b_3)_n}{(a_2+b_2+b_3)_{m+n}\ m!\ n!}\ x^m[\rho(y-z)]^n$$

$$\cdot \left\{ \frac{\Gamma(m-n)\Gamma(a_2+b_2+b_3+m+n)}{\Gamma(a_2+m)\Gamma(b_2+b_3+m)} \sum_{k=0}^{m-n-1} \frac{(a_2+n)_k(b_2+b_3+n)_k}{k!\ (1-m+n)_k}\ (\rho y)^k \right.$$

$$+\frac{\Gamma(a_2+b_2+b_3+m+n)}{\Gamma(a_2+n)\Gamma(b_2+b_3+n)}(-\rho y)^{m-n}\sum_{k=0}^{\infty}\frac{(a_2+m)_k(b_2+b_3+m)_k}{k!\;(1)_{m-n+k}}(\rho y)^k$$

$$\left.\cdot\;[\psi(k+1)+\psi(m-n+k+1)-\psi(a_2+m+k)-\psi(b_2+b_3+m+k)-\log\rho y]\right\},$$

$$S_2=\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}\frac{(a_1)_m(a_2)_n(b_1)_m(b_3)_n}{(a_2+b_2+b_3)_{m+n}\;m!\;n!}\,x^m[\rho(y-z)]^n \tag{3.13}$$

$$\cdot\left\{\frac{\Gamma(n-m)\Gamma(a_2+b_2+b_3+m+n)}{\Gamma(a_2+n)\Gamma(b_2+b_3+n)}(\rho y)^{m-n}\sum_{k=0}^{n-m-1}\frac{(a_2+m)_k(b_2+b_3+m)_k}{k!\;(1-n+m)_k}(\rho y)^k\right.$$

$$+(-1)^{n-m}\frac{\Gamma(a_2+b_2+b_3+m+n)}{\Gamma(a_2+m)\Gamma(b_2+b_3+m)}\sum_{k=0}^{\infty}\frac{(a_2+n)_k(b_2+b_3+n)_k}{k!\;(1)_{n-m+k}}(\rho y)^k$$

$$\left.\cdot\;[\psi(k+1)+\psi(n-m+k+1)-\psi(a_2+n+k)-\psi(b_2+b_3+n+k)-\log\rho y]\right\}.$$

The series S_1 is further partitioned into six series:

$$S_1=\frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3)}\left\{\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{k=0}^{p-1}\right. \tag{3.14}$$

$$\cdot\;\frac{(a_1)_{p+n}(b_1)_{p+n}(b_3)_n(a_2)_{n+k}(b_2+b_3)_{n+k}(p-k-1)!}{(1)_{p+n}(a_2)_{p+n}(b_2+b_3)_{p+n}(b_2+b_3)_n\;n!\;k!}$$

$$\cdot\;x^{p+n}[\rho(y-z)]^n(-\rho y)^k$$

$$+\sum_{n,p,k=0}^{\infty}\frac{(a_1)_{n+p}(b_1)_{n+p}(b_3)_n(a_2)_{n+p+k}(b_2+b_3)_{n+p+k}}{(1)_{n+p}(b_2+b_3)_n(a_2)_{n+p}(b_2+b_3)_{n+p}(1)_{p+k}\;n!\;k!}$$

$$\cdot\, x^{p+n}[\rho(y-z)]^n(-\rho y)^p(\rho y)^k$$

$$\left.\cdot\,[\psi(k+1)+\psi(p+k+1)-\psi(a_2+p+n+k)-\psi(b_2+b_3+p+n+k)-\log \rho y]\right\}\Bigg]$$

$$=\frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3)}\{S_{11}+S_{12}+S_{13}-S_{14}-S_{15}-S_{16}\}.$$

Again, by rounding the terms involving ρ into the order $\mathcal{O}(\rho)$, each of the series S_{11} to S_{16} is calculated as follows:

$$(3.15)\qquad S_{11}=\frac{a_1b_1}{a_1(b_2+b_3)}\,x\sum_{n,p=0}^{\infty}\;\sum_{k=0}^{p}$$

$$\cdot\,\frac{(a_1+1)_{p+n}(b_1+1)_{p+n}(b_3)_n(a_2)_{n+k}(b_2+b_3)_{n+k}(p-k)!}{(2)_{p+n}(a_2+1)_{p+n}(b_2+b_3+1)_{p+n}(b_2+b_3)_n\;n!\;k!}\,x^{p+n}[\rho(y-z)]^n(-\rho y)^k$$

$$=\frac{a_1b_1}{a_2(b_2+b_3)}\,x\sum_{q=0}^{\infty}\frac{(a_1+1)_q(b_1+1)_q(1)_q}{(2)_q(a_2+1)_q(b_2+b_3+1)_q}\,x^q+\mathcal{O}(\rho)$$

$$=\frac{a_1b_1}{a_2(b_2+b_3)}\,x\;{}_4F_3\left[\begin{matrix}a_1+1,\ b_1+1,\ 1,\ 1;\\ \\ a_2+1,\ b_2+b_3+1,\ 2;\end{matrix}\ x\right]+\mathcal{O}(\rho),\qquad(\rho\longrightarrow 0+);$$

$$(3.16) \qquad S_{12} = \sum_{n,p,k=0}^{\infty} \frac{(a_1)_{n+p}(b_1)_{n+p}(b_3)_n(a_2)_{n+p+k}(b_2+b_3)_{n+p+k}}{(1)_{n+p}(b_2+b_3)_n(a_2)_{n+p}(b_2+b_3)_{n+p}(1)_{p+k}\, n!\; k!}$$

$$\cdot \left[-\gamma + \sum_{q=0}^{k-1} \frac{(1)_q}{(2)_q}\right] [\rho x(y-z)]^n(-\rho xy)^p(\rho y)^k$$

$$= -\gamma + \mathcal{O}(\rho), \qquad (\rho \longrightarrow 0+);$$

$$(3.17) \qquad S_{13} = -\gamma + \mathcal{O}(\rho), \qquad (\rho \longrightarrow 0+);$$

$$(3.18) \qquad S_{14} = \psi(a_2) + \mathcal{O}(\rho), \qquad (\rho \longrightarrow 0+);$$

$$(3.19) \qquad S_{15} = \psi(b_2+b_3) + \mathcal{O}(\rho), \qquad (\rho \longrightarrow 0+);$$

$$(3.20) \qquad S_{16} = \log \rho y + \mathcal{O}(\rho \log \rho), \qquad (\rho \longrightarrow 0+),$$

by virtue of the relation (2.11).

The series S_2 also has the form:

$$(3.21) \quad S_2 = \frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3+1)}\, b_3\left[1-\frac{z}{y}\right]\ {}_3F_2\left[\begin{matrix} b_3+1,\ 1,\ 1; \\ \\ b_2+b_3+1,\ 2; \end{matrix}\ 1-\frac{z}{y}\right] + \mathcal{O}(\rho), \quad (\rho \longrightarrow 0+).$$

Thus we have the formula (S3).

Proof of (S4). Setting $a_1+b_1 = a_2+b_2$ and writing $1-\rho y$ instead of y in (S2), we have

$$(3.22)\quad F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_1+b_1;\ 1-\rho x,\ 1-\rho y,\ z)$$

$$= \frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\, a_2 b_2\ {}_4F_3\left[\begin{matrix} a_2+1,\ b_2+1,\ 1,\ 1; \\ \\ a_1+1,\ b_1+1,\ \ 2; \end{matrix}\ 1-\rho y\right]$$

$$+ \frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\, a_2 b_3 z\ F^{2:1;2}_{2:0;1}\left[\begin{matrix} a_2+1,\ 1:\ b_2;\ b_3+1,\ 1; \\ \\ a_1+1,\ b_1+1:\ \text{—};\qquad 2; \end{matrix}\ 1-\rho y, z\right]$$

$$- \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\,[2\gamma + \psi(a_1) + \psi(b_1) + \log \rho x] + o(1), \qquad (\rho \to 0+).$$

Here ${}_4F_3$ and $F^{2:1;2}_{2:0;1}$ in (3.22) are calculated as follows by using the formulas (2.14), (2.16), and (2.13):

$$(3.23)\quad {}_4F_3 = -\frac{\Gamma(a_1+1)\Gamma(b_1+1)}{\Gamma(a_2+1)\Gamma(b_2+1)}\,[2\gamma + \psi(a_2+1) + \psi(b_2+1) + \log \rho y]$$

$$+ \frac{\Gamma(a_1+1)\Gamma(b_1+2)}{\Gamma(a_2+2)\Gamma(b_2+2)}\, a_1\ {}_4F_3\left[\begin{matrix} a_1+1,\ b_1+2,\ 1,\ 1; \\ \\ a_2+2,\ b_2+2,\quad 2; \end{matrix}\ 1\right]$$

$$+ \frac{b_1}{b_1+1}\ {}_4F_3\left[\begin{matrix} a_2+1,\ b_2+1,\ 1,\ 1; \\ \\ a_1+1,\ b_1+2,\quad 2; \end{matrix}\ 1\right] + o(1), \qquad (\rho \to 0+);$$

$$(3.24)\quad F^{2:1;2}_{2:0;1} = \sum_{m=0}^{\infty} \frac{(a_2+1)_m(b_3+1)_m(1)_m}{(a_1+1)_m(b_1+1)_m(2)_m}\ {}_3F_2\left[\begin{matrix} a_2+1+m,\ 1+m,\ b_2; \\ \\ a_1+1+m,\ \ b_1+1+m; \end{matrix}\ 1-\rho y\right] z^m$$

$$= -\frac{\Gamma(a_1+1)\Gamma(b_1+1)}{\Gamma(a_2+1)\Gamma(b_2)} \sum_{m=0}^{\infty} \frac{(b_3+1)_m}{(2)_m}\Big[2\gamma + \psi(a_2) + \psi(b_2) + \log \rho y$$

$$+ \frac{1}{a_2}\sum_{k=0}^{m} \frac{(a_2)_k}{(a_2+1)_k}\Big] z^m$$

$$+ \frac{\Gamma(a_1+1)\Gamma(b_1+1)}{\Gamma(a_2+2)\Gamma(b_2+1)}\, a_1 b_1 \sum_{m=0}^{\infty} \frac{(a_2+1)_m(b_3+1)_m}{(a_2+2)_m(2)_m}$$

$$\cdot\ {}_4F_3\left[\begin{matrix} a_1+1,\ b_1+1,\ 1,\ 1; \\ \\ a_2+2+m,\ b_2+1,\ \ 2; \end{matrix}\ 1\right] z^m + o(1)$$

$$= -\frac{\Gamma(a_1+1)\Gamma(b_1+1)}{\Gamma(a_2+1)\Gamma(b_2)}\ {}_2F_1(1,b_3+1;2;z)\,[2\gamma + \psi(a_2) + \psi(b_2) + \log \rho y]$$

$$-\frac{\Gamma(a_1+1)\Gamma(b_1+1)}{\Gamma(a_2+1)\Gamma(b_2)}\ \frac{1}{a_2}\ F^{1:2;1}_{1:1;0}\left[\begin{matrix} b_3+1:\ a_2,\ 1;\ 1; \\ \\ 2:\ \ a_2+1;\ -; \end{matrix}\ z,z\right]$$

$$+\frac{\Gamma(a_1+1)\Gamma(b_1+1)}{\Gamma(a_2+2)\Gamma(b_2+1)}\, a_1 b_1$$

$$\cdot\ F^{0:4;3}_{1:2;1}\left[\begin{matrix} \text{—}:\ a_1+1,\ b_1+1,\ 1,\ 1;\ a_2+1,\ b_3+1,\ 1; \\ \\ a_2+2:\qquad b_2+1,\ 2;\qquad 2; \end{matrix}\ 1,z\right]$$

$$+ o(1), \qquad (\rho \to 0+).$$

Finally, by using the relation (2.3), we have

$$(3.25) \qquad {}_2F_1(1,\ b_3+1;\ 2;\ z) = \frac{1}{b_3 z}\left\{(1-z)^{-b_3} - 1\right\},$$

and the proof of the formula (S4) is completed.

Proof of (S5). If we put $a_1 + b_1 = a_2 + b_2 + b_3$ and replace x by $1 - \rho x$ in the relation (S3), we have

$$(3.26) \quad F_S(a_1,a_2,a_2,b_1,b_2,b_3;\ a_1+b_1;\ 1-\rho x,\ 1-\rho y, 1-\rho z)$$

$$= \frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2+1)\Gamma(b_2+b_3+1)}\, a_1 b_1\ {}_4F_3\left[\begin{matrix} a_1+1,\ b_1+1,\ 1,\ 1; \\ \\ a_2+1,\ b_2+b_3+1,\ 2; \end{matrix}\ 1-\rho x\right]$$

$$+ \frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3+1)}\, b_3\left[1 - \frac{z}{y}\right]\ {}_3F_2\left[\begin{matrix} b_3+1,\ 1,\ 1; \\ \\ b_2+b_3+1,\ 2; \end{matrix}\ 1 - \frac{z}{y}\right]$$

$$- \frac{\Gamma(a_2+b_2+b_3)}{\Gamma(a_2)\Gamma(b_2+b_3)}\,[2\gamma + \psi(a_2) + \psi(b_2+b_3) + \log \rho y] + o(1), \quad (\rho \to 0+).$$

The series ${}_4F_3$ occurring in (3.26) may be rewritten in the form:

$$(3.27)\qquad {}_4F_3 = -\frac{\Gamma(a_2+1)\Gamma(b_2+b_3+1)}{\Gamma(a_1+1)\Gamma(b_1+1)}\,[2\gamma + \psi(a_1+1) + \psi(b_1+1) + \log \rho x]$$

$$+\frac{\Gamma(a_2+1)\Gamma(b_2+b_3+2)}{\Gamma(a_1+2)\Gamma(b_1+2)}\,a_2\;{}_4F_3\left[\begin{matrix} a_2+1,\ b_2+b_3+2,\ 1,\ 1; \\ \\ a_1+2,\ b_1+2,\ 2; \end{matrix}\ 1\right]$$

$$+\frac{b_2+b_3}{b_2+b_3+1}\;{}_4F_3\left[\begin{matrix} a_1+1,\ b_1+1,\ 1,\ 1; \\ \\ a_2+1,\ b_2+b_3+2,\ 2; \end{matrix}\ 1\right] + o(1), \qquad (\rho \to 0+),$$

in view of (2.14). The formula (S5) follows readily from (3.26) and (3.27).

Proof of (T1). If we take into account the formulas (2.28) and (2.4), we have

$$(3.28)\quad F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_2+b_1;\ x,\ y,\ 1-\rho)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(a_2+b_1)_{m+n}\ m!\ n!}\ {}_2F_1(a_2+n, b_1+m;\ a_2+b_1+m+n;\ 1-\rho)\ x^m y^n$$

$$= \frac{\Gamma(a_2+b_1)}{\Gamma(a_2)\Gamma(b_1)} \sum_{m,n,k=0}^{\infty} \frac{(a_2)_{n+k}(b_1)_{m+k}(a_1)_m(b_2)_n}{(a_2)_n(b_1)_m\ m!\ n!\ (k!)^2}\, x^m y^n \rho^k$$

$$\cdot\ [2\psi(k+1) - \psi(a_2+n+k) - \psi(b_1+m+k) - \log \rho]$$

$$= \frac{\Gamma(a_2+b_1)}{\Gamma(a_2)\Gamma(b_1)}\,\{2S_1 - S_2 - S_3 - S_4\},$$

where the series S_1 to S_4 are represented as follows:

$$(3.29) \qquad S_1 = \sum_{m,n,k=0}^{\infty} \frac{(a_2)_{n+k}(b_1)_{m+k}(a_1)_m(b_2)_n}{(a_2)_n(b_1)_m \; m! \; n! \; (k!)^2} x^m y^n \rho^k \left[-\gamma + \sum_{\ell=0}^{k-1} \frac{(1)_\ell}{(2)_\ell}\right]$$

$$= -\gamma(1-x)^{-a_1}(1-y)^{-b_2} + O(\rho), \qquad (\rho \to 0+);$$

$$(3.30) \qquad S_2 = \sum_{m,n,k=0}^{\infty} \frac{(a_2)_{n+k}(b_1)_{m+k}(a_1)_m(b_2)_n}{(a_2)_n(b_1)_m \; m! \; n! \; (k!)^2} x^m y^n \rho^k$$

$$\cdot \left[\psi(a_2) + \frac{1}{a_2} \sum_{\ell=0}^{n+k-1} \frac{(a_2)_\ell}{(a_2+1)_\ell}\right]$$

$$= \psi(a_2) \sum_{m,n=0}^{\infty} \frac{(a_1)_m(b_2)_n}{m! \; n!} x^m y^n$$

$$+ \frac{1}{a_2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \frac{(a_1)_m(b_2)_n(a_2)_\ell}{(a_2+1)_\ell \; m! \; n!} x^m y^n + O(\rho)$$

$$= \psi(a_2)\,(1-x)^{-a_1}(1-y)^{-b_2}$$

$$+ \frac{b_2}{a_2} y(1-x)^{-a_1} F^{1:2;1}_{1:1;0}\left[\begin{matrix} b_2+1: \; a_2, \; 1; \; 1; \\ \\ 2: \; a_2+1; \; -; \end{matrix} \; y, y\right] + O(\rho), \qquad (\rho \to 0+).$$

Similar calculations for S_3 and S_4 lead to the relation (T1).

Proof of (T2). The formulas (2.30) and (2.5) imply

$$(3.31)\quad F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_1+b_1;\ 1-\rho,y,z)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a_2)_{m+n}(b_2)_m(b_1)_n}{(a_1+b_1)_{m+n}\ m!\ n!}\ {}_2F_1(a_1,b_1+n;\ a_1+b_1+m+n;\ 1-\rho)\ y^m z^n$$

$$= \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} \sum_{m,n=0}^{\infty} \sum_{k=0}^{m} \frac{(a_2)_{m+n+1}(b_2)_{m+1}(1)_m(a_1)_k(b_1)_{n+k}}{(2)_m(a_1)_{m+1}(b_1)_{m+n+1}(-m)_k\ n!\ k!}\, y^{m+1}\ z^n \rho^k$$

$$+ \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} \sum_{m,n,k=0}^{\infty} \frac{(b_2)_m(a_2)_{m+n}(a_1)_{m+k}(b_1)_{m+n+k}}{(a_1)_m(b_1)_{m+n}\ k!\ (m+k)!\ m!\ n!}\,(-\rho y)^m\ z^n \rho^k$$

$$\cdot\ [\psi(k+1) + \psi(m+k+1) - \psi(a_1+m+k) - \psi(b_1+n+m+k) - \log\rho]$$

$$= \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\{S_1 + S_2 + S_3 - S_4 - S_5 - S_6\}.$$

The series S_1, S_2, and S_6 are written in the forms:

$$(3.32)\qquad S_1 = \frac{a_2b_2}{a_1b_1}\,y \sum_{m,n=0}^{\infty} \sum_{k=0}^{m} \frac{(a_2+1)_{m+n}(b_1)_{n+k}(b_2+1)_m(a_1)_k(m-k)!}{(b_1+1)_{m+n}(a_1+1)_m(2)_m\ n!\ k!}\, y^m z^n (-\rho)^k$$

$$= \frac{a_2b_2}{a_1b_1}\,y \sum_{m,n=0}^{\infty} \frac{(a_2+1)_{m+n}(b_1)_n(b_2+1)_m(1)_m}{(b_1+1)_{m+n}(a_1+1)_m(2)_m\ n!}\, y^m z^n + O(\rho)$$

$$= \frac{a_2 b_2}{a_1 b_1}\, y\, F^{1:3;1}_{1:2;0}\left[\begin{matrix} a_2+1: & b_2+1,\ 1,\ 1; & b_1; \\ & & \\ b_1+1: & a_1+1,\ 2; & —; \end{matrix}\ y,z\right] + \mathcal{O}(\rho), \qquad (\rho \longrightarrow 0+);$$

$$\text{(3.33)} \qquad S_2 = \sum_{n,m,k=0}^{\infty} \frac{(b_1)_{n+m+k}(a_2)_{n+m}(a_1)_{m+k}(b_2)_m}{(b_1)_{n+m}(1)_{m+k}(a_1)_m\ n!\ m!\ k!} z^n(-\rho y)^m \rho^k \left[-\gamma + \sum_{\ell=0}^{k-1} \frac{(1)_\ell}{(2)_\ell}\right]$$

$$= -\gamma(1-z)^{-a_2} + \mathcal{O}(\rho), \qquad (\rho \longrightarrow 0+);$$

$$\text{(3.34)} \qquad S_5 = \sum_{n,m,k=0}^{\infty} \frac{(b_1)_{n+m+k}(a_2)_{n+m}(a_1)_{m+k}(b_2)_m}{(b_1)_{n+m}(1)_{m+k}(a_1)_m\ n!\ m!\ k!} z^n(-\rho y)^m \rho^k$$

$$\cdot \left[\psi(b_1) + \frac{1}{b_1} \sum_{\ell=0}^{n+m+k-1} \frac{(b_1)_\ell}{(b_1+1)_\ell}\right]$$

$$= \psi(b_1)(1-z)^{-a_2} + \frac{a_2}{b_1} z \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \frac{(a_2+1)_n (b_1)_\ell}{(2)_n (b_1+1)_\ell} z^n + \mathcal{O}(\rho), \qquad (\rho \longrightarrow 0+).$$

The series S_3, S_4, and S_6 are similarly calculated. Thus we obtain the relation (T2).

Proof of (T3). If we apply the relations (2.31), (2.19), and (2.6), we obtain

$$\text{(3.35)} \qquad F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_2+b_1+b_2;\ x,1-\rho y,1-\rho z)$$

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m(b_1)_m}{(a_2+b_1+b_2)_m\ m!} F_1(a_2,b_2,b_1+m;\ a_2+b_1+b_2+m;\ 1-\rho y,\ 1-\rho z)\, x^m$$

$$= \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_{m+n}}{(a_2+b_1+b_2)_{m+n}\, m!\; n!}$$

$$\cdot\ {}_2F_1(a_2+n, b_1+b_2+m+n;\ a_2+b_1+b_2+m+n;\ 1-\rho y)\ x^m [\rho(y-z)]^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_{m+n}}{(a_2+b_1+b_2)_{m+n}\, m!\; n!}\, x^m [\rho(y-z)]^n$$

$$\cdot \left\{ \frac{\Gamma(n)\Gamma(a_2+b_1+b_2+m+n)}{\Gamma(a_2+n)\Gamma(b_1+b_2+m+n)} (\rho y)^{-n} \sum_{k=0}^{n-1} \frac{(a_2)_k (b_1+b_2+m)_k}{k!\; (1-n)_k} (\rho y)^k \right.$$

$$+ (-1)^n \frac{\Gamma(a_2+b_1+b_2+m+n)}{\Gamma(a_2)\Gamma(b_1+b_2+m)} \sum_{k=0}^{\infty} \frac{(a_2+n)_k (b_1+b_2+m+n)_k}{k!\; (1)_{n+k}} (\rho y)^k$$

$$\left. \cdot\ [\psi(k+1) + \psi(n+k+1) - \psi(a_2+n+k) - \psi(b_1+b_2+m+n+k) - \log \rho y] \right\}$$

$$= \frac{\Gamma(a_2+b_1+b_2)}{\Gamma(a_2)\Gamma(b_1+b_2)} \{S_1 + S_2 + S_3 - S_4 - S_5 - S_6\},$$

where the series S_1 to S_6 may be written in the forms:

$$(3.36) \qquad S_1 = \frac{b_1}{b_1+b_2} \left[1 - \frac{z}{y}\right] \sum_{m,n=0}^{\infty} \sum_{k=0}^{n} \frac{(a_1)_m (b_1+1)_{m+n} (a_2)_k (b_1+b_2)_{m+k} (n-k)!}{(b_1+b_2+1)_{m+n} (b_1+b_2)_m (2)_n\, m!\; k!}$$

$$\cdot\ x^m \left[1 - \frac{z}{y}\right]^n (-\rho y)^k$$

$$= \frac{b_1}{b_1+b_2}\left[1-\frac{z}{y}\right] F^{1:1;2}_{1:0;1}\left[\begin{matrix} b_1+1:\ a_1;\ 1,\ 1; \\ \\ b_1+b_2+1:\ —;\quad 2; \end{matrix}\ x,\ 1-\frac{z}{y}\right] + \mathcal{O}(\rho), \quad (\rho \to 0+);$$

(3.37)
$$S_2 = \sum_{m,n,k=0}^{\infty} \frac{(a_1)_m (a_2)_{n+k} (b_1)_{m+n} (b_1+b_2)_{m+n+k}}{(b_1+b_2)_m (b_1+b_2)_{m+n} (1)_{n+k}\ m!\ n!\ k!} x^m [\rho(z-y)]^n (\rho y)^k$$

$$\cdot \left[-\gamma + \sum_{\ell=0}^{k-1} \frac{(1)_\ell}{(2)_\ell}\right]$$

$$= -\gamma\ {}_2F_1(a_1, b_1;\ b_1+b_2;\ x) + \mathcal{O}(\rho), \qquad (\rho \to 0+);$$

(3.38)
$$S_5 = \sum_{m,n,k=0}^{\infty} \frac{(a_1)_m (a_2)_{n+k} (b_1)_{m+n} (b_1+b_2)_{m+n+k}}{(b_1+b_2)_m (b_1+b_2)_{m+n} (1)_{n+k}\ m!\ n!\ k!} x^m [\rho(z-y)]^n (\rho y)^k$$

$$\cdot \left[\psi(b_1+b_2) + \frac{1}{b_1+b_2} \sum_{\ell=0}^{m+n+k-1} \frac{(b_1+b_2)_\ell}{(b_1+b_2+1)_\ell}\right]$$

$$= \psi(b_1+b_2)\ {}_2F_1(a_1, b_1;\ b_1+b_2;\ x)$$

$$+ \frac{a_1 b_1}{(b_1+b_2)^2}\ x\ F^{2:2;1}_{2:1;0}\left[\begin{matrix} a_1+1,\ b_1+1:\ b_1+b_2,\ 1;\ 1; \\ \\ b_1+b_2+1,\ 2:\ b_1+b_2+1;\ -; \end{matrix}\ x, x\right]$$

$$+ \mathcal{O}(\rho), \quad (\rho \to 0+).$$

The series S_3, S_4, and S_6 may be calculated similarly. Thus, by gathering these results, we have the formula (T3).

Proof of (T4). If we set $a_2 + b_2 = a_1 + b_1$ in the formula (T2), we have

$$(3.39)\quad F_T(a_1,a_2,a_2,b_1,b_2,b_1;\ a_1+b_1;\ 1-\rho x,\ 1-\rho y, z)$$

$$= -\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}(1-z)^{-a_2}\,[2\gamma + \psi(a_1) + \psi(b_1) + \log\rho x]$$

$$+\frac{\Gamma(a_1+b_1)}{\Gamma(a_1+1)\Gamma(b_1+1)}a_2b_2\ F^{1:3;1}_{1:2;0}\left[\begin{array}{lr} a_2+1:\ b_2+1,\ 1,\ 1;\ b_1; & \\ & 1-\rho y, z \\ b_1+1:\quad a_1+1,\ 2;\ —; & \end{array}\right]$$

$$-\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1+1)}a_2\ F^{1:2;1}_{1:1;0}\left[\begin{array}{lr} a_2+1:\ b_1,\ 1;\ 1; & \\ & z,z \\ 2:\ b_1+1;\ -; & \end{array}\right] + o(1),\quad (\rho \to 0+).$$

Now, by virtue of the formulas (2.16) and (2.14), the series $F^{1:3;1}_{1:2;0}$ occurring in (3.39) is represented by

$$(3.40)\qquad F^{1:3;1}_{1:2;0} = \sum_{m=0}^{\infty}\frac{(a_2+1)_m(b_1)_m}{(b_1+1)_m\ m!}\ {}_4F_3\left[\begin{array}{lr} a_2+1+m,\ b_2+1,\ 1,\ 1; & \\ & 1-\rho y \\ b_1+1+m,\ a_1+1,\ 2; & \end{array}\right] z^m$$

$$= -\frac{\Gamma(a_1+1)\Gamma(b_1+1)}{\Gamma(a_2+1)\Gamma(b_2+1)}\sum_{m=0}^{\infty}\frac{(b_1)_m}{m!}\Bigg[2\gamma + \psi(a_2) + \psi(b_2) + \log\rho y$$

$$+\frac{1}{a_2}\sum_{k=0}^{m}\frac{(a_2)_k}{(a_2+1)_k} + \frac{1}{b_2}\Bigg] z^m$$

$$+\frac{\Gamma(a_1+1)\Gamma(b_1+2)}{\Gamma(a_2+2)\Gamma(b_2+2)}\,a_1\sum_{m=0}^{\infty}\frac{(a_2+1)_m(b_1+2)_m(b_1)_m}{(a_2+2)_m(b_1+1)_m\; m!}$$

$$\cdot\,{}_4F_3\left[\begin{matrix} b_1+2+m, & a_1+1, & 1, & 1; & \\ & & & & 1 \\ a_2+2+m, & b_2+2, & 2; & & \end{matrix}\right] z^m$$

$$+\frac{b_1}{b_1+1}\sum_{m=0}^{\infty}\frac{(a_2+1)_m(b_1+1)_m}{(b_1+2)_m\; m!}\;{}_4F_3\left[\begin{matrix} a_2+1+m, & b_2+1, & 1, & 1; & \\ & & & & 1 \\ b_1+2+m, & a_1+1, & 2; & & \end{matrix}\right] z^m + o(1),$$

$$(\rho \longrightarrow 0+).$$

Thus the relation (T4) is obtained.

4. APPLICATIONS AND FURTHER EXTENSIONS

The various asymptotic formulas for the triple hypergeometric series F_S and F_T, which we have presented in this paper, are potentially useful in a wide variety of problems in several seemingly diverse fields of physical and engineering sciences, and indeed also of statistics and operations research. For the purpose of illustration of the usefulness of our results, we choose to mention here some areas of applications which lead naturally to the triple hypergeometric series F_S and F_T. We shall also indicate the possibility of unifying (and generalizing) our work to hold true for a much wider class of triple hypergeometric series.

First of all, in the theory of differential equations, it is not difficult to verify that the solutions of the system of second-order partial differential equations (*cf.* [15] and [16]):

(4.1a)
$$\left\{x\frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\gamma-1\right]\right.$$
$$\left.-x\left[x\frac{\partial}{\partial x}+\alpha_1\right]\left[x\frac{\partial}{\partial x}+\beta_1\right]\right\}W=0,$$

(4.1b)
$$\left\{y\frac{\partial}{\partial y}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\gamma-1\right]\right.$$
$$\left.-y\left[y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\alpha_2\right]\left[y\frac{\partial}{\partial y}+\beta_2\right]\right\}W=0,$$

(4.1c)
$$\left\{z\frac{\partial}{\partial z}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\gamma-1\right]\right.$$
$$\left.-z\left[y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\alpha_2\right]\left[z\frac{\partial}{\partial z}+\beta_3\right]\right\}W=0,$$

are expressible in terms of the triple hypergeometric series F_S defined by (2.20). Similarly, the solutions of the system of partial differential equations (*cf.* [15] and [16])

(4.2a)
$$\left\{x\frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\gamma-1\right]\right.$$
$$\left.-x\left[x\frac{\partial}{\partial x}+\alpha_1\right]\left[x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}+\beta_1\right]\right\}W=0,$$

(4.2b)
$$\left\{x\frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\gamma-1\right]\right.$$
$$\left.-y\left[y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\alpha_2\right]\left[y\frac{\partial}{\partial y}+\beta_2\right]\right\}W=0,$$

$$(4.3c)\qquad \left\{x\frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\gamma-1\right]\right.$$
$$\left. -z\left[y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+\alpha_2\right]\left[x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}+\beta_1\right]\right\}W=0,$$

can be expressed in terms of the triple hypergeometric series F_T defined by (2.21). Thus the asymptotic formulas of Section 3 are applicable whenever the behaviors of these solutions near one or the other of the various singular boundary points of the unit cube Ω are needed in a detailed study of the above systems of partial differential equations.

In order to describe another substantially different (but relevant) situation leading to a rather conspicuous presence of the triple hypergeometric series F_T, let the random variable U_j $(j=1,2)$ have a Gamma distribution with space parameter φ_j and scale parameter β_j. The ratio

$$(4.3)\qquad X=\frac{U_1}{U_2},$$

where U_1 and U_2 are independent, is called a generalized F variate, and the distribution of X is called a generalized F distribution and has the probability density function:

$$(4.4)\qquad f(x)=\frac{\alpha^p}{B(p,m-p)}\,\frac{x^{p-1}}{(1+\alpha x)^m}$$

$$(x>0;\ \alpha>0;\ m>p>0),$$

where

(4.5) $$\alpha = \frac{\beta_2}{\beta_1}, \quad p = \varphi_1, \quad m = \varphi_1 + \varphi_2,$$

and $B(\alpha,\beta)$ is the Beta function given, in terms of the Gamma functions, by

(4.6) $$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\beta,\alpha).$$

Following Dyer [2] we write

(4.7) $$X \sim \mathcal{GI}(p,m,\alpha)$$

to indicate the fact that a random variable X has its probability density function given by (4.4). The distribution function of such a random variable X is easily expressed in terms of the Gaussian hypergeometric series ${}_2F_1$ involved in the known asymptotic formulas (2.4), (2.5), (2.6), and (2.8). On the other hand, the distribution function of the random variable

(4.8) $$Y = X_1 + X_2 \quad (X_j \sim \mathcal{GI}(p_j,m_j,\alpha_j);\ j = 1,2),$$

where X_1 and X_2 are independent, is given by (*cf.* [2, p. 185]; see also [20, p. 52])

(4.9) $$F(y) = \Lambda \omega_1^{p_1} \omega_2^{p_2} F_T\Big[p_1-m_1+1, p_2, p_2, p_1, p_2-m_2+1, p_1;$$
$$p_1+p_2+1, p_1+p_2+1, p_1+p_2+1; \omega_1,\omega_2, 1-(1-\omega_1)(1-\omega_2)\Big],$$

where, for convenience,

$$(4.10) \qquad \Lambda = \frac{\Gamma(m_1)\Gamma(m_2)}{\Gamma(p_1+p_2+1)\Gamma(m_1-p_1)\Gamma(m_2-p_2)}$$

and

$$(4.11) \qquad \omega_j = \frac{\alpha_j y}{1+\alpha_j y} \qquad (j = 1,2).$$

For another instance of occurrence of a multiple hypergeometric series in probability theory, see (for example) Srivastava and Kashyap [21, p. 264].

Just as Srivastava's series H_C defined by (*cf.*, *e.g.*, [19, p. 99]; see also [3, p. 74])

$$(4.12) \qquad H_C(\alpha,\beta,\beta';\gamma;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta')_{n+p}}{(\gamma)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}$$

$$((x,y,z) \in \Omega),$$

the triple series F_S and F_T are two important elements of Lauricella's set of hypergeometric series in three variables (*cf.* [4, p. 114] and [20, p. 43]). Moreover, since [19, p. 112]

$$(4.13)\qquad H_C(\alpha,\beta,\beta';\,\gamma;\,x,y,z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\beta')_n}{(\gamma)_{2n}} \frac{(xy)^n}{n!}$$

$$\cdot\, F_T(\beta+n,\beta'+n,\beta'+n,\alpha+n,\beta+n,\alpha+n;\; \gamma+2n,\gamma+2n,\gamma+2n;\; x,y,z),$$

and since [23, p. 835]

$$(4.14)\qquad H_C(\alpha,\beta,\rho+\rho'-1;\,\gamma;\,x,y,z) = \frac{\Gamma(\rho)\Gamma(\rho')\Gamma(2-\rho-\rho')}{(2\pi i)^2}$$

$$\cdot \int_P (-\zeta)^{-\rho}(\zeta-1)^{-\rho'}\, F_T\left[\rho,\alpha,\alpha,\beta,\rho',\beta;\; \gamma;\; \frac{y}{\zeta}, \frac{z}{1-\zeta}, x\right] d\zeta,$$

where the contour of integration is Pochhammer's double-loop (1+,0+,1−,0−) described (for example) by Whittaker and Watson [24, p. 256], a knowledge of the asymptotic behaviors of F_T will facilitate a similarly detailed study of the widely occurring (and symmetrical) triple series H_C (see, amongst other places, [3], [19], [20], and [23]).

Finally, with a view to unifying (and generalizing) our work presented in this paper, as also in the earlier papers [10] and [11], we should consider Srivastava's general triple hypergeometric series $F^{(3)}[x,y,z]$ defined by (*cf.* [18, p. 428]; see also [3, p. 108] and [20, p. 44])

$$(4.15)\qquad F^{(3)}[x,y,z] = \sum_{m,n,p=0}^{\infty} \Phi(m,n,p)\, \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

where the coefficients $\Phi(m,n,p)$ contain quotients of essentially arbitrary number of Pochhammer symbols with subscripts

$$m+n+p,\ m+n,\ n+p,\ p+m,\ m,\ n,\ \text{and}\ p,$$

it being understood that the triple series in (4.15) converges absolutely. As already pointed out in the literature, the general triple hypergeometric series $F^{(3)}[x,y,z]$ provides a unification (and generalization) of Lauricella's fourteen hypergeometric series $F_1,\cdots,F_{14}$ in three variables (including, for example, F_7 or F_S *and* F_{13} or F_T), the additional series H_A, H_B, H_C introduced by Srivastava [19], and many other triple hypergeometric series studied by Srivastava and Karlsson [20, Chapter 3]. An investigation of the asymptotic behavior of $F^{(3)}[x,y,z]$, analogous to the work presented here, is expected to be fairly involved in the general case. Nevertheless, in view of the demonstrated importance of such asymptotic behaviors, it seems worthwhile to work toward a more general treatment for some wider classes of triple hypergeometric series, contained in $F^{(3)}[x,y,z]$ defined by (4.15), which would include the series F_S and F_T as particular cases.

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