

ORDINARY AND BASIC (OR q -) HYPERGEOMETRIC
FUNCTIONS AND ASSOCIATED ORTHOGONAL
POLYNOMIALS

by

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Abstract

In this survey article the author presents some important recent developments in the theory of ordinary and basic hypergeometric functions (of one and more variables) and associated orthogonal polynomials. Several additional references, relevant to the work presented here, are also provided.

1. Introduction

The object of this article is to present a brief survey of some recent works connected with such interesting special cases of the (ordinary) hypergeometric function ${}_rF_s$ as the classical orthogonal polynomials including, for example, the Hermite polynomials, the Jacobi polynomials (and, of course, the Gegenbauer or ultraspherical polynomials, the Legendre or spherical polynomials, and the Tchebycheff polynomials of the first and second kinds), and the Laguerre polynomials, and the various generalizations and discrete analogues of these polynomials. We shall also consider some recent developments in the theory of orthogonal polynomials associated similarly with the basic (or q -) hypergeometric function ${}_r\Phi_s$. Finally, we indicate a number of recent results involving ordinary and basic hypergeometric functions of one and more variables.

2. Definitions, Notations, and Preliminaries

Making use of the Pochhammer symbol $(\lambda)_n$ given by

$$(2.1) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, 3, \dots\}, \end{cases}$$

an ordinary hypergeometric function ${}_rF_s$ (with r numerator and s denominator parameters) is defined by

$$(2.2) \quad {}_rF_s \left[\begin{matrix} \alpha_1, \dots, \alpha_r; \\ \beta_1, \dots, \beta_s; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$

$$(\beta_j \neq 0, -1, -2, \dots; \quad j = 1, \dots, s),$$

where, for convergence, $r \leq s$ and $|z| < \infty$, or

$$r = s + 1 \quad \text{and} \quad z \in \mathcal{U} = \{z : |z| < 1\}.$$

Indeed, if we set

$$(2.3) \quad \omega = \sum_{j=1}^s \beta_j - \sum_{j=1}^r \alpha_j,$$

the series in (2.2), with $r = s + 1$, can be seen to be

- (i) absolutely convergent for $|z| = 1$ if $\text{Re}(\omega) > 0$;
 - (ii) conditionally convergent for $|z| = 1$ ($z \neq 1$) if $-1 < \text{Re}(\omega) \leq 0$;
- and
- (iii) divergent for $|z| = 1$ if $\text{Re}(\omega) \leq -1$.

Furthermore, since

$$(2.4) \quad (-N)_n = \begin{cases} \frac{(-1)^n N!}{(N-n)!}, & \text{if } n = 0, 1, \dots, N, \\ 0, & \text{if } n = N+1, N+2, N+3, \dots, \end{cases}$$

the series in (2.2) would terminate when one (or more) of the numerator parameters $\alpha_1, \dots, \alpha_r$ is zero or a negative integer, and the question of convergence will not arise. If, for example, one of $\alpha_1, \dots, \alpha_r$ is a nonpositive integer $-N$, and there

are no zeros in the denominator of (2.2), the function ${}_rF_s(z)$ would reduce to what may be called a *hypergeometric polynomial* of degree N in z . Some of the most important hypergeometric polynomials include the classical orthogonal polynomials, and their generalizations and discrete analogues, which we describe below.

Let us begin by considering an orthogonal system of functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ such that the inner product

$$(2.5) \quad (\varphi_m, \varphi_n) = \int_a^b \varphi_m(x) \varphi_n(x) d\mu(x) = \lambda_n \delta_{m,n}$$

$$(m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where $\delta_{m,n}$ is the Kronecker delta, (a, b) is a finite, one-sided infinite, or two-sided infinite interval on the real axis, and $d\mu(x)$ is a distribution along that interval. Here

$$(2.6) \quad \lambda_n = \|\varphi_n\|^2 = (\varphi_n, \varphi_n) \quad (n \in \mathbb{N}_0),$$

and $\mu(x)$ is a non-decreasing function; if $\mu(x)$ is absolutely continuous, we may set

$$\mu'(x) = w(x),$$

and refer to $w(x)$ as the weight function of the orthogonal system $\{\varphi_n(x)\}_{n=0}^{\infty}$.

The family of the classical orthogonal polynomials forms a special type of the orthogonal system $\{\varphi_n(x)\}_{n=0}^{\infty}$ defined by (2.5). This family is led by the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ which indeed are the most general of the three classes of orthogonal polynomials mentioned in Section 1. These polynomials can be defined by a Rodrigues type formula:

$$(2.7) \quad P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n (1-x)^{-\alpha} (1+x)^{-\beta}}{2^n n!}$$

$$\cdot D_x^n \left\{ (1-x)^{a+n} (1+x)^{\beta+n} \right\}, \quad D_x = \frac{d}{dx},$$

which may be employed to derive a number of explicit hypergeometric representations for the Jacobi polynomials. For example, we have

$$(2.8) \quad P_n^{(a, \beta)}(x) = \sum_{k=0}^n \begin{bmatrix} a+n \\ n-k \end{bmatrix} \begin{bmatrix} \beta+n \\ k \end{bmatrix} \left[\frac{x-1}{2} \right]^k \left[\frac{x+1}{2} \right]^{n-k},$$

or, equivalently,

$$(2.9) \quad P_n^{(a, \beta)}(x) = \begin{bmatrix} a+n \\ n \end{bmatrix} {}_2F_1 \left[\begin{matrix} -n, a+\beta+n+1; \\ a+1; \end{matrix} \frac{1-x}{2} \right].$$

When $\min\{\operatorname{Re}(a), \operatorname{Re}(\beta)\} > -1$, these polynomials are orthogonal with respect to the Beta distribution on $[-1, 1]$:

$$(2.10) \quad \int_{-1}^1 (1-x)^a (1+x)^\beta P_m^{(a, \beta)}(x) P_n^{(a, \beta)}(x) dx \\ = \frac{2^{a+\beta+1} \Gamma(a+n+1) \Gamma(\beta+n+1)}{n! (a+\beta+2n+1) \Gamma(a+\beta+n+1)} \delta_{m,n} \\ (m, n \in \mathbb{N}_0; \quad \min\{\operatorname{Re}(a), \operatorname{Re}(\beta)\} > -1).$$

Various other members of the family, which are *special* cases of the Jacobi polynomials, include the Gegenbauer (or ultraspherical) polynomials $C_n^\nu(x)$, where

$$(2.11) \quad C_n^{a+\frac{1}{2}}(x) = \begin{bmatrix} a+n \\ n \end{bmatrix}^{-1} \begin{bmatrix} 2a+n \\ n \end{bmatrix} P_n^{(a, a)}(x) \\ = \sum_{k=0}^n \frac{(a+\frac{1}{2})_k (a+\frac{1}{2})_{n-k}}{k! (n-k)!} e^{i(n-2k)\theta} \quad (x = \cos \theta),$$

the relatively more familiar Legendre (or spherical) polynomials:

$$(2.12) \quad P_n(x) = P_n^{(0,0)}(x) = C_n^{\frac{1}{2}}(x),$$

and the Tchebycheff polynomials (of the first and second kinds):

$$(2.13) \quad \begin{cases} T_n(x) = \begin{bmatrix} n-\frac{1}{2} \\ n \end{bmatrix}^{-1} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) = \frac{1}{2} n C_n^0(x), \\ U_n(x) = \frac{1}{2} \begin{bmatrix} n+\frac{1}{2} \\ n+1 \end{bmatrix}^{-1} P_n^{(\frac{1}{2}, \frac{1}{2})}(x) = C_n^1(x), \end{cases}$$

where, by definition,

$$(2.14) \quad C_n^0(x) = \lim_{\lambda \rightarrow 0} \left\{ \lambda^{-1} C_n^\lambda(x) \right\}.$$

Two other important members of the family of the classical orthogonal polynomials are the Hermite polynomials:

$$(2.15) \quad \begin{aligned} H_n(x) &= \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k! (n-2k)!} (2x)^{n-2k} \\ &= (2x)^n {}_2F_0 \left[\begin{matrix} -\frac{1}{2}n, & -\frac{1}{2}n+\frac{1}{2}; \\ & -\frac{1}{x^2} \end{matrix} \right] \end{aligned}$$

and the Laguerre polynomials:

$$(2.16) \quad L_n^{(a)}(x) = \sum_{k=0}^n \begin{bmatrix} a+n \\ n-k \end{bmatrix} \frac{(-x)^k}{k!}$$

$$= \begin{bmatrix} \alpha+n \\ n \end{bmatrix} {}_1F_1 \left[\begin{matrix} -n; \\ \alpha+1; \end{matrix} x \right].$$

Indeed, since

$$(2.17) \quad H_n(x) = (-1)^n 2^{n/2} n! \lim_{\alpha \rightarrow \infty} \left\{ \alpha^{-n/2} L_n^{(\alpha)}(\alpha + x\sqrt{2\alpha}) \right\}$$

and

$$(2.18) \quad L_n^{(\alpha)}(x) = \lim_{\beta \rightarrow \infty} \left\{ P_n^{(\alpha, \beta)} \left[1 - \frac{2x}{\beta} \right] \right\},$$

many of the properties of the Hermite and Laguerre polynomials can be deduced from those involving the classical Jacobi polynomials.

Yet another interesting class of orthogonal polynomials is provided by the generalized Bessel polynomials:

$$(2.19) \quad y_n(x, \alpha, \beta) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} \alpha+n+k-2 \\ k \end{bmatrix} k! \left[\frac{x}{\beta} \right]^k$$

$$= {}_2F_0 \left[\begin{matrix} -n, \alpha+n-1; \\ \hline \end{matrix} -\frac{x}{\beta} \right],$$

which were studied systematically by Krall and Frink [76] (and, more recently, by Grosswald [58]). In view of the relationships:

$$(2.20) \quad y_n(x, \alpha, \beta) = \lim_{\lambda \rightarrow \infty} \left\{ \frac{n!}{(\lambda)_n} P_n^{(\lambda-1, \alpha-\lambda-1)} \left[1 + \frac{2\lambda x}{\beta} \right] \right\}$$

and

$$(2.21) \quad y_n(x, \alpha, \beta) = n! \left[-\frac{x}{\beta} \right]^n L_n^{(1-\alpha-2n)} \left[\frac{\beta}{x} \right],$$

the Bessel polynomials are also recoverable from the classical Jacobi and Laguerre polynomials.

The classical Jacobi, Laguerre, and Hermite polynomials, and many of their aforementioned relatives, are often characterized by one or the other of a number of properties which they have in common. Given a system of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$, the three most important ones of these characteristics of the classical Jacobi, Laguerre, and Hermite polynomials may be recalled as follows:

PROPERTY 1 (Sonine [110], Hahn [59], and Krall [75]). *The derivatives $\{p'_n(x)\}_{n=1}^{\infty}$ form a system of orthogonal polynomials.*

PROPERTY 2 (Bochner [29]). *The polynomials $p_n(x)$ satisfy a differential equation of the form:*

$$(2.22) \quad A(x) \frac{d^2 y}{dx^2} + B(x) \frac{dy}{dx} + \lambda_n y = 0,$$

where $A(x)$ and $B(x)$ are independent of n , and λ_n is independent of x .

PROPERTY 3 (Tricomi [147]). *The polynomials $p_n(x)$ are given by a generalized Rodrigues formula [cf. Equation (2.7)]*

$$(2.23) \quad p_n(x) = \frac{1}{K_n w(x)} D_x^n \{w(x) [X(x)]^n\},$$

where K_n is a constant, $X(x)$ is a polynomial in x whose coefficients are independent of n , and $w(x)$ is independent of n .

There are a number of important instances of orthogonal polynomials of a *discrete* variable, that is, orthogonal polynomials for which the distribution function $\mu(x)$ in (2.5) is a *jump function*. A study of such orthogonal polynomials was initiated by Tchebycheff ([144], [145], and [146]) who also introduced a general system of orthogonal polynomials of a discrete variable. These polynomials (popularly known as Hahn polynomials) may be defined by (cf. Tchebycheff [146] and Hahn [60]; see also Karlin and McGregor [73])

$$(2.24) \quad \mathbf{Q}_n(x; \alpha, \beta, N) = {}_3F_2 \left[\begin{matrix} -n, \alpha + \beta + n + 1, -x; \\ \alpha + 1, -N; \end{matrix} \right]$$

and satisfy the (discrete) orthogonality relation:

$$(2.25) \quad \sum_{x=0}^N \begin{bmatrix} \alpha + x \\ x \end{bmatrix} \begin{bmatrix} \beta + N - x \\ N - x \end{bmatrix} \mathbf{Q}_m(x; \alpha, \beta, N) \mathbf{Q}_n(x; \alpha, \beta, N) = 0,$$

$$m \neq n \in \{0, 1, \dots, N\}.$$

The following limiting cases of the Hahn polynomials are worthy of note.

Krawtchouk polynomials:

$$(2.26) \quad \mathbf{K}_n(x; p, N) = {}_2F_1 \left[\begin{matrix} -n, -x; \\ -N; \end{matrix} \frac{1}{p} \right]$$

$$= \lim_{\alpha \rightarrow \infty} \left\{ \mathbf{Q}_n \left[x; \alpha, \frac{(1-p)\alpha}{p}, N \right] \right\};$$

Meixner polynomials:

$$(2.27) \quad \mathbf{M}_n(x; \beta, c) = (\beta)_n {}_2F_1 \left[\begin{matrix} -n, -x; \\ \beta; \end{matrix} 1 - \frac{1}{c} \right]$$

$$= (\beta)_n \lim_{N \rightarrow \infty} \left\{ \mathbf{Q}_n \left[x; \beta - 1, \frac{(1-c)N}{c}, N \right] \right\};$$

Poisson–Charlier polynomials:

$$(2.28) \quad c_n(x; a) = {}_2F_0 \left[\begin{matrix} -n, -x; \\ \text{---}; \end{matrix} -\frac{1}{a} \right]$$

$$= \lim_{N \rightarrow \infty} \left\{ Q_n(x; N-1, N^2 a^{-1}, N) \right\}.$$

All these orthogonal polynomials of a discrete variable are characterized by (for example) a finite–difference analogue of Property 3 above (see, *e.g.*, Hahn [60], and Weber and Erdélyi [148]).

Since

$$(2.29) \quad P_n^{(\alpha, \beta)}(1-2x) = \begin{bmatrix} \alpha+n \\ n \end{bmatrix} \lim_{N \rightarrow \infty} \left\{ Q_n(xN; \alpha, \beta, N) \right\},$$

some of the properties of the classical Jacobi polynomials can be deduced from those involving the Hahn polynomials defined by (2.24). We also note the relationships:

$$(2.30) \quad c_n(a; x) = n! \left[-\frac{1}{x} \right]^n L_n^{(a-n)}(x)$$

between Poisson–Charlier and Laguerre polynomials, and

$$(2.31) \quad M_n(x; \beta, c) = n! P_n^{(\beta-1, -\beta-n-x)} \left[\frac{2}{c} - 1 \right]$$

between Meixner and Jacobi polynomials.

One further set of orthogonal polynomials of the type described above is the set of Meixner–Pollaczek polynomials defined by (*cf.* Meixner [79] and Pollaczek [85])

$$(2.32) \quad P_n^{(\lambda)}(x; \varphi) = \begin{bmatrix} 2\lambda+n-1 \\ n \end{bmatrix} e^{in\varphi} {}_2F_1 \left[\begin{matrix} -n, \lambda+ix; \\ 2\lambda; \end{matrix} 1 - e^{-2i\varphi} \right]$$

$$(\lambda > 0; 0 < \varphi < \pi),$$

which satisfy the orthogonality property (2.5) with

$$(2.33) \quad -a = b = \infty \quad \text{and} \quad w(x) = \mu'(x) = e^{(2\varphi-\pi)x} |\Gamma(\lambda+ix)|^2.$$

The subject of orthogonal polynomials is treated, in a lucid and systematic manner, by Szegő [143] who also considered some of the aforementioned discrete orthogonal polynomials. Other useful references on this subject include Erdélyi *et al.* [49, Chapter 10], Rainville [96], Luke [77, Chapter 11], Chihara [41], and Nikiforov *et al.* ([82], [83], and [84]). In particular, Nikiforov and Uvarov (*cf.*, *e.g.*, [84, Section 13]) present a detailed construction of a theory of basic (or q -) extensions of the various discrete orthogonal polynomials including those mentioned above. The basic (or q -) extensions of the classical orthogonal polynomials and of the various generalizations and discrete analogues of these polynomials, are expressible in terms of the basic (or q -) hypergeometric function ${}_r\Phi_s$ defined by (*cf.* Bailey [23, Chapter 8], Slater [109, Chapter 3], Exton [50], and Srivastava and Karlsson [130, p. 347])

$$(2.34) \quad {}_r\Phi_s \left[\begin{matrix} a_1, \dots, a_r; \\ b_1, \dots, b_s; \end{matrix} \middle| q, z \right] \\ = \sum_{n=0}^{\infty} (-1)^{(1-r+s)n} q^{(1-r+s)n(n-1)/2} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \frac{z^n}{(q; q)_n},$$

where (and throughout the rest of this article) we find it to be convenient to write

$$(2.35) \quad (\lambda_1, \dots, \lambda_\kappa; q)_\mu = (\lambda_1; q)_\mu \cdots (\lambda_\kappa; q)_\mu,$$

and, for convergence of the infinite series in (2.34),

$$|q| < 1 \quad \text{and} \quad |z| < \infty \quad \text{when} \quad r \leq s$$

or

$$\max\{|q|, |z|\} < 1 \quad \text{when} \quad r = s + 1,$$

provided that no zeros appear in the denominator of (2.34). Here, for real or complex q ($|q| < 1$), we put

$$(2.36) \quad (\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j)$$

and let $(\lambda; q)_\mu$ be defined by

$$(2.37) \quad (\lambda; q)_\mu = \frac{(\lambda; q)_\infty}{(\lambda q^\mu; q)_\infty}$$

for arbitrary (real or complex) parameters λ and μ , so that [cf. Equation (2.1)]

$$(2.38) \quad (\lambda; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q) \cdots (1-\lambda q^{n-1}), & \text{if } n \in \mathbb{N}. \end{cases}$$

Since, by l'Hôpital's rule,

$$(2.39) \quad \lim_{q \rightarrow 1} \left\{ \frac{(q^\lambda; q)_n}{(q^\mu; q)_n} \right\} = \frac{(\lambda)_n}{(\mu)_n} \quad (n \in \mathbb{N}_0),$$

it is not difficult to recover the ordinary hypergeometric function ${}_rF_s$ as (appropriate) limiting cases of the basic (or q -) hypergeometric function ${}_r\Phi_s$. Observe, in particular, that

$$(2.40) \quad \lim_{q \rightarrow 1} \left\{ {}_{s+1}\Phi_s \left[\begin{matrix} q^{\alpha_0}, \dots, q^{\alpha_s}; \\ q^{\beta_1}, \dots, q^{\beta_s}; \end{matrix} \middle| q, z \right] \right\} = {}_{s+1}F_s \left[\begin{matrix} \alpha_0, \dots, \alpha_s; \\ \beta_1, \dots, \beta_s; \end{matrix} \middle| z \right] \quad (z \in \mathcal{U}),$$

which follows readily by appealing to (2.39).

3. Racah's ${}_4F_3$ Polynomials and Their Further Generalizations

The hypergeometric ${}_4F_3$ polynomials

$$(3.1) \quad p_n(\lambda(x)) = {}_4F_3 \left[\begin{matrix} -n, a+\beta+n+1, -x, \gamma+\sigma+x+1; \\ a+1, \beta+\delta+1, \gamma+1; \\ 1 \end{matrix} \right]$$

$$(\lambda(x) = x(\gamma+\delta+x+1))$$

are orthogonal on $x = 0, 1, \dots, N$ when $a+1$, $\beta+\delta+1$ or $\gamma+1$ is $-N$ ($N \in \mathbb{N}_0$). Popularly known as Racah coefficients or 6- j symbols in quantum mechanics, these polynomials are widely used in the coupling of angular momenta in atomic physics (*cf.* Racah [86]; see also Biedenharn *et al.* [28]).

The Racah polynomials defined by (3.1) contain, as their limiting cases, the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite, and their numerous discrete analogues which go under the names of Hahn, Krawtchouk, Meixner, and Poisson–Charlier polynomials defined in the preceding section.

Explicit representations and orthogonality relations are given by Wilson [149] for the hypergeometric ${}_4F_3$ polynomials:

$$(3.2) \quad p_n(x^2) = p_n(x^2; a, b, c, d)$$

$$= (a+b)_n (a+c)_n (a+d)_n$$

$$\cdot {}_4F_3 \left[\begin{matrix} -n, a+b+c+d+n-1, a-x, a+x; \\ a+b, a+c, a+d; \\ 1 \end{matrix} \right]$$

which include, as their special or limiting cases, the Racah polynomials defined by (3.1) and their aforementioned consequences, as well as the Meixner–Pollaczek polynomials defined by (2.32). Indeed, when

$$\min\{a, b, c, d\} > 0,$$

the orthogonality relation for these hypergeometric polynomials is given by (*cf.*

Wilson [149, p. 694, Equation (3.1)])

$$\begin{aligned}
 (3.3) \quad & \frac{1}{2\pi} \int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2 \\
 & \cdot p_m(-x^2; a, b, c, d) p_n(-x^2; a, b, c, d) dx \\
 & = \frac{n!(a+b+c+d+n-1)_n}{\Gamma(a+b+c+d+2n)} \\
 & \cdot \Gamma(a+b+n)\Gamma(a+c+n)\Gamma(a+d+n)\Gamma(b+c+n)\Gamma(b+d+n)\Gamma(c+d+n) \delta_{m,n}.
 \end{aligned}$$

In the general case when a , b , c , and d are suitably restricted complex numbers, (3.3) may be replaced by a complex orthogonality relation involving a Mellin–Barnes contour integral (*cf.* Wilson [149, p. 691, Equation (2.1)]).

The proofs of the aforementioned orthogonality relations for the ${}_4F_3$ polynomials defined by (3.2), though not overly unmotivated, are fairly involved and depend rather heavily on Mellin–Barnes contour integrals and several hypergeometric summation theorems including, for example, the following known limiting case of Dougall's theorem (*cf.* Bailey [23, p. 27, Equation 4.4(1)])

$$\begin{aligned}
 (3.4) \quad & {}_5F_4 \left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d; \\ & & & & 1 \end{matrix} \right] \\
 & = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \\
 & \quad (\operatorname{Re}(b+c+d-a) < 1),
 \end{aligned}$$

which incidentally provides the sum of a nonterminating† very well–poised

†A terminating version of the hypergeometric summation theorem (3.4) when $d = -n$ ($n \in \mathbb{N}_0$) was used by Gasper [52] in his *shorter* proof of a Jacobi polynomial inequality [18, p. 713, Theorem 3] whose special case was required by de Branges [46] in proving, among other results, the celebrated Bieberbach conjecture of 1916 for univalent functions. See also Gasper [53] for q –extensions of some of these inequalities.

hypergeometric ${}_5F_4$ series (see also Slater [109, p. 56, Equation (2.3.4.5)]). For an interesting algebraic derivation of the orthogonality relation (3.3), *without* using the hypergeometric summation theorem (3.4), one should refer to a recent paper by Miller [80]. In fact, as already pointed out by Miller [80, p. 1226], the local symmetry (recurrence relation) techniques used in his paper would yield a purely algebraic derivation of the discrete orthogonality for the Racah polynomials defined by (3.1).

Askey and Wilson [21] introduced the q -Racah polynomials defined by

$$(3.5) \quad R_n(\mu(x)) \equiv R_n(\mu(x); \alpha, \beta, \gamma, \delta; q) = {}_4\Phi_3 \left[\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1}; \\ \alpha q, \beta\delta q, \gamma q; \end{matrix} \right]_{q, q}$$

$$(\mu(x) \equiv q^{-x} + \gamma\delta q^{x+1})$$

which are orthogonal on the set $\{0, 1, \dots, N\}$ when (for example) one of

$$\alpha q, \beta\delta q \text{ or } \gamma q \text{ is } q^{-N} \quad (N \in \mathbb{N}_0),$$

and satisfy the following three-term recurrence relation (*cf.* [21, p. 1015, Equation (4.6)]):

$$(3.6) \quad -(1-q^{-x})(1-\gamma\delta q^{x+1})R_n(\mu(x))$$

$$= A_n R_{n+1}(\mu(x)) - (A_n + C_n)R_n(\mu(x)) + C_n R_{n-1}(\mu(x)), \quad n \in \mathbb{N}_0,$$

where $R_{-1}(\mu(x)) = 0$, $R_0(\mu(x)) = 1$,

$$(3.7) \quad A_n = \frac{(1-\alpha\beta q^{n+1})(1-\alpha q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})}$$

and

$$(3.8) \quad C_n = \frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}.$$

In fact, when one of αq , $\beta\delta q$ or γq is q^{-N} ($N \in \mathbb{N}_0$), the orthogonality relation for the q -Racah polynomials defined by (3.5) is (*cf.* [21, p. 1014])

$$(3.9) \quad \sum_{x=0}^N w(x) R_m(\mu(x); \alpha, \beta, \gamma, \delta: q) R_n(\mu(x); \alpha, \beta, \gamma, \delta: x) \\ = h_n \delta_{m,n} \quad (m, n \in \{0, 1, \dots, N\}),$$

where

$$(3.10) \quad w(x) = \frac{(1-\gamma\delta q^{2x+1})(\alpha q; q)_x (\beta\delta q; q)_x (\gamma q; q)_x (\gamma\delta q; q)_x (\alpha\beta q)^{-x}}{(1-\gamma\delta q)(\gamma\delta q/\alpha; q)_x (\gamma q/\beta; q)_x (\delta q; q)_x (q; q)_x}$$

and

$$(3.11) \quad h_n = \frac{(1-\alpha\beta q)(q; q)_n (\alpha q/\delta; q)_n (\alpha\beta q/\gamma; q)_n (\beta q; q)_n (\gamma\delta q)^n}{(1-\alpha\beta q^{2n+1})(\alpha q; q)_n (\alpha\beta q; q)_n (\beta\delta q; q)_n (\gamma q; q)_n} \\ \cdot \frac{(\gamma\delta q^2; q)_\infty (\gamma/(\alpha\beta); q)_\infty (\delta/\alpha; q)_\infty (1/\beta; q)_\infty}{(\gamma\delta q/\alpha; q)_\infty (\gamma q/\beta; q)_\infty (\delta q; q)_\infty (1/(\alpha\beta q); q)_\infty}.$$

Since one of αq , $\beta\delta q$ or γq is q^{-N} , it is not difficult to observe that all of the infinite products in (3.11) reduce actually to finite products. Thus the orthogonality relation holds true both for $|q| < 1$ and $|q| > 1$, and indeed also for $|q| = 1$ as long as no zeros are introduced in denominators.

Many interesting special cases of the q -Racah polynomials are scattered in the literature. For $\gamma = q^{-N-1}$ and $\delta = 0$, the ${}_4\Phi_3$ polynomials in (3.5) reduce immediately to the q -Hahn polynomials (*cf.* [60] and [22]):

$$(3.12) \quad Q_n(x; \alpha, \beta, N: q) = {}_3\Phi_2 \left[\begin{matrix} q^{-n}, a\beta q^{n+1}, q^{-x}; \\ q, q \\ aq, q^{-N}; \end{matrix} \right].$$

On the other hand, the ${}_4\Phi_3$ polynomials with $\alpha = 0$ and $\gamma = q^{-N-1}$ (and with δ replaced by γ) yield the so-called *dual q-Hahn polynomials* (cf. [61] and [22]):

$$(3.13) \quad \mathcal{Q}_n(x; \beta, \gamma, N: q) = {}_3\Phi_2 \left[\begin{matrix} q^{-n}, q^{-x}, \gamma q^{x-N}; \\ q, q \\ q^{-N}, \beta\gamma q; \end{matrix} \right].$$

The orthogonality relations for q-Hahn polynomials and dual q-Hahn polynomials (not contained in Hahn's papers [60] and [61] referred to above) would follow readily from the orthogonality relation (3.9) for the q-Racah polynomials.

Making use of a different variable, parameters, and normalization, Askey and Wilson [22] studied the q-Racah (or ${}_4\Phi_3$) polynomials (3.5) in the *alternative* form:

$$(3.14) \quad p_n(x) \equiv p_n(x; a, b, c, d | q)$$

$$= a^{-n} (ab, ac, ad; q)_n$$

$$= {}_4\Phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta}; \\ q, q \\ ab, ac, ad; \end{matrix} \right] (x = \cos \theta),$$

which obviously are symmetric in the parameters $a, b, c,$ and d . [For real values of θ , these polynomials (of degree n in $x = \cos \theta$) are analytic functions of $a, b, c,$ and d , and are real-valued when $a, b, c,$ and d are real or, if complex, occur in conjugate papers.]

The Askey-Wilson polynomials defined by (3.14) provide a q-extension of Wilson's ${}_4F_3$ polynomials defined by (3.2); hence these polynomials are also known in the literature as the q-Wilson polynomials. In essence, these polynomials are the absolutely continuous version of the q-Racah polynomials defined by (3.5).

For the ${}_4\Phi_3$ polynomials in (3.14), Askey and Wilson [22] gave a number of important properties including, for example, the orthogonality relation, three-term recurrence relation, and difference equation. Since the recurrence relation (3.6) for q -Racah polynomials continues to hold true without the restriction that a denominator parameter is q^{-N} ($N \in \mathbb{N}_0$) by merely translating it into the notation for $p_n(x)$, they obtain the three-term recurrence relation (*cf.* [22, p. 5, Equation (1.24)])

$$(3.15) \quad 2x p_n(x) = \mathcal{A}_n p_{n+1}(x) + \mathcal{B}_n p_n(x) + \mathcal{C}_n p_{n-1}(x), \quad n \in \mathbb{N}_0,$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$,

$$(3.16) \quad \mathcal{A}_n = \frac{1 - abcd q^{n-1}}{(1-abcd q^{2n-1})(1-abcd q^{2n})},$$

$$(3.17) \quad \mathcal{C}_n = (1-q^n)(1-ab q^{n-1})(1-ac q^{n-1})(1-ad q^{n-1}) \\ \cdot \frac{(1-bc q^{n-1})(1-bd q^{n-1})(1-cd q^{n-1})}{(1-abcd q^{2n-2})(1-abcd q^{2n-1})},$$

and

$$(3.18) \quad \mathcal{B}_n = a + a^{-1} - \mathcal{A}_n (1-ab q^n)(1-ac q^n)(1-ad q^n) a^{-1} \\ - a \mathcal{C}_n \left\{ (1-ab q^{n-1})(1-ac q^{n-1})(1-ad q^{n-1}) \right\}^{-1}.$$

Moreover, the orthogonality relation for $p_n(x)$ is given by (*cf.* [22, p. 11, Theorem 2.2])

$$(3.19) \quad \int_{-1}^1 \frac{w(x)}{\sqrt{1-x^2}} p_m(x; a, b, c, d|q) p_n(x; a, b, c, d|q) dx \\ = \lambda_n \delta_{m,n} \quad (m, n \in \mathbb{N}_0),$$

where

$$(3.20) \quad w(x) = \frac{\prod_{j=0}^{\infty} \{1 - 2(2x^2 - 1)q^j + q^{2j}\}}{h(x, a; q)h(x, b; q)h(x, c; q)h(x, d; q)}$$

with

$$(3.21) \quad \begin{aligned} h(x, \kappa; q) &= \prod_{j=0}^{\infty} (1 - 2\kappa x q^j + \kappa^2 q^{2j}) \\ &= (\kappa e^{i\theta}, \kappa e^{-i\theta}; q)_{\infty} \quad (x = \cos \theta), \end{aligned}$$

and

$$(3.22) \quad \lambda_n = \frac{2\pi(abcd q^{n-1}; q)_n (abcd q^{2n}; q)_{\infty}}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_{\infty}},$$

provided that

$$(3.23) \quad \max(|q|, |a|, |b|, |c|, |d|) < 1.$$

A more general orthogonality relation for the polynomials $p_n(x)$ was also proven by Askey and Wilson [22, p. 13, Theorem 2.4] using contour integration. Kalnins and Miller [72], on the other hand, made use of symmetry (recurrence relation) techniques to give a rather elementary proof of the orthogonality relation (3.19). (See also Askey [17], Rahman [89], and Ismail *et al.* [65], [66]).

The importance and usefulness of the q -Racah and q -Wilson polynomials cannot be overemphasized. Indeed they are the most extensive known generalization of the classical orthogonal polynomials, and deserve to be widely investigated. With these points in view, numerous publications have emerged since the very conception of the Askey–Wilson monograph [22]. See, for instance, Andrews and Askey [12], Chihara [40], Gasper and Rahman ([55], [56]), Ismail [64], Ismail and Wilson [67], Nassrallah and Rahman [81], and Rahman ([88], [90], [91], and [92]).

Making use of the recurrence relation (3.6), Jain and Srivastava [70] gave the following Christoffel–Darboux formula for the q -Racah polynomials defined by (3.5):

$$(3.24) \quad \sum_{k=0}^n g_k^{-1} R_k^*(\mu(x)) R_k^*(\mu(y)) \\ = \frac{h_n}{g_n h_{n+1}} \{f(x, y; q)\}^{-1} \left\{ R_{n+1}^*(\mu(x)) R_n^*(\mu(y)) - R_n^*(\mu(x)) R_{n+1}^*(\mu(y)) \right\},$$

where, for convenience,

$$(3.25) \quad R_n^*(\mu(x)) = \frac{(aq; q)_n}{(q; q)_n} R_n(\mu(x)),$$

$$(3.26) \quad f(x, y; q) = (q^{-x} - q^{-y}) (1 - \gamma \delta q^{x+y+1}),$$

$$(3.27) \quad g_n = \frac{(\gamma \delta q)^n (aq, \beta q, a\beta q/\gamma, aq/\delta; q)_n}{(1 - a\beta q^{2n+1}) (q, a\beta q, \gamma q, \beta \delta q; q)_n},$$

and

$$(3.28) \quad h_n = \frac{(a\beta q; q)_{2n}}{(q, a\beta q, \gamma q, \beta \delta q; q)_n}.$$

For $y = 0$, the Christoffel–Darboux formula (3.24) reduces easily to the following connection (or summation) formula for the q -Racah polynomials defined by (3.5):

$$(3.29) \quad \sum_{k=0}^n \frac{(1 - a\beta q^{2k+1}) (aq, \gamma q, a\beta q, \beta \delta q; q)_k}{(1 - a\beta q) (\gamma \delta q)^k (q, \beta q, a\beta q/\gamma, aq/\delta; q)_k} R_k(\mu(x); a, \beta, \gamma, \delta: q)$$

$$= \frac{(\alpha q^2, \gamma q^2, \alpha \beta q^2, \beta \delta q^2; q)_n}{(q, \beta q, \alpha \beta q / \gamma, \alpha q / \delta; q)_n} (\gamma \delta q^2)^{-n} R_n(\mu(x-1); \alpha q, \beta, \gamma q, \delta q; q),$$

which is due to Chihara [40, p. 193, Equation (2.11)].

Some interesting applications of the Christoffel–Darboux formula (3.24) are worthy of mention. Indeed, by specializing the various parameters involved, (3.24) can be applied to derive the corresponding results for several simpler classes of q -orthogonal polynomials such as the ones recorded above and each of the following classes.

The little q -Jacobi polynomials (cf. [60, p. 29] and [11, p. 11, Equation (3.1)]):

$$(3.30) \quad P_n^{(\alpha, \beta)}(x; q) = \frac{(\alpha q; q)_n}{(q; q)_n} {}_2\Phi_1 \left[\begin{matrix} q^{-n}, \alpha \beta q^{n+1}; \\ \alpha q; \end{matrix} q, qx \right];$$

The big q -Jacobi polynomials [12, p. 47, Equation (3.28)]:

$$(3.31) \quad P_n^{(\alpha, \beta)}(x; \gamma, \delta; q) = \frac{(\alpha q, -\delta \alpha q / \gamma; q)_n}{(q, -q; q)_n} \left[\frac{\gamma}{\alpha q} \right]^n \cdot {}_3\Phi_2 \left[\begin{matrix} q^{-n}, \alpha \beta q^{n+1}, \alpha x q / \gamma; \\ \alpha q, -\delta \alpha q / \gamma; \end{matrix} q, q \right];$$

The continuous q -Jacobi polynomials (cf. [22, p. 23]; see also [87]):

$$(3.32) \quad P_n^{(\alpha, \beta)}(x|q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n}$$

$$\cdot {}_4\Phi_3 \left[\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{(2\alpha+1)/4} e^{i\theta}, q^{(2\alpha+1)/4} e^{-i\theta}; \\ q^{\alpha+1}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2}; \\ q, q \end{matrix} \right]$$

($x = \cos \theta$);

$$(3.33) \quad P_n^{(\alpha, \beta)}(x; q) = \frac{(q^{\alpha+1}, -q^{\beta+1}; q)_n}{(q, -q; q)_n}$$

$$\cdot {}_4\Phi_3 \left[\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, \sqrt{q} e^{i\theta}, \sqrt{q} e^{-i\theta}; \\ q^{\alpha+1}, -q^{\beta+1}, -q; \\ q, q \end{matrix} \right] \quad (x = \cos \theta)$$

$$= \frac{(-q^{\alpha+\beta+1}; q)_n}{(-q; q)_n} q^{-an} P_n^{(\alpha, \beta)}(x|q^2);$$

The q -Laguerre polynomials (cf. [68, p. 57] and [60, p. 29]; see also [50, p. 188, Equation (5.5.1.9)]):

$$(3.34) \quad L_n^{(\alpha)}(x; q) = \frac{(\alpha q; q)_n}{(q; q)_n} {}_1\Phi_1 \left[\begin{matrix} q^{-n}; \\ q, -xq^n \\ \alpha q; \end{matrix} \right].$$

In particular, the little q -Jacobi polynomials in (3.30) may be recovered from $R_n^*(\mu(x))$ upon setting $\gamma q = q^{-N}$, replacing x by $N-x$, letting $N \rightarrow \infty$ and $\delta \rightarrow 0$, and then writing x for q^x . Thus (3.24) yields the Christoffel–Darboux formula:

$$(3.35) \quad \sum_{k=0}^n \frac{(1-\alpha\beta q^{2k+1})(q, \alpha\beta q; q)_k}{(1-\alpha\beta q)(\alpha q)^k (q, \beta q; q)_k} P_k^{(\alpha, \beta)}(x; q) P_k^{(\alpha, \beta)}(y; q)$$

$$= \frac{(q; q)_{n+1} (\alpha\beta q^2; q)_n (x-y)^{-1}}{(1-\alpha\beta q^{2n+2}) \alpha^n (\alpha q, \beta q; q)_n} \\ \cdot \left\{ p_{n+1}^{(\alpha, \beta)}(x; q) p_n^{(\alpha, \beta)}(y; q) - p_n^{(\alpha, \beta)}(x; q) p_{n+1}^{(\alpha, \beta)}(y; q) \right\}$$

Furthermore, in view of the limit relationship [*cf.* Equations (3.30) and (3.34)]:

$$(3.36) \quad L_n^{(\alpha)}(x; q) = \lim_{\beta \rightarrow \infty} \left\{ p_n^{(\alpha, \beta)} \left[-\frac{x}{\alpha\beta q^2}; q \right] \right\},$$

it is not difficult to further specialize the Christoffel–Darboux formula (3.35) to the case of the q -Laguerre polynomials $L_n^{(\alpha)}(x; q)$.

Now replace x and y in (3.35) by xt and yt , respectively, multiply both sides by

$$t^{\lambda-1} (tq; q)_{\mu-\lambda-1},$$

and q -integrate with respect to t , using the familiar result (*cf. e.g.*, [15, p. 135, Equation (5.7)]; see also [16, p. 257, Equation (1.9)]):

$$(3.37) \quad \frac{\Gamma_q(\alpha) \Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} = \int_0^1 t^{\alpha-1} (tq; q)_{\beta-1} d(t; q),$$

where the (Jackson's) q -Gamma function $\Gamma_q(z)$ is defined by

$$(3.38) \quad \Gamma_q(z) = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1-q)^{1-z},$$

and the measure $d(t; q)$ is given by (Jackson's) q -integrals (*cf.* [15] and [16]):

$$(3.39) \quad \int_0^a f(t) d(t; q) = a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n,$$

$$(3.40) \quad \int_a^\infty f(t) d(t; q) = a(1-q) \sum_{n=1}^{\infty} f(aq^{-n}) q^{-n},$$

and

$$(3.41) \quad \int_0^\infty f(t) d(t; q) = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n.$$

Upon writing λ for q^λ , and μ for q^μ , Jain and Srivastava [70] thus find from (3.35) that

$$(3.42) \quad \sum_{k=0}^n \frac{(1-\alpha\beta q^{2k+1})(\alpha q, \alpha\beta q; q)_k}{(1-\alpha\beta q)(\alpha q)^k (q, \beta q; q)_k} \\ \cdot \Phi_{1:1;1}^{1:2;2} \left[\begin{matrix} \lambda: q^{-k}, \alpha\beta q^{k+1}; q^{-k}, \alpha\beta q^{k+1}; \\ \mu: \alpha q; \alpha q; \end{matrix} \middle| q; x, y \right] \\ = \frac{(1-\mu q^{-1})(\alpha q; q)_{n+1}(\alpha\beta q^2; q)_n (x-y)^{-1}}{(1-\lambda q^{-1})(1-\alpha\beta q^{2n+2})\alpha^n (q, \beta q; q)_n} \\ \cdot \Phi_{1:1;1}^{1:2;2} \left[\begin{matrix} \lambda q^{-1}: q^{-n}, \alpha\beta q^{n+1}; q^{-n-1}, \alpha\beta q^{n+2}; \\ \mu q^{-1}: \alpha q; \alpha q; \end{matrix} \middle| q; x, y \right] + x \leftrightarrow y,$$

where $x \leftrightarrow y$ indicates the presence of a second term which originates from the first by interchanging x and y , and $\Phi_{S:h;v}^{r:p;u}$ denotes a general double q -hypergeometric function (*cf.* [130, p. 349]).

For $\beta = 0$, (3.42) immediately reduces to the elegant form:

$$\begin{aligned}
(3.43) \quad & \sum_{k=0}^n \frac{(\alpha q; q)_k}{(\alpha q)^k (q; q)_k} \Phi_{1:1;1}^{1:1;1} \left[\begin{array}{c} \lambda: q^{-k}; q^{-k}; \\ q; x, y \\ \mu: \alpha q; \alpha q; \end{array} \right] \\
& = \frac{(1-\mu q^{-1})(\alpha q; q)_{n+1}}{(1-\lambda q^{-1})\alpha^n (q; q)_n} (x-y)^{-1} \Phi_{1:1;1}^{1:1;1} \left[\begin{array}{c} \lambda q^{-1}: q^{-n}; q^{-n-1}; \\ q; x, y \\ \mu q^{-1}: \alpha q; \alpha q; \end{array} \right] + x \leftrightarrow y.
\end{aligned}$$

Formulas (3.42) and (3.43) provide q -extensions of certain finite summation formulas due to Srivastava [114, p. 5, Equation (3.4); p. 4, Equation (2.4)]. More generally, by repeating the aforementioned process of q -integration using (3.3) and some manipulations, Jain and Srivastava [70] derive the q -summation formula:

$$\begin{aligned}
(3.44) \quad & \sum_{k=0}^n \frac{(1-\alpha\beta q^{2k+1})(\alpha q, \alpha\beta q; q)_k}{(1-\alpha\beta q)(\alpha q)^k (q, \beta q; q)_k} \\
& \cdot \Phi_{s:1;1}^{r:2;2} \left[\begin{array}{c} \lambda_1, \dots, \lambda_r: q^{-k}, \alpha\beta q^{k+1}; q^{-k}, \alpha\beta q^{k+1}; \\ q; x, y \\ \mu_1, \dots, \mu_s: \alpha q; \alpha q; \end{array} \right] \\
& = \frac{(1-\mu_1 q^{-1}) \cdots (1-\mu_s q^{-1})(\alpha q; q)_{n+1} (\alpha\beta q^2; q)_n (x-y)^{-1}}{(1-\lambda_1 q^{-1}) \cdots (1-\lambda_r q^{-1})(1-\alpha\beta q^{2n+2})\alpha^n (q, \beta q; q)_n} \\
& \cdot \Phi_{s:1;1}^{r:2;2} \left[\begin{array}{c} \lambda_1 q^{-1}, \dots, \lambda_r q^{-1}: q^{-n}, \alpha\beta q^{n+1}; q^{-n-1}, \alpha\beta q^{n+2}; \\ q; x, y \\ \mu_1 q^{-1}, \dots, \mu_s q^{-1}: \alpha q; \alpha q; \end{array} \right] + x \leftrightarrow y,
\end{aligned}$$

which reduces to (3.42) when $r = s = 1$, and (for $\beta = 0$) yields a similar generalization of (3.43).

Finite summation formulas involving (ordinary) double hypergeometric functions, corresponding to (3.44) and its forementioned special case when $\beta = 0$, were also given by Srivastava [114, p. 6, Equation (3.6); p. 3, Equation (2.1)]. In fact, as already observed by Jain and Srivastava [70], it is not difficult to prove

q-extensions of Srivastava's finite summation formulas [121, p. 315, Equations (2.1) and (2.2)] involving (ordinary) triple hypergeometric functions.

Finally, corresponding to a bounded complex sequence $\{\Omega_n\}_{n=0}^\infty$, Jain and Srivastava [70] proved the following class of generating functions for the q-Racah polynomials defined by (3.5) and (3.25):

$$(3.45) \quad \sum_{n \geq 0} \frac{(\gamma q, \beta \delta q; q)_n}{(a q, \beta q; q)_n} \Omega_n R_n^*(\mu(x); a, \beta, \gamma, \delta; q) t^n$$

$$= \sum_{\ell, m \geq 0} \Omega_{\ell+m} \frac{(a q^{x+1}, \gamma \delta q^{x+1}; q)_\ell (\beta q^{-x}/\gamma, q^{-x}/\delta; q)_m}{(a q; q)_\ell (\beta q; q)_m} \frac{(t q^{-x})^\ell}{(q; q)_\ell} \frac{(\gamma \delta t q^{x+1})^m}{(q; q)_m}.$$

Formula (3.45) and its numerous special or limiting cases would provide q-extensions of a considerably large number of known generating functions for classical orthogonal polynomials, which are presented systematically by Srivastava and Manocha [133, Chapter 2 *et seq.*]. In particular, if in (3.45) we let

$$\Omega_n = 1 \quad (n \in \mathbb{N}_0) \quad \text{and} \quad \delta \rightarrow 0,$$

and replace q^{-x} by x , we shall arrive at a known generating function (due to Ismail and Wilson [67, p. 46, Equation (1.16)]) for certain big q-Jacobi polynomials which are normalized slightly differently from (3.31).

4. *Contributions to the Theory and Applications of Generating Functions*

We choose to survey some important recent contributions to the theory and applications of generating functions under the following three classes:

4.1 Linear Generating Functions

Formula (3.45) already provides an example of a class of linear q-generating

functions. Several other families of linear q -generating functions in one and more variables can be found in the work of Srivastava [120, Section 3] (see also Srivastava and Agarwal [125], Agarwal *et al.* [2], and Srivastava and Jain [129]).

An interesting unification (and generalization) of a number of generating functions for the classical Laguerre polynomials was given by Carlitz [31] in the form:

$$(4.1) \quad \sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x) t^n = \frac{(1+v)^{\alpha+1}}{1-\lambda v} \exp(-xv),$$

where α and λ are arbitrary (real or complex) numbers, and v is a function of t defined by

$$(4.2) \quad v = t(1+v)^{\lambda+1}, \quad v(0) = 0.$$

We also recall a subsequent generalization of Carlitz's result (4.1) due to Srivastava and Singhal [139, p. 749, Equation 980]:

$$(4.3) \quad \sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n, \beta+\mu n)}(x) t^n \\ = (1+\xi)^{\alpha+1} (1+\eta)^{\beta+1} \{1 - \lambda\xi - \mu\eta - (1+\lambda+\mu)\xi\eta\}^{-1},$$

where ξ and η satisfy

$$(4.4) \quad (x+1)^{-1}\xi = (x-1)^{-1}\eta = \frac{1}{2} t(1+\xi)^{\lambda+1} (1+\eta)^{\mu+1}.$$

Motivated by (4.1) and (4.3), Carlitz [33] derived generating functions for certain general one- and two-parameter coefficients. Subsequently, Srivastava [112] presented several generalizations of Carlitz's theorem and applied his general multivariable (and multiparameter) generating functions to various systems of polynomials in one and more variables.

The generating functions (4.1) and (4.3), and their numerous special cases, have been recorded systematically by Hansen [62], and (more recently) by Srivastava and Manocha [133]. As a matter of fact, Srivastava and Manocha [133, Chapter 7] have also included a systematic presentation of the aforementioned works of Carlitz [33] and Srivastava [112]. For the sake of completeness, we recall Srivastava's multiparameter and multivariable extension of Carlitz's theorem as

THEOREM 1 (Srivastava [112]; see also Srivastava and Manocha [133, p. 378, Theorem 12]). *Let $A(z)$, $B(z)$, and $z^{-1}C(z)$ be arbitrary functions which are analytic in a neighborhood of the origin, and assume that*

$$(4.5) \quad A(0) = B(0) = C'(0) = 1.$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}$ by means of

$$(4.6) \quad A(z) [B(z)]^\alpha \exp(xC(z)) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{z^n}{n!},$$

where α and x are arbitrary complex numbers independent of z .

Then, for arbitrary parameters λ and y independent of z ,

$$(4.7) \quad \sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny) \frac{t^n}{n!} \\ = \frac{A(\zeta) [B(\zeta)]^\alpha \exp(x C(\zeta))}{1 - \zeta \{ \lambda [B'(\zeta)/B(\zeta)] + y C'(\zeta) \}},$$

where

$$(4.8) \quad \zeta = t [B(\zeta)]^\lambda \exp(yC(\zeta)).$$

More generally, if $A(z)$, $B_i(z)$, and $z^{-1}C_j(z)$ are analytic about the origin such that

$$(4.9) \quad A(0) = B_i(0) = C'_j(0) = 1 \quad (i = 1, \dots, r; j = 1, \dots, s),$$

and if

$$(4.10) \quad A(z) \prod_{i=1}^r \{ [B_i(z)]^{\alpha_i} \} \exp \left[\sum_{j=1}^s x_j C_j(z) \right] \\ = \sum_{n=0}^{\infty} g_n^{(\alpha_1, \dots, \alpha_r)} (x_1, \dots, x_s) \frac{z^n}{n!},$$

then, for arbitrary parameters

$$\alpha_i, \gamma_i, x_j, \text{ and } y_j \quad (i = 1, \dots, r; j = 1, \dots, s)$$

independent of z ,

$$(4.11) \quad \sum_{n=0}^{\infty} g_n^{(\alpha_1 + \lambda_1 n, \dots, \alpha_r + \lambda_r n)} (x_1 + ny_1, \dots, x_s + ny_s) \frac{t^n}{n!}$$

$$= \frac{A(w) \prod_{i=1}^r \{ [B_i(w)]^{\alpha_i} \} \exp \left[\sum_{j=1}^s x_j C_j(w) \right]}{1 - w \left\{ \sum_{i=1}^r \lambda_i [B'_i(w)/B_i(w)] + \sum_{j=1}^s y_j C'_j(w) \right\}},$$

where

$$(4.12) \quad w = t \prod_{i=1}^r \{ [B_i(w)]^{\lambda_i} \} \exp \left[\sum_{j=1}^s y_j C_j(w) \right].$$

REMARK 1. The choice of 1 in the conditions (4.5) and (4.9) is merely a convenient one; in fact, as already observed in the literature (*cf.* Srivastava and Manocha [133, p. 379]), any nonzero constant values may be assumed for

$$A(0), B(0), C'(0), B_i(0), \text{ and } C'_j(0)$$

$$(i = 1, \dots, r; j = 1, \dots, s).$$

REMARK 2. For $x = y = 0$, the generating function (4.7) reduces to Carlitz's result [33, p. 521, Theorem 1].

REMARK 3. The general result (4.11) with

$$r = 2 \quad \text{and} \quad x_j = y_j = 0 \quad (j = 1, \dots, s)$$

is essentially the same as a generating function for certain two-parameter coefficients due to Carlitz [33, p. 521, Equation (2.10)].

Carlitz [33] applied his aforementioned results involving general one- and two-parameter coefficients to prove the following class of *mixed* generating functions for the Laguerre polynomials:

$$(4.13) \quad \sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x+ny) t^n = \frac{(1-\zeta)^{-\alpha-1} \exp\left[-\frac{x\zeta}{1-\zeta}\right]}{1 - \zeta(1-\zeta)^{-1} [\lambda - y(1-\zeta)^{-1}]},$$

where

$$(4.14) \quad \zeta = t(1-\zeta)^{-\lambda} \exp\left[-\frac{y\zeta}{1-\zeta}\right].$$

Setting $\zeta = w/(1+w)$, (4.13) assumes the form (*cf.* Carlitz [33, p. 525, Equation (5.5)])

$$(4.15) \quad \sum_{n=0}^{\infty} L_n^{(\alpha+\lambda n)}(x+ny) t^n = \frac{(1+w)^{\alpha+1} \exp(-xw)}{1 - w[\lambda - y(1+w)]},$$

where

$$(4.16) \quad w = t(1+w)^{\lambda+1} \exp(-yw).$$

For $y = 0$, both (4.13) and (4.15) reduce immediately to Carlitz's earlier result (4.1).

Cigler [45] showed that some of Carlitz's results in [33] can alternatively be derived by using the theory of Sheffer sets (*cf.* Rota *et al.* [104], [105]).

Srivastava [112] applied Theorem 1 to derive *mixed* generating functions for various classes of polynomials. In particular, he considered the Srivastava–Singhal generating function [136, p. 78, Equation (3.2)]:

$$(4.17) \quad \sum_{n=0}^{\infty} G_n^{(a)}(x^{1/r}, r, p, k) z^n = (1-kz)^{-a/k} \\ \cdot \exp(px[1 - (1-kz)^{-r/k}]),$$

where $G_n^{(a)}(x, r, p, k)$ are the polynomials considered by Srivastava and Singhal [136] in an attempt to present a unified study of various known generalizations of the classical Laguerre and Hermite polynomials, the parameters a , p , k , and r being arbitrary (with, of course, $k, r \neq 0$). Thus it follows from the assertion (4.7) of Theorem 1 that

$$(4.18) \quad \sum_{n=0}^{\infty} G_n^{(a+\lambda n)}([x+ny]^{1/r}, r, p, k) t^n \\ = \frac{(1-\zeta)^{-a/k} \exp(px[1 - (1-\zeta)^{-r/k}])}{1 - k^{-1} \zeta(1-\zeta)^{-1} [\lambda - rpy(1-\zeta)^{-r/k}]},$$

where

$$(4.19) \quad \zeta = kt(1-\zeta)^{-\lambda/k} \exp(py[1 - (1-\zeta)^{-r/k}]).$$

Putting $\zeta = w/(1+w)$ in (4.18), we have [112, p. 474, Equation (3.6)]

$$(4.20) \quad \sum_{n=0}^{\infty} G_n^{(\alpha+\lambda n)}([x+ny]^{1/r}, r, p, k) t^n \\ = \frac{(1+w)^{\alpha/k} \exp(px[1 - (1+w)^{r/k}])}{1 - k^{-1} w[\lambda - rpy(1+w)^{r/k}]},$$

where

$$(4.21) \quad w = kt(1+w)^{1+\lambda/k} \exp(py[1 - (1+w)^{r/k}]).$$

By employing the known relationships [136]

$$(4.22) \quad G_n^{(\alpha+1)}(x, 1, 1, 1) = L_n^{(\alpha)}(x)$$

and

$$(4.23) \quad G_n^{(0)}(x, 2, 1, -1) = \frac{(-x)^n}{n!} H_n(x) = G_n^{(1-n)}(x, 2, 1, 1),$$

it is not difficult to deduce, from (4.18) and (4.20), the corresponding *mixed* generating functions for the classical Laguerre and Hermite polynomials. More importantly, since (*cf.*, *e.g.*, Srivastava and Lavoie [131, p. 315, Equation (83)])

$$(4.24) \quad G_n^{(\alpha+1)}(x, 1, 1, k) = k^n Y_n^\alpha(x; k),$$

where $Y_n^\alpha(x; k)$ are one class of the biorthogonal polynomials considered by Konhauser [74] for $\alpha > -1$ and $k \in \mathbb{N}$, (4.18) readily yields the *mixed* generating

function:

$$(4.25) \quad \sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n} (x+ny; k) t^n$$

$$= \frac{(1-\zeta)^{-(\alpha+1)/k} \exp(x[1 - (1-\zeta)^{-1/k}])}{1 - k^{-1} \zeta(1-\zeta)^{-1} [\lambda - y(1-\zeta)^{-1/k}]},$$

where

$$(4.26) \quad \zeta = t(1-\zeta)^{-\lambda/k} \exp(y[1 - (1-\zeta)^{-1/k}]),$$

while (4.20) reduces to the form:

$$(4.27) \quad \sum_{n=0}^{\infty} Y_n^{\alpha+\lambda n} (x+ny; k) t^n$$

$$= \frac{(1+w)^{(\alpha+1)/k} \exp(x[1 - (1+w)^{1/k}])}{1 - k^{-1} w[\lambda - y(1+w)^{1/k}]},$$

where

$$(4.28) \quad w = t(1+w)^{1+\lambda/k} \exp(y[1 - (1+w)^{1/k}]).$$

It should be noted in passing that

$$(4.29) \quad Y_n^\alpha(x; 1) = L_n^{(\alpha)}(x) \quad (\alpha > -1; n \in \mathbb{N}_0),$$

and that the polynomials $Y_n^\alpha(x; 2)$ were encountered earlier by Spencer and Fano [111] in certain analytical calculations involving the penetration of Gamma rays through matter (see also Srivastava [115]).

For further applications and analogues of Theorem 1, see the works by Srivastava [112], Srivastava and Manocha [133, Chapter 7], Srivastava *et al.* [132], and Chandel *et al.* ([35] to [38]). More recently, Carlitz and Srivastava [34] applied Theorem 1 and its known consequences to obtain several new generating functions for the Hermite polynomials.

Each of the generating functions considered or referred to above has tremendous potential for applications. For example, the generating function (4.3) was applied recently by Chen and Ismail [39] in order to determine the asymptotic behavior of the Jacobi polynomials

$$P_n^{(\alpha+\lambda n, \beta+\mu n)}(x)$$

when $n \rightarrow \infty$ and $\alpha, \beta, \lambda, \mu$, and x remain fixed; in fact, these authors similarly applied Carlitz's generating function (4.13) for Laguerre polynomials. On the other hand, Strehl [141] has presented an interesting combinatorial proof of the Srivastava–Singhal result (4.3). Furthermore, several applications of Carlitz's result (4.15), in its *equivalent* form:

$$(4.29) \quad \sum_{n, k=0}^{\infty} \binom{\alpha + (\lambda+1)(n+k)}{n} [-x - y(n+k)]^k \frac{t^{n+k}}{k!}$$

$$= \frac{(1+w)^{\alpha+1} \exp(-xw)}{1 - w[\lambda - y(1+w)]},$$

with w given by (4.16), appear in the works of Raina ([97] to [100]).

4.2 Bilinear and Bilateral Generating Functions

In their 1984 treatise on the subject of generating functions, Srivastava and Manocha [133, Sections 8.2 to 8.5] presented a systematic account of the earlier works on bilinear and bilateral generating functions stemming from the following generating function of Singhal and Srivastava [108, p. 755, Equation (1)]:

$$(4.30) \quad \sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x) t^n = f(x,t) \{g(x,t)\}^{-m} S_m(h(x,t)),$$

where $m \in \mathbb{N}_0$, the coefficients $A_{m,n}$ are arbitrary (real or complex) constants, and f, g, h are suitable functions of x and t . To the various relevant references, provided already by Srivastava and Manocha [133], one may add the recent works by Barik [25] and Chongdar ([42], [43], and [44]) who considered some obvious variations of the Singhal–Srivastava definition (4.30).

Making use of group–theoretic methods (see, *e.g.*, [133, Chapter 6]), Shrivastava and Kaur [107] established two theorems on bilateral generating functions involving the (Gould–Hopper) generalized Hermite polynomials (*cf.* [57]):

$$(4.31) \quad H_n^\gamma(x, \alpha, \beta) = (-1)^n x^{-\alpha} \exp(\beta x^\gamma) D_x^n \{x^\alpha \exp(-\beta x^\gamma)\}$$

$$\left[D_x = \frac{d}{dx} \right],$$

the Laguerre polynomials, and the Gegenbauer (or ultraspherical) polynomials. In an attempt to give relatively simpler proofs of these results, without using group–theoretic methods, Hubbell and Srivastava [63] were led to much more general classes of bilateral generating functions for such sequences of functions as defined by (4.30). We recall here one of the Hubbell–Srivastava results which they applied to the Bessel function $J_\nu(z)$ and the Srivastava–Singhal polynomials $G_n^{(\alpha)}(x, r, p, k)$ involved in (4.17). Furthermore, since [136, p. 76, Equation (1.6)]

$$(4.32) \quad \begin{aligned} G_n^{(\alpha)}(x, \gamma, \beta, -1) &= \frac{(-x)^\alpha}{n!} H_n^\gamma(x, \alpha, \beta) \\ &= G_n^{(\alpha-n+1)}(x, \gamma, \beta, 1), \end{aligned}$$

Theorem 2 below applies also to the Gould–Hopper polynomials $H_n^\gamma(x, \alpha, \beta)$ defined by (4.31), and indeed to the Konhauser biorthogonal polynomials $Y_n^\alpha(x; k)$ given by (4.24).

THEOREM 2 (Hubbell and Srivastava [63]). *Corresponding to the functions $S_n(x)$, generated by (4.30), let*

$$(4.33) \quad \Theta_N[x, y, t] = \sum_{n=0}^{\infty} a_n S_n(x) L_N^{(\lambda+n)}(y) t^n \quad (a_n \neq 0),$$

where λ is an arbitrary (real or complex) parameter. Suppose also that

$$(4.34) \quad \theta_{m,n}(z, w, x) = \sum_{k=0}^{\min(m,n)} \frac{(-1)^k}{k!} a_{m-k} A_{m-k, n-k} z^{m-k} w^{n-k} S_{m+n-2k}(x).$$

Then

$$(4.35) \quad \sum_{m,n=0}^{\infty} \theta_{m,n}(z, w, x) L_N^{(\lambda+m)}(y) t^m \\ = \exp(-t) f(x, w) \Theta_N[h(x, w), y+t, zt/g(x, w)],$$

provided that each member exists.

It should be pointed out that, by setting

$$(4.36) \quad a_n = b_n T_n(z_1, \dots, z_s) \quad (b_n \neq 0; n \in \mathbb{N}_0),$$

where $T_n(z_1, \dots, z_s)$ is a nonvanishing function of s variables z_1, \dots, z_s ($s \in \mathbb{N}$), Theorem 1 (and each of its consequences discussed already by Hubbell and Srivastava [63]) can be applied to derive various classes of mixed multilateral generating functions analogous to those considered in the literature (*cf.* [113] and [133, Sections 8.4 and 8.5]). Furthermore, as already observed by Srivastava [113, p. 222], the definition (4.30) can easily be transformed to include cases when m is an arbitrary complex parameter.

The importance of a systematic study of various properties and characteristics of the classical Jacobi polynomials (including, for example, linear, bilinear, and bilateral generating functions (*cf.* [78] and [133]) associated with them) led

Srivastava and Popov [135] recently to a class of mixed multilateral generating functions for the Jacobi polynomials, which can indeed be suitably applied to derive numerous further results involving Jacobi polynomials and some of their aforementioned relatives.

A familiar unification of each of the classes of orthogonal polynomials named above, and of their various known generalizations studied in the literature (*cf.*, *e.g.*, [133]), is provided by the sequence [134, p. 307, Equation (5)]:

$$(4.37) \quad S_n^{(\alpha, \beta)} [x, a, b, c, d; \gamma, \delta; w(x)] \\ = \frac{(ax+b)^{-\alpha} (cx+d)^{-\beta}}{n! w(x)} D_x^n \left\{ (ax+b)^{\gamma+n+\alpha} (cx+d)^{\delta n+\beta} w(t) \right\} \\ \left[D_x = \frac{d}{dx}; \quad n \in \mathbb{N}_0 \right],$$

where the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$ are arbitrary constants, real or complex, and $w(x)$ is independent of n and differentiable any number of times. For example, it is easily observed from (4.37) and the known Rodrigues formulas for the Jacobi and Bessel polynomials that

$$(4.38) \quad S_n^{(\alpha, \beta)} [x, a, -a, c, c; 1, 1; C] = (2ca)^n P_n^{(\alpha, \beta)}(x)$$

for any nonzero constant C , and

$$(4.39) \quad S_n^{(\alpha-2, 0)} [x, a, 0, c, d; 2, 0; \exp(-\beta/x)] = \frac{a^{2n} \beta^n}{n!} y_n(x, \alpha, \beta).$$

Indeed, in each of these and other cases of reducibility, we can have numerous alternative sets of choices for the various parameters involved.

For the sequence of functions defined by (4.37), various classes of linear, bilinear, bilateral, and mixed multilateral generating functions are given in the literature. The most general results on bilateral generating functions for the sequence defined by (4.37) are given by Theorems 3, 4, and 5 below:

THEOREM 3 (cf. Srivastava and Handa [127]; see also [123]). *Corresponding to a nonvanishing function $\Omega_\mu(y_1, \dots, y_S)$ of s variables y_1, \dots, y_S ($s \in \mathbb{N}$) and of (complex) order μ , let*

$$(4.40) \quad F_{m,p,q}^{(\rho,\sigma)}[x; y_1, \dots, y_S; t] = \sum_{n=0}^{\infty} a_n S_{m+qn}^{(\alpha-\rho qn, \beta-\sigma qn)}[x, a, b, c, d; \gamma, \delta; w(x)] \\ \cdot \Omega_{\mu+pn}(y_1, \dots, y_S) t^n \quad (a_n \neq 0),$$

where p and q are positive integers, and ρ and σ are complex parameters. Also, for an arbitrary integer $m \geq 0$, let

$$(4.41) \quad \Theta_{n,m,p,q}^{\alpha,\beta,\rho,\sigma}(x; y_1, \dots, y_S; z) = \sum_{k=0}^{[n/q]} \begin{bmatrix} m+n \\ n-qk \end{bmatrix} a_k \\ \cdot S_{m+n}^{(\alpha-\gamma n+\rho qk, \beta-\delta n+\sigma qk)}[x, a, b, c, d; \gamma, \delta; w(x)] \Omega_{\mu+pk}(y_1, \dots, y_S) z^k.$$

Then

$$(4.42) \quad \sum_{n=0}^{\infty} \Theta_{n,m,p,q}^{\alpha,\beta,\rho,\sigma}(x; y_1, \dots, y_S; z) t^n = \left[1+at(ax+b)^{\gamma-1}(cx+d)^{\delta} \right] a \\ \cdot \left[1+ct(ax+b)^{\gamma}(cx+d)^{\delta-1} \right] \frac{\beta}{w(x)} F_{m,p,q}^{(\gamma-\rho, \delta-\sigma)} \left[\xi; y_1, \dots, y_S; \right. \\ \left. zt^q \left\{ 1+at(ax+b)^{\gamma-1}(cx+d)^{\delta} \right\}^{(\rho-\gamma)q} \left\{ 1+ct(ax+b)^{\gamma}(cx+d)^{\delta-1} \right\}^{(\sigma-\delta)q} \right],$$

provided that each side exists; here, for convenience,

$$(4.43) \quad \xi \equiv \xi(x, t) = x + t(ax+b)^{\gamma}(cx+d)^{\delta}.$$

THEOREM 4 (cf. Srivastava and Handa [127]; see also [123]). *Under the hypotheses of Theorem 1, let*

$$(4.44) \quad G_{m,p,q}^{(\rho)}[x; y_1, \dots, y_S; t] = \sum_{n=0}^{\infty} a_n S_{m+qn}^{(\alpha-\rho qn, \beta)}[x, a, b, c, d; \gamma, 1; w(x)] \\ \cdot \Omega_{\mu+pn}(y_1, \dots, y_S) t^n \quad (a_n \neq 0),$$

and define

$$(4.45) \quad \Phi_{n,m,p,q}^{\alpha, \beta, \rho}(x; y_1, \dots, y_S; z) = \sum_{k=0}^{[n/q]} \begin{bmatrix} m+n \\ n-qn \end{bmatrix} a_k \\ \cdot S_{m+n}^{(\alpha-\gamma n + \rho qk, \beta)}[x, a, b, c, d; \gamma, 1; w(x)] \Omega_{\mu+pk}(y_1, \dots, y_S) z^k$$

for an arbitrary integer $m \geq 0$.

Then

$$(4.46) \quad \sum_{n=0}^{\infty} \Phi_{n,m,p,q}^{\alpha, \beta, \rho}(x; y_1, \dots, y_S; z) t^n = \left[1 + t(ad-bc)(ax+b)^{\gamma-1} \right]^{\alpha} \\ \cdot \left[1 - ct(ax+b)^{\gamma} \right]^{-\alpha-\beta-m-1} \frac{w(\eta)}{w(x)} G_{m,p,q}^{(\gamma-\rho)} \left[\eta; y_1, \dots, y_S; \right. \\ \left. zt^q \left\{ 1 + t(ad-bc)(ax+b)^{\gamma-1} \right\}^{(\rho-\gamma)q} \left\{ 1 - ct(ax+b)^{\gamma} \right\}^{(\gamma-\rho-1)q} \right],$$

where

$$(4.47) \quad \eta \equiv \eta(x, t) = [x + dt(ax+b)^{\gamma}] / [1 - ct(ax+b)^{\gamma}].$$

THEOREM 5 (cf. Srivastava and Handa [127]; see also [123]). *Under the hypotheses of Theorem 1, let*

$$(4.48) \quad \mathbb{H}_{m,p,q}^{(\sigma)}[x; y_1, \dots, y_S; t] = \sum_{n=0}^{\infty} a_n S_{m+qn}^{(\alpha, \beta - \sigma qn)}[x, a, b, c, d; 1, \delta; w(x)] \\ \cdot \Omega_{\mu+pn}(y_1, \dots, y_S) t^n \quad (a_n \neq 0),$$

and define

$$(4.49) \quad \Psi_{n,m,p,q}^{\alpha, \beta, \sigma}(x; y_1, \dots, y_S; z) = \sum_{k=0}^{\lfloor n/q \rfloor} \begin{bmatrix} m+n \\ n-qn \end{bmatrix} a_k \\ \cdot S_{m+n}^{(\alpha, \beta - \delta n + \sigma qk)}[x, a, b, c, d; 1, \delta; w(x)] \Omega_{\mu+pk}(y_1, \dots, y_S) z^k$$

for an arbitrary integer $m \geq 0$.

Then

$$(4.50) \quad \sum_{n=0}^{\infty} \Psi_{n,m,p,q}^{\alpha, \beta, \sigma}(x; y_1, \dots, y_S; z) t^n = \left[1 + t(bc-ad)(cx+d)^{\delta-1} \right]^{\beta} \\ \cdot \left[1 - at(cx+d)^{\delta} \right]^{-\alpha - \beta - m - 1} \frac{w(\zeta)}{w(x)} \mathbb{H}_{m,p,q}^{(\delta - \sigma)}[\zeta; y_1, \dots, y_S; \\ zt^q \left\{ 1 + t(bc-ad)(cx+d)^{\delta-1} \right\}^{(\sigma - \delta)q} \left\{ 1 - at(cx+d)^{\delta} \right\}^{(\delta - \sigma - 1)q}]$$

where

$$(4.51) \quad \zeta \equiv \zeta(x, t) = [x + bt(cx+d)^{\delta}] / [1 - at(cx+d)^{\delta}].$$

Each of these general results (Theorems 3, 4, and 5 above) can indeed be applied to derive the corresponding generating functions for various systems of orthogonal polynomials and for numerous other polynomials associated with them.

Several interesting proofs are given in the literature for Bailey's bilinear generating function for the classical Jacobi polynomials [24, p. 9, Equation

(2.1)]. One of the recent proofs was given by Stanton [140]; it makes use of the orthogonality property (2.10) and a known quadratic transformation for a well-poised hypergeometric ${}_3F_2$ series. Indeed, as remarked by Stanton [140, p. 399], this technique applies *mutatis mutandis* to yield a q -extension of Bailey's result in the form:

$$(4.52) \quad \sum_{n=0}^{\infty} \frac{(q; q)_n (\alpha\beta q; q)_n}{(\alpha q; q)_n (\beta q; q)_n} (\beta q)^n q^{n(n-1)/2} p_n^{(\alpha, \beta)}(x; q) p_n^{(\alpha, \beta)}(y; q) t^n$$

$$= \frac{(-\alpha\beta qt; q)_{\infty}}{(-t; q)_{\infty}} \Phi_{2:1;1}^{4:0;2} \left[\begin{array}{c} \sqrt{\alpha\beta q}, -\sqrt{\alpha\beta q}, q\sqrt{\alpha\beta}, -q\sqrt{\alpha\beta}: -; \beta qx, \beta qy; \\ q; \beta q^2 xy, q \\ -\alpha\beta qt, -q/t: \alpha q; \quad \beta q; \end{array} \right],$$

where $p_n^{(\alpha, \beta)}(x; q)$ are the little q -Jacobi polynomials defined by (3.30).

Making use of some simple ideas, Srivastava [116] gave an elementary proof of Bailey's result as well as of its q -extension (4.52). Srivastava [117], on the other hand, presented yet another proof of (4.52).

Finally, Jain and Srivastava [70] proved a bilinear generating function for the q -Konhauser polynomials (*cf.* [6, p. 3, Equation (3.1)])

$$(4.53) \quad Z_n^{(\alpha)}(x, k|q) = \frac{(\alpha q; q)_{nk}}{(q^k; q^k)_n} \sum_{j=0}^n \frac{(q^{-nk}; q^k)_j}{(\alpha q; q)_j} q^{kj(n+1)+kj(kj-1)/2} \frac{(ax)^{kj}}{(q^k; q^k)_j},$$

which corresponds, when $k = 1$, to the familiar q -Laguerre polynomials defined by (3.34). They also introduced a new pair of q -Konhauser biorthogonal polynomials which were motivated by a particular case ($\beta = 0$) of a system of biorthogonal polynomials studied by Al-Salam and Verma [7, p. 275, Equations (2.1) and (2.2)].

4.3 Multilinear Generating Functions

In his important memoirs on expansions of certain infinite products, L.J. Rogers ([101], [102], and [103]) introduced the continuous q -Hermite polynomials $H_n(x|q)$ and the continuous q -ultraspherical polynomials $C_n(x; \nu|q)$, defined by

$$(4.54) \quad \sum_{n=0}^{\infty} H_n(x|q) \frac{t^n}{(q;q)_n} = \frac{1}{h(x,t)}$$

and

$$(4.55) \quad \sum_{n=0}^{\infty} C_n(x;\nu|q) t^n = \frac{h(x,\nu t)}{h(x,t)},$$

where [*cf.* Equation (2.35)]

$$(4.56) \quad h(\cos \theta, \tau) = (\tau e^{i\theta}, \tau e^{-i\theta}; q)_{\infty}.$$

As a matter of fact, Rogers used his results involving these polynomials in order to prove the celebrated Rogers–Ramanujan identities.

Another set of q -Hermite polynomials defined explicitly by (*cf.*, *e.g.*, [4] and [142])

$$(4.57) \quad h_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} = \begin{bmatrix} n \\ n-k \end{bmatrix},$$

are closely related to the continuous q -Hermite polynomials *via*

$$(4.58) \quad H_n(\cos \theta|q) = e^{-in\theta} h_n(e^{2i\theta}|q),$$

so that any given result involving one set of q -Hermite polynomials would apply also to the other set.

For these q -Hermite polynomials, Ismail and Stanton [65] have derived several interesting multilinear generating functions with the help of certain integrals which they studied systematically. Combinatorial proofs of some of Ismail–Stanton results were given by Ismail, Stanton, and Viennot [66]. By applying the transformation theory of the q -hypergeometric series (2.34), Srivastava and Jain [128] derived the following general multilinear generating function for the q -Hermite polynomials:

$$\begin{aligned}
(4.59) \quad & \sum_{n_1, \dots, n_k=0}^{\infty} h_{n_1+\dots+n_k+s}(x|q) h_{m_1+n_1}(x_1|q) \cdots h_{m_k+n_k}(x_k|q) \\
& \cdot \frac{t_1^{n_1}}{(q; q)_{n_1}} \cdots \frac{t_k^{n_k}}{(q; q)_{n_k}} \\
& = \sum_{r_1=0}^{m_1} \cdots \sum_{r_k=0}^{m_k} q^{m_1 r_1 + \dots + m_k r_k} \prod_{j=1}^k \left\{ \frac{(q^{-m_j}; q)_{r_j}}{(q; q)_{r_j}} (x_j t_j)^{r_j} h_{m_j - r_j}(x_j|q) \right\} \\
& \cdot \left\{ \frac{1}{(x, t_1, x_1 t_1, \dots, t_k, x_k t_k; q)_{\infty}} {}_{2k} \Phi_{2k-1} \left[\begin{matrix} t_1, x_1 t_1, \dots, t_k, x_k t_k; \\ q, q^{1+s+R} \end{matrix} \right] \right. \\
& \quad \left. + \frac{x^s}{(1/x, x t_1, x x_1 t_1, \dots, x t_k, x x_k t_k; q)_{\infty}} \right. \\
& \quad \left. \cdot {}_{2k} \Phi_{2k-1} \left[\begin{matrix} x t_1, x x_1 t_1, \dots, x t_k, x x_k t_k; \\ q, q^{1+s+R} \end{matrix} \right] \right\},
\end{aligned}$$

where, for convenience, $R = r_1 + \dots + r_k$.

For $m_1 = \dots = m_k = 0$, the multilinear generating function (4.59) simplifies considerably, and we thus obtain (*cf.* Srivastava and Jain [128])

$$(4.60) \quad \sum_{n_1, \dots, n_k=0}^{\infty} h_{n_1+\dots+n_k+s}(x|q) h_{n_1}(x_1|q) \cdots h_{n_k}(x_k|q) \frac{t_1^{n_1}}{(q; q)_{n_1}} \cdots \frac{t_k^{n_k}}{(q; q)_{n_k}}$$

$$\begin{aligned}
&= \frac{1}{(x, t_1, x_1 t_1, \dots, t_k, x_k t_k; q)_\infty} {}_2\phi_{2k-1} \left[\begin{matrix} t_1, x_1 t_1, \dots, t_k, x_k t_k; \\ q, q^{1+s} \end{matrix} \right] \\
&+ \frac{x^s}{(1/x, x t_1, x x_1 t_1, \dots, x t_k, x x_k t_k; q)_\infty} {}_2\phi_{2k-1} \left[\begin{matrix} x t_1, x x_1 t_1, \dots, x t_k, x x_k t_k; \\ q, q^{1+s} \end{matrix} \right],
\end{aligned}$$

which, for $s = 0$, corresponds to a result due to Ismail and Stanton [65, p. 1042, Equation (5.14)]. Observe that, since s is not an index of summation, (4.60) is markedly different from another result presented by Ismail and Stanton [65, p. 1042, Equation (5.15)].

Formula (1.12) may be looked upon as a q -analogue of a multilinear generating function for the classical Hermite polynomials, which was given by Srivastava and Singhal (*cf.* [137, p. 140, Equation (4.6) with $s = 0$]; see also [138, p. 1239, Equation (4)] and [133, p. 496, Problems 11 and 12]).

Srivastava and Jain [128] developed their proof of the general result (4.59) in many stages. Indeed, in this process, they derived several interesting generating functions for the q -Hermite polynomials. For example, they proved in this way the following alternative form of Carlitz's bilinear generating function for q -Hermite polynomials (*cf.* [32, p. 96, Equation (4.1)]):

$$\begin{aligned}
(4.61) \quad \sum_{n=0}^{\infty} h_{n+s}(x|q) h_n(y|q) \frac{t^n}{(q; q)_n} &= \frac{(xyt^2; q)_\infty}{(t, xt, yt, xyt; q)_\infty} \\
&\cdot \frac{(xt; q)_s}{(xyt^2; q)_s} {}_2\phi_1 \left[\begin{matrix} yt, q^{-s}; \\ q^{1-s}/xt; \end{matrix} \right] q, q/t
\end{aligned}$$

or, equivalently,

$$(4.62) \quad \sum_{n=0}^{\infty} h_{n+s}(x|q) h_n(y|q) \frac{t^n}{(q; q)_n}$$

$$= \frac{1}{(x, t, yt; q)_\infty} {}_2\Phi_1 \left[\begin{matrix} t, yt; \\ q, q^{1+s} \\ q/x; \end{matrix} \right] + \frac{x^s}{(1/x, xt, xyt; q)_\infty} {}_2\Phi_1 \left[\begin{matrix} xt, xyt; \\ q, q^{1+s} \\ qx; \end{matrix} \right],$$

where s is a nonnegative integer (see also Al-Salam and Ismail [5, Section 2]).

5. Polynomial Expansions for Functions of Several Variables

Motivated by the remarkably vast literature on polynomial expansions for various special functions of one and more variables (*cf.*, *e.g.*, [130, pp. 339–343]), Srivastava ([118], [119]) has recently given several general classes of q -polynomial expansions for a multivariable function which is defined by a multiple power series with essentially arbitrary terms. The proof of one of these classes of q -polynomial expansions was based upon the q -combinatorial identity:

$$(5.1) \quad \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{(\rho/q^k; \sigma)_n}{(\rho/q^k; \sigma)_1} q^{k(k-1)/2} = \frac{\delta_{n,0}}{(\rho; \sigma)_1} \quad (\rho \neq 1),$$

where $\delta_{m,n}$ is the Kronecker delta, and [*cf.* also Equation (4.57)]

$$(5.2) \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k} = \begin{bmatrix} n \\ n-k \end{bmatrix} \quad (0 \leq k \leq n),$$

so that, by l'Hôpital's rule as (for instance) in Equation (2.39),

$$(5.3) \quad \lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{(n-k)! k!} \quad (0 \leq k \leq n).$$

The q -combinatorial identity (5.1), and its equivalent form:

$$(5.4) \quad \left[1 - \frac{a}{q} \right] \sum_{k=0}^n \frac{(-1)^k (ap^k; q)_{n-1}}{(p; p)_k (p; p)_{n-k}} p^{(n-k)(n-k-1)/2} = \delta_{n,0},$$

were proven by Srivastava [119], and Al-Salam and Verma [8], respectively, using the simple fact that the n th q -difference (or q -derivative) of a polynomial of degree less than n is zero.

The q -identity (5.1) or (5.4) corresponds to the $b \rightarrow 0$ limit case of the *bibasic* summation formula:

$$(5.5) \quad \left[1 - \frac{a}{p}\right] \left[1 - \frac{b}{p}\right] \sum_{k=0}^n \frac{(-1)^k (1-ap^{2k/b}) (ap^k, bp^{-k}; q)_{n-1}}{(p; p)_k (p; p)_{n-k} (ap^k/b; p)_{n+1}} p^{k(k-1)/2} = \delta_{n,0},$$

which is due to Gasper [54, p. 265, Equation (3.5)]. As a matter of fact, by using Jackson's ${}_8\Phi_7$ summation formula [109, p. 247, Equation (IV.8)], Gasper proved the general bibasic summation formula [54, p. 260, Equation (1.14)]:

$$(5.6) \quad \sum_{k=0}^n \frac{(1-ap^k q^k) (1-bp^k q^{-k})}{(1-a)(1-b)} \frac{(a, b; p)_k (c, a/bc; q)_k}{(q, aq/b; q)_k (ap/c, bcp; p)_k} q^k \\ = \frac{(ap, bp; p)_n (cq, aq/bc; q)_n}{(q, aq/b; q)_n (ap/c, bcp; p)_n},$$

and gave its several interesting applications and consequences.

An interesting unification (and generalization) of the remaining classes of the aforementioned q -polynomial expansions of Srivastava ([118], [119]) was proved recently by Jain and Srivastava [69] by making use of a known terminating version of the ${}_6\Phi_5$ summation formula [109, p. 96, Equation (3.3.1.3)]. We recall their result as

THEOREM 6 (Jain and Srivastava [69]). *In terms of the bounded complex coefficients $\Lambda(k_1, \dots, k_r)$ and Ω_n , $\forall n, k_j \in \mathbb{N}_0, j = 1, \dots, r$, define the multivariable function $F(z_1, \dots, z_r)$ by*

$$(5.7) \quad F(z_1, \dots, z_r) = \sum_{k_1, \dots, k_r=0}^{\infty} \Lambda(k_1, \dots, k_r) \Omega_M z_1^{k_1} \cdots z_r^{k_r},$$

where, and in what follows,

$$(5.8) \quad M = m_1 k_1 + \cdots + m_r k_r$$

for arbitrary positive integers m_1, \dots, m_r .

Then

$$(5.9) \quad F(\omega^{m_1} z_1, \dots, \omega^{m_r} z_r) = \sum_{n=0}^{\infty} \frac{(\mu; q)_n}{(\lambda q^n; q)_n} \frac{(\omega/\mu)^n}{(q; q)_n} \\ \cdot \sum_{\ell=0}^{\infty} \Omega_{n+\ell} \frac{(1/\mu; q)_\ell}{(\lambda q^{2n+1}; q)_\ell} \frac{\omega^\ell}{(q; q)_\ell} \sum_{k_1, \dots, k_r=0}^{M \leq n} \frac{(q^{-n}, \lambda q^n; q)_M (\lambda q^{n+\ell+M}/\mu; q)_{n-M}}{(q^{1-n}/\mu; q)_M (\lambda q^{2M+1}/\mu; q)_{n-M}} \\ \cdot \Lambda(k_1, \dots, k_r) (q^{m_1} z_1)^{k_1} \cdots (q^{m_r} z_r)^{k_r},$$

provided that the parameters λ and μ , and the variables ω, z_1, \dots, z_r , are so constrained that each side of the expansion formula (5.9) exists.

Jain and Srivastava [69, Section 3] applied Theorem 6 and its consequences to derive various bilinear formulas for the little q -Jacobi, q -Laguerre, and q -Hahn polynomials, and to obtain certain summation (or multiplication) formulas involving many of the q -Lauricella functions $\Phi_A^{(r)}$, $\Phi_B^{(r)}$, $\Phi_C^{(r)}$, and $\Phi_D^{(r)}$ of r variables.

6. Other Miscellaneous Developments

Srivastava ([122], [124]) considered the q -summation formula:

$$(6.1) \quad {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ cq^n, abq^{1-N}/c; \end{matrix} \middle| q, q \right] = \frac{(cq^n/a; q)_{N-n} (cq^n/b; q)_{N-n}}{(cq^n; q)_{N-n} (c/ab; q)_N}$$

$$\sum_{k=0}^n \binom{n}{k} \frac{(a;q)_k (b;q)_k (c/ab;q)_{n-k}}{(cq^N;q)_k} \left[\frac{c}{ab} \right]^k$$

($n, N \in \mathbb{N}_0$).

In particular, Srivastava [122] presented two independent proofs of the fact that

$$(6.2) \quad f(n, N) = \frac{(c;q)_N (c/ab;q)_N}{(c/a;q)_N (c/b;q)_N} {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ q, q \\ cq^n, abq^{1-N}/c; \end{matrix} \right]$$

($n, N \in \mathbb{N}_0$)

is a symmetric function of n and N . Subsequently, Srivastava and Damjanović [126] gave a rather elementary proof of the following slightly more general ${}_3\Phi_2$ transformation:

$$(6.3) \quad \frac{(cq^\nu;q)_N (c/ab;q)_N}{(cq^\nu/a;q)_N (cq^\nu/b;q)_N} {}_3\Phi_2 \left[\begin{matrix} a, b, q^{-N}; \\ q, q \\ cq^\nu, abq^{1-N}/c; \end{matrix} \right]$$

$$= q^{-\nu N} {}_3\Phi_2 \left[\begin{matrix} q^{1-\nu-N}/c, q^{-\nu}, q^{-N}; \\ q, \frac{abq}{c} \\ aq^{1-\nu-N}/c, bq^{1-\nu-N}/c; \end{matrix} \right],$$

where N is a nonnegative integer, as before, but ν is unrestricted, in general. Observe that the second member of (6.3) is symmetric in ν and N . Thus, in its special case when

$$\nu = n \quad (n \in \mathbb{N}_0),$$

the left-hand side of (6.3) leads immediately to the assertion that $f(n, N)$ defined by (6.2) is a symmetric function of n and N .

For $n = 0$, (6.1) reduces immediately to Jackson's ${}_3\Phi_2$ summation formula [109, p. 97, Equation (3.3.2.2)]. Furthermore, upon replacing a , b , and c by q^a , q^b , and q^c , respectively, and letting $q \rightarrow 1$, the general result (6.1) yields a ${}_3F_2$ transformation which is at least as old as Sheppard [106, p. 476, Equation (18)].

A number of linear and bilinear generating functions, and connection formulas, were proved recently by Srivastava and Jain [129] for q -Jacobi polynomials and for various q -orthogonal polynomials associated with them. They also studied the relationships between different q -extensions of the classical Gegenbauer (or ultraspherical) polynomials rather systematically (see also Askey and Ismail [19], [20]).

The subject of q -series (and associated q -polynomials and q -integrals) is rapidly growing at the present time. It is not practicable to report all these recent developments in an article like this one. Thus it would not seem to be out of place to draw the reader's attention toward a number of important recent publications including, for example, those by Adiga *et al.* [1], Agarwal [3], Andrews ([9], [10]), Andrews *et al.* [13], Aomoto [14], Berndt ([26], [27]), Bressoud [30], Dehesa [47], Dunkl [48], Fine [51], Kadell [71], Rahman ([93], [94]), Rahman and Verma [95], and Zeilberger [150]. Several important further references can be found in each of the works cited in this article.

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REFERENCES

1. C. Adiga, B.C. Berndt, S. Bhargava, and G.N. Watson, *Chapter 16 of Ramanujan's Second Notebook: Theta Functions and q-Series*, Mem. Amer. Math. Soc., Vol. 53 (No. 315), Amer. Math. Soc., Providence, Rhode Island, 1985.
2. A.K. Agarwal, E.G. Kalnins, and W. Miller, Canonical equations and symmetry techniques for q-series, *SIAM J. Math. Anal.*, **18**(1987), 1519–1538.
3. R.P. Agarwal, Glimpses of the theory of generalized hypergeometric functions, *Ganita Sandesh*, **2**(1988), 87–93.
4. W.A. Al-Salam and L. Carlitz, Some orthogonal q-polynomials, *Math. Nachr.*, **30**(1965), 47–61.
5. W.A. Al-Salam and M.E.H. Ismail, q-Beta integrals and the q-Hermite polynomials, *Pacific J. Math.*, **135**(1988), 209–221.
6. W.A. Al-Salam and A. Verma, q-Konhauser polynomials, *Pacific J. Math.*, **108**(1983), 1–7.
7. W.A. Al-Salam and A. Verma, A pair of biorthogonal sets of polynomials, *Rocky Mountain J. Math.*, **13**(1983), 273–279.
8. W.A. Al-Salam and A. Verma, On quadratic transformations of basic series, *SIAM J. Math. Anal.*, **15**(1984), 414–421.
9. G.E. Andrews, Applications of basic hypergeometric functions, *SIAM Rev.*, **16**(1974), 441–484.
10. G.E. Andrews, Problems and prospects for basic hypergeometric functions, in *Theory and Application of Special Functions* (R. Askey, Editor), pp. 191–224, Academic Press, New York, San Francisco, and London, 1975.
11. G.E. Andrews and R. Askey, Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials, in *Higher Combinatorics* (M. Aigner, Editor), pp. 3–26, D. Reidel Publishing Company, Dordrecht and Boston, 1977.
12. G.E. Andrews and R. Askey, Classical orthogonal polynomials, in *Polynômes Orthogonaux et Applications* (C. Brezinski, A. Draux, A.P. Magnus, P. Maroni, et A. Ronveaux, Editors), pp. 36–62, Springer-Verlag, Berlin and New York, 1985.
13. G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan, and R.A. Rankin (Editors), *Ramanujan Revisited*, Academic Press, New York, London, Sydney, Tokyo, and Toronto, 1988.
14. K. Aomoto, Jacobi polynomials associated with Selberg integrals, *SIAM J. Math. Anal.*, **18**(1987), 545–549.

15. R. Askey, The q -Gamma and q -Beta functions, *Applicable Anal.*, **8**(1978), 125–141.
16. R. Askey, A q -extension of Cauchy's form of the Beta integral, *Quart. J. Math. Oxford Ser. (2)*, **32**(1981), 255–266.
17. R. Askey, An elementary evaluation of a Beta type integral, *Indian J. Pure Appl. Math.*, **14**(1983), 892–895.
18. R. Askey and G. Gasper, Positive Jacobi polynomial sums. II, *Amer. J. Math.*, **98**(1976), 709–737.
19. R. Askey and M.E.H. Ismail, A generalization of ultraspherical polynomials, in *Studies in Pure Mathematics* (P. Erdős, Editor), pp. 55–78, Birkhäuser, Basel and Boston, 1983.
20. R. Askey and M. Ismail, *Recurrence Relations, Continued Fractions and Orthogonal Polynomials*, Mem. Amer. Math. Soc., Vol. 49 (No. 300), Amer. Math. Soc., Providence, Rhode Island, 1984.
21. R. Askey and J. Wilson, A set of orthogonal polynomials that generalize the Racah coefficients or $6-j$ symbols, *SIAM J. Math. Anal.*, **10**(1979), 1008–1021.
22. R. Askey and J. Wilson, *Some Basic Hypergeometric Orthogonal Polynomials that Generalize Jacobi Polynomials*, Mem. Amer. Math. Soc., Vol. 54 (No. 319), Amer. Math. Soc., Providence, Rhode Island, 1985.
23. W.N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, Cambridge University Press, Cambridge, London, and New York, 1935; Reprinted by Stechert-Hafner Service Agency, New York and London, 1964.
24. W.N. Bailey, The generating function of Jacobi polynomials, *J. London Math. Soc.*, **13**(1938), 8–12.
25. N. Barik, Some general generating functions for certain special functions, *Pure Math. Manuscript*, **5**(1986), 97–114.
26. B.C. Berndt, *Ramanujan's Notebooks*, Vol. I, Springer-Verlag, New York and Berlin, 1985.
27. B.C. Berndt, *Ramanujan's Notebooks*, Vol. II, Springer-Verlag, New York and Berlin, 1988.
28. L.C. Biedenharn, J.M. Blatt, and M.E. Rose, Some properties of the Racah and associated coefficients, *Rev. Modern Phys.*, **24**(1952), 249–257.
29. S. Bochner, Über Sturm-Liouvillesche Polynomsysteme, *Math. Zeitschr.*, **29**(1929), 730–736.
30. D.M. Bressoud, Almost poised basic hypergeometric series, *Proc. Indian Acad. Sci. Math. Sci.*, **96**(1987), 61–66.

31. L. Carlitz, Some generating functions for Laguerre polynomials, *Duke Math. J.*, **35**(1968), 825–827.
32. L. Carlitz, Generating functions for certain q -orthogonal polynomials, *Collect. Math.*, **23**(1972), 91–104.
33. L. Carlitz, A class of generating functions, *SIAM J. Math. Anal.*, **8**(1977), 518–532.
34. L. Carlitz and H.M. Srivastava, Some new generating functions for the Hermite polynomials, *J. Math. Anal. Appl.* (to appear).
35. R.C.S. Chandel and S. Sahgal, Further applications and extensions of the addition theorems of Srivastava, Lavoie and Tremblay, *Indian J. Pure Appl. Math.*, **18**(1987), 830–834.
36. R.C.S. Chandel and S. Sahgal, Binomial analogues of the class of addition theorems of Srivastava, Lavoie and Tremblay, *Indian J. Math.* (to appear).
37. R.C.S. Chandel and H.C. Yadava, Applications of Srivastava's theorem, *Indian J. Pure Appl. Math.*, **15**(1984), 1313–1318.
38. R.C.S. Chandel and H.C. Yadava, Additional applications of a binomial analogue of Srivastava's theorem, *Indian J. Math.*, **27**(1985), 137–141.
39. L.-C. Chen and M.E.H. Ismail, On asymptotics of Jacobi polynomials, *SIAM J. Math. Anal.*, **20**(1989).
40. L. Chihara, On the zeros of the Askey–Wilson polynomials, with applications to coding theory, *SIAM J. Math. Anal.*, **18**(1987), 191–207.
41. T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, London, and Paris, 1978.
42. A.K. Chongdar, On a class of bilateral generating functions for certain special functions, *Proc. Indian Acad. Sci. Math. Sci.*, **95**(1986), 133–140.
43. A.K. Chongdar, On the unification of a class of bilateral generating functions for certain special functions, *Tamkang J. Math.*, **18**(1987), no. 3, 53–59.
44. A.K. Chongdar, A general theorem on the unification of a class of bilateral generating functions for certain special functions, *Bull. Calcutta Math. Soc.*, **81**(1989), 46–53.
45. J. Cigler, A note on a paper by L. Carlitz, *SIAM J. Math. Anal.*, **10**(1979), 78–79.
46. L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.*, **154**(1985), 137–152.
47. J.S. Dehesa, On a general system of orthogonal q -polynomials, *J. Comput. Appl. Math.*, **5**(1979), 37–45.

48. C.F. Dunkl, Orthogonal polynomials in two variables of q -Hahn and q -Jacobi type, *SIAM J. Algebraic Discrete Methods*, **1**(1980), 137–151.
49. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, Toronto, and London, 1953.
50. H. Exton, *q -Hypergeometric Functions and Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Brisbane, Chichester, and Toronto, 1983.
51. N.J. Fine, *Basic Hypergeometric Series and Applications*, Mathematical Surveys and Monographs, Vol. 27, Amer. Math. Soc., Providence, Rhode Island, 1988.
52. G. Gasper, A short proof of an inequality used by de Branges in his proof of the Bieberbach, Robertson and Milin conjectures, *Complex Variables Theory Appl.*, **7**(1986), 45–50.
53. G. Gasper, q -Extensions of Clausen's formula and of the inequalities used by de Branges in his proof of the Bieberbach, Robertson, and Milin conjectures, *SIAM J. Math. Anal.*, **20**(1989), 1019–1034.
54. G. Gasper, Summation, transformation, and expansion formulas for bibasic series, *Trans. Amer. Math. Soc.*, **312**(1989), 257–277.
55. G. Gasper and M. Rahman, Nonnegative kernels in product formulas for q -Racah polynomials, *J. Math. Anal. Appl.*, **95**(1983), 304–318.
56. G. Gasper and M. Rahman, Product formulas of Watson, Bailey and Bateman types and positivity of the Poisson kernel for q -Racah polynomials, *SIAM J. Math. Anal.*, **15**(1984), 768–789.
57. H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.*, **29**(1962), 51–63.
58. E. Grosswald, *Bessel Polynomials*, Springer-Verlag, Berlin, Heidelberg, and New York, 1978.
59. W. Hahn, Über die Jacobischen Polynome und zwei verwandte Polynomklassen, *Math. Zeitschr.*, **39**(1935), 634–638.
60. W. Hahn, Über Orthogonalpolynome, die q -Differenzgleichungen genügen, *Math. Nachr.*, **2**(1949), 4–34.
61. W. Hahn, Über Polynome, die gleichzeitig zwei verschiedenen Orthogonalsystemen angehören, *Math. Nachr.*, **2**(1949), 263–278.
62. E.R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, New Jersey, 1975.
63. J.H. Hubbell and H.M. Srivastava, Certain theorems on bilateral generating functions involving Hermite, Laguerre, and Gegenbauer polynomials, *J. Math. Anal. Appl.* (to appear).

64. M.E.H. Ismail, Asymptotics of the Askey–Wilson and q -Jacobi polynomials, *SIAM J. Math. Anal.*, **17**(1986), 1475–1482.
65. M.E.H. Ismail and D. Stanton, On the Askey–Wilson and Rogers polynomials, *Canad. J. Math.*, **40**(1988), 1025–1045.
66. M.E.H. Ismail, D. Stanton, and G. Viennot, The combinatorics of q -Hermite polynomials and the Askey–Wilson integral, *European J. Combin.*, **8**(1987), 379–392.
67. M.E.H. Ismail and J.A. Wilson, Asymptotic and generating relations for the q -Jacobi and ${}_4\phi_3$ polynomials, *J. Approx. Theory*, **36**(1982), 43–54.
68. F.H. Jackson, Basic double hypergeometric functions (II), *Quart. J. Math. Oxford Ser.*, **15**(1944), 49–61.
69. V.K. Jain and H.M. Srivastava, Some general q -polynomial expansions for functions of several variables and their applications to certain q -orthogonal polynomials and q -Lauricella functions, *Bull. Soc. Roy. Sci. Liège*, **58**(1989), 13–24.
70. V.K. Jain and H.M. Srivastava, New results involving a certain class of q -orthogonal polynomials (to appear).
71. K.W.J. Kadell, *Path Functions and Generalized Basic Hypergeometric Functions*, Mem. Amer. Math. Soc., Vol. 65 (No. 360), Amer. Math. Soc., Providence, Rhode Island, 1987.
72. E.G. Kalnins and W. Miller, Symmetry techniques for q -series: Askey–Wilson polynomials, *Rocky Mountain J. Math.*, **19**(1989).
73. S. Karlin and J.L. McGregor, The Hahn polynomials, formulas and an application, *Scripta Math.*, **26**(1961), 33–46.
74. J.D.E. Konhauser, Biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J. Math.*, **21**(1967), 303–314.
75. H.L. Krall, On derivatives of orthogonal polynomials, *Bull. Amer. Math. Soc.*, **42**(1936), 423–428.
76. H.L. Krall and O. Frink, A new class of orthogonal polynomials: The Bessel polynomials, *Trans. Amer. Math. Soc.*, **65**(1949), 100–115.
77. Y.L. Luke, *Mathematical Functions and Their Approximations*, Academic Press, New York, San Francisco, and London, 1975.
78. E.B. McBride, *Obtaining Generating Functions*, Springer Tracts in Natural Philosophy, Vol. 21, Springer-Verlag, New York, Heidelberg, and Berlin, 1971.
79. J. Meixner, Orthogonale Polynomsysteme mit einer besonderen Gestalt de erzeugenden Funktion, *J. London Math. Soc.*, **9**(1934), 6–13.

80. W. Miller, A note on Wilson polynomials, *SIAM J. Math. Anal.*, **18**(1987), 1221–1226.
81. B. Nassrallah and M. Rahman, Projection formulas, a reproducing kernel and generating function for q -Wilson polynomials, *SIAM J. Math. Anal.*, **16**(1985), 186–197.
82. A.F. Nikiforov and S.K. Suslov, *Classical Orthogonal Polynomials* (Russian), "Znanie", Moscow, 1985.
83. A.F. Nikiforov, S.K. Suslov, and V.B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable* (Russian), "Nauka", Moscow, 1985.
84. A.F. Nikiforov and V.B. Uvarov, *Special Functions of Mathematical Physics: A Unified Introduction with Applications* (Translated from the Russian by R.P. Boas), Birkhäuser, Basel and Boston, 1988.
85. F. Pollaczek, Sur une famille de polynômes orthogonaux à quatre paramètres, *C.R. Acad. Sci. Paris*, **232**(1950), 2254–2256.
86. G. Racah, Theory of complex spectra. II, *Phys. Rev.*, **62**(1942), 438–462.
87. M. Rahman, The linearization of the product of continuous q -Jacobi polynomials, *Canad. J. Math.*, **33**(1981), 255–284.
88. M. Rahman, Reproducing kernels and bilinear sums for q -Racah and q -Wilson polynomials, *Trans. Amer. Math. Soc.*, **273**(1982), 483–508.
89. M. Rahman, A simple evaluation of Askey and Wilson's q -Beta integral, *Proc. Amer. Math. Soc.*, **92**(1984), 413–417.
90. M. Rahman, q -Wilson functions of the second kind, *SIAM J. Math. Anal.*, **17**(1986), 1280–1286.
91. M. Rahman, A projection formula for the Askey–Wilson polynomials and an application, *Proc. Amer. Math. Soc.*, **103**(1988), 1099–1107.
92. M. Rahman, Some extensions of the Askey–Wilson's q -Beta integral and the corresponding orthogonal systems, *Canad. Math. Bull.*, **33**(1988), 467–476.
93. M. Rahman, A note on the biorthogonality of q -Bessel functions, *Canad. Math. Bull.* (to appear).
94. M. Rahman, Some cubic summation formulas for basic hypergeometric series, *Proc. Amer. Math. Soc.* (to appear).
95. M. Rahman and A. Verma, Infinite sums of products of continuous q -ultraspherical functions, *Rocky Mountain J. Math.*, **17**(1987), 371–384.
96. E.D. Rainville, *Special Functions*, Macmillan, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
97. R.K. Raina, On certain generating relations, *Math. Japon.*, **27**(1982), 591–595.

98. R.K. Raina, Some new generating functions, *Mat. Vesnik*, **37**(1985), 305–310.
99. R.K. Raina, Some new generating relations for certain polynomial systems involving the H-functions, *Indian J. Pure Appl. Math.*, **16**(1985), 498–503.
100. R.K. Raina, On certain generating functions associated with the H-functions, *Bull. Inst. Math. Acad. Sinica*, **13**(1985), 149–155.
101. L.J. Rogers, On the expansion of some infinite products, *Proc. London Math. Soc.*, **24**(1893), 337–352.
102. L.J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.*, **25**(1894), 318–343.
103. L.J. Rogers, Third memoir on the expansion of certain infinite products, *Proc. London Math. Soc.*, **26**(1895), 15–32.
104. G.–C. Rota, *Finite Operator Calculus*, Academic Press, New York and London, 1975.
105. G.–C. Rota, D. Kahaner, and A. Odlyzko, Finite operator calculus, *J. Math. Anal. Appl.*, **42**(1973), 684–760.
106. W.F. Sheppard, Summation of the coefficients of some terminating hypergeometric series, *Proc. London Math. Soc.* (2), **10**(1912), 469–478.
107. P.N. Shrivastava and B. Kaur, Some theorems, associated with bilateral generating functions involving Hermite, Laguerre and Gegenbauer polynomials, *J. Indian Math. Soc. (N.S.)*, **49**(1985), 179–183.
108. J.P. Singhal and H.M. Srivastava, A class of bilateral generating functions for certain classical polynomials, *Pacific J. Math.*, **42**(1972), 755–762.
109. L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, London, and New York, 1966.
110. N. Ja. Sonine, Über die angenäherte Berechnung der bestimmten Integrale unter über die dabei workommenden ganzen Funktionen (Russian), *Warsaw Univ. Izv.*, **18**(1887), 1–76.
111. L. Spencer and U. Fano, Penetration and diffusion of X-rays. Calculation of spatial distribution by polynomial expansion, *J. Res. Nat. Bur. Standards*, **46**(1951), 446–461.
112. H.M. Srivastava, Some generalizations of Carlitz's theorem, *Pacific J. Math.*, **85**(1979), 471–477.
113. H.M. Srivastava, Some bilateral generating functions for a certain class of special functions. I and II, *Nederl. Akad. Wetensch. Indag. Math.*, **42**(1980), 221–233 and 234–246.

114. H.M. Srivastava, Some finite summation formulas involving generalized hypergeometric functions, *Indian J. Math.* (Prof. C.T. Rajagopal Memorial Vol.), **22**(1980), 1–7.
115. H.M. Srivastava, Some biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J. Math.*, **98**(1982), 235–250.
116. H.M. Srivastava, An elementary proof of Bailey's bilinear generating function for Jacobi polynomials and of its q -analogue, *IMA J. Appl. Math.*, **29**(1982), 275–280.
117. H.M. Srivastava, A q -extension of Bailey's bilinear generating function for the Jacobi polynomials, *Rocky Mountain J. Math.*, **13**(1983), 461–465.
118. H.M. Srivastava, Certain q -polynomial expansions for functions of several variables, *IMA J. Appl. Math.*, **30**(1983), 315–323.
119. H.M. Srivastava, Certain q -polynomial expansions for functions of several variables. II, *IMA J. Appl. Math.*, **33**(1984), 205–209.
120. H.M. Srivastava, A family of q -generating functions, *Bull. Inst. Math. Acad. Sinica*, **12**(1984), 327–336.
121. H.M. Srivastava, A class of finite summation formulas involving generalized hypergeometric functions, *Boll. Un. Mat. Ital. A* (6), **5**(1986), 311–320.
122. H.M. Srivastava, A transformation for an n -balanced ${}_3\Phi_2$, *Proc. Amer. Math. Soc.*, **101**(1987), 108–112.
123. H.M. Srivastava, A unified presentation of certain bilateral generating functions for a class of polynomials, in *Topics in Mathematical Analysis* (Th. M. Rassias, Editor), pp. 829–840, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1989.
124. H.M. Srivastava, An extension of the q -Saalschützian theorem, *Acta Math. Hungar.*, **53**(1989), 115–118.
125. H.M. Srivastava and A.K. Agarwal, Generating functions for a class of q -polynomials, *Ann. Mat. Pura Appl.* (4), **157**(1989), 101–121.
126. H.M. Srivastava and S.Lj. Damjanović, An elementary proof of a certain transformation for an n -balanced hypergeometric ${}_3\Phi_2$ series, *Proc. Japan Acad. Ser. A Math. Sci.*, **64**(1988), 198–200.
127. H.M. Srivastava and S. Handa, Some general theorems on generating functions and their applications, *Appl. Math. Lett.*, **1**(1988), 391–394.
128. H.M. Srivastava and V.K. Jain, Some multilinear generating functions for q -Hermite polynomials, *J. Math. Anal. Appl.*, **144**(1989), 101–121.
129. H.M. Srivastava and V.K. Jain, Some formulas involving q -Jacobi and related polynomials, *Ann. Mat. Pura Appl.* (4) (to appear).

130. H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1985.
131. H.M. Srivastava and J.-L. Lavoie, A certain method of obtaining bilateral generating functions, *Nederl. Akad. Wetensch. Indag. Math.*, **37**(1975), 304–320.
132. H.M. Srivastava, J.-L. Lavoie, and R. Tremblay, A class of addition theorems, *Canad. Math. Bull.*, **26**(1983), 438–445.
133. H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984.
134. H.M. Srivastava and R. Panda, On the unified presentation of certain classical polynomials, *Boll. Un. Mat. Ital.* (4), **12**(1975), 306–314.
135. H.M. Srivastava and B.S. Popov, A class of generating functions for the Jacobi and related polynomials, *Studies in Appl. Math.*, **78**(1988), 177–182.
136. H.M. Srivastava and J.P. Singhal, A class of polynomials defined by generalized Rodrigues' formula, *Ann. Mat. Pura Appl.* (4), **90**(1971), 75–85.
137. H.M. Srivastava and J.P. Singhal, Some extensions of the Mehler formula, *Proc. Amer. Math. Soc.*, **31**(1972), 135–141.
138. H.M. Srivastava and J.P. Singhal, Some formulas involving the products of several Jacobi or Laguerre polynomials, *Acad. Roy. Belg. Bull. Cl. Sci.* (5), **58**(1972), 1238–1247.
139. H.M. Srivastava and J.P. Singhal, New generating functions for Jacobi and related polynomials, *J. Math. Anal. Appl.*, **41**(1973), 748–752.
140. D. Stanton, A short proof of a generating function for Jacobi polynomials, *Proc. Amer. Math. Soc.*, **80**(1980), 398–400.
141. V. Strehl, Combinatorics of Jacobi-configurations. II: The Srivastava–Singhal generating function revisited, *Discrete Math.*, **73**(1988), 221–232.
142. G. Szegő, Ein Beitrag zur Theorie der Thetafunktionen, *Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl.*, **19**(1926), 242–252.
143. G. Szegő, *Orthogonal Polynomials*, Fourth ed., Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, Rhode Island, 1975.
144. P.L. Tchebycheff, Sur les fractions continues, in *Oeuvres*, Tome I, pp. 203–230, Chelsea Publishing Company, New York, 1962.
145. P.L. Tchebycheff, Sur une nouvelle série, in *Oeuvres*, Tome I, pp. 381–384, Chelsea Publishing Company, New York, 1962.

146. P.L. Tchebycheff, Sur l'interpolation des valeurs équidistantes, in *Oeuvres*, Tome II, pp. 219–242, Chelsea Publishing Company, New York, 1962.
147. F.G. Tricomi, *Serie Ortogonali di Funzioni*, Torino, 1948.
148. M. Weber and A. Erdélyi, On the finite difference analogue of Rodrigues' formula, *Amer. Math. Monthly*, **59**(1952), 163–168.
149. J.A. Wilson, Some hypergeometric orthogonal polynomials, *SIAM J. Math. Anal.*, **11**(1980), 690–701.
150. D. Zeilberger, A Stembridge–Stanton style elementary proof of the Habsieger–Kadell q -Morris identity, *Discrete Math.* (to appear).

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