Supercritical Phase Transitions from Number Theory

by

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B.Sc., University of Victoria, 2021

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

in the Department of Mathematics and Statistics

We acknowledge and respect the ləkʷəŋən peoples on whose traditional territory the university stands and the Songhees, Esquimalt, and W̱SÁNEĆ peoples whose historical relationships with the land continue to this day.

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ABSTRACT

We classify the KMS$_\beta$ states of the right $ax + b$ C*-dynamical system of $\mathbb{N}$ in the supercritical range $\beta \in (0, 1]$, thus completing the classification of KMS$_\beta$ states initiated in [19]. We show that the simplex of KMS$_\beta$ states is affinely isomorphic to the simplex of subconformal measures on the circle. We then provide explicit formulas for the extremal subconformal measures and corresponding KMS$_\beta$ states in terms of classical arithmetic functions. For $\beta \in (0, 1]$, our measures are parameterized by the compact space $\mathbb{N}^\times \cup \{\infty\}$, and in particular, demonstrate phase transition at each value of β, a novel feature among C*-dynamical systems related to number theory.

Another new feature of the right $ax + b$ system is the existence of equivariant quotients, corresponding to the quotient rings $\mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{N}^\times$. We provide a classification of the KMS$_\beta$ states of the quotient C*-dynamical systems, and show that the quotient systems exhibit spontaneous symmetry-breaking with respect to the group of units $(\mathbb{Z}/m\mathbb{Z})^*$. We then use this action to compute the type of the high-temperature KMS$_\beta$ states with parameter belonging to $\mathbb{N}^\times \subseteq \mathbb{N}^\times \cup \{\infty\}$.

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ACKNOWLEDGEMENTS

I would like to thank my advisor Marcelo Laca for his mentorship. I am grateful for your guidance in both mathematics and acadamia, and for always lending an encouraging ear when I have a new idea to share.

I would also like to thank the other member of my supervisory committee, Stephen Scully, and my external examiner, Aidan Sims.

Finally, I would like to thank my family – Mom, Dad, Nick, and of course, Annie. I will always be grateful for your love and support.

This research was funded in part by the Natural Sciences and Engineering Research Council of Canada through an Alexander Graham Bell CGS-M award (2021).

DEDICATION

For Mom and Dad.

Chapter 1
Introduction

For centuries, analysis and number theory have shared a fascinating and bountiful interplay - one has frequently motivated developments in the other. A key aspect of this connection is the notion of the *zeta function*, a meromorphic function whose growth and decay is dictacted by the distribution of norms. The connections between the Riemann zeta function (first introduced by Euler) and the distribution of prime numbers were a motivating factor in Riemann's development of complex analysis, which eventually led to the first proofs of the Prime Number Theorem by Hadamard [16] and de la Vallée Poussin [37]. A century later, the theory of Fourier analysis was marvelously employed by Tate [36] in his analysis of zeta functions for local and global fields, proving their functional equations and meromorphic continuations. In recent years, investigations initiated by Connes in the 90's have examined the relationships between algebraic structures appearing in number theory and operator algebras, and how quantum statistical mechanics can be used to couple these structures with zeta functions [5, 8, 9, 10, 11, 30]. This thesis constitutes a modest step forward in this program.

The first association between quantum statistical mechanics and number theory arose in the independent work of physicists Spector [35] and Julia [21]. They investigated the model of a Bosonic Fock space in which prime numbers play the role of elementary states. The prime p is assigned the energy $\ln(p)$, resulting in a system with partition function equal to the Riemann zeta function $\zeta(\beta)$, the parameter β being the inverse-temperature of the system. This quickly caught the attention of operator algebraists, including Bost and Connes, who proposed another system based on the arithmetic of Hecke algebras exhibiting the same partition function with a richer space of equilibrium states. For this system, there is *phase transition* (plurality of equilibrium states) at low temperatures ($\beta > 1$), where extremal states are parameterized by faithful characters of the group $\mathbb{Q}/\mathbb{Z}$. (Superpositions of these *extremal* equilibrium states are also equilibrium states, yielding a simplex of equilibria; more on this later.) At high temperatures ($\beta \leq 1$), there is a unique equilibrium state.

Another remarkable feature of this system are its equivariant homomorphisms, which includes a group naturally identified with the absolute Galois group of $\mathbb{Q}$. The equivariant homomorphisms induce an action on the space of equilibrium states at a fixed temperature, and this action is free and transitive at low temperatures and trivial at high temperatures, leaving the equilibrium states invariant – a phenomenon known as *spontaneous symmetry-breaking* in quantum statistical mechanics. Spontaneous symmetry-breaking was first observed in models of magnetic materials, wherein rotational symmetries are spontaneously lost as the material cools below the Curie temperature, and here it is observed in models of... arithmetic? The ramifications of this observation are still an active subject of research.

Bost and Connes' work has inspired a multitude of new examples over the last 30 years, exhibiting some or all of the same interesting features. The two examples most relevant to us are the $ax + b$ system considered by Laca and Raeburn [26], and the opposite $ax + b$ system considered by an Huef, Laca, and Raeburn [19], though we prefer the names "left $ax + b$ system" and "right $ax + b$ system."

The underlying algebra of the left $ax + b$ system is generated from an isometry S representing addition and isometries V_a for $a \in \mathbb{N}^\times$ representing multiplication, subject to the relations $V_a S = S^a V_a$ (among others). These operators constitute a *Nica-covariant* representation of the $ax + b$ monoid $\mathbb{N} \rtimes \mathbb{N}^\times$ (now referred to as the *left $ax + b$* monoid), wherein the projections $S^n V_a V_a^* S^{n*}$ are ordered according to left-divisibility in the monoid $\mathbb{N} \rtimes \mathbb{N}^\times$. In the $ax + b$ system, the isometry S leaves energy invariant, while the isometry V_a increases the energy by $\log(a)$. Despite the energy levels being analogous to those in the Bost-Connes system, Laca and Raeburn showed that the partition function of this system is in fact $\zeta(\beta - 1)$. There is also phase transition for this system at low temperatures ($\beta > 2$), a unique equilibrium state at intermediate temperatures ($1 \leq \beta \leq 2$), and no equilibrum at high temperatures ($\beta < 1$) [26, Theorem 7.1]. The extremal equilibrium states at low temperatures are parameterized by characters on the group $\mathbb{Z}$, or equivalently, by the circle $\mathbb{T}$. However, unlike in the Bost-Connes system, this system does not admit non-trivial symmetries.

The right $ax+b$ system was then considered by an Huef, Laca, and Raeburn, where the relations between the addition and multiplication operators is replaced with the new relation $SV_a = V_a S^a$. This is again a Nica-covariant representation, only this time of the opposite $ax+b$ monoid, $(\mathbb{N} \rtimes \mathbb{N}^\times)^{\mathrm{op}} \cong \mathbb{N}^\times \ltimes \mathbb{N}$ (the *right $ax+b$* monoid), wherein projections $V_a S^n S^{n*} V_a^*$ are ordered by left-divisibility in $\mathbb{N}^\times \ltimes \mathbb{N}$. The partition function is once again $\zeta(\beta)$, shedding the shift of the inverse temperature that was present in the left system. At low temperatures, the extremal equilibrium states again exhibits phase transition parameterized by $\mathbb{T}$ [19, Theorem 8.1]. At high temperatures, a new and fascinating behaviour was observed: phase transition at the critical temperature ($\beta = 1$). The authors demonstrated this with three examples, each arising as a limit of subcritical equilibrium states as $\beta \searrow 1$: one from the identity element of $\mathbb{T}$, one from the element of order two, and one from Lebesgue measure (which at low temperatures is a uniform superposition of the extremal equilibria) [19, Section 9]. However, beyond these three examples, nothing of the high temperature equilibrium states was known: there was no indication of how the phase transition at the critical temperature appeared or if there existed equilibrium states at higher temperatures. It was these two questions which formed the motivation behind this thesis.

Before proceeding, we should first explain the choice of C*-algebra in this thesis. An Huef, Laca, and Raeburn considered the Toeplitz algebra of the monoid $\mathbb{N}^\times \ltimes \mathbb{N}$, but in the analysis of its KMS states, they showed that the states all factored through the so-called *additive boundary quotient* [19, Theorem 7.1]. This quotient is obtained by imposing the relation $SS^* = 1$, making the addition operator unitary. The resulting algebra is also a Toeplitz algebra, generated by the left-regular representation of the monoid $\mathbb{N}^\times \ltimes \mathbb{Z}$. Since we are only interested in analyzing the KMS states, we prefer to work the with the monoid $\mathbb{N}^\times \ltimes \mathbb{Z}$ and its Toeplitz algebra for our right $ax + b$ system. We write U for the addition unitary and V_a for the multiplication isometries.

A useful fact (shown in Chapter 2) is that the Toeplitz algebra $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ is equal to the closed linear span of monomials of the form $V_a U^k V_b^*$, for $a, b \in \mathbb{N}^\times$ and $k \in \mathbb{Z}$. The dynamics on the algebra are determined by their definition on these monomials,

$$\sigma_t(V_a U^k V_b^*) = (a/b)^{it} V_a U^k V_b^*,$$

as are the KMS$_\beta$ states on $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma)$.

In order to state our main result, we recall the definition of two classical arithmetic functions. The *Euler totient function* $\varphi(n)$ counts the number of integers between 1 and n which are relatively prime to n. Equivalently, this is the order of the group $(\mathbb{Z}/n\mathbb{Z})^*$ of units in the ring $\mathbb{Z}/n\mathbb{Z}$. Euler gave an explicit formula for this function, namely

$$\varphi(n) = n \prod_{p|n} \frac{p-1}{p} = n \prod_{p|n} 1 - p^{-1}.$$

Our formulas involve a *generalized totient function* $\varphi_\beta(n)$, given by

$$\varphi_\beta(n) = n^\beta \prod_{p|n} 1 - p^{-\beta}.$$

We also make use of the Möbius function $\mu : \mathbb{N}^\times \to \{-1, 0, 1\}$, where $\mu(n)$ is $(-1)^k$ when n is square-free where k is the number of prime divisors of n, and $\mu(n) = 0$ if n is not square-free.

Our main result is the following explicit formulas for extremal KMS$_\beta$ states:

Theorem 1.1. *For each* $0 < \beta \leq 1$*, there is a Choquet simplex of* KMS$_\beta$ *states of the system* $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma)$ *with extreme points* $\psi_{n,\beta}$ *parameterized by* $n \in \mathbb{N}^\times \cup \{\infty\}$ *that* *are given by the formulas*

$$\psi_{n,\beta}(V_a U^k V_b^*) = \delta_{a,b} a^{-\beta} \left(\frac{n}{\gcd(n,k)} \right)^{-\beta} \sum_{d \mid \frac{n}{\gcd(n,k)}} \mu(d) \frac{\varphi_\beta(d)}{\varphi(d)} \tag{1.1}$$

for $n < \infty$ and

$$\psi_{\infty,\beta}(V_a U^k V_b^*) = \delta_{a,b} \delta_{k,0} a^{-\beta}.$$

For each $n < \infty$, the state $\psi_{n,\beta}$ is a factor state of type III_1*; in particular, these states are also extremal in the full simplex of* KMS_β *states.*

The KMS_1 states of [19] correspond to the states $\psi_{1,1}$, $\psi_{2,1}$, and $\psi_{\infty,1}$, in the above parameterization. We examine this in closer detail in the introduction to Chapter 5.

By [19, Prop 7.2], a state ψ on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ satisfies the KMS_β condition if and only if

$$\psi(V_a U^k V_b^*) = \delta_{a,b} a^{-\beta} \psi(U^k), \qquad k \in \mathbb{Z},\ a, b \in \mathbb{N}^\times, \tag{1.2}$$

and hence every KMS_β state is uniquely determined by its restriction to $C^*(U) \cong C(\mathbb{T})$, corresponding to a probability measure on $\mathbb{T}$. At first-glance, formula (1.2) appears to give a correspondence between KMS_β states and probability measures on $\mathbb{T}$, extending states on $C^*(U)$ to monomials in $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. However, there are two obstacles: (1) the linear functional appearing in (1.2) is initially defined on monomials, and may fail to extend to a linear functional on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$; (2) the extension of a state on $C^*(U)$ to $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, if it does exist, may fail to be a positive linear functional, and hence not a state. These issues are resolved precisely when a measure is *β-subconformal*, meaning that it satisfies a series of positivity conditions, appearing in Chapter 4.

The problem of classifying KMS_β states is therefore equivalent to classifying the β-subconformal measures on $\mathbb{T}$. Indeed, the formulas for $\psi_{n,\beta}$ in Theorem 1.1 are realized as the Fourier moments of extremal β-subconformal measure $\nu_{n,\beta}$ on $\mathbb{T}$ through this correspondence.

The technical core of this thesis lies in the classification of β-subconformal measures for $\beta \leq 1$, in Chapter 5. Subconformality of a measure is inherited by its atomic and non-atomic components, so we divide our efforts into these two cases. When $\beta \leq 1$, the divergence of the ζ-series implies that the atomic β-subconformal measures are supported on the torsion subgroup of $\mathbb{T}$. Fourier analysis on subgroups of order n reveals that the β-subconformal measures are obtained from a "twisting" of uniform measures through the wrapping action of $\mathbb{N}^\times$ on $\mathbb{T}$ by endomorphisms, and convex combinations of these yield all of the atomic subconformal measures. The non-atomic 1-subconformal measures are treated through a limiting procedure, whereby the measures are realized as limits of β-subconformal measures as β decreases to 1, using simple properties of functions appearing in analytic number theory. Under a limit hypothesis (which we believe could hold for all non-atomic 1-subconformal measures), the only non-atomic measure appearing in this limit is Lebesgue measure, and the classification of atomic 1-subconformal measures ensures that this is the only non-atomic 1-subconformal measure satisfying our hypothesis. Our conjecture that all non-atomic 1-subconformal measures satisfy the limit hypothesis is equivalent to the simplex of KMS_1 states in Theorem 1.1 containing every KMS_1 state on $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma)$. This argument extends to $\beta < 1$ by a result of [3], so that our conjecture would imply that Theorem 1.1 provides a complete description of the KMS_β states for all $0 < \beta \leq 1$.

In Section 6, we introduce new quotients of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, which we call the *modular quotients*, obtained by imposing the relation $U^m = 1$ on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. These quotients are isomorphic to the Toeplitz algebra of the monoid $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ for $m \in \mathbb{N}^\times$, which we show is left-cancellative and right LCM (c.f. Section 2.1). We define natural dynamics on $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ under which the quotient map is equivariant, and use this to give a complete description of the KMS_β states of $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$. The remarkable fact about $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ is that the associated C*-dynamical system exhibits spontaneous symmetry breaking, under an action of the group $(\mathbb{Z}/m\mathbb{Z})^*$. This group is a finite quotient of the Galois group appearing in the Bost-Connes system, and is itself the Galois group of the cyclotomic extension of $\mathbb{Q}$ for the polynomial $x^m - 1$. The actions on the the Bost-Connes system and the modular quotient systems are related by an equivariant *-homomorphism, though this avenue of research is not explored in this thesis and is reserved for future investigations.

We close with an analysis of the GNS representations of our extremal KMS_β states for $\beta \leq 1$. We show that the GNS representations admit *spatial symmetries* related to the symmetries of the modular quotients in Chapter 6. This provides us with the necessary tools to prove that the GNS representation of the state $\psi_{n,\beta}$ has type III_1 when $n < \infty$. Using the spatial symmetries, we produce a conditional expectation onto the type III_1 von Neumann factor of the Riemann gas at inverse temperature β, a type III_1 factor by [5, Proposition 8]. We conjecture that the von Neumann factor of the GNS representation for $\psi_{\infty,\beta}$ is also of type III_1, but as of yet, we do not know of a proof.

Chapter 2
Preliminaries

2.1 Toeplitz algebras

Consider a left-cancellative monoid: a set P with a binary operation $\cdot : P \times P \to P$ satsfying the following 3 properties:

> Associativity: for any $p, q, r \in P$, $p \cdot (q \cdot r) = (p \cdot q) \cdot r$.
>
> Identity: there exists $1 \in P$ such that $p \cdot 1 = 1 \cdot p = p$ for all $p \in P$.
>
> Left-cancellative: for any $p, q, r \in P$, $p \cdot q = p \cdot r$ if and only if $q = r$.

We occasionally write pq instead of $p \cdot q$. We also write pP for the set of elements of P of the form $p \cdot q$, $q \in P$. These are called the *right-multiples* of p.

An element u of P is called a *unit* if there exists another element $v \in P$ such that $uv = vu = 1$. We denote the set of units in P by P^*. This set is always non-empty, as it contains 1.

The left-cancellative property is important to the representation theory of a monoid. A left-cancellative monoid admits a representation by isometries on the Hilbert space $\ell^2(P)$, defined initially on the standard orthonormal basis $\{\delta_p \in \ell^2(P) : p \in P\}$ and extended by linearity, as demonstrated in the following lemma:

Lemma 2.1. *For a left-cancellative monoid P, the functions defined on the standard orthonormal basis of $\ell^2(P)$ by*

$$\lambda_p \delta_q = \delta_{p \cdot q}$$

extend linearly to isometries of $\ell^2(P)$. The resulting operators on $\ell^2(P)$ are isometries which satisfy $\lambda_p \lambda_q = \lambda_{pq}$.

Proof. It suffices to show that λ_p preserves orthogonality of the basis vectors. We have

$$\langle \lambda_p \delta_q, \lambda_p \delta_r \rangle = \langle \delta_{pq}, \delta_{pr} \rangle = \begin{cases} 1 & \text{if } pq = pr, \\ 0 & \text{if } pq \neq pr. \end{cases}$$

The first case is equivalent to $q = r$ and the second case is equivalent to $q \neq r$, since P is left-cancellative. It follows that λ_p preserves orthogonality and the norm of the standard basis, and hence extends linearly to an isometry of $\ell^2(P)$. The claim that $\lambda_p \lambda_q = \lambda_{pq}$ is immediate. □

The range of λ_p is clearly spanned by the basis vectors of the form δ_{pq}, which we naturally identify with $\ell^2(pP)$. For $p \in P$, the adjoint λ_p^* is a co-isometry which acts on the standard basis according to the formula

$$\lambda_p^* \delta_q = \begin{cases} \delta_r, & \text{if } pr = q \\ 0, & \text{if } q \notin pP. \end{cases}$$

It is easily verified that $\lambda_p^* \lambda_p$ is the identity operator and $\lambda_p \lambda_p^*$ is the projection to the subspace $\ell^2(pP)$.

Definition 2.2. The *Toeplitz algebra of* P is the C*-subalgebra of bounded operators on $\ell^2(P)$ generated by $\{\lambda_p : p \in P\}$. This algebra is denoted $\mathcal{T}(P)$.

In this thesis, we only consider left-cancellative monoids with the *right LCM (least common multiple)* property. This asserts that for $p, q \in P$, the set $pP \cap qP$ is either empty or is itself of the form rP for some $r \in P$. This choice of r is essentially unique, as the following lemma demonstrates:

Lemma 2.3. *If* $pP = qP$*, then* $p = qs$ *for some unit* $s \in P^*$*.*

Proof. The hypothesis implies that there exist elements $s, t \in P$ such that $p = qs$ and $q = pt$. Combining these yields $q = pt = qst$, and by left-cancellativity, $st = 1$. Similarly, $ts = 1$. □

The right LCM property plays an important role in the behaviour of the projections $\lambda_p \lambda_p^*$.

Lemma 2.4. *For* $p, q \in P$*, if* $pP \cap qP = rP$*, then one has* $\lambda_p \lambda_p^* \lambda_q \lambda_q^* = \lambda_r \lambda_r^*$*. If* $pP \cap qP = \emptyset$*, then* $\lambda_p \lambda_p^* \lambda_q \lambda_q^* = 0$*.*

Proof. Suppose that $pP \cap qP = rP$. For $v \in rP$, one has $v \in pP$ and $v \in qP$, so that

$$\lambda_p \lambda_p^* \lambda_q \lambda_q^* \delta_v = \lambda_p \lambda_p^* \delta_v = \delta_v.$$

Conversely, if $v \notin rP$, then either $v \notin pP$ or $v \notin qP$. In the second case, one has

$$\lambda_p \lambda_p^* \lambda_q \lambda_q^* \delta_v = \lambda_p \lambda_p^* 0 = 0,$$

and similarly in the second case. Thus, the identity holds.

The proof in the case where $pP \cap qP = \emptyset$ is identitical to the proof of the converse above. □

Corollary 2.5. *The operators $\lambda_p\lambda_p^*$ span a commutative subalgebra of $\mathcal{T}(P)$.*

2.2 KMS states

In this section, we review some of the standard facts regarding KMS$_\beta$ states. The proofs of these can be found in [6],which is an excellent technical reference for the study of KMS states. It also outlines some of the relationships between the study of KMS states and statistical mechanics.

A C*-dynamical system will refer to a C*-algebra A with a strongly-continuous $\mathbb{R}$-action α by *-homomorphisms. One can consider the subspace A^{an} of analytic elements of A. These are elements $x \in A$ such that the function $F_x(t) = \alpha_t(x)$ (defined initially for $t \in \mathbb{R}$) can be analytically continued to an entire function of $t \in \mathbb{C}$.

Proposition 2.6. *The subspace A^{an} forms a dense subalgebra of A.*

Proof. That A^{an} is a subalgebra is easily seen: the analytic function $F_{xy}(t)$ is equal to $F_x(t)F_y(t)$ for $t \in \mathbb{R}$ and, since the product of analytic functions is analytic, this formula analytically continues to $\mathbb{C}$. The density of A^{an} is proven in [38, Section 8.12.1]. $\square$

For $\beta \in (0, \infty)$, a state ϕ on A satisfies the α-KMS$_\beta$ condition (or KMS$_\beta$ condition, when α is clear) if

$$\phi(xy) = \phi(y\alpha_{i\beta}(x)) \qquad \forall x, y \in A^{an}.$$

This property originally appeared in the study of quantum statistical mechanics. Typically, $\beta \leq 0$ is also included, but we exclude this case from our discussion for simplicity. In all of the examples we give, it is known that there are no KMS$_\beta$ states for $\beta < 0$, and the only remaining case is $\beta = 0$ (which is qualitatively different from $\beta \neq 0$). There are also the related notions of KMS$_\infty$ state and ground states, which we do not touch on here.

We summarize some of the key properties of KMS$_\beta$ state in the following proposition.

Proposition 2.7. *Suppose that A is a separable C*-algebra, and α a strongly-continuous $\mathbb{R}$-action on A by *-automorphisms. Then the following hold:*

1. *every α-KMS$_\beta$ state is also α-invariant [6, Proposition 5.3.3];*
2. *the set of α-KMS$_\beta$ states of A forms a compact Choquet simplex K_β (in the weak*-topology) [6, Theorem 5.3.30];*
3. *K_β is affinely isomorphic to the simplex of (Borel) probability measures on Ω_β, the set of extreme points in K_β;*
4. *elements of Ω_β are factor states [6, Theorem 5.3.30].*

The isomorphism in (3) is realized by the following assignment, sending a probability measure on Ω_β to a KMS_β *state on A:*

$$\mu \mapsto \left(x \mapsto \int_{\Omega_\beta} \phi(x) d\mu(\phi) \right).$$

See the discussion in [6] following the statement of Theorem 5.3.30.

Now let P be a left-cancellative right LCM and $N : P \to \mathbb{R}$ a monoid homomorphism. (This is typically called a *weight* on the monoid when the condition $N(p) \geq 0$ is imposed.) Using this data, we describe a C*-dynamical system on the Toeplitz algebra $\mathcal{T}(P)$.

We may define 1-parameter family of unitary operators U_t on $\ell^2(P)$ where $U_t\delta_p = e^{itN(p)}\delta_p$. These form a strongly continuous unitary representation of $\mathbb{R}$ on $\ell^2(P)$, that is, $t \mapsto U_t\xi$ is continuous for every $\xi \in \ell^2(P)$ and $U_tU_s = U_{t+s}$ for every $t, s \in \mathbb{R}$.

Proposition 2.8. *The adjoint action $Ad_{U_t}(x) = U_t x U_t^*$ gives rise to a strongly continuous action of $\mathbb{R}$ on $\mathcal{T}(P)$.*

Proof. We must verify that the adjoint action leaves the closed subspace $\mathcal{T}(P)$ invariant. Since Ad_{U_t} is a *-homomorphism, it suffices to show that it leaves the linear span of the generating isometries λ_p invariant.

For any $p, q \in P$ and $t \in \mathbb{R}$, one has

$$\begin{aligned} U_t\lambda_p U_t^*\delta_q &= e^{-itN(q)}U_t\lambda_p\delta_q = e^{-itN(q)}U_t\delta_{pq} \\ &= e^{it(N(pq)-N(q))}\delta_{pq} = e^{itN(p)}\lambda_p\delta_q, \end{aligned}$$

where $N(pq) - N(q) = N(p) + N(q) - N(q) = N(p)$. This implies that $\mathrm{Ad}_{U_t}(\lambda_p) = e^{itN(p)}\lambda_p$. Therefore, the action extends to an action on $\mathcal{T}(P)$ by *-homomorphisms. □

Numerous examples of Toeplitz C*-dynamical systems have been examined. In the next two sections, we discuss two such examples, where the monoids involved are $\mathbb{Z} \rtimes \mathbb{N}^\times$ and $\mathbb{N}^\times \ltimes \mathbb{Z}$.

2.3 The left $ax + b$ system

The left $ax + b$ monoid $\mathbb{Z} \rtimes \mathbb{N}^\times$ consists of the set $\mathbb{Z} \rtimes \mathbb{N}^\times$ (where $\mathbb{N}^\times$ denotes the non-zero natural numbers), with the binary operation

$$(x, a) \cdot (y, b) = (x + ay, ab).$$

It is standard to refer to this as the $ax+b$ monoid; we have chosen the convention of *left* $ax+b$ monoid, as monoids of the form $G \rtimes P$ arise from a left-action of a monoid P on a group G. In our case, the monoid $\mathbb{N}^\times$ is is commutative, so every left-action is also a right-action, but we maintain the distinction for consistency with literature on semi-direct products.

Lemma 2.9.

(i) The monoid $\mathbb{Z} \rtimes \mathbb{N}^\times$ is left-cancellative and right LCM.

(ii) The function $N : \mathbb{Z} \rtimes \mathbb{N}^\times \to \mathbb{R}$, $N(x,a) = \ln(a)$ is a monoid homomorphism.

Proof. *(i)* Left-cancellative follows from the fact that $\mathbb{Z} \rtimes \mathbb{N}^\times$ embeds in the semi-direct product group $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$, by the natural inclusions $\mathbb{Z} \subseteq \mathbb{Q}$ and $\mathbb{N}^\times \subseteq \mathbb{Q}_+^\times$. To see that $\mathbb{Z} \rtimes \mathbb{N}^\times$ is right LCM, we calculate generic right-multiples of (x,a) and (y,b):

$$(x,a) \cdot (z,c) = (x+az, ac), \qquad (y,b) \cdot (z,c) = (y+bz, bc).$$

The common multiples must have a multiple of $\operatorname{lcm}(a,b)$ in the second entry. Note that in the first entry of the right-multiples of (x,a) appear all integers congruent to x modulo a (independent of c), while the first entry of the right-multiples of (y,b) are all integers congruent to y modulo b. Thus, there exist common multiples of (x,a) and (y,b) if and only if there exists an integer w which is congruent to x modulo a and y modulo b. When such a w exists, any other choice of w differs by a multiple of $\operatorname{lcm}(a,b)$. In this case, we have that the common multiples of (x,a) and (y,b) are all of the form

$$(w + \operatorname{lcm}(a,b)z, \operatorname{lcm}(a,b)c) = (w, \operatorname{lcm}(a,b)) \cdot (z,c).$$

Therefore, $\mathbb{Z} \rtimes \mathbb{N}^\times$ is right LCM.

(ii) It is clear that $N(0,1) = 0$. We check that N intertwines the binary operations:

$$N(x+ay, ab) = \ln(ab) = \ln(a) + \ln(b) = N(x,a) + N(y,b). \quad \square$$

The *left $ax+b$ system* is the C*-dynamical system consisting of the C*-algebra $\mathcal{T}(\mathbb{Z} \rtimes \mathbb{N}^\times)$ and the $\mathbb{R}$-action from Section 2.2 for the homomorphism $N : \mathbb{Z} \rtimes \mathbb{N}^\times \to \mathbb{R}$, $N(x,a) = \ln(a)$.

For this monoid, we make a point of expressing the generating isometries in the form $\lambda_{(x,a)} = U^x V_a$, where $U^x = \lambda_{(x,1)} = \lambda_{(1,1)}^x$ is a unitary and $V_a = \lambda_{(0,a)}$ is an isometry. By Corollary 2.5, the elements of the form $U^x V_a V_b^* U^{-y}$ span a dense subalgebra of $\mathcal{T}(\mathbb{Z} \rtimes \mathbb{N}^\times)$. Consequently, a state on $\mathcal{T}(\mathbb{Z} \rtimes \mathbb{N}^\times)$ is determined by its values on these spanning elements. The dynamics determined by N are determined by the formulas $\sigma_t(U^x) = U^x$ and $\sigma_t(V_a) = a^{it} V_a$.

The left $ax+b$ system was first considered by Laca and Raeburn in [26]. They provided formulas for the KMS_β states at all $\beta \in \mathbb{R}$, which we summarize in the following theorem:

Theorem 2.10 (LR Theorem 7.1)**.** *Let σ denote the dynamics on $\mathcal{T}(\mathbb{Z} \rtimes \mathbb{N}^\times)$ from the homomorphism N above.*

1. *For $\beta \in [0,1)$, there are no σ-KMS_β states.*
2. *For $\beta \in [1,2]$, there is a unique σ-KMS_β state characterized by*

$$\psi_\beta(U^x V_a V_b^* U^{-y}) = \begin{cases} 0 & \text{if } a \neq b \text{ or } x \neq y \\ a^{-\beta} & \text{if } a = b \text{ and } x = y. \end{cases}$$

3. *For $\beta \in (2,\infty)$, the simplex of σ-KMS_β states is affinely isomorphic to the simplex of probability measures on $\mathbb{T}$. For $z \in \mathbb{T}$, the extremal KMS_β state $\psi_{\beta,z}$ corresponding to the point mass δ_z is a type I factor state satisfying*

$$\psi_{z,\beta}(U^x V_a V_b^* U^{-y}) = \begin{cases} 0 & \text{if } a \neq b \text{ or } x \not\equiv y \pmod a \\ \dfrac{1}{a\zeta(\beta-1)} \displaystyle\sum_{\{d : a|d|(x-y)\}} d^{1-\beta} z^{(x-y)/d} & \text{if } a = b \text{ and } x \equiv y \pmod a. \end{cases}$$

It was later shown by Laca and Neshveyev that the KMS_β states for $1 \le \beta \le 2$ are all of type III_1, c.f. [29, Theorem 3.2].

In terms of measures on $\mathbb{T}$, this theorem asserts that the function sending a probability measure η to the state

$$\psi_{\eta,\beta}(U^x V_a V_b^* U^{-y}) = \begin{cases} 0 & \text{if } a \neq b \text{ or } x \not\equiv y \pmod a \\ \dfrac{1}{a\zeta(\beta-1)} \displaystyle\sum_{\{d : a|d|(x-y)\}} d^{1-\beta} \int_{\mathbb{T}} z^{(x-y)/d} d\eta & \text{if } a = b \text{ and } x \equiv y \pmod a \end{cases}$$

is an affine isomorphism onto the simplex of KMS_β states when $\beta > 2$.

2.4 The right $ax+b$ system

The right $ax+b$ monoid $\mathbb{N}^\times \ltimes \mathbb{Z}$ consists of the set $\mathbb{N}^\times \times \mathbb{Z}$, with the binary operation

$$(a,x)\cdot(b,y) = (ab, bx+y).$$

Lemma 2.11.

(i) The monoid $\mathbb{N}^\times \ltimes \mathbb{Z}$ is left-cancellative and directed.

(ii) The function $N : \mathbb{Z} \rtimes \mathbb{N}^\times \to \mathbb{R}$, $N(x, a) = \ln(a)$ is a monoid homomorphism.

Proof. *(i)* Once again, this monoid embeds in a group, $\mathbb{Q}_+^\times \ltimes \mathbb{Q}$, whence left-cancellativity follows. For directedness, consider the generic right-multiples of pairs (a, x) and (b, y):

$$(a, x) \cdot (c, z) = (ac, cx + z), \qquad (b, y) \cdot (c, z) = (bc, cy + z).$$

Reparameterizing, the right-multiples of (a, x) are of the form (ac, z) and right-multiples of (b, y) are of the form (bc, z), for $c \in \mathbb{N}^\times$ and $z \in \mathbb{Z}$. Thus, the common right-multiples are of the form $(\mathrm{lcm}(a, b)c, z)$, which are right-multiples of $(\mathrm{lcm}(a, b), 0)$.

(ii) Analogous to that of Lemma 2.9 *(ii)*. ☐

The proof of directedness in the previous lemma illustrates a key difference between the monoids $\mathbb{N}^\times \ltimes \mathbb{Z}$ and $\mathbb{Z} \rtimes \mathbb{N}^\times$. The common right-multiples no longer depend on congruence modulo a, and this leads to fascinating new behaviour. The first difference we will see is a shift (or lack thereof) in critical temperature of the C*-dynamical system, and a simpler non-degeneracy condition for KMS states.

The *right $ax + b$ system* is the C*-dynamical system consisting of the C*-algebra $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ and the $\mathbb{R}$-action from Section 2.2 for the homomorphism $N : \mathbb{N}^\times \ltimes \mathbb{Z} \to \mathbb{R}$, $N(a, x) = \ln(a)$. Just as for the left $ax + b$ system, we separate the generators as $\lambda_{(a,x)} = V_a U^x$, with $a \in \mathbb{N}^\times$ and $x \in \mathbb{Z}$. The dynamics determined by N are determined by the same formulas as in the left case, namely $\sigma_t(U^x) = U^x$ and $\sigma_t(V_a) = a^{it} V_a$.

The discussion in Section 2 of [19] regarding the left/right Toeplitz algebras of $\mathbb{N}^\times \ltimes \mathbb{N}$ carries over analogously for the monoid $\mathbb{N}^\times \ltimes \mathbb{Z}$. The Toeplitz algebra $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ (generated by the left-regular representation) is isomorphic to the C*-algebra generated by the coisometries $W_{(a,x)}$ on $\ell^2(\mathbb{Z} \rtimes \mathbb{N}^\times)$, with

$$W_{(x,a)} \delta_{(y,b)} = \begin{cases} \delta_{(y - xa^{-1}b, a^{-1}b)} & \text{if } (y, b) \in \mathbb{Z} \rtimes \mathbb{N}^\times \cdot (x, a), \\ 0 & \text{otherwise} \end{cases}$$

(sometimes referred to as the *right*-regular representation). The C*-algebra generated by these operators is referred to as the *right* Toeplitz algebra. It is a matter of preference whether one works with the *right*-regular representation of the *left $ax + b$* monoid, or the *left*-regular representation of the *right $ax + b$* monoid. We choose the latter, keeping with [19].

The following universal characterization of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ will be useful for our purposes.

Proposition 2.12. *There is an isomorphism from $\mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z})$ onto $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, where $\mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z})$ is the universal C*-algebra generated by isometries $\{v_a : a \in \mathbb{N}^\times\}$ and a unitary u satisfying the relations*

(T0) $v_a^* v_a = u^* u = u u^* = 1$,

(T1) $uv_a = v_a u^a$,

(T2) $v_a v_b = v_{ab}$

(T3) $v_a v_b^* = v_b^* v_a$ *if* $\gcd(a,b) = 1$.

The isomorphism sends $v_a \mapsto V_A$ *and* $u \mapsto U$.

Proof. It is easy to verify that V_a and U satisfy the uppercase analogues of (T0)–(T3) (c.f. [19, Example 3.9]). Hence, there is a surjective *-homomorphism $\pi : \mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z}) \to \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. We will show that this *-homomorphism is faithful.

The embedding $\mathbb{N}^\times \ltimes \mathbb{Z} \to \mathbb{Q}_+^\times \ltimes \mathbb{Q}$ induces a coaction $\delta : \mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z}) \to \mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z}) \otimes_{\max} C^*(\mathbb{Q}_+^\times \ltimes \mathbb{Q})$ that satisfies $\delta(u) = u \otimes \varepsilon_{(1,1)}$ and $\delta(v_a) = v_a \otimes \varepsilon_{(a,0)}$. Letting τ_0 denote the trace on $C^*(\mathbb{Q}_+^\times \ltimes \mathbb{Q})$ determined by $\tau_0(\varepsilon_{(a,x)}) = \delta_{a,1}\delta_{x,0}$, we obtain a conditional expectation

$$E_\delta := (\mathrm{id} \otimes \tau_0) \circ \delta : \mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z}) \to \mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z}), \qquad E_\delta(v_a u^x v_b^*) = \delta_{a,b}\delta_{0,x} v_a v_a^*.$$

Since $\mathbb{Q}_+^\times \ltimes \mathbb{Q}$ is amenable, E_δ is faithful, c.f. [23, Lemma 6.5]. The range of E_δ is the closed linear span of the projections $e_a = v_a v_a^*$. These satisfy $e_a e_b = e_{\mathrm{lcm}(a,b)}$ and $\prod_{i=1}^n (e_a - e_{z_i}) \neq 0$ whenever $a < z_i$, so it follows from [23, Proposition 1.3] that the range of E_δ is isomorphic to the algebra $B_{\mathbb{N}^\times}$ appearing in Section 1 of [23]. This isomorphism sends $1_a \mapsto e_a$.

We also have a conditional expectation E on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, defined spatially over $\ell^2(\mathbb{N}^\times \ltimes \mathbb{Z})$ by

$$E(V_a U^x V_b^*) = \sum_{(c,y) \in \mathbb{N}^\times \ltimes \mathbb{Z}} P_{(c,y)} V_a U^x V_b^* P_{(c,y)} = \delta_{a,b}\delta_{0,x} V_a V_a^*.$$

The range of this conditional expectation is also isomorphic to $B_{\mathbb{N}^\times}$, by the linear map defined on the projections by $1_a \mapsto V_a V_a^*$ (existence and faithfulness of this map is proven as above). Therefore, $\pi(v_a v_a^*) = V_a V_a^*$ is also an isomorphism.

We now have a commuting square, where the vertical maps are conditional expectations, and the horizontal maps are *-homomorphisms:

$$\begin{tikzcd}
\mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z}) \arrow[r, "\pi"] \arrow[d, "E_\delta"] & \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) \arrow[d, "E"] \\
\mathrm{Ran}(E_\delta) \arrow[r, "\cong"] & \mathrm{Ran}(E)
\end{tikzcd}$$

Since E_δ is faithful, it follows that $E \circ \pi$ is also faithful. Therefore, π is a faithful surjective *-homomorphism, and hence an isomorphism. ☐

The algebra $\mathcal{T}_u(\mathbb{N}^\times \ltimes \mathbb{Z})$ is the *additive boundary quotient* considered in [19], c.f. [19, Proposition 3.8]. Explicit formulas for the low-temperature (subcritical) KMS_β states of $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma_t)$ were given in Section 8 of [19], which we summarize in the following theorem:

Theorem 2.13 ([19, Theorem 8.1])**.** *Let σ denote the dynamics on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ from the homomorphism N above. For $\beta > 1$, the simplex of σ-KMS_β states is affinely isomorphic to the simplex of probability measures on $\mathbb{T}$. For $z \in \mathbb{T}$, the extremal KMS_β state $\psi_{z,\beta}$ corresponding to the point mass δ_z is a type I factor state satisfying*

$$\psi_{z,\beta}(V_a U^x V_b^*) = \begin{cases} 0 & \textit{if } a \neq b \\ \dfrac{a^{-\beta}}{\zeta(\beta)} \displaystyle\sum_{c=1}^{\infty} c^{-\beta} z^{cx} & \textit{if } a = b \end{cases}$$

In terms of measures on $\mathbb{T}$, this theorem asserts that the function sending a probability measure η to the state

$$\psi_{\eta,\beta}(V_a U^x V_b^*) = \begin{cases} 0 & \text{if } a \neq b \\ \dfrac{a^{-\beta}}{\zeta(\beta)} \displaystyle\sum_{c=1}^{\infty} c^{-\beta} \int_{\mathbb{T}} z^{cx} d\eta & \text{if } a = b \end{cases}$$

is an affine isomorphism onto the simplex of KMS_β states when $\beta > 1$.

A consequence of our work in Chapter 4 is an alternative proof of this theorem, via a characterization of KMS_β states in terms of *subconformal measures* on $\mathbb{T}$, followed by a characterization of such measures. The advantage of this recharacterization is its applicability to the larger inverse temperature space $\beta \in (0, \infty)$, in contrast to the above theorem.

Besides the explicit formulas in Theorem 2.13, [19] also provides a characterization of the states satisfying the KMS_β in terms of a simpler condition:

Proposition 2.14 ([19, Proposition 7.2])**.** *Suppose that $\beta > 0$ and ϕ is a state of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. Then ϕ is a σ-KMS_β state if and only if*

$$\phi(V_a U^n V_b^*) = \delta_{a,b} a^{-\beta} \phi(U^n) \qquad \textit{for all } n \in \mathbb{Z}, a, b \in \mathbb{N}^\times.$$

This proposition will prove to be quite useful, as it provides a method of assigning, to each σ-KMS_β state on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, a state on $C^*(U) \cong C(\mathbb{T})$. Our first objective is to describe the states on $C(\mathbb{T})$ (i.e. probability measures on $\mathbb{T}$) which arise in this way.

Chapter 3
The gauge action and its fixed-point subalgebra

3.1 The gauge action and $\mathfrak{D}$

The $\mathbb{R}$-action on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ described in the previous chapter is part of an action of a larger group $\widehat{\mathbb{Q}}_+^*$, the group of characters on the topological group $\mathbb{Q}_+^\times$ (with the discrete topology). Given a character $\chi : \mathbb{Q}_+^\times \to \mathbb{T}$, we define a unitary U_χ on $\ell^2(\mathbb{N}^\times \times \mathbb{Z})$ by letting

$$U_\chi \delta_{(a,x)} = \chi(a)\delta_{(a,x)} \qquad \text{for all } a \in \mathbb{N}^\times, x \in \mathbb{Z}.$$

By the same arguments as in the proof of Proposition 2.8, $g_\chi := \mathrm{Ad}_{U_\chi}$ is a *-isomorphism of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, and the assignment $\chi \mapsto g_\chi$ is a strongly-continuous action of $\widehat{\mathbb{Q}}_+^*$. This is called the *gauge action*. On the generating isometries, it looks like

$$g_\chi(V_a) = \chi(a)V_a \quad \text{for all } a \in \mathbb{N}^\times, \qquad g_\chi(U) = U;$$

a standard computation also yields g_χ on the co-isometries,

$$g_\chi(V_a^*) = (g_\chi(V_a))^* = (\chi(a)V_a)^* = \overline{\chi(a)}V_a^* = \chi(a^{-1})V_a^*.$$

The dynamics on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ are obtained from the above action of $\widehat{\mathbb{Q}}_+^*$ by composition with the group homomorphism mapping $t \in \mathbb{R}$ to the character $\chi_t(q) = q^{it}$.

The fixed-point subalgebras of the $\mathbb{R}$-action and the gauge action are the same. The spanning monomials which are fixed by either action are of the form $V_a U^n V_a^*$, as

$$g_\chi(V_a U^n V_b^*) = g_\chi(V_a) g_\chi(U^n) g_\chi(V_b^*) = \chi(a)\chi(b^{-1}) V_a U^n V_b^* = \chi\left(\frac{a}{b}\right) V_a U^n V_b^*,$$

and $\chi\left(\frac{a}{b}\right) = 1$ for all $\chi \in \widehat{\mathbb{Q}}_+^*$ only if $\frac{a}{b} = 1$. Thus, the fixed-point subalgebra is the closed linear span of these monomials,

$$\mathfrak{D} = \overline{\operatorname{span}}\{V_a U^n V_a^* : a \in \mathbb{N}^\times, n \in \mathbb{Z}\}.$$

The advantage of the gauge action over the $\mathbb{R}$-action is that the acting group is compact: the group $\mathbb{Q}_+^\times$ is discrete, and thus the group of characters $\widehat{\mathbb{Q}}_+^*$ is a compact abelian group. This means that $\widehat{\mathbb{Q}}_+^*$ admits a Haar probability measure, and integrating against this measure yields a bounded linear map.

The following is a standard result for compact groups acting on C*-algebras, but we include its proof for those who may be unfamiliar.

Lemma 3.1. *The map $\Phi : \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) \to \mathfrak{D}$, defined by*

$$\Phi(x) = \int_{\widehat{\mathbb{Q}}_+^*} g_\chi(x) d\chi,$$

is a faithful conditional expectation of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ onto the fixed-point subalgebra $\mathfrak{D}$. For a monomial of the form $V_a U^n V_b^*$, one has

$$\Phi(V_a U^n V_b^*) = \delta_{a,b} V_a U^n V_a^*$$

Proof. Clearly Φ is linear and $\Phi(1) = 1$. For each $\chi \in \widehat{\mathbb{Q}}_+^*$, g_χ is a positive map. Thus, for each $x \geq 0$ and state φ on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, one has

$$\varphi(\Phi(x)) = \varphi\left(\int_{\widehat{\mathbb{Q}}_+^*} g_\chi(x) d\chi\right) = \int_{\widehat{\mathbb{Q}}_+^*} \varphi(g_\chi(x)) d\chi \geq 0.$$

Since $\varphi(\Phi(x)) \geq 0$ for all states φ on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, it follows that $\Phi(x) \geq 0$, so Φ is positive. That Φ is contractive follows from the fact that g_χ is an isometry and $d\chi$ is a probability measure, since

$$|\Phi(x)| \leq \left|\int_{\widehat{\mathbb{Q}}_+^*} g_\chi(x) d\chi\right| \leq \int_{\widehat{\mathbb{Q}}_+^*} |g_\chi(x)| d\chi = \int_{\widehat{\mathbb{Q}}_+^*} |x| d\chi = |x|.$$

To see that $\Phi(x)$ is faithful: since the action is strongly continuous, the function $\chi \mapsto \varphi(g_\chi(x))$ is continuous for any state φ and $x \geq 0$. If $x \neq 0$, then there exists a state φ such that $\varphi(x) > 0$; by continuity, there is then a neighborhood A of the identity in $\widehat{\mathbb{Q}}_+^*$ such that $\varphi(g_\chi(x))$ is non-zero for all $\chi \in A$. Since every open subset of $\widehat{\mathbb{Q}}_+^*$ has positive Haar measure, it follows that

$$\varphi(\Phi(x)) = \int_{\widehat{\mathbb{Q}}_+^*} \varphi(g_\chi(x)) d\chi \geq \int_A \varphi(g_\chi(x)) d\chi > 0,$$

and thus Φ is faithful.

Let $a, b \in \mathfrak{D}$. Then $g_\chi(a) = a$ and $g_\chi(b) = b$, and we compute:

$$\Phi(axb) = \int_{\widehat{\mathbb{Q}}_+^*} g_\chi(axb)d\chi = \int_{\widehat{\mathbb{Q}}_+^*} a g_\chi(x) b d\chi = a \int_{qd} g_\chi(x) d\chi b = a\Phi(x)b.$$

Lastly, recall that if $q \in \mathbb{Q}_+^\times$ is not equal to 1, then $\int_{\widehat{\mathbb{Q}}_+^*} \chi(q)d\chi = 0$. We can use this to verify the behaviour of Φ on monomials:

$$\Phi(V_a U^n V_b^*) = \int_{\widehat{\mathbb{Q}}_+^*} g_\chi(V_a U^n V_b^*)d\chi = \int_{\widehat{\mathbb{Q}}_+^*} \chi\left(\frac{a}{b}\right) d\chi V_a U^n V_b^* = \begin{cases} 0 & \text{if } a \neq b \\ V_a U^n V_a^* & \text{if } a = b \end{cases}$$

□

A conditional expectation for the gauge action is extremely useful, as it implies that every gauge-invariant functional on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ factors through this expectation. Only slightly more difficult is the following:

Corollary 3.2. *If φ is a continuous linear functional such that $\varphi(\sigma_t(x)) = \varphi(x)$ for all $x \in \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ and $t \in \mathbb{R}$, then $\varphi = \varphi|_{\mathfrak{D}} \circ \Phi$.*

Proof. We show that this is true for monomials. For $a, b \in \mathbb{N}^\times$, $n \in \mathbb{Z}$, and $t \in \mathbb{R}$, one has

$$\varphi(V_a U^n V_b^*) = \varphi(\sigma_t(V_a U^n V_b^*)) = \varphi\left(\left(\frac{a}{b}\right)^{it} V_a U^n V_b^*\right) = \left(\frac{a}{b}\right)^{it} \varphi(V_a U^n V_b^*).$$

In order for these to be equal for every $t \in \mathbb{R}$, one requires that either $\frac{a}{b} = 1$ (in which case $V_a U^n V_b^* = \Phi(V_a U^n V_b^*)$) or that $\varphi(V_a U^n V_b^*) = 0$. When $\frac{a}{b} \neq 1$, we have that $\Phi(V_a U^n V_b^*) = 0$, and so regardless of a, b, and n, $\varphi(V_a U^n V_b^*) = \varphi(\Phi(V_a U^n V_b^*))$. This shows that $\varphi = \varphi|_{\mathfrak{D}} \circ \Phi$ on a dense subspace of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, whence it follows that the identity holds in general. □

This corollary will be particularly useful to us in classifying KMS$_\beta$ states of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, since these are σ-invariant states. We would like a characterization of those states φ on $\mathfrak{D}$ such that $\varphi \circ \Phi$ is a KMS$_\beta$ state on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. Before we can answer this question, we require a better understanding of the algebra $\mathfrak{D}$.

3.2 $\mathfrak{D}$ as an inductive limit

The product of monomials in $\mathfrak{D}$ is given by the formula

$$(V_b U^n V_b^*)(V_c U^m V_c^*) = V_{\operatorname{lcm}(b,c)} U^{nc' + mb'} V_{\operatorname{lcm}(b,c)}, \tag{3.1}$$

where $b' = \frac{b}{\gcd(b,c)}$ and $c' = \frac{c}{\gcd(b,c)}$. This product is commutative, so that $\mathfrak{D}$ is isomorphic to the C*-algebra of continuous functions on its spectrum. Our goal in the next few sections is to obtain a clear picture of this spectrum. The first step towards this is expressing $\mathfrak{D}$ as an inductive limit of C*-algebras of continuous functions on disjoint circles. This provides $\mathfrak{D}$ with a quasi-lattice grading, in the sense of [3, Section 4].

Lemma 3.3. *Let $\omega_n : \mathbb{T} \to \mathbb{T}$ be the map $z \mapsto z^n$, and for each $f \in C(\mathbb{T})$ and each $a \in \mathbb{N}^\times$ define operators $\pi(f)$ and w_a on $\mathcal{H} := \ell^2(\mathbb{N}^\times) \otimes L^2(\mathbb{T})$ by*

$$\pi(f)(\delta_n \otimes g) = \delta_n \otimes (f \circ \omega_n)g \qquad \text{and} \qquad w_a(\delta_n \otimes g) = \delta_{an} \otimes g.$$

Let $\mathfrak{z} \in C(\mathbb{T})$ denote the inclusion $\mathbb{T} \to \mathbb{C}$. Then

1. *the maps $(1,1) \mapsto \pi(\mathfrak{z})$ and $(a,0) \mapsto w_a$ give a Nica-covariant representation of $\mathbb{N}^\times \ltimes \mathbb{Z}$;*
2. *there is an isomorphism $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) \cong C^*(\pi, w)$ such that $U \mapsto \pi(\mathfrak{z})$ and $V_a \mapsto w_a$;*
3. *this isomorphism restricts to an isomorphism*

$$\mathfrak{D} \cong \overline{\operatorname{span}}\{w_a \pi(f) w_a^* : f \in C(\mathbb{T}), a \in \mathbb{N}^\times\}. \tag{3.2}$$

Proof. It is possible to show by a direct computation that the unitary $\pi(\mathfrak{z})$ and the isometries w_a satisfy the relations from [19, Prop. 3.8], and this gives a surjective homomorphism of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ to $C^*(\pi, w)$ from which a Nica-covariant representation of $\mathbb{N}^\times \ltimes \mathbb{Z}$ can be extracted. Instead, we shall prove part (2) and observe that part (1) follows easily.

Let $\mathcal{F} : \ell^2(\mathbb{N}^\times) \otimes \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N}^\times) \otimes L^2(\mathbb{T})$ denote the Fourier transform on the second coordinate, determined by

$$\mathcal{F}(\delta_n \otimes \delta_k) = \delta_n \otimes \mathfrak{z}^k, \qquad n \in \mathbb{N}^\times,\ k \in \mathbb{Z}.$$

Then

$$\pi(\mathfrak{z})\mathcal{F}(\delta_n \otimes \delta_k) = \pi(\mathfrak{z})(\delta_n \otimes \mathfrak{z}^k) = (\delta_n \otimes \mathfrak{z}^n \mathfrak{z}^k) = \mathcal{F}(\delta_n \otimes \delta_{n+k}) = \mathcal{F}U(\delta_n \otimes \delta_k)$$

and similarly

$$w_a \mathcal{F}(\delta_a \otimes \delta_k) = w_a(\delta_n \otimes \mathfrak{z}^k) = \delta_{an} \otimes \mathfrak{z}^k = \mathcal{F}(\delta_{an} \otimes \delta_k) = \mathcal{F}V_a(\delta_n \otimes \delta_k).$$

This shows that $\operatorname{Ad}_{\mathcal{F}} : \mathbb{B}(\ell^2(\mathbb{N}^\times) \otimes \ell^2(\mathbb{Z})) \to \mathbb{B}(\ell^2(\mathbb{N}^\times) \otimes L^2(\mathbb{T}))$ restricts to the isomorphism of part (2).

It is clear from the presentation of $\mathfrak{D}$ that its image under $\operatorname{Ad}_{\mathcal{F}}$ is contained in the closed span of operators of the form $w_a\pi(f)w_a^*$, $a \in \mathbb{N}^\times$, $f \in C(\mathbb{T})$. To see that these sets are equal, we first note that the linear map extending the function $u^n \mapsto U^n$, $n \in \mathbb{Z}$ is an embedding of the C*-algebra $C^*(\mathbb{Z})$ generated by a single unitary u into $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. Indeed, this follows from the fact that the left-regular representation of $C^*(\mathbb{Z})$ is faithful, and is a subrepresentation of the left-regular representation of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ on $\ell^2(\mathbb{N}^\times \ltimes \mathbb{Z})$. Since the Fourier transform is an isomorphism between $C^*(\mathbb{Z})$ and $C(\mathbb{T})$, the map $\operatorname{Ad}_{\mathcal{F}} : U \mapsto \pi(\mathfrak{z})$ therefore linearly extends to an isomorphism between $C^*(U) \subseteq \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ and $\pi(C(\mathbb{T}))$, so there exists some $x \in \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ such that $\operatorname{Ad}_{\mathcal{F}}(x) = f$. Therefore,

$$\operatorname{Ad}_{\mathcal{F}}(V_a x V_a^*) = \operatorname{Ad}_{\mathcal{F}}(V_a)\operatorname{Ad}_{\mathcal{F}}(x)\operatorname{Ad}_{\mathcal{F}}(V_a)^* = w_a\pi(f)w_a^*.$$

Since $\mathfrak{D}$ is closed and $\operatorname{Ad}_{\mathcal{F}}$ is isometric, it follows that the image of $\mathfrak{D}$ under $\operatorname{Ad}_{\mathcal{F}}$ is equal to the closed linear span appearing in (3.2). □

To simplify the notation, we write $V_a f V_a^*$ for the element of $\mathfrak{D}$ corresponding to $w_a\pi(f)w_a^*$. For each fixed $a \in \mathbb{N}^\times$ define

$$\mathfrak{D}_a := \operatorname{span}\{V_d f V_d^* : d|a,\ f \in C(\mathbb{T})\}.$$

It follows easily from (3.1) that

$$V_c f V_c^*\, V_d g V_d^* = V_{c\vee d}(f \circ \omega_{d'})(g \circ \omega_{c'})V_{c\vee d}^*, \tag{3.3}$$

where we have written $c \vee d$ for $\operatorname{lcm}(c,d)$ to streamline notation. Hence $\mathfrak{D}_a$ is a C*-subalgebra of $\mathfrak{D}$. Moreover, $\mathfrak{D}_a$ is contained in $\mathfrak{D}_b$ whenever $a|b$; we denote the inclusion map by $\iota_{a,b} : \mathfrak{D}_a \hookrightarrow \mathfrak{D}_b$. By Lemma 3.3 (3), $\bigcup_{a\in\mathbb{N}^\times} \mathfrak{D}_a$ is dense in $\mathfrak{D}$, so that $\mathfrak{D}$ is the inductive limit of the directed system $(\mathfrak{D}_a, \iota_{a,b})_{a\in\mathbb{N}^\times}$.

Next we give a realization of this system, in the form of continuous functions on topological spaces consisting of disjoint unions of circles, parameterized by the sets $\Delta_a = \{d \in \mathbb{N}^\times : d|a\}$. This presentation has useful applications in Chapter 4. On transposing the inclusions $\iota_{a,b} : \mathfrak{D}_a \hookrightarrow \mathfrak{D}_b$, we also obtain a dual projective system $\iota_{a,b}^* : \operatorname{Spec}\mathfrak{D}_b \to \operatorname{Spec}\mathfrak{D}_a$ whose limit is $\operatorname{Spec}\mathfrak{D}$. This dual picture (with the explicit spaces from the following lemma) is explored further in Section 3.4.

Proposition 3.4. *Fix $a \in \mathbb{N}^\times$ and let $X_a := \mathbb{T} \times \Delta_a$. For each $b|a$ and each $f \in C(\mathbb{T})$ let $\gamma_b(f)$ be the function on X_a defined by $\gamma_b(f)(z,d) = \delta_{b,d}f(z)$, where $\delta_{b,d}$ is the usual Kronecker delta. Then the map*

$$\Gamma_a : V_b f V_b^* \mapsto \sum_{d|\frac{a}{b}} \gamma_{bd}(f \circ \omega_d)$$

extends linearly to an isomorphism from $\mathfrak{D}_a$ *to* $C(X_a)$ *whose inverse is given by*

$$\Gamma_a^{-1} : \gamma_b(f) \mapsto \sum_{d|\frac{a}{b}} \mu(d) V_{bd}(f \circ \omega_d) V_{bd}^*.$$

Here μ denotes the Möbius function, defined in the introduction.

Proof. We identify $C(X_a)$ with $\bigoplus_{d|a} C(\mathbb{T})$, via the unique linear map which identifies the function $\gamma_b(f)$ with the tuple $(f_d)_{d|a}$ where $f_d(z) = \gamma_b(f)(z, d)$.

For each $d|a$, there is a unique linear map which is defined on the spanning elements by

$$\varepsilon_d : \mathfrak{D}_a \to C(\mathbb{T}), \qquad \varepsilon_d(V_b f V_b^*) = \begin{cases} f \circ \omega_{d/b} & \text{if } b|d \\ 0 & \text{otherwise.} \end{cases}$$

The map Γ_a is the direct sum of these maps under the previous identification, $\bigoplus_{d|a} \varepsilon_d : \mathfrak{D} \to \bigoplus_{d|a} C(\mathbb{T}) \cong C(X_a)$. We will show that ε_d is a *-homomorphism for each $d|a$, whence it follows that Γ_a is a *-homomorphism.

It is clear that ε_d preserves the adjoint; it only remains to show that ε_d preserves multiplication. Given two monomials $V_b f V_b^*$ and $V_c g V_c^*$, and letting $b' = \frac{b}{\gcd(b,c)} = \frac{b \vee c}{c}$ and $c' = \frac{b \vee c}{b}$, one has

$$\begin{aligned} \varepsilon_d(V_b f V_b^* V_c g V_c^*) &= \varepsilon_d(V_{b\vee c}(f \circ \omega_{c'})(g \circ \omega_{b'}) V_{b\vee c}^*) \\ &= \begin{cases} (f \circ \omega_{c'} \circ \omega_{d/(b\vee c)})(g \circ \omega_{b'} \circ \omega_{d/(b\vee c)}) = (f \circ \omega_{d/b})(g \circ \omega_{d/c}) & \text{if } (b \vee c)|d \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The first case implies that $b|d$ and $c|d$, so this equals $\varepsilon_d(V_b f V_b^*)\varepsilon_d(V_c g V_c^*)$. The second case implies (WLOG) that $b \nmid d$, so $\varepsilon_d(V_b f V_b^*) = 0 = \varepsilon_d(V_b f V_b^*)\varepsilon_d(V_c g V_c^*)$.

Next, we use a modification of the proof of the classical Möbius inversion formula to show that our formula for Γ_a^{-1} does in fact give an inverse of Γ_a. We start by pointing out that Γ^{-1} is a well-defined linear map, by the universal property of the direct sum. As before, it is enough to carry out the computations for spanning elements $V_b f V_b^*$:

$$\begin{aligned} \Gamma_a^{-1} \circ \Gamma_a(V_b f V_b^*) &= \Gamma_a^{-1} \sum_{d|\frac{a}{b}} \gamma_{bd}(f \circ \omega_d) \\ &= \sum_{d|\frac{a}{b}} \sum_{e|\frac{a}{bd}} \mu(e) V_{bde}(f \circ \omega_{de}) V_{bde}^* \\ &= \sum_{c|\frac{a}{b}} V_{bc}(f \circ \omega_c) V_{bc}^* \sum_{e|c} \mu(e), \end{aligned}$$

where we have made the substitution $c = de$ and rearranged terms. Using the classical identity $\sum_{e|c} \mu(e) = \delta_{c,1}$, we arrive at $\Gamma_a^{-1} \circ \Gamma_a(V_b f V_b^*) = V_b f V_b^*$. A similar computation shows that $\Gamma_a \circ \Gamma_a^{-1}(\gamma_b(f)) = \gamma_b(f)$, completing the proof. $\square$

The following proposition provides the dual picture, presenting the spectrum $\mathfrak{D}$ as a projective limits of disjoint circles. This is not strictly necessary for future chapters, but illustrates the geometric nature of the objects at hand. This presentation is explored further in Sections 3.3 and 3.4.

Proposition 3.5. *For each $a, b \in \mathbb{N}^\times$ such that $a|b$, define $\psi_{a,b} : X_b \to X_a$ by*

$$\psi_{a,b}(z, d) := (z^{d/\gcd(a,d)}, \gcd(a, d)) \qquad \textit{for each } z \in \mathbb{T} \textit{ and } d|b.$$

Then $\psi = (X_a, \psi_{a,b})_{a\in\mathbb{N}^\times}$ is a projective system that is topologically conjugate to the system $\iota^* = (\operatorname{Spec}\mathfrak{D}_a, \iota^*_{a,b})_{a\in\mathbb{N}^\times}$ under the transformations $\Gamma^*_a : X_a \to \operatorname{Spec}\mathfrak{D}_a$, and this gives a homeomorphism

$$\operatorname{proj}\lim_a (X_a, \psi_{a,})_{a\in\mathbb{N}^\times} \cong \operatorname{Spec}\mathfrak{D}.$$

Proof. It suffices to show that $\iota^*_{a,b}\circ\Gamma^*_b = \Gamma^*_a\circ\psi_{a,b}$ on X_b, or dually, that $\Gamma_b\circ\iota_{a,b} = \psi^*_{a,b}\circ\Gamma_a$ on $\mathfrak{D}_a$. Evaluating the first expression at $V_c f V_c^*$ with $c|a$ gives

$$\Gamma_b \circ \iota_{a,b}(V_c f V_c^*) = \Gamma_b(V_c f V_c^*) = \sum_{d|\frac{b}{c}} \gamma_{cd}(f \circ \omega_d) = \sum_{d|\frac{a}{c}} \sum_{\substack{e|\frac{b}{c} \\ \gcd(ce,a)=cd}} \gamma_{ce}(f \circ \omega_e),$$

where in the last step, we have split the sum according to the greatest common divisor with $\frac{a}{c}$. On the other hand, the second expression gives

$$\psi^*_{a,b}\circ\Gamma_a(V_c f V_c^*) = \psi^*_{a,b}\left(\sum_{d|\frac{a}{c}} \gamma_{cd}(f \circ \omega_d)\right) = \sum_{d|\frac{a}{c}} \psi^*_{a,b}\left(\gamma_{cd}(f \circ \omega_d)\right) = \sum_{d|\frac{a}{c}} \gamma_{cd}(f\circ\omega_d)\circ\psi_{a,b}$$

For a fixed index $d|\frac{a}{c}$, the corresponding summand is a function taking a point $(z, k) \in X_b$ to $(z^{k/\gcd(a,k)}, \gcd(a, k))$, then maps this to 0 if $\gcd(a, k) \neq cd$ and to $f(z^{k/c})$ otherwise. In particular, this summand is non-zero only if $k = ce$ for some $e|\frac{b}{c}$, allowing for the expression

$$\psi^*_{a,b} \circ \Gamma_a(V_c f V_c^*) = \sum_{d|\frac{a}{c}} \sum_{\substack{e|\frac{b}{c} \\ \gcd(ce,a)=cd}} \gamma_{ce}(f \circ \omega_e)$$

$\square$

3.3 Fibre-multiplicative projective systems

In this section we introduce the notion of a fibre-multiplicative projective system over $\mathbb{N}^\times$. We show that such systems exhibit an "Euler-factorization," whereby the projective limit of the system can be expressed as a fibre product indexed by primes, with factors given by the projective limit of the restricted projective system along prime powers. We then provide a number of examples of fibre-multiplicative systems, leading to a description of $\operatorname{Spec}\mathfrak{D}$ and related spaces. This material is not used in the remaining chapters, but is included out of independent interest.

We partially order $\mathbb{N}^\times$ by division, i.e., $a \leq b$ means $a|b$. We continue to use the notation $a \vee b = \operatorname{lcm}(a, b)$. Our projective systems will consist of some set of objects $\{X_a\}_{a \in S}$ and morphisms $\{\alpha_{a,b} : X_b \to X_a\}_{a,b \in S, a|b}$, which we shorten to $(X_a, \alpha_{a,b})$ when the indexing set S is clear from context (and assumed to be $\mathbb{N}^\times$ by default).

Definition 3.6. A projective system $(X_a, \alpha_{a,b})$ is *fibre-multiplicative* if $(X_{a\vee b}, \alpha_{a,a\vee b}, \alpha_{b,a\vee b})$ is the fibre product of the morphisms $\alpha_{1,a}$ and $\alpha_{1,b}$ whenever $a, b \in \mathbb{N}^\times$ satisfy $a \wedge b = 1$. That is, for any space Z with maps $f_a : Z \to X_a$ and $f_b : Z \to X_b$ satisfying $\alpha_{1,a} \circ f_a = \alpha_{1,b} \circ f_b$, there is a unique map $f_{a\vee b} : Z \to X_{a\vee b}$ such that $f_a = \alpha_{a,a\vee b} \circ f_{a\vee b}$ and $f_b = \alpha_{b,a\vee b} \circ f_{a\vee b}$.

Proposition 3.7. *Let $(X_n, \alpha_{n,m})$ be a fibre-multiplicative projective system over $\mathbb{N}^\times$ in a category which is closed under projective limits that are either finite or countable and well-ordered. For each prime p let $(X^p, \alpha^p_{p^e} : X^p \to X_{p^e})$ denote the projective limit of the subsystem $(X_{p^e}, \alpha_{p^e,p^f})$ obtained by restriction to the submonoid $p^{\mathbb{N}} \subset \mathbb{N}^\times$. Then the projective limit of $(X_n, \alpha_{n,m})$ exists if and only if the fibre product of the maps $\alpha^p_1 : X^p \to X_1$ exists, in which case they are naturally isomorphic.*

Proof. Let $(X, \alpha^p : X \to X^p)$ denote the fibre product of the maps $\alpha^p_1 : X^p \to X_1$. Let $\alpha_{p^e} = \alpha^p_{p^e} \circ \alpha^p$. By our assumption on the system $(X_n, \alpha_{n,m})$, there exists a unique map $\alpha_n : X \to X_n$ such that $\alpha_{m,n} \circ \alpha_n = \alpha_m$ for all $n, m \in \mathbb{N}^\times$ (argued inductively on the number of prime divisors of n). We will show that (X, α_n) satisfies the universal property of the projective limit.

Suppose that $\delta_n : Z \to X_n$ is a family of maps such that $\alpha_{m,n} \circ \delta_n = \delta_m$. Then by the universal property of the projective limit, we have a unique map $\delta^p : Z \to X^p$ such that $\delta_{p^e} = \alpha^p_{p^e} \circ \delta^p$ for each prime p. By the universal property of the fibre product, we also have a unique map $\delta : Z \to X$ such that $\delta^p = \alpha^p \circ \delta$. For each prime power, we then have that

$$\alpha_{p^e} \circ \delta = \alpha^p_{p^e} \circ \alpha^p \circ \delta = \alpha^p_{p^e} \circ \delta^p = \delta_{p^e}$$

Since $(X_a, \alpha_{a,b})$ is fibre-multiplicative, δ_n is uniquely determined by the maps $\delta_{p_i^{e_i}}$, where $n = \prod p_i^{e_i}$ is the prime factorization of n, and $\alpha_n \circ \delta$ is uniquely determined by $\alpha_{p_i^{e_i}} \circ \delta$. It follows that $\alpha_n \circ \delta = \delta_n$ for all n.

The last thing we need to show is uniqueness of δ. Suppose that $\gamma : Z \to X$ is a map such that $\alpha_n \circ \gamma = \delta_n$ for all n. The map δ^p is uniquely determined by the property that $\alpha^p_{p^e} \circ \delta^p = \delta_{p^e}$; since $\delta_{p^e} = \alpha_{p^e} \circ \gamma = \alpha^p_{p^e} \circ \alpha^p \circ \gamma$, it follows that $\alpha^p \circ \gamma = \delta^p$. Since δ is uniquely determined by the property that $\alpha^p \circ \delta = \delta^p$, we conclude that $\delta = \gamma$. Therefore, there is a unique map $\delta : Z \to X$ such that $\alpha_n \circ \delta = \delta_n$, so (X, α_n) satisfies the universal property of the projective limit of $(X_n, \alpha_{n,m})$.

A similar argument shows that if (X, α_n) is the projective limit of $(X_n, \alpha_{n,m})$, then there exist maps $\alpha^p : X \to X^p$ (as were constructed for Z) such that (X, α^p) satisfies the universal property of the fibre product of (X^p, α_1^p). $\square$

In number theory, a function f on $\mathbb{N}^\times$ such that $f(ab) = f(a)f(b)$ whenever a and b are relatively prime is called a multiplicative function (functions for which the factorization holds for every a and b are typically called completely multiplicative). Proposition 3.7 can be thought of as a categorical version of the Euler factorization of Dirichlet series associated to multiplicative arithmetic functions. The analogue of multiplicative arithmetic functions are fibre-multiplicative functors on the partial order category $(\mathbb{N}^\times, \cdot|\cdot)$, and the analogue of Dirichlet series are projective limits in the target category.

For the remainder of this section we focus on projective systems in the category of compact Hausdorff spaces.

Example 3.8. Let $X_n = \Delta_n = \{d \in \mathbb{N}^\times : d|n\}$ and define $\alpha_{n,m} : X_m \to X_n$ by $\alpha_{n,m}(d) = \gcd(n, d)$. This system satisfies the assumptions of Proposition 3.7. The projective limit along the powers of a prime p is homeomorphic to the one-point compactification of $\mathbb{N}$, which we write $\{1, p, p^2, ..., p^\infty\}$. Since $X_1 = \{1\}$, the fibre product is in fact the direct product, so the projective limit is homeomorphic to

$$X \cong \prod_{p \in \mathcal{P}} \{1, p, p^2, p^3, ..., p^\infty\}.$$

The latter is the familiar space of *supernatural numbers*, which we denote by $\overline{\mathbb{N}}$. It is homeomorphic to the Cantor set.

There is an embedding $\mathbb{N}^\times \to \overline{\mathbb{N}}$ mapping a number to its prime factorization, and the image of this map is dense. We can extend the partial order of division on $\mathbb{N}^\times$ to $\overline{\mathbb{N}}$ by letting $(p^{e_p})_{p\in\mathcal{P}} | (p^{f_p})_{p\in\mathcal{P}}$ whenever $e_p \leq f_p$ for all $p \in \mathcal{P}$.

Example 3.9. Let $X_n = \mathbb{Z}/n\mathbb{Z}$ and let $\alpha_{n,m} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the quotient map. The limit of this system is known as the profinite completion of $\mathbb{Z}$, and is typically denoted $\hat{\mathbb{Z}}$ (although we reserve the "hat" to denote the group of characters). Once again, this system satisfies the assumptions of Proposition 3.7, and since $X_1 = \{0\}$, the fibre product is a direct product. The limit along powers of p is the ring $\mathbb{Z}_p$ of p-adic integers, so we recover the known isomorphism

$$\hat{\mathbb{Z}} \cong \prod_{p\in\mathcal{P}} \mathbb{Z}_p$$

For an action of $\mathbb{Z}$ on a space Y, we write $x + a$ for the image of $x \in Y$ under the action of $a \in \mathbb{Z}$, and $x + a\mathbb{Z}$ to denote the orbit of x under the action of the subgroup $a\mathbb{Z} \subseteq \mathbb{Z}$.

Proposition 3.10. *Given a free and proper action of* $(\mathbb{Z},+)$ *on a space* Y*, define*

$$X_a = \bigsqcup_{c|a} Y/c\mathbb{Z}, \qquad \alpha_{a,b}(x + c\mathbb{Z}) = x + (a\mathbb{Z} + c\mathbb{Z}).$$

Then $(X_a, \alpha_{a,b})$ *is a fibre-multiplicative projective system.*

Proof. Recall that by the Chinese remainder theorem, $\gcd(a,b) = 1$ if and only if $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$. Thus, we need to check that if $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$, then X_{ab} is the fibre product of X_a and X_b.

Suppose that Z is a space with maps $f_a : Z \to X_a$ and $f_b : Z \to X_b$ such that $\alpha_{1,a} \circ f_a = \alpha_{1,b} \circ f_b$. For any $z \in Z$, pick some $x_1, x_2 \in Y$ and $c|a$, $d|b$ such that $f_a(z) = x_1 + c\mathbb{Z}$ and $f_b(z) = x_2 + d\mathbb{Z}$. Then since $\alpha_{1,a}(x_1 + c\mathbb{Z}) = \alpha_{1,b}(x_2 + d\mathbb{Z})$, there exists some $k \in \mathbb{Z}$ such that $x_1 + k = x_2$. Since $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$, this must also be true for $c\mathbb{Z}$ and $d\mathbb{Z}$, so we can write $k = c' + d'$ for $c' \in c\mathbb{Z}$ and $d' \in d\mathbb{Z}$. Rearranging, we find that $x_3 := x_1 + c' = x_2 - d'$ belongs to both $x_1 + c\mathbb{Z}$ and $x_2 + d\mathbb{Z}$. If we define $f_{ab} : Z \to X_{ab}$, $f_{ab}(z) = x_3 + cd\mathbb{Z}$, then the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, "f_b"] \arrow[d, "f_a"] \arrow[dr, "f_{ab}"] & X_b \\
X_a & X_{ab} \arrow[l] \arrow[u]
\end{tikzcd}$$

The Chinese remainder theorem implies that $a\mathbb{Z} + cd\mathbb{Z} = c\mathbb{Z}$ and $b\mathbb{Z} + cd\mathbb{Z} = d\mathbb{Z}$. In order for the diagram to commute, we require that $f_{ab}(z)$ belongs to both $(x_1 + c\mathbb{Z}) = (x_3 + c\mathbb{Z})$ and $(x_2 + d\mathbb{Z}) = (x_3 + d\mathbb{Z})$. Suppose that $y \in (x_3 + c\mathbb{Z}) \cap (x_3 + d\mathbb{Z})$, so that $y = x_3 + e = x_3 + f$ for some $e \in c\mathbb{Z}$ and $f \in d\mathbb{Z}$. Since the action is free, we must have $e = f$, so $y \in x_3 + c\mathbb{Z} \cap d\mathbb{Z} = x_3 + cd\mathbb{Z}$ (again by the Chinese Remainder Theorem). This shows that the function f_{ab} is unique.

The last thing to check is continuity of f_{ab}. It suffices to show that the maps $\alpha_{a,ab}$ and $\alpha_{b,ab}$ are local homeomorphisms. But this follows from the fact that our action is free and proper, since then the maps $Y \to Y/a\mathbb{Z}$ are all local homeomorphisms. $\square$

3.4 $\operatorname{Spec}\mathfrak{D}$ as a fibre product

We now aim to show that our projective system from Proposition 3.5 is fibre-multiplicative and thus realize $\operatorname{Spec}\mathfrak{D}$ as a fibre product. The result is derived from Proposition 3.10, after an appropriate rescaling of circles.

Lemma 3.11. *The action of* $\mathbb{Z}$ *on* $\mathbb{R}$ *by translation is free and proper and the maps*

$$\bigsqcup_{d|a} \mathbb{R}/d\mathbb{Z} \to \mathbb{T} \times \Delta_a, \qquad x + d\mathbb{Z} \mapsto (e^{2\pi i x/d}, d)$$

implement a natural isomorphism of the projective system defined in Proposition 3.10 to that appearing in Proposition 3.5.

Corollary 3.12. *For each prime p let* $(X^p, \psi^p_{p^e})$ *be the projective limit of the subsystem* $(X_{p^e}, \psi_{p^e,p^f})$*, where* X_{p^e} *and* ψ_{p^e,p^f} *are the spaces and maps from Proposition 3.5. Then* $\operatorname{Spec}\mathfrak{D}$ *is isomorphic to the fibre product of the maps* $\psi^p_1 : X^p \to X_1$.

Proof. Proposition 3.5 says that $\operatorname{Spec}\mathfrak{D}$ is isomorphic to the projective limit of $(X_a, \psi_{a,b})$. By Proposition 3.10, this system is fibre multiplicative. The result then follows from Proposition 3.7. □

Next we show that $\operatorname{Spec}(\mathfrak{D})$ is a fibration over $\overline{\mathbb{N}}$ in which the fibre over each point is a solenoid determined by the divisibility of the point. We note that, with the appropriate modifications, the next proposition holds more generally for the systems considered in Proposition 3.10.

Proposition 3.13. *The projections* $X_a = \mathbb{T} \times \Delta_a \to \Delta_a$, $a \in \mathbb{N}^\times$ *form a natural transformation from the projective system* $(X_a, \psi_{a,b})$ *to* $\Delta_a, \alpha_{a,b})$*, and thus induce a surjective map* $\pi : \operatorname{Spec}(\mathfrak{D}) \to \overline{\mathbb{N}}$*. The fibre* $\pi^{-1}(\{N\})$ *over* $N \in \overline{\mathbb{N}}$ *is homeomorphic to the space Z that is the projective limit over* $S = \{a \in \mathbb{N}^\times : a|N\}$ *of the maps* $\omega_{a/b} : \mathbb{T} \to \mathbb{T}$, $\omega_a(z) = z^a$ *for* $a, b \in \mathbb{N}^\times$, $b|a|N$*; the homeomorphism is obtained from the universal property of Z applied to the maps*

$$\varphi_a : \pi^{-1}(N) \to \mathbb{T}, \qquad \varphi_a(x) = z \text{ where } \psi_a(x) = (z, c), \qquad a \in S.$$

Proof. Let $(X_a, \psi_{a,b})$ be the projective system from Proposition 3.5, and let (X, ψ_a) be the associated projective limit. Let $(\Delta_a, \alpha_{a,b})$ be the projective system from Example 3.8 with the limit $(\overline{\mathbb{N}}, \alpha_a)$. Letting $f_a : X_a \to \Delta_a$, $f_a(z, c) = c$, then when $\psi_b(x) = (z, c)$, one has

$$\alpha_{a,b} \circ f_b \circ \psi_b(x) = \alpha_{a,b}(c) = \gcd(a, c)$$

and

$$f_a \circ \psi_a(x) = f_a \circ \psi_{a,b} \circ \psi_b(x) = f_a\left(z^{c/\gcd(a,c)}, \gcd(a, c)\right) = \gcd(a, c),$$

so the maps $\delta_a = f_a \circ \psi_a$ satisfy $\alpha_{a,b} \circ \delta_b = \delta_a$. By the universal property we obtain a map $\pi : X \to \overline{\mathbb{N}}$ satisfying $\alpha_a \circ \pi = \delta_a$.

We can identify X with the subspace of $\prod_{a\in\mathbb{N}^\times} X_a$ of sequences (x_a) such that $\psi_{a,b}(x_b) = x_a$ for every $b|a$. The fibre $\pi^{-1}(N)$ then consists of sequences (x_a) such that each x_a is of the form $(z_a, \gcd(N, a))$ for some $z_a \in \mathbb{T}$ satisfying $z_b^{\gcd(N,b)/\gcd(N,a)} = z_a$. In particular, $z_a = z_{\gcd(N,a)}$ for every $a \in \mathbb{N}^\times$, so it follows that $\pi^{-1}(N)$ is precisely the projective limit of the directed system $(Z_a = \mathbb{T}, \omega_{a/b})_{a|N}$. □

Let $Z_m = \{z \in \mathbb{T} : z^m = 1\}$ denote the set of m^{th} roots of unity in $\mathbb{T}$. We consider the projective system consisting of the subspaces ${}_mX_a = Z_m \times \Delta_a \subseteq X_a$ and restricted maps ${}_m\psi_{a,b} = \psi_{a,b}|_{{}_mX_a}$.

Lemma 3.14. *For every $m \in \mathbb{N}^\times$, the projective system $({}_mX_a,\ {}_m\psi_{a,b})$ is fibre-multiplicative.*

Proof. Fix $m \in \mathbb{N}^\times$; let a and b be coprime numbers, and let $c = \gcd(a, m)$, $d = \gcd(b, m)$. We will show that $(Z_m, \omega_b, \omega_a)$ is the fibre product of the maps $\omega_a : Z_m \to Z_m$ and $\omega_b : Z_m \to Z_m$, whence it follows that $({}_mX_{ab},\ {}_m\psi_{a,ab},\ {}_m\psi_{b,ab})$ is the fibre product of ${}_m\psi_{1,a}$ and ${}_m\psi_{1,b}$.

It is useful to identify Z_m with the set $\mathbb{Z}/m\mathbb{Z}$ by fixing a character χ on this group with trivial kernel. We then have that $\omega_a = \chi \circ \mu_a \circ \chi^{-1}$ regardless of the choice of χ, where $\mu_a : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is multiplication by a. We will show that $(\mathbb{Z}/m\mathbb{Z}, \mu_b, \mu_a)$ is the fibre product of the maps (μ_a, μ_b).

Suppose that Z is a space with maps $\delta_a : Z \to \mathbb{Z}/m\mathbb{Z}$ and $\delta_b : Z \to \mathbb{Z}/m\mathbb{Z}$ such that $\mu_a \circ \delta_a = \mu_b \circ \delta_b$. Let $c = \gcd(a, m)$ and $d = \gcd(b, m)$; then the range of μ_a is the set $c\mathbb{Z}/m\mathbb{Z}$, and the range of μ_b is $d\mathbb{Z}/m\mathbb{Z}$. Since c and d are relatively prime, the intersection of these ranges (which contains the shared range of $\mu_a \circ \delta_a = \mu_b \circ \delta_b$) is $cd\mathbb{Z}/m\mathbb{Z}$, by the Chinese remainder theorem. Pulling this back along μ_a shows that the range of δ_a is contained in $d\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\frac{m}{d}\mathbb{Z}$, and similarly, the range of δ_b is $c\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\frac{m}{c}\mathbb{Z}$. For any $z \in Z$, it follows from another application of the Chinese remainder theorem that there is a unique $x \in \mathbb{Z}/(\frac{m}{c}\mathbb{Z} \cap \frac{m}{d}\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$ such that $\mu_b(x) = \delta_a(z)$ and $\mu_a(x) = \delta_b(z)$. Letting $\delta_{ab}(z) = x$ gives us a (unique) map such that $\mu_b \circ \delta_{ab} = \delta_a$ and $\mu_a \circ \delta_{ab} = \delta_b$, so $(\mathbb{Z}/m\mathbb{Z}, \mu_b, \mu_a)$ is the fibre product of (μ_a, μ_b).

Since χ is a bijection, we conclude that $(Z_m, \omega_b, \omega_a)$ is the fibre product of ω_a and ω_b on Z_m. This yields the desired result. ☐

3.5 Quasi-lattice gradings

In this section, we consider the more general setting of quasi-lattice graded C*-algebras, [3, Section 4]. We prove analogues of the results from 3.2. These comprise a slight refinement of the results in Section 4 of [3], where the definition of quasi-lattice graded C*-algebras was first introduced.

A *quasi-lattice ordered set* is a partially ordered set $(I, \leq)$ with the property that for each $p, q \in I$, if the set of all $r \in I$ with $p \leq r$ and $q \leq r$ is non-empty, then there exists a unique element of I denoted $p \vee q$ such that $p \vee q \leq r$ for all such r. If this set is empty, then we write $p \vee q = \infty$.

Definition 3.15. A quasi-lattice graded C*-algebra is a pair $(\mathcal{F}, (B_p)_{p \in I})$ consisting of a C*-algebra $\mathcal{F}$ and a family of C*-subalgebras $B_p \subseteq \mathcal{F}$ indexed by a quasi-lattice ordered set I, such that

1. $\mathcal{F} = \overline{\left(\sum_{p\in I} B_p\right)}$.

2. for each $p \in I$ and finite $J \subseteq I\backslash\{p\}$, $B_p \cap \left(\sum_{q\in J} B_q\right) = \{0\}$,

3. if $p \vee q = r \in I$, then $B_pB_q \subseteq B_r$, otherwise $B_pB_q = \{0\}$.

We refer to the collection $(B_p)_{p\in I}$ as a quasi-lattice grading on $\mathcal{F}$, and the subalgebras B_p as the homogeneous subalgebras of the grading. When $q \leq p$, there is a natural *-homomorphism $\iota_q^p : B_q \to M(B_p)$ defined by $\iota_q^p(x)(y) = xy$ for every $x \in B_q$, $y \in B_p$. These maps satisfy $\iota_p^r \circ \iota_q^p(x) = \iota_q^r(x)$ and $xy = \iota_q^{p\vee q}(x)\iota_p^{p\vee q}(y)$.

By [39, Corollary 2.51], ι_q^p extends uniquely to a *-homomorphism $\iota_q^p : M(B_q) \to M(B_p)$. We will not distinguish between ι_q^p and its extension, in general.

If I is a quasi-lattice, then the set $I^+ = I \cup \{\infty\}$ is a directed set with the relations $p \leq_{I^+} q$ if $p, q \in I$ and $p \leq_I q$, and $p \leq_{I^+} \infty$ for all $p \in I$. If $(B_p)_{p\in I}$ is a quasi-lattice grading of $\mathcal{F}$ over I, then we can construct a grading of $\mathcal{F}$ over I^+ by including $B_\infty = \{0\}$, the zero C*-algebra. If $p \vee q = \infty$ in I^+, then $B_pB_q \subseteq \{0\} = B_\infty$, so this is a quasi-lattice grading. We will not distinguish between the I grading and the I^+ grading in general. The I^+-grading has the advantage that there is a map ι_q^∞ sending every element of B_q to 0, often avoiding the need to explicitly consider the cases where p and q have no common upper bound separately.

If λ is a finite $\vee$-closed subset of I, then $B_\lambda = \sum_{p\in\lambda} B_p$ is a C*-subalgebra of $\mathcal{F}$ [3, Lemma 4.2]. For every $p \in \lambda$, linear independence of the subspaces B_q (Definition 3.15, 2) implies that there is a unique linear map satisfying

$$\gamma_p : B_\lambda \to M(B_p), \quad \forall x \in B_q,\ \gamma_p(x) = \begin{cases} \iota_q^p(x) & \text{if } q \leq p \\ 0 & \text{if } q \not\leq p. \end{cases}$$

Lemma 3.16. *The map γ_p is a *-homomorphism.*

Proof. It is clear that γ_p preserves the adjoint, so it only remains to show that multiplication is preserved. Given $x \in B_q$ and $y \in B_r$, we have

$$\gamma_p(xy) = \begin{cases} \iota_{q\vee r}^p(xy) = \iota_{q\vee r}^p(\iota_q^{q\vee r}(x)\iota_r^{q\vee r}(y)) = \iota_q^p(x)\iota_r^p(y) & \text{if} \quad q, r \leq p \\ 0 = \gamma_p(x)\gamma_p(y) & \text{if} \quad q \not\leq p \text{ or } r \not\leq p \end{cases} \qquad \square$$

From these we obtain an injective *-homomorphism

$$\Gamma_\lambda = \bigoplus_{p\in\lambda} \gamma_p : B_\lambda \longrightarrow \bigoplus_{p\in\lambda} M(B_p) =: M_\lambda^\oplus.$$

Under mild assumptions on the maps ι_q^p, we will prove that Γ is an isomorphism to the direct sum $\bigoplus_{p\in\lambda} B_p$, analogous to Proposition 3.4.

For $J \subseteq I$ and $p \in I$, let $S(p, J)$ be the (possibly empty) set of minimal elements in the partially ordered set $\{q \in J : p < q\}$. For a subset $K \subseteq I$ and $p \in I$, let $q_{p,K} = \bigvee(K \cup \{p\})$. Note that if $K \subseteq S(p, I)$, then $q_{p,K} = \bigvee K$ when K is non-empty and p otherwise.

Proposition 3.17. *Suppose that the image of* ι_q^p *is contained in* B_p *for all* $q \leq p$. *Then the image of* Γ_λ *is* $\bigoplus_{p\in\lambda} B_p$, *and* Γ_λ *is a *-isomorphism onto its image with inverse given on the spanning elements by the formula*

$$\Gamma_\lambda^{-1}(\gamma_p(x)) = \sum_{K\subseteq S(p,\lambda)} (-1)^{|K|} \iota_p^{q_{p,K}}(x), \qquad \forall x \in B_p,\ p \in \lambda. \tag{3.4}$$

Proof. The image of Γ is contained in $\bigoplus_{p\in\lambda} B_p$ by our assumptions on ι_q^p; likewise, the formula for Γ_λ^{-1} is gives a function to B_λ on the spanning elements. By the linear independence of the direct summands, the formula for Γ_λ^{-1} extends uniquely to a linear function on $\bigoplus_{p\in\lambda} B_p$.

We will first check that $\Gamma_\lambda \circ \Gamma_\lambda^{-1} = \mathrm{id}_{\bigoplus B_p}$. It suffices to check this for $\gamma_p(x)$ for $x \in B_p$, $p \in \lambda$.

$$\begin{aligned}\Gamma_\lambda \circ \Gamma_\lambda^{-1}(\gamma_p(x)) &= \bigoplus_{q\in\lambda} \sum_{K\subseteq S(p,\lambda)} (-1)^{|K|} \gamma_q\left(\iota_p^{q_{p,K}}(x)\right) \\ &= \bigoplus_{q\in\lambda} \gamma_q(x) \sum_{\substack{K\subseteq S(p,\lambda)\\ q_{p,K}\leq q}} (-1)^{|K|}\end{aligned}$$

If $q_{p,K} \leq q$ and $q_{p,J} \leq q$, then $q_{p,J\cup K} = q_{p,J} \vee q_{p,K} \leq q$. In particular, the set of $K \subseteq S(p, \lambda)$ with $q_{p,K} \leq q$ forms a directed set. The sum is then 0 if this set is empty ($p \not\leq q$), $\gamma_q(x)$ if the it contains only $\emptyset$ ($q = p$), and 0 if it contains more than one element ($p < q$). Therefore, we have $\Gamma \circ \Gamma^{-1}(\gamma_p(x)) = \gamma_p(x)$.

Since $\Gamma_\lambda \circ \Gamma_\lambda^{-1}$ is the identity, Γ_λ is surjective on $\bigoplus_{p\in\lambda} B_p$. It only remains to show that Γ is injective. Let $x = \sum_{p\in\lambda} x_p$, $x_p \in B_p$, be a non-zero element of B_λ, and choose some $r \in \lambda$ such that $x_r \neq 0$ which is minimal with respect to such elements of λ. Then the r-component of $\Gamma_\lambda(x)$ is given by

$$\sum_{p\in\lambda} \gamma_r(x_p) = \sum_{\substack{p\in\lambda\\ p\leq r}} \iota_p^r(x_p) = x_r \neq 0$$

by minimality of r. Therefore, Γ_λ has trivial kernel, and we conclude that it is a *-isomorphism. □

Chapter 4
KMS states via subconformal measures

As we saw in Chapter 3, the KMS_β states on $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma)$ all factor through the conditional expection $\Phi : \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) \to \mathfrak{D}$ of the gauge action, by virtue of σ-invariance of the states. The restriction of a KMS_β state ψ to $\mathfrak{D}$ is also a state, $\phi = \psi|_{\mathfrak{D}}$, and the KMS_β condition provides us with a conformal relation on ϕ: $\phi \circ \upsilon_a = a^{-\beta}\phi$ for all $a \in \mathbb{N}^\times$, where υ_a is the *-homomorphism $\upsilon_a(x) = V_a x V_a^*$ (c.f. [19, Proposition 8.3]).

We also saw in Chapter 3 that $\mathfrak{D}$ contains the subalgebra $\mathfrak{D}_1 = C^*(U) \cong C(\mathbb{T})$. While there is typically no canonical way to extend a linear functional on $\mathfrak{D}_1$ to $\mathfrak{D}$, the conformal relation on ϕ allows us to recover it uniquely from $\phi|_{\mathfrak{D}_1}$. In particular, for any monomial $V_a f V_a^*$ in $\mathfrak{D}$, we have

$$\phi(V_a f V_a^*) = \phi(\alpha_a(f)) = a^{-\beta}\phi(f) \qquad \text{for all } a \in \mathbb{N}^\times,\ f \in C(\mathbb{T}),$$

which extends by linearity to a formula for ϕ in terms of $\phi|_{\mathfrak{D}_1}$ on a dense subspace of $\mathfrak{D}$.

The discussion above demonstrates that the assignment $\psi \mapsto \psi|_{\mathfrak{D}_1}$ provides an injective affine map from the simplex of KMS_β states to the simplex of states on $\mathfrak{D}_1$. This suggests the following question: can one provide a description of the KMS_β states in terms of their images under this affine map?

For each state ϕ_1 on $\mathfrak{D}_1$, we can construct a densely-defined linear functional φ on $\mathfrak{D}$,

$$\varphi(V_a f V_a^*) = a^{-\beta}\phi_1(f), \qquad a \in \mathbb{N}^\times,\ f \in C(\mathbb{T}), \tag{4.1}$$

which satisfies $\varphi \circ \alpha_a = a^{-\beta}\varphi$ on the subspace of finite linear combinations of monomials. If φ extends to a state ϕ on $\mathfrak{D}$, then by Proposition 2.14, the linear functional $\phi \circ \Phi$ is a KMS_β state on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. For instance, if ψ is a KMS_β state and $\phi_1 = \psi|_{\mathfrak{D}_1}$, then $\phi = \psi|_{\mathfrak{D}}$ is a state extending φ to $\mathfrak{D}$, and we know that $\psi = \phi \circ \Phi$.

The question asked above regarding the range of the affine map $\psi \mapsto \psi|_{\mathfrak{D}_1}$ may be restated as follows: for which states ϕ_1 on $\mathfrak{D}_1$ does the linear functional φ of (4.1) extend to a state on $\mathfrak{D}$? This chapter is dedicated to answering this question in terms of so-called *subconformal measures* on $\mathbb{T}$, characterized by a positivity condition introduced in Section 4.1. We will show that the assignment of KMS$_\beta$ states to subconformal measures is an affine isomorphism between simplices. In Section 4.2, we reframe the subconformal condition in terms of certain operators on the space $\mathcal{M}(\mathbb{T})$, which admit a factorization reminiscent of the Euler product.

We conclude the chapter with Section 4.3, proving the analogous positivity condition for quasi-lattice graded C*-algebras. We show that the linear functionals ψ_p constructed in [3, Section 4] can be factored through the *-homomorphism Γ_λ of Section 3.5 in a natural way, and that this factorization implies the positivity condition. This offers a slight refinement of the condition from [3], that relies on a slightly more precise application of the principle of inclusion-exclusion.

A large motivation for this chapter is [3], wherein Afsar, Larsen, and Neshveyev considered a similar problem for product systems over quasi-lattice ordered monoids. The positivity condition we obtain is a particular case of condition (2.1) in [3]: indeed, the Toeplitz algebra $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ is the C*-algebra of a product system over $\mathbb{N}^\times$ by work of Kwaśniewski and Larsen, [25], and Theorem 5.1 of [3] yields the same condition as our own. We have chosen to provide a self-contained proof, for two main reasons. First, our version and proof of the positivity condition makes use of a different approach involving *-homomorphisms. Second, the proof of the positivity condition in our particular case is very direct, and our simple example may provide further insight into the more general setting. Nonetheless, this chapter is heavily influenced by the versatile (and, as we demonstrate, effective) technique introduced in [3].

4.1 The positivity condition and subconformal measures

Recall from Proposition 2.14 that a state ψ on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ satisfies the KMS$_\beta$ condition if and only if it satisfies

$$\psi(V_a U^k V_b^*) = \delta_{a,b} a^{-\beta} \psi(U^k) \quad \text{for all } a, b \in \mathbb{N}^\times \text{ and } k \in \mathbb{Z}. \tag{4.2}$$

As we remarked above, this provides a formula for lifting a state on $C^*(U)$ to a densely defined linear functional on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, which may or may not extend to all of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. Our first objective is to characterize the states which do admit an extension.

We work at the level of the diagonal algebra $\mathfrak{D}$ of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. Since $C^*(U) \cong C(\mathbb{T})$, the linear functionals on $C^*(U)$ are given by integration over $\mathbb{T}$ against regular Borel measures ν on $\mathbb{T}$. The push-forward measure $\omega_{a*}\nu$ on $\mathbb{T}$ is given by $\omega_{a*}\nu(E) = \nu(\omega_a^{-1}(E))$, for each measurable subset E.

Lemma 4.1. *Let $\beta > 0$ and suppose ν is a finite measure on $\mathbb{T}$. Then there exists a unique linear functional $\phi_{\nu,\beta,a} : \mathfrak{D}_a \to \mathbb{C}$ for each $a \in \mathbb{N}^\times$ such that*

$$\phi_{\nu,\beta,a}(V_b f V_b^*) := b^{-\beta} \int_{\mathbb{T}} f d\nu \qquad b|a,\ f \in C(\mathbb{T}), \tag{4.3}$$

and $(\phi_{\nu,\beta,a})_{a\in\mathbb{N}^\times}$ is a coherent family for the inductive system $(\mathfrak{D}_a, \iota_{a,b})_{a\in\mathbb{N}^\times}$.

If, in addition, ν satisfies

$$\sum_{d|n} \mu(d) d^{-\beta} \omega_{d*}(\nu) \geq 0 \qquad \forall n \in \mathbb{N}^\times, \tag{4.4}$$

then each $\phi_{\nu,\beta,a}$ is positive, and there is a unique positive linear functional $\phi_{\nu,\beta} : \mathfrak{D} \to \mathbb{C}$ such that

$$\phi_{\nu,\beta}(V_b f V_b^*) = b^{-\beta} \int_{\mathbb{T}} f d\nu \qquad b \in \mathbb{N}^\times,\ f \in C(\mathbb{T}). \tag{4.5}$$

Proof. From Lemma 3.4 we know that $\mathfrak{D}_a$ is the linear space direct sum of the subspaces $V_b C(\mathbb{T}) V_b^*$ over the divisors b of a, and hence (4.3) defines a unique linear functional on $\mathfrak{D}_a$. The resulting family $(\phi_{\nu,\beta,a})_{a\in\mathbb{N}^\times}$ of linear functionals is coherent with respect to inclusion because the right hand side does not depend on a explicitly.

Notice that condition (4.4) with $n = 1$ says that ν is positive, so we may as well assume it is a probability measure. We will show next that (4.4) implies that $\phi_{\nu,\beta,a}$ is a state of $\mathfrak{D}_a$ for each $a \in \mathbb{N}^\times$. After this is established, we apply the standard fact that a coherent family of states on a directed system uniquely determines a state on the direct limit C*-algebra.

The isomorphism $\Gamma_a : \mathfrak{D}_a \cong C(X_a)$ from Lemma 3.4 establishes a bijection between positive cones. Since the positive cone of $C(X_a) = C(\bigsqcup_{b|a}(\mathbb{T} \times \{b\}))$ is the direct sum of positive cones of the $C(\mathbb{T} \times \{b\})$, the functional $\phi_{\nu,\beta,a}$ is positive if and only if $\phi_{\nu,\beta,a}(\Gamma_a^{-1}(\gamma_b(f))) \geq 0$ for every $b|a$ and every $f \in C(\mathbb{T})^+$. We verify the latter condition by the following direct computation:

$$\phi_{\nu,\beta,a}(\Gamma_a^{-1}(\gamma_b(f))) = \phi_{\nu,\beta,a}\Big(\sum_{d|\frac{a}{b}} \mu(d) V_{bd}(f \circ \omega_d) V_{bd}^*\Big) = \sum_{d|\frac{a}{b}} \mu(d)(bd)^{-\beta} \int_{X_a} \gamma_{bd}(f \circ \omega_d) d\nu$$

$$= \sum_{d|\frac{a}{b}} \mu(d)(bd)^{-\beta} \int_{\mathbb{T}\times\{b\}} (f \circ \omega_d) d\nu = b^{-\beta} \int_{\mathbb{T}} f d\Big(\sum_{d|\frac{a}{b}} \mu(d) d^{-\beta} \omega_d^* \nu\Big).$$

This shows that $\phi_{\nu,\beta,a}$ is positive if and only if condition (4.4) holds for all divisors of a. Computing at the identity shows that $\phi_{\nu,\beta,a}$ is a state. □

We refer to measures satisfying (4.4) as β-subconformal measures, as this condition is similar to that of a conformal measure. For a measurable map T on a measure space

$(X, \mathcal{B})$, a measure μ is conformal with respect to the function $\varphi : X \to \mathbb{R}_+$ if for every $E \in \mathcal{B}$,

$$\mu(T(E)) = \int_E \varphi(x) d\mu(x).$$

When $n = p^e$ is a prime power, our positivity condition may be written $p^{-\beta}\nu(\omega_p^{-1}(E)) \leq \nu(E)$, which is similar to the conformal condition for the function $\varphi(z) = p^\beta$ but with two important changes: the obvious inequality, and the use of the push-forward measure rather than the pull-back measure. The positivity condition when n is not a prime power combines the various subconformal conditions for prime divisors of n using an inclusion-exclusion-like formula; for example, when $n = pq$ is a product of two primes,

$$p^{-\beta}\nu(\omega_p^{-1}(E)) + q^{-\beta}\nu(\omega_q^{-1}(E)) - (pq)^{-\beta}\nu(\omega_{pq}^{-1}(E)) \leq \nu(E).$$

The name "subconformal measure" is not conventional and may be disputed. The intention of its use is to highlight the interesting connection between the two settings of (quantum) KMS states and (classical) thermodynamical formalism.

Theorem 4.2. *For every $0 < \beta < \infty$, the map $\nu \mapsto \phi_{\nu,\beta} \circ \Phi$ is an affine isomorphism of the simplex of probability measures ν on $\mathbb{T}$ satisfying* (4.4) *onto the simplex of* KMS_β *states of $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma)$. The inverse map sends a* KMS_β *state ψ to the measure ν_ψ on $\mathbb{T}$ defined by*

$$\int_{\mathbb{T}} f d\nu_\psi = \psi(f) \qquad f \in C(\mathbb{T}). \tag{4.6}$$

Proof. Suppose ν is a probability measure on $\mathbb{T}$ satisfying (4.4). Then Lemma 4.1 gives a state $\phi_{\nu,\beta}$ of $\mathfrak{D}$ satisfying (4.2), whence $\phi_{\nu,\beta} \circ \Phi$ is a KMS_β state whose restriction to $\mathfrak{D}_1 \cong C(\mathbb{T})$ is implemented by integration with respect to ν.

Conversely, suppose ψ is a KMS_β state. Then the measure ν_ψ on $\mathbb{T}$ defined by (4.6) is a probability measure, as $\mathfrak{D}_1$ is a unital subalgebra of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. Moreover, for each positive function $f \in C(\mathbb{T})$,

$$\begin{aligned}
\int_{\mathbb{T}} f d\Big(\sum_{d|n} \mu(d) d^{-\beta} \omega_{d*}\nu_\psi\Big) &= \sum_{d|n} \mu(d) d^{-\beta} \int_{\mathbb{T}} (f \circ \omega_d) d\nu_\psi \\
&= \sum_{d|n} \mu(d) d^{-\beta} \psi(f \circ \omega_d) = \sum_{d|n} \mu(d) \psi(V_d (f \circ \omega_d) V_d^*) \\
&= \sum_{d|n} \mu(d) \psi(V_d V_d^* f V_d V_d^*) = \sum_{d|n} \mu(d) \psi(f V_d V_d^*) \\
&= \psi\Big(f \prod_{p|n} (1 - V_p V_p^*)\Big) \geq 0,
\end{aligned}$$

since $f \prod_{p|n}(1 - V_p V_p^*)$ is a positive element of $\mathfrak{D}$. This shows that the map $\varphi \mapsto \nu_\varphi$ sends

KMS$_\beta$ states to probability measures satisfying (4.4). As observed at the beginning of the proof $\nu_{\phi_{\nu,\beta}\circ\Phi} = \nu$, so this map is surjective. Since φ and $\phi_{\nu_\varphi,\beta}\circ\Phi$ are KMS$_\beta$ states with the same restriction to $\mathfrak{D}_1$, injectivity follows by (4.2). $\square$

4.2 Subconformal operators

The positivity condition (4.4) can be expressed in terms of operators on the Banach space $\mathcal{M}(\mathbb{T})$ of finite complex Borel measures on $\mathbb{T}$, with its cone $\mathcal{M}(\mathbb{T})^+$ of positive measures. This results in products over subsets of primes, resembling the Euler product formula for the Riemann zeta function.

Lemma 4.3. *For each $n \in \mathbb{N}$ and $\beta > 0$, view the left hand side of the positivity condition* (4.4) *as defining an operator $A_{n,\beta} := \sum_{d|n}\mu(d)d^{-\beta}\omega_{d*}$ acting on the space $\mathcal{M}(\mathbb{T})$ of measures on the circle. Then*

$$A_{n,\beta} = \prod_{p|n,\, p\in\mathcal{P}} (1 - p^{-\beta}\omega_{p*}),$$

and a measure $\nu \in \mathbb{T}$ satisfies (4.4) *if and only if*

$$\prod_{p\in F}(1 - p^{-\beta}\omega_{p*})\nu \geq 0 \qquad \text{for all finite } F \subset \mathcal{P}.$$

Proof. When p is prime, we have $A_{p,\beta} = 1 - p^{-\beta}\omega_{p*}$. Since the operators $1 - p^{-\beta}\omega_{p*}$ pairwise commute, the usual inclusion-exclusion formula for the product $\prod_{p|n}(1 - p^{-\beta}\omega_{p*})$ gives the result for squarefree n, which suffices since the Möbius function eliminates repeated primes. Condition (4.4) says $A_{n,\beta}\nu \in \mathcal{M}(\mathbb{T})^+$ for every $n \in \mathbb{N}^\times$ so the last assertion follows immediately. $\square$

Next we see that $A_{p,\beta}$ is invertible for each prime and $\beta > 0$, and hence so is $A_{n,\beta}$, for every $n \in \mathbb{N}^\times$.

Lemma 4.4. *For every prime p and every $\beta > 0$, the series $\sum_{n=0}^\infty p^{-\beta n}\omega_{p^n *}$ converges in norm to a positive operator (i.e. mapping positive measures to positive measures) that is the inverse of $A_{p,\beta} = 1 - p^{-\beta}\omega_{p*}$.*

Proof. The operator ω_{p*} is positive and has norm 1 because

$$\|\omega_{p*}\nu\| = (\omega_{p*}\nu)(X) = \nu(\omega_p^{-1}(X)) = \nu(X) = \|\nu\|, \qquad \nu \in \mathcal{M}(\mathbb{T})^+$$

and

$$\|\nu\| = \|\nu_+\| + \|\nu_-\| + \|\nu_i\| + \|\nu_{-i}\|, \qquad \nu \in \mathcal{M}(\mathbb{T}),$$

where $\nu = \nu_+ - \nu_- + i\nu_i - i\nu_{-i}$ is the complex Hahn-Jordan decomposition of ν. Thus, $\|p^{-\beta}\omega_{p*}\| = p^{-\beta} < 1$, and since $\omega_{p*}^n = \omega_{p^n*}$, the inverse of $A_{p,\beta}$ is given by the well-known Neumann series $\sum_{n=0}^{\infty} p^{-\beta n}\omega_{p^n*}$ of positive terms in the Banach algebra $B(\mathcal{M}(\mathbb{T}))$. $\square$

Remark 4.5. For $\beta > 1$, the series $T_\beta = \frac{1}{\zeta(\beta)}\sum_{c=1}^{\infty} c^{-\beta}\omega_{c*}$ defines a bounded linear transformation on $\mathcal{M}(\mathbb{T})$. The lemma shows that T_β^{-1} is related to the $A_{n,\beta}$ operators by the formula

$$\zeta(\beta)T_\beta^{-1} = \lim_{n\in\mathbb{N}^\times} A_{n,\beta} = \prod_{p \text{ prime}} A_{p,\beta}.$$

By [19, Theorem 8.1] and Theorem 4.2, T_β is an affine isomorphism between the simplex of probability measures on $\mathbb{T}$ and the simplex of probability measures on $\mathbb{T}$ satisfying (4.4). This fact will be useful in Section 5.2, when we consider limits of the measures $T_\beta\nu$ as $\beta \searrow 1$

The following shows that for each fixed $\beta > 0$, the simplex of KMS$_\beta$ states is equal to the intersection of a decreasing family of sets. This will be used in the computation of the atomic subconformal measures in the following chapter.

Proposition 4.6. *The collection of sets* $(A_{n,\beta}^{-1}\mathcal{M}(\mathbb{T})^+)_{n\in\mathbb{N}^\times}$ *is decreasing in the sense that* $A_{m,\beta}^{-1}\mathcal{M}(\mathbb{T})^+ \supseteq A_{n,\beta}^{-1}\mathcal{M}(\mathbb{T})^+$ *whenever* $m|n$, *and the intersection* $\bigcap_n A_{n,\beta}^{-1}\mathcal{M}(\mathbb{T})^+$ *is the set of measures satisfying* (4.4).

Proof. Suppose $m|n$ and let n' be the product of the primes that divide n but not m. By Lemma 4.3

$$A_{n,\beta} = \prod_{p|n}(1 - p^{-\beta}\omega_{p*}) = \prod_{p|n'}(1 - p^{-\beta}\omega_{p*})\prod_{p|m}(1 - p^{-\beta}\omega_{p*}) = A_{n',\beta}A_{m,\beta}.$$

Then $A_{n,\beta}^{-1} = A_{m,\beta}^{-1}A_{n',\beta}^{-1}$. By Lemma 4.4, the operator $A_{n',\beta}^{-1}$ is positive, so

$$A_{n,\beta}^{-1}\mathcal{M}(\mathbb{T})^+ = A_{m,\beta}^{-1}(A_{n',\beta}^{-1}\mathcal{M}(\mathbb{T})^+) \subseteq A_{m,\beta}^{-1}\mathcal{M}(\mathbb{T})^+.$$

The assertion that $\nu \in A_{n,\beta}^{-1}\mathcal{M}(\mathbb{T})^+$ for every $n \in \mathbb{N}^\times$ is simply a reformulation of (4.4). $\square$

4.3 The positivity condition for quasi-lattice gradings

We conclude this section with the analogue of Lemma 4.1 for general quasi-lattice graded C*-algebras. We resume the conventions and notation of Section 3.5: I is a quasi-lattice and $\mathcal{F}$ is a quasi-lattice graded C*-algebra with grading $\bigoplus_{p\in I} B_p$.

Given a family of positive linear functionals $\phi_p, p \in I$, we would like to determine when the functional

$$\phi\left(\sum_{p\in I} x_p\right) = \sum_{p\in I} \phi_p(x_p)$$

extends to a positive linear functional on $\mathcal{F}$. Since $\mathcal{F} = \varinjlim B_\lambda$, this is equivalent to positivity of the restriction

$$\phi_\lambda = \phi|_{B_\lambda} : \sum_{p\in\lambda} x_p \mapsto \sum_{p\in\lambda} \phi_p(x)$$

for every finite $\vee$-closed subset $\lambda \subseteq I$.

Note that ϕ_p defines a linear functional ϕ_p^0 on $M_\lambda^\oplus$ by $\phi_p^0((x_q)_{q\in\lambda}) = \phi_p(x_p)$. Then $\phi_p^0 \circ \Gamma_\lambda = \psi_p$ as defined in [3, Section 4].

Theorem 4.7. *Given a finite $\vee$-closed subset $\lambda \subseteq I$, let $\phi_p, p \in \lambda$ be a family linear functionals on B_p such that, for every $x \in B_p^+$, one has*

$$\sum_{K\subseteq S(p,\lambda)} (-1)^{|K|} \phi_{q_{p,K}}(\iota_p^{q_{p,K}}(x)) \geq 0. \tag{4.7}$$

Then $\phi_\lambda = \sum_{p\in\lambda} \phi_p$ is a positive linear functional on B_λ.

Before giving the proof, we recall some basics of the theory of functions on quasi-lattice orders. Given a finite quasi-lattice λ, we define a pair of operations G, S on the set of complex-valued functions $f : \lambda \to \mathbb{C}$ by

$$G[f](p) = \sum_{\substack{q\in\lambda \\ p\leq q}} f(q), \qquad S[f](p) = \sum_{K\subseteq S(p,\lambda)} (-1)^{|K|} f(q_{p,K}).$$

The *inversion formula* for quasi-lattice orders says that $G[S[f]] = S[G[f]] = f$. This is a consequence of the principle of inclusion-exclusion.

Proof of Theorem 4.7. For each $p \in \lambda$, let Φ_p denote the linear functional defined by the left-hand side of (4.7), and recall that Φ_p^0 denotes the extension of Φ_p to $M_\lambda^\oplus$ that vanishes on the direct summands of $q \neq p$. By our assumption (4.7), the linear functional $\phi_\lambda^\oplus = \sum_{p\in\lambda} \Phi_p^0$ is positive on $M_\lambda^\oplus$. We will show that $\phi_\lambda = \phi_\lambda^\oplus \circ \Gamma_\lambda$; it suffices to show that $\phi_\lambda(x) = \phi_\lambda^\oplus \circ \Gamma_\lambda(x)$ for each $x \in B_p^+$, $p \in \lambda$.

Define functions on $C_{p,\lambda} = \{q \in \lambda : p \leq q\}$ by

$$f(q) = \phi_\lambda^\oplus \circ \Gamma_\lambda(\iota_p^q(x)), \qquad g(q) = \phi_\lambda(\iota_p^q(x)).$$

For any $q \in C_{p,\lambda}$, we have the formula:

$$\begin{aligned} f(q) &= \sum_{\substack{r\in\lambda \\ q\leq r}} \Phi_r(\iota_p^r(x)) = \sum_{\substack{r\in\lambda \\ q\leq r}} \sum_{K\subseteq S(r,\lambda)} (-1)^{|K|}\phi_{q_{r,K}}(\iota_r^{q_{r,K}}(\iota_p^r(x))) \\ &= \sum_{\substack{r\in\lambda \\ q\leq r}} \sum_{K\subseteq S(r,\lambda)} (-1)^{|K|}\phi_\lambda(\iota_p^{q_{r,K}}(x)) = G[S[g]](q) = g(q) \end{aligned}$$

Therefore, $\phi_\lambda^\oplus \circ \Gamma_\lambda(x) = f(p) = g(p) = \phi_\lambda(x)$. $\square$

Chapter 5
High temperature equilibrium

In this chapter, we compute the β-subconformal measures for $\beta \leq 1$. This will be accomplished in two parts: first for atomic measures, then for non-atomic measures. If $\nu = \nu_a + \nu_{na}$ is the decomposition of a measure into its atomic and non-atomic parts, then ν is positive if and only if ν_a and ν_{na} are positive. One obtains the same decomposition of subconformality, since the functions ω_d are finite-to-1, and hence preserve the countability/uncountability of sets.

When $\beta > 1$, the atomic measures satisfying (4.4) are plentiful. The measures $T_\beta \delta_z = \frac{1}{\zeta(\beta)} \sum_{c=1}^{\infty} c^{-\beta} \delta_{z^c}$ for $z \in \mathbb{T}$ are all atomic, supported on the orbit of z under the action of $\mathbb{N}^\times$. However, when one considers $\beta \leq 1$, this series is no longer convergent. Our first observation is that atomic β-subconformal measures are supported on the torsion subgroup of $\mathbb{T}$, also known as the *roots of unity* in $\mathbb{T}$. This subgroup is isomorphic to the group $\mathbb{Q}/\mathbb{Z}$ through e.g. $q \mapsto e^{2\pi i q}$ (though there are many other isomorphisms, c.f. Chapter 6).

The group of roots of unity is an inductive limit of the subgroups $Z_k = \{z \in \mathbb{T} : z^k = 1\}$, with inclusions $Z_k \to Z_\ell$ whenever $k|\ell$. These subgroups are invariant under the action of $\omega_m, m \in \mathbb{N}^\times$, and the subconformal condition may be reasonably restricted to these subgroups. Our approach to the atomic case considers the restrictions of arbitrary subconformal measures to the subgroups Z_k, which we show are also subconformal. With a bit of work, this reduces our problem in the atomic setting to a computation of subconformal meaures on Z_k for $k \in \mathbb{N}^\times$, which is carried out in Section 5.1

As a set, Z_k may be viewed as a disjoint union of multiplicative groups. Indeed, when $d|k$, we identify the subset $Z_d^* \subseteq Z_k$ of primitive d^{th} roots of unity with the group $(\mathbb{Z}/\frac{k}{d}\mathbb{Z})^*$ by the assignment $n \mapsto e^{2\pi i \frac{nd}{k}}$. Our key observation is that the functions ω_m, which are m-to-1 on $\mathbb{T}$, restrict to bijections on the sets Z_d^* when $\gcd(m,k) = 1$. With the above identification, we have an isomorphism of Hilbert spaces $\ell^2(Z_d^*) \cong \ell^2((\mathbb{Z}/\frac{k}{d}\mathbb{Z})^*)$, and the dual action is simply the regular representation of $(\mathbb{Z}/\frac{k}{d}\mathbb{Z})^*$, enabling a straightforward application of techniques from Fourier analysis. Using the conditional convergence of Dirichlet L-functions for $\Re(s) > 0$, we show that the subconformal operators $A_{n,\beta}$ with $\gcd(n,k) = 1$ (or, more precisely, scalar multiples of these operators) converge to the projection of $\ell^2(Z_k)$ onto the subspace spanned by trivial characters in the Fourier-transformed setting. When we apply the remaining subconformal operator $A_{k,\beta}$ to this subspace, we obtain explicit formulas for the weights of the subconformal measures on Z_k. In this way we obtain one extremal subconformal measure $\nu_{n,\beta}$ for each positive integer $k \in \mathbb{N}^\times$, corresponding to the trivial character on $(\mathbb{Z}/k\mathbb{Z})^*$.

Our analysis of the non-atomic case in Section 5.2 is much simpler. We have already seen one non-atomic subconformal measure, Lebesgue measure on $\mathbb{T}$. Indeed, this measure is β-subconformal for all $\beta > 0$, since

$$A_{n,\beta}\lambda = \sum_{d|n} \mu(d) d^{-\beta} \omega_{d*}\lambda = \left(\sum_{d|n} \mu(d) d^{-\beta} \right) \lambda = \left(\prod_{p|n} 1 - p^{-\beta} \right) \lambda \geq 0.$$

Our aim is to show that this is the only non-atomic β-subconformal measure when $\beta \leq 1$. This claim may appear dubious at first; after all, a limit of subconformal measures may converge to a measure which is non-atomic, even when the measures in the sequence are atomic. However, our description of the extremal atomic subconformal measures precludes the possibility of a limit of their convex combinations being non-atomic, except in the limit of $\nu_{n,\beta}$ as $n \to \infty$. One can directly check that this limit is Lebesgue measure, but this will also follow indirectly from our analysis.

Despite the absence of more exotic non-atomic measures as limits of our atomic measures, we must still justify the claim that Lebesgue is the only non-atomic subconformal measure. Our approach makes use of an interesting result of [3], which says that measures which are β'-subconformal are also β-subconformal when $\beta \geq \beta'$. This allows us to decompose a β'-subconformal measure ν as a (generalized) convex combination of extremal β-subconformal measures with $\beta > 1$, which of course are of the form $T_\beta \delta_z$, $z \in \mathbb{T}$. The limit of these measures as $\beta \to 1^+$ may be computed using basic properties of the *periodic zeta functions*, which are defined, for $\beta > 1$ and $z \in \mathbb{T}$, by the formula

$$\zeta(\beta, z) = \sum_{c=1}^{\infty} c^{-\beta} z^c.$$

(Alternatively, we make use of the *Hurwitz zeta function* in the case where z is a root of unity, though the two approaches are equivalent.) If we were to naively take the limit as $\beta \to 1^+$, we would obtain normalized Haar measure; however, to make this precise, we require a "limit hypothesis" on the measure ν. We suspect that this hypothesis holds for any non-atomic 1-subconformal measure, but do not currently have a proof of this.

Our main theorem, Theorem 1.1, is proven in Section 5.3. We show that the measures $\nu_{n,\beta}$ for $n \in \mathbb{N}^\times$ and λ constitute the extremal points in a simplex of β-

subconformal measures. By the results of the previous chapter, the proof of the main theorem boils down to a computation of the Fourier moments of these measures.

We close this introduction with a few examples, generalizing those of [19]. Recall that the Euler totient function, denoted $\varphi(n)$, is a multiplicative function counting the number of residues that are units modulo n. It can be expressed as

$$\varphi(n) = n \prod_{p|n} \frac{p-1}{p} = n \prod_{p|n} (1 - p^{-1}).$$

We define the *generalized totient function* $\varphi_\beta : \mathbb{N}^\times \to \mathbb{R}_+^*$ by

$$\varphi_\beta(n) = n^\beta \prod_{p|n} (1 - p^{-\beta}).$$

Example 5.1. For prime p, let $\xi_p = e^{2\pi i/p}$ and let $\nu_{p,\beta}$ be the measure on $\mathbb{T}$ defined by

$$\nu_{p,\beta}(\{1\}) = p^{-\beta}, \qquad \nu_{p,\beta}(\{\xi_p^k\}) = \frac{1-p^{-\beta}}{p-1} \quad \text{for } 0 < k < p.$$

We claim that $\nu_{p,\beta}$ satisfies condition (4.4). Indeed, for every prime $q \neq p$, we have that $\omega_{q*}\nu_{p,\beta} = \nu_{p,\beta}$ while $\omega_{p*}\nu_p = \delta_1$, so

$$A_{q,\beta}\nu_{p,\beta} = \begin{cases} (1-q^{-\beta})\nu_p & \text{if } q \neq p \\ \nu_p - p^{-\beta}\delta_1 & \text{if } q = p. \end{cases}$$

Using the factorization $A_{n,\beta} = \prod_{q|n} A_{q,\beta}$ from Lemma 4.3, and placing the factor $A_{p,\beta}$ on the left of the product in the case $p|n$, we see that

$$A_{n,\beta}\nu_{p,\beta} = \prod_{q|n} A_{q,\beta}\nu_{p,\beta} = \begin{cases} \prod_{q|n}(1-q^{-\beta})\nu_{p,\beta} & \text{if } p \nmid n \\ \prod_{\substack{q|n \\ q\neq p}}(1-q^{-\beta})(\nu_{p,\beta} - p^{-\beta}\delta_1) & \text{if } p|n, \end{cases}$$

from which it is clear that $A_{n,\beta}\nu_p \geq 0$ for every n. Integrating $\nu_{p,\beta}$ against the function $\mathfrak{z}^k$ recovers the formula for $\psi_{p,\beta}(U^k)$ in Theorem 1.1, so that $\phi_{\nu_{p,\beta},\beta} \circ \Phi = \psi_{p,\beta}$.

Positivity also holds for the measure $\nu_1 = \delta_1$, namely the unit point mass at $1 \in \mathbb{T}$. It is easy to see that the KMS$_1$ state obtained from ν_1 through Theorem 4.2 is the state ψ_{1,δ_1} from [19, Example 9.2], and the KMS$_1$ state obtained from $\nu_{2,1} = (1/2)\delta_1 + (1/2)\delta_{-1}$ is the state $\psi_{1,\delta_{-1}}$ from [19, Example 9.3]. □

5.1 Atomic measures on the circle.

Recall that a probability measure ν on $\mathbb{T}$ is atomic if its support is a countable set. Equivalently, if we can express ν as a convex combination of point masses on $\mathbb{T}$,

$$\nu = \sum_{z\in\mathbb{T}} \lambda_z \delta_z,$$

where λ_z is zero for all but countably many $z \in \mathbb{T}$.

Proposition 5.2. *If ν is a probability measure on $\mathbb{T}$ satisfying (4.4) for $\beta \le 1$ and $\nu(\{z\}) > 0$, then z is a root of unity.*

Proof. For prime $n = p$, condition (4.4) says that $\nu \ge p^{-\beta}\omega_{p*}\nu$. In particular, for any $a \in \mathbb{N}^\times$,

$$\nu(\{z^{ap}\}) \;\ge\; p^{-\beta}\nu(\omega_p^{-1}(\{z^{ap}\})) \;=\; p^{-\beta} \sum_{s:s^p=(z^a)^p} \nu(\{s\}) \;\ge\; p^{-\beta}\nu(\{z^a\}).$$

Iterating this procedure, we see that $\nu(\{z^{ap^k}\}) \ge p^{-k\beta}\nu(\{z^a\})$ for every $k \in \mathbb{N}$, and, more generally, using the prime factorization $n = \prod_{p|n} p^{e_p(n)}$, we conclude that

$$\nu(\{z^n\}) \ge n^{-\beta}\nu(\{z\}).$$

Since $\beta \le 1$, the series $\sum_n n^{-\beta}$ diverges, so the map $n \mapsto z^n$ cannot be injective. Hence there exist $n_1 \ne n_2$ such that $z^{n_1} = z^{n_2}$ and z is an $(n_1 - n_2)$ root of unity. □

For $z \in \mathbb{T}$ a root of unity, we will write $\operatorname{ord}(z)$ for the order of z as an element of the group $(\mathbb{T}, \cdot)$. That is, $\operatorname{ord}(z)$ is the smallest $n \in \mathbb{N}^\times$ such that $z^n = 1$. When z is not a root of unity, we write $\operatorname{ord}(z) = \infty$.

Theorem 5.3. *For every $0 < \beta \le 1$, the atomic measures on $\mathbb{T}$ satisfying condition (4.4) form a simplex with extreme points parameterized by $\mathbb{N}^\times$ by the formula*

$$\nu_{n,\beta}(\{z\}) = \begin{cases} n^{-\beta}\frac{\varphi_\beta(\operatorname{ord}(z))}{\varphi(\operatorname{ord}(z))}, & \text{if } \operatorname{ord}(z)|n, \\ 0, & \text{otherwise.} \end{cases}$$

If ν is an atomic measure on $\mathbb{T}$ satisfying condition (4.4), then $\nu = \sum_n \lambda_n \nu_{n,\beta}$, where

$$\lambda_n = n^\beta \sum_{d\in\mathbb{N}^\times} \mu(d)\frac{1}{\varphi_\beta(nd)}\nu(Z^*_{nd}),$$

where Z^*_k denotes the set of primitive k^{th} roots of unity

An interesting consequence of our parameterization which will be useful in the proof of Theorem 1.1 and later in Section 7.1 is the following "equivariance" relation:

Lemma 5.4. *For every* $n, k \in \mathbb{N}^\times$*, let* $n' = \frac{n}{\gcd(n,k)}$*. Then* $\omega_{k*}\nu_{n,\beta} = \nu_{n',\beta}$*.*

Proof. By factoring ω_{k*}, it suffices to consider the case where $k = p$ is prime. In order to show $\omega_{p*}\nu_{n,\beta}(\{z\}) = \nu_{n',\beta}(\{z\})$, we consider two cases:

(i) $p|\mathrm{ord}(z)$: then $\omega_p^{-1}(z)$ is a set of p elements of order $\mathrm{ord}(z) \cdot p$. The measure of this set under $\omega_{p*}\nu_{n,\beta}$ is equal to 0 if either $p \nmid n$ or $\mathrm{ord}(z) \nmid \frac{n}{p}$, both of which imply that $\nu_{n',\beta}(\{z\}) = 0$. If $\omega_{p*}\nu_{n,\beta}(\{z\}) \neq 0$, then $\mathrm{ord}(z)|\frac{n}{p}$, and we have

$$\omega_{p*}\nu_{n,\beta}(\{z\}) = \sum_{s : s^p = z} \nu_{n,\beta}(\{s\}) = p \cdot n^{-\beta} \frac{\varphi(p \cdot \mathrm{ord}(z))}{\varphi(p \cdot \mathrm{ord}(z))} = p \cdot n^{-\beta} \frac{p^\beta \cdot \varphi_\beta(\mathrm{ord}(z))}{p \cdot \varphi(\mathrm{ord}(z))},$$

where we have factored p^β from φ_β using $p|\mathrm{ord}(z)$. This is equal to $\nu_{n',\beta}(\{z\})$, since $n' = \frac{n}{p}$.

(ii) $p \nmid \mathrm{ord}(z)$: then $\omega_p^{-1}(z)$ is a set of $p - 1$ elements of order $\mathrm{ord}(z) \cdot p$ and one element of order $\mathrm{ord}(z)$. The measure of this set under $\omega_{p*}\nu_{n,\beta}$ is zero if and only if $\mathrm{ord}(z) \nmid n$; since $p \nmid \mathrm{ord}(z)$, this is equivalent to $\mathrm{ord}(z) \nmid n'$. When $\mathrm{ord}(z)|n'$ and $p \nmid n$, then we immediately have that $\omega_{p*}\nu_{n,\beta}(\{z\}) = \nu_{n,\beta}(\{z\})$. If $\mathrm{ord}(z)|n'$ and $p|n$, then

$$\begin{aligned}
\omega_{p*}\nu_{n,\beta}(\{z\}) &= n^{-\beta} \frac{\varphi_\beta(\mathrm{ord}(z))}{\varphi(\mathrm{ord}(z))} + (p-1) \cdot n^{-\beta} \frac{\varphi_\beta(p \cdot \mathrm{ord}(z))}{\varphi(p \cdot \mathrm{ord}(z))} \\
&= p^{-\beta} \cdot (n')^{-\beta} \frac{\varphi_\beta(\mathrm{ord}(z))}{\varphi(\mathrm{ord}(z))} + (p-1) \cdot p^{-\beta} \cdot (n')^{-\beta} \frac{(p^\beta - 1)\varphi_\beta(\mathrm{ord}(z))}{(p-1)\varphi(\mathrm{ord}(z))} \\
&= (n')^{-\beta} \frac{\varphi_\beta(\mathrm{ord}(z))}{\varphi(\mathrm{ord}(z))} = \nu_{n',\beta}(\{z\}),
\end{aligned}$$

where we have used that φ_β and φ are multiplicative functions and $p \nmid \mathrm{ord}(z)$. □

For $k \in \mathbb{N}^\times$, let ν^k denote the restriction of a measure ν to the set Z_k of k^{th} roots of unity, i.e. $\nu^k(A) := \nu(Z_k \cap A)$. The formulas above show that $\nu^k_{n,\beta}$ is equal to $\left(\frac{n}{\gcd(n,k)}\right)^{-\beta} \nu_{\gcd(n,k),\beta}$, so that ν^k satisfies condition (4.4) when ν does, by the second half of Theorem 5.3. This fact can also be proven without the use of Theorem 5.3, and will be used in its proof.

Lemma 5.5. *If* ν *is an atomic measure satisfying condition* (4.4)*, then* ν^k *also satisfies* (4.4)

Proof. Let z be a primitive r^{th} root of unity for some $r|k$, and suppose that $p|r$ is a prime such that $p \nmid \frac{k}{r}$. Then $\omega_p^{-1}(z)$ does not contain any k^{th} roots of unity, so $A_{p,\beta}A_{n,\beta}\nu^k(\{z\}) = A_{n,\beta}\nu^k(\{z\}$. Thus, it suffices to check condition (4.4) where the prime divisors of n either divide $\frac{k}{r}$ or are relatively prime to k.

If $\ell|\frac{k}{r}$, then $\omega_\ell^{-1}(z)$ consists of only k^{th} roots of unity. If m is relatively prime to k, then $\omega_m^{-1}(z)$ consists of exactly one k^{th} root of unity, which we denote by $z^{1/m}$. If $n = \ell m$ where $\ell|\frac{k}{r}$ and $\gcd(m,k) = 1$ (and WLOG n is square-free), we have that

$$A_{n,\beta}\nu^k(\{z\}) = A_{\beta,n}\nu(\{z\}) - \sum_{1\neq d|m} \mu(d)d^{-\beta}A_{\ell,\beta}\nu(\omega_d^{-1}(z) - \{z^{1/d}\})$$

The first term $A_{\beta,n}\nu$ is positive by assumption, and the measure $A_{\ell,\beta}\nu$ satisfies condition (4.4) for every integer relatively prime to ℓ. In order to conclude the lemma, we will show that

$$-\sum_{1\neq d|m} \mu(d)d^{-\beta}\eta(\omega_d^{-1}(z) - \{z^{1/d}\}) \geq 0,$$

whenever η is a measure satisfying $A_{d,\beta}\eta \geq 0$ for each $d|m$. We prove this by induction on the number of prime divisors of m (the result is trivial when m is prime). If $p|m$ is prime, then we can write

$$\begin{aligned}
&-\sum_{1\neq d|m} \mu(d)d^{-\beta}\eta(\omega_d^{-1}(z) - \{z^{1/d}\}) \\
&= p^{-\beta}\eta(\omega_p^{-1}(z) - \{z^{1/p}\}) \\
&\qquad - \sum_{1\neq d|\frac{m}{p}} \mu(d)d^{-\beta}\left[\eta(\omega_d^{-1}(z) - \{z^{1/d}\}) - p^{-\beta}\eta(\omega_{pd}^{-1}(z) - \{z^{1/pd}\})\right] \\
&= p^{-\beta}\eta(\omega_p^{-1}(z) - \{z^{1/p}\}) - \sum_{1\neq d|\frac{m}{p}} \mu(d)d^{-\beta}A_{p,\beta}\eta(\omega_d^{-1}(z) - \{z^{1/d}\}).
\end{aligned}$$

The measure $A_{p,\beta}\eta$ satisfies $A_{d,\beta}(A_{p,\beta}\eta) \geq 0$ for each divisor $d|\frac{m}{p}$, so the claim follows. □

Our approach to Theorem 5.3 consists of computing the simplex of measures ν^k explicitly and establishing the converse of Lemma 5.5. To this end, let W_k denote the subspace of $\mathcal{M}(\mathbb{T})$ consisting of the measures supported on the set of k^{th} roots of unity, and let $W_k^+ = \mathcal{M}(\mathbb{T})^+ \cap W_k$ denote the positive measures. W_k is invariant under ω_{n*}, and so $\nu \in W_k$ satisfies condition (4.4) if and only if $\nu \in \bigcap_{n\in\mathbb{N}^\times} A_{n,\beta}^{-1}W_k^+$ (c.f. Proposition 4.6).

Lemma 5.6. *If* $A_\beta = \lim_{n\in\mathbb{N}^\times}\prod_{p|n}(1-p^{-\beta})A_{p,\beta}^{-1}|_{W_k}$ *exists, then* $\bigcap_{n\in\mathbb{N}^\times} A_{n,\beta}^{-1}W_k^+ = A_\beta W_k^+$.

Proof. It follows from Proposition 4.6 that $A_\beta W_k^+ \subseteq A_{n,\beta}^{-1} W_k^+$ for each n; it remains to be shown that $\bigcap_{n\in\mathbb{N}^\times} A_{n,\beta}^{-1} W_k^+ \subseteq A_\beta W_k^+$. Let $P_{n,\beta} = \prod_{p|n}(1-p^{-\beta})A_{p,\beta}^{-1}|_{W_k}$; note that for any $\nu \in W_k^+$, we have

$$\|P_{n,\beta}\nu\| = (P_{n,\beta}\nu)(X) = \nu(X) = \|\nu\|$$

For any $y \in \bigcap_{n\in\mathbb{N}^\times} A_{n,\beta}^{-1} W_k^+$, we have that $\|P_{n,\beta}^{-1} y\| = \|y\|$, so the sequence $P_{n,\beta}^{-1} y$ has an accumulation point x. Let $P_{n_k,\beta}^{-1} y$ be a subsequence converging to x. For any $\varepsilon_1, \varepsilon_2 > 0$, choose some K such that $\|P_{n_K,\beta} - A\| < \varepsilon_1$ and $\|P_{n_k,\beta}^{-1} y - x\| < \varepsilon_2$ for all k such that $K|k$. This gives:

$$\begin{aligned}\|y - A_\beta x\| &\leq \|y - A_\beta P_{n_k,\beta}^{-1} y\| + \|A_\beta P_{n_k,\beta}^{-1} y - A_\beta x\| \\ &\leq \|P_{n_k,\beta} - A_\beta\| \cdot \|P_{n_k,\beta}^{-1} y\| + \|A_\beta\| \cdot \|P_{n_k,\beta}^{-1} y - x\| \\ &< \|y\|\varepsilon_1 + \|A\|\varepsilon_2.\end{aligned}$$

Since ε_1 and ε_2 are arbitrary, it follows that $A_\beta x = y$. $\square$

Computing the elements of W_k satisfying condition (4.4) therefore amounts to computing the operator A explicitly. This is accomplished via Fourier analysis on W_k.

Lemma 5.7. *Identify W_k with $\ell^2(\mathbb{Z}/k\mathbb{Z})$ by the map $\delta_{\xi^j} \mapsto \delta_{[j]}$, where $\xi = e^{2\pi i/k}$. If $d|k$, then*

$$V_d := \operatorname{span}\{\delta_{[j]} \in \ell^2(\mathbb{Z}/k\mathbb{Z}) : \gcd(j,k) = d\}$$

is an invariant subspace of ω_{p*} for every prime $p \nmid k$.

Proof. The result follows from the observation that $\omega_{p*}\delta_{[j]} = \delta_{[jp]}$ and $\gcd(pj,k) = \gcd(j,k)$. $\square$

Let $C = \sqcup_{d|k}((\mathbb{Z}/d\mathbb{Z})^\times)\hat{}$, where $((\mathbb{Z}/d\mathbb{Z})^\times)\hat{}$ is the set of characters on the group of units of $\mathbb{Z}/d\mathbb{Z}$.

Lemma 5.8. *There is an orthogonal basis $\mathcal{B}$ of $\ell^2(\mathbb{Z}/k\mathbb{Z})$ and a bijection $\hat{\chi} : \mathcal{B} \to C$, $\hat{\chi}(\vec{v}) = \hat{\chi}_{\vec{v}}$ such that $\omega_{p*}\vec{v} = \hat{\chi}_{\vec{v}}(p)\vec{v}$ for every $\vec{v} \in \mathcal{B}$ and prime $p \nmid k$.*

Proof. Consider the subspaces V_d from the previous lemma. To prove the desired result, it suffices to show that there is a basis of eigenvectors for each V_d and a bijection with the characters on $(\mathbb{Z}/d\mathbb{Z})^\times$. The operator $\omega_{p*}|_{V_d}$ is unitarily equivalent to the operator ω_{p*} on $\ell^2((\mathbb{Z}/k'\mathbb{Z})^\times)$, where $k' = \frac{k}{d}$. From this, it suffices to construct a basis and bijection for V_1.

Since $(\mathbb{Z}/k\mathbb{Z})^\times$ is its own Pontryagin dual (i.e. there is a group isomorphism $[j] \mapsto \chi_{[j]}$), the Fourier transform gives us a unitary $\mathcal{F}$ on V_1 such that

$$\mathcal{F}^*\omega_{p*}\mathcal{F}\delta_{[j]} = \chi_{[j]}([p])\delta_{[j]}$$

Applying $\mathcal{F}$ to the standard basis of $\ell^2((\mathbb{Z}/k\mathbb{Z})^\times)$ gives us a basis and bijection $\mathcal{F}\delta_{[j]} \mapsto \chi_{[j]}$ with the desired properties. $\square$

Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be the subset of basis vectors corresponding to trivial characters under $\hat{\chi}$. That is, $\hat{\chi}$ restricts to a bijection between $\mathcal{B}_0$ and the set of characters

$$\chi_0 : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{T}, \quad \chi_0([j]) = 1.$$

Lemma 5.9. *The infinite product*

$$A'_\beta = \prod_{p\nmid k}(1-p^{-\beta})A_{p,\beta}^{-1}|_{W_k}$$

converges to the orthogonal projection to $\operatorname{span}\mathcal{B}_0$ *for all* $\beta \leq 1$.

Proof. For $\vec{v} \in \mathcal{B}_0$, one has $\omega_{p*}\vec{v} = \vec{v}$ for all $p \nmid k$. Thus,

$$\lim_{\ell\in\mathbb{N}^\times} \prod_{\substack{p|\ell \\ p\nmid n}}(1-p^{-\beta})A_{p,\beta}^{-1}\vec{v} = \vec{v}.$$

For $\vec{v} \in \mathcal{B}\backslash\mathcal{B}_0$, the range of $\hat{\chi}(\vec{v})$ is the set of r^{th} roots of unity for some integer $r \geq 2$. For arbitrary p, we have

$$|1-p^{-\beta}\hat{\chi}_{\vec{v}}(p)| \geq 1-p^{-\beta}.$$

Since $r \geq 2$, we have $\Re(\hat{\chi}_{\vec{v}}(p)) < 0$ for some residue classes of p modulo r; in particular,

$$|1-p^{-\beta}\hat{\chi}_{\vec{v}}(p)| > 1.$$

This can be used to obtain a bound on $\|A'_\beta\vec{v}\|_1$:

$$\lim_{\ell\in\mathbb{N}^\times}\Big\|\prod_{\substack{p|\ell \\ p\nmid k}}(1-p^{-\beta})A_{p,\beta}^{-1}\vec{v}\Big\|_1 = \lim_{\ell\in\mathbb{N}^\times}\prod_{\substack{p|\ell \\ p\nmid k}}\left|\frac{1-p^{-\beta}}{1-\chi(p)p^{-\beta}}\right| \leq \lim_{\ell\in\mathbb{N}^\times}\prod_{\substack{p|\ell \\ \Re(\hat{\chi}_{\vec{v}}(p))<0}} 1-p^{-\beta}.$$

This limit converges to 0 by the prime number theorem for arithmetic progressions, so we conclude that $A'_\beta\vec{v} = 0$. Since $\mathcal{B}$ is a mutually orthogonal set, it follows that A'_β is the orthogonal projection to the span of $\mathcal{B}_0$. $\square$

Lemma 5.10. *The infinite product*

$$A_\beta = \prod_{p}(1-p^{-\beta})A_{p,\beta}^{-1}|_{W_k} \tag{5.1}$$

converges in $\mathbb{B}(W_k)$ *for all* $\beta > 0$*. For* $\beta \le 1$*,* $\xi = e^{2\pi i/k}$*, and* $n|k$*,*

$$\langle \delta_z, A_\beta \delta_{\xi^n} \rangle = \begin{cases} \frac{n^{-\beta}}{\varphi(n)} \frac{\varphi_\beta(\mathrm{ord}(z))}{\varphi(\mathrm{ord}(z))}, & \textit{if } \mathrm{ord}(z)|n, \\ 0, & \textit{otherwise.} \end{cases}$$

In particular, the measure $A_\beta \delta_{\xi^n}$ *is positive, and* $\nu_{n,\beta} = \varphi(\frac{k}{n}) A_\beta \delta_{\xi^n}$ *is a probability measure satisfying* (4.4).

Proof. Let

$$A''_\beta = \prod_{p|k} (1 - p^{-\beta}) A^{-1}_{p,\beta},$$

so that

$$A_\beta = \prod_p (1 - p^{-\beta}) A^{-1}_{p,\beta}|_{W_k} = A''_\beta \prod_{p\nmid k} (1 - p^{-\beta}) A^{-1}_{p,\beta}|_{W_k}.$$

By Lemma 5.9, this product converges, so that A_β is a bounded operator on W_k and $A_\beta = A'_\beta A''_\beta$.

For $d|k$, let $\vec{v}_d \in \mathcal{B}_0$ be the vector such that $\hat{\chi}_{\vec{v}_d}$ is the trivial character on $(\mathbb{Z}/d\mathbb{Z})^\times$. For $n|k$, we have

$$\left\langle \vec{v}_d, A'_\beta \delta_{\xi^n} \right\rangle = \langle \vec{v}_d, \delta_{\xi^n} \rangle = \begin{cases} \frac{1}{\varphi(n)}, & \text{if } n = d \\ 0, & \text{otherwise,} \end{cases}$$

since A'_β is the projection to $\operatorname{span} \mathcal{B}_0$. This implies that $A'_\beta \delta_{\xi^n} = \frac{1}{\varphi(n)} \vec{v}_n$, and thus

$$\langle \delta_z, A_\beta \delta_{\xi^n} \rangle = \frac{1}{\varphi(n)} \left\langle \delta_z, A''_\beta \vec{v}_n \right\rangle = \left(\frac{1}{\varphi(n)} \prod_{p|k} 1 - p^{-\beta} \right) \left\langle \delta_z, \prod_{p|k} A^{-1}_{p,\beta} \vec{v}_n \right\rangle. \tag{5.2}$$

Recall that the operator $A^{-1}_{p,\beta}$ can be expressed as

$$A^{-1}_{p,\beta} = \sum_{m=0}^{\infty} p^{-\beta m} \omega_{p^m *}.$$

The operator $\omega_{p^m *}$ sends $\vec{v}_n$ to $\vec{v}_{n/p^e}$, where $p^e = \gcd(n, p^m)$. Letting $e = e_p(n)$ be the largest integer where $p^e|n$, we have

$$A^{-1}_{p,\beta} \vec{v}_n = \sum_{m=1}^{e-1} p^{-\beta m} \vec{v}_{n/p^m} + \frac{p^{-\beta e}}{1 - p^{-\beta}} \vec{v}_{n/p^e}.$$

Thus, we have

$$\left\langle \delta_z, \prod_{p|k} A_{p,\beta}^{-1}\vec{v}_n \right\rangle = \begin{cases} \left(\frac{n}{\operatorname{ord}(z)}\right)^{-\beta} \frac{1}{\varphi(\operatorname{ord}(z))} \prod_{p\nmid \operatorname{ord}(z)} \frac{1}{1-p^{-\beta}}, & \text{if } \operatorname{ord}(z)|n \\ 0, & \text{otherwise.} \end{cases}$$

Combining this with (5.2) yields the desired result. □

Proof of Theorem 5.3. By Lemmas 5.6 and 5.10, $\{\nu_{d,\beta} : d|k\}$ is the set of extreme points of the measures satisfying condition (4.4) which are supported on the k^{th} roots of unity. It only remains to show that every atomic measure ν satisfying (4.4) can be expressed uniquely as a convex combination of the $\nu_{n,\beta}$; we will exhibit a convex combination of $\nu_{n,\beta}$ which yields the measure ν, and show that this combination is unique in the proof of Proposition 5.15.

We let Z_k^* denote the primitive k^{th} roots of unity. This contains $\varphi(k)$ points of order k, so that one has

$$\nu_{d,\beta}(Z_k^*) = \begin{cases} d^{-\beta}\varphi_\beta(n) & \text{if } k|d \\ 0 & \text{otherwise.} \end{cases}$$

For an arbitrary atomic measure ν satisfying (4.4), recall that ν^k denotes the measure

$$\nu^k(A) = \nu(A \cap Z_k),$$

where Z_k denotes the set of k^{th} roots of unity. By Lemma 5.5, this measure also satisfies (4.4), and hence decomposes uniquely as $\sum_{n|k} \lambda_{n,k}\nu_{n,\beta}$, where $\lambda_{n,k} \geq 0$ and $\sum \lambda_{n,k} = \nu^k(\mathbb{T})$. For each divisor $n|k$,

$$\nu(Z_n^*) = \sum_{d|\frac{k}{n}} \lambda_{nd,k}\nu_{nd,\beta}(Z_n^*) = \sum_{d|\frac{k}{n}} \lambda_{nd,k}(nd)^{-\beta}\varphi_\beta(n) = \sum_{d|\frac{k}{n}} \lambda_{\left(\frac{nd}{k}\right)k,k} \left(\frac{nd}{k}\right)^{-\beta} k^{-\beta}\varphi_\beta(n).$$

The map $d \mapsto \frac{k}{nd}$ is a bijection on the divisors of $\frac{k}{n}$, and reindexing the sum yields

$$\frac{1}{\varphi_\beta(n)}\nu(Z_n^*) = \sum_{d|\frac{k}{n}} \lambda_{k/d,k} \left(\frac{k}{d}\right)^{-\beta}.$$

Applying the Möbius inversion formula gives

$$\lambda_{n,k} = n^\beta \sum_{d|\frac{k}{n}} \mu(d) \frac{1}{\varphi_\beta(nd)} \nu(Z_{nd}^*) \tag{5.3}$$

Taking the limit of equation (5.3) along the directed set $k \in n\mathbb{N}^\times$, we obtain the formula

$$\lambda_n = n^\beta \sum_{d \in \mathbb{N}^\times} \mu(d) \frac{1}{\varphi_\beta(nd)} \nu(Z^*_{nd}).$$

This series converges absolutely, since $\frac{1}{\varphi_\beta(nd)} \leq 1$ and ν is a probability measure.

It only remains to verify that $\nu = \sum_n \lambda_n \nu_{n,\beta}$. If z is primitive k^{th} root of unity, then

$$\begin{aligned}
\left(\sum_{n \in \mathbb{N}^\times} \lambda_n \nu_{n,\beta}\right)(\{z\}) &= \sum_{\substack{n \in \mathbb{N}^\times \\ k|n}} \left(n^\beta \sum_{d \in \mathbb{N}^\times} \mu(d) \frac{1}{\varphi(nd)} \nu(Z^*_{nd})\right) \left(n^{-\beta} \frac{\varphi_\beta(k)}{\varphi(k)}\right) \\
&= \frac{\varphi_\beta(k)}{\varphi(k)} \sum_{\substack{n,d \in \mathbb{N}^\times \\ k|n}} \mu(d) \frac{1}{\varphi_\beta(nd)} \nu(Z^*_{nd}) \\
&= \frac{\varphi_\beta(k)}{\varphi(k)} \sum_{\substack{m \in \mathbb{N}^\times \\ k|m}} \frac{1}{\varphi_\beta(m)} \nu(Z^*_m) \sum_{d|\frac{m}{k}} \mu(d)
\end{aligned}$$

The sum over divisors of $\frac{m}{k}$ is 1 if $m = k$ and 0 if $m > k$. Since $\nu(\{z\}) = \nu^k(\{z\})$ depends only on the order k of z and $|Z^*_k| = \varphi(k)$, we conclude that

$$\nu(\{z\}) = \frac{1}{\varphi(k)} \nu(Z^*_k) = \frac{\varphi_\beta(k)}{\varphi(k)} \cdot \frac{1}{\varphi_\beta(k)} \nu(Z^*_k) = \left(\sum_{n \in \mathbb{N}^\times} \lambda_n \nu_{n,\beta}\right)(\{z\}). \qquad \square$$

An alternative to $\nu(Z^*_k)$ in the formula for λ_n is to use the inclusion-exclusion principle on $Z^*_k = \bigcap_{d|k} Z_k \backslash Z_{k/d}$,which yields

$$\nu(Z^*_k) = \sum_{d|k} \mu(d) \omega_{k/d*} \nu(\{1\}).$$

This shows that λ_n can be obtained from ν using only the twists ω_{d*} and the point $1 \in \mathbb{T}$.

5.2 Non-atomic measures

We now turn our attention to non-atomic measures satisfying condition (4.4). Recall from Remark 4.5 that, for $\beta > 1$, we define an operator $T_\beta = \frac{1}{\zeta(\beta)} \sum_{c=1}^\infty c^{-\beta} \omega_{c*}$ which is an affine isomorphism between the simplex of probability measures on $\mathbb{T}$ and the simplex of β-subconformal measures on $\mathbb{T}$. Our computation depends on the following theorem:

Theorem 5.11. *The weak* limit* $T_{1^+}\delta_z := \lim_{\beta\to 1^+} T_\beta\delta_z$ *exists for every* $z \in \mathbb{T}$*. If* z *is a primitive* n^{th} *root of unity, then* $T_{1^+}\delta_z$ *is equal to the measure* $\nu_{n,1}$ *from Theorem 5.3. If* z *has infinite order in* $\mathbb{T}$*, then* $T_{1^+}\delta_z$ *is equal to normalized Haar measure on* $\mathbb{T}$*.*

Proof. Let $\zeta(\beta,a) = \sum_{n=0}^{\infty}(n+a)^{-\beta}$, the Hurwitz zeta function with parameter a. This series converges for $\Re(\beta) > 1$ and holomorphically extends to $\beta \in \mathbb{C}\backslash\{1\}$; the extension has a simple pole at $\beta = 1$ with residue 1. If z is a primitive n^{th} root of unity, then

$$T_\beta\delta_z = \frac{1}{\zeta(\beta)}\sum_{k=1}^{n}\sum_{\substack{c\geq 1\\ n|(c-k)}} c^{-\beta}\delta_{z^k} = \frac{n^{-\beta}}{\zeta(\beta)}\sum_{k=1}^{n}\zeta\left(\beta,\frac{k}{n}\right)\delta_{z^k}$$

Taking the limit as $\beta \to 1^+$, we find

$$T_{1^+}\delta_z = n^{-1}\sum_{k=1}^{n}\delta_{z^k}.$$

We can check directly that this agrees with the measure ν_n from Theorem 5.3 (we write $d = \gcd(n,k)$):

$$\nu_n(\{z^k\}) = \frac{1}{d}\cdot\varphi\left(\frac{n}{d}\right)^{-1}\cdot\prod_{p|\frac{n}{d}} 1-p^{-1} = \frac{1}{d}\cdot\varphi\left(\frac{n}{d}\right)^{-1}\cdot\frac{d}{n}\cdot\varphi\left(\frac{n}{d}\right) = \frac{1}{n}$$

Now suppose that z has infinite order in the group $\mathbb{T}$. For each $n \in \mathbb{Z}$, we have that

$$T_\beta\delta_z(\mathfrak{z}^n) = \frac{1}{\zeta(\beta)}\sum_{c=1}^{\infty} c^{-\beta}\delta_{z^c}(\mathfrak{z}^n) = \frac{1}{\zeta(\beta)}\sum_{c=1}^{\infty} c^{-\beta}z^{cn} = \frac{\mathrm{Li}_\beta(z^n)}{\zeta(\beta)}$$

where $\mathrm{Li}_\beta(z) = \sum_{c=1}^{\infty} z^c c^{-\beta}$ is the polylogarithm function. Since z has infinite order, $z^n \neq 1$ for $n \neq 0$, so by [28, Theorem 2.2], $\mathrm{Li}_\beta(z^n)$ has a well-defined limit as $\beta \to 1$. This implies that $T_\beta\delta_z(\mathfrak{z}^n)$ converges to 0 when $n \neq 0$, so the limit $T_{1^+}\delta_z$ exists and is equal to normalized Haar measure. □

As mentioned in the introduction to this chapter, our approach to the non-atomic β-subconformal measures in Theorem 5.14 will make use of the following adaptation of [3, Corollary 9.5]:

Lemma 5.12. *If* ν *satisfies condition* (4.4) *for some* $\beta = \beta' > 0$*, then* ν *satisfies* (4.4) *for every* $\beta \geq \beta'$*.*

Proof. Follows directly from the proof of Corollary 9.5 of [3]. □

We also require a version of Jensen's inequality for locally convex spaces, due to M. D. Perlman [34, Theorem 3.6]. We will restate Perlman's theorem for our context and verify its assumptions in the following lemma.

Lemma 5.13. *Let d be a metric on the space $\mathbb{P}(\mathbb{T})$ of probability measures on $\mathbb{T}$ that induces the weak* topology. Let*

$$E = \{T_{1^+}\delta_z : z \in \mathbb{T}\},$$

denote the closed, convex hull of E as usual by $\overline{\text{co}}(E)$, and let $D : \mathbb{P}(\mathbb{T}) \to \mathbb{R}$ be the function

$$D(\mu) = d(\mu, \overline{\text{co}}(E)).$$

For each $\beta > 1$, let $t_\beta : \mathbb{T} \to \mathbb{P}(\mathbb{T})$ be the function $t_\beta(z) = T_\beta \delta_z$. Then $(X, C, f, Y) = (t_\beta, \mathbb{P}(\mathbb{T}), D, \mathbb{R})$ satisfy the assumptions of [34, Theorem 3.6], and thus

$$D\left(\int_{\mathbb{T}} t_\beta(z) d\mu(z)\right) \leq \int_{\mathbb{T}} D(t_\beta(z)) d\mu$$

for every $\mu \in \mathbb{P}(\mathbb{T})$.

Proof. Most of the assumptions of [34, Theorem 3.6] are immediate; we will verify that t_β and $D \circ t_\beta$ are Pettis integrable on $(\mathbb{T}, \mu)$. Since we are considering $\mathbb{P}(\mathbb{T}) \subseteq \mathcal{M}(\mathbb{T})$ in the weak* topology, the dual of $\mathcal{M}(\mathbb{T})$ is isomorphic to $C(\mathbb{T})$ in the weak topology. The statement that t_β is Pettis integrable is then equivalent to the following three conditions being satisfied: (a) the function $t_\beta(z)(f)$ is measurable for every $f \in C(\mathbb{T})$; (b) $\int t_\beta(z)(f) d\mu(z)$ exists for every $f \in C(\mathbb{T})$; and (c) there exists a measure μ_β such that $\int f(z) d\mu_\beta(z) = \int t_\beta(z)(f) d\mu(z)$. Conditions (a) and (b) are immediate, since the function $t_\beta(z)$ is continuous, and thus $t_\beta(z)(f)$ is also continuous and integrable. For (c), note that

$$t_\beta(z)(f) = \frac{1}{\zeta(\beta)} \sum_{c=1}^{\infty} c^{-\beta} f(z^c) = (T_{\beta *} f)(z),$$

where $T_{\beta *}$ is the operator

$$T_{\beta *} f = \frac{1}{\zeta(\beta)} \sum_{c=1}^{\infty} c^{-\beta} f \circ \omega_c$$

which is a bounded operator on $C(\mathbb{T})$ in the norm topology. Thus, taking the adjoint $(T_{\beta *})^* = T_\beta$ yields the expression

$$\int t_\beta(z)(f) d\mu(z) = \int (T_{\beta *} f)(z) d\mu(z) = \int f d(T_\beta \mu),$$

whence $\mu_\beta = T_\beta \mu$ satisfies (c).

To see that $D \circ t_\beta(z)$ is Pettis integrable, we simply note that $t_\beta(z)$ and D are continuous functions, so the composition is continuous and integrable. $\square$

We will now show that normalized Haar measure (i.e. Lebesgue measure) is the only β'-subconformal measure for each $0 < \beta' \leq 1$ satisfying a limit hypothesis. To state this hypothesis, let ν be a non-atomic probability measure satisfying (4.4) for $\beta = \beta'$. By Lemma 5.12, we have that ν also satisfies (4.4) for every $\beta > 1$. In particular, there are probability measures ν_β such that

$$\nu = \int_{\mathbb{T}} T_\beta \delta_z d\nu_\beta(z),$$

since $T_\beta \delta_z$ form the extreme points of the simplex of β-subconformal measures when $\beta > 1$.

Theorem 5.14. *For every* $0 < \beta' \leq 1$*, normalized Haar measure is the only non-atomic probability measure* ν *on* $\mathbb{T}$ *satisfying the positivity condition* (4.4) *for* $\beta = \beta'$ *and for which*

$$\lim_{\beta \to 1^+} \int_{\mathbb{T}} D(t_\beta(z)) d\nu_\beta(z) = 0.$$

Proof. Fix $0 < \beta' \leq 1$, let ν be a β'-subconformal measure. For each $\varepsilon > 0$, choose some $1 < \beta$ such that

$$\int_{\mathbb{T}} D(t_\beta(z)) d\nu_\beta(z) < \varepsilon$$

Then Lemma 5.13 gives

$$D(\nu) = D\left(\int_{\mathbb{T}} t_\beta(z) d\nu_\beta(z)\right) \leq \int_{\mathbb{T}} D(t_\beta(z)) d\nu_\beta < \varepsilon.$$

In particular, ν is arbitrarily close to $\overline{\text{co}}(E)$, and therefore $\nu \in \overline{\text{co}}(E)$.

By Theorem 5.11, E consists of the measures $\nu_{n,1}$ for every $n \in \mathbb{N}^\times$ and Haar measure. Since ν is non-atomic and E is countable, ν must be a convex combination of the non-atomic measures in E. There is only one such measure, normalized Haar measure, so the result follows. □

5.3 Proof of the main theorem

Proposition 5.15. *For every* $0 < \beta \leq 1$*, the set*

$$E_\beta := \{\lambda, \nu_{n,\beta} : n \in \mathbb{N}^\times\}$$

is the set of extreme points of the simplex $\overline{\text{co}}(E_\beta)$*, where* λ *is normalized Haar measure on* $\mathbb{T}$ *and* $\nu_{n,\beta}$ *is the measure from Theorem 5.3.*

Proof. Since $E_\beta \backslash \{\lambda\}$ is countable and consists of atomic measures, it follows that λ cannot belong to $\overline{\text{co}}(E_\beta \backslash \{\lambda\})$. Hence, λ is extremal in $\overline{\text{co}}(E_\beta)$.

Suppose that $\nu_{n,\beta}$ is not extremal in $\overline{\text{co}}(E_\beta)$, so that $\nu_{n,\beta}$ is a convex combination of the elements of $E_\beta \backslash \{\nu_{n,\beta}, \lambda\}$. This set can be partitioned as:

$$E_\beta \backslash \{\nu_{n,\beta}, \lambda\} = A \cup B, \quad A = \{\nu_{1,\beta}, \nu_{2,\beta}, ..., \nu_{n-1,\beta}\}, \quad B = \{\nu_{n+1,\beta}, \nu_{n+2,\beta}, ...\}.$$

We can then write $\nu_{n,\beta} = \lambda\nu_A + (1-\lambda)\nu_B$ for some $0 \leq \lambda \leq 1$, $\nu_A \in \overline{\text{co}}(A)$ and $\nu_B \in \overline{\text{co}}(B)$. Since $\nu_{k,\beta}(\xi_j) = 0$ for all $k \leq n < j$, we arrive at the equation

$$(1-\lambda)\nu_B(\{\xi_j\}) = \nu_{n,\beta}(\{\xi_j\}) = 0$$

for each $j > n$. Since $\nu_{j,\beta}(\xi_j) > 0$, this then implies that $\lambda = 1$. However, we also have

$$\nu_A(\{\xi_n\}) = \nu_{n,\beta}(\{\xi_n\}) > 0,$$

which contradicts $\nu_{k,\beta}(\{\xi_n\}) = 0$ for any $k < n$. Therefore, $\nu_{n,\beta}$ is extremal in E_β. □

It only remains to show that the formulas in Theorem 1.1 are obtained from the measures in E_β.

Proof of Theorem 1.1. The extremality of $\psi_{\infty,\beta}$ follows from extremality of λ in E_β. In order to show that $\psi_{n,\beta}$ is both a KMS$_\beta$ state and extremal, it suffices to show that

$$\psi_{n,\beta}(U^k) := (n')^{-\beta} \sum_{d|n'} \mu(d) \frac{\varphi_\beta(d)}{\varphi(d)} = \int_{\mathbb{T}} \mathfrak{z}^k d\nu_{n,\beta},$$

where $n' = n/\gcd(n,k)$. By Lemma 5.4, we have that

$$\begin{aligned}\int_{\mathbb{T}} \mathfrak{z}^k d\nu_{n,\beta} &= \int_{\mathbb{T}} \mathfrak{z}\, d\omega_{k*}\nu_{n,\beta} = \int_{\mathbb{T}} \mathfrak{z}\, d\nu_{n',\beta} \\ &= (n')^{-\beta} \sum_{d|n'} \frac{\varphi_\beta(d)}{\varphi(d)} \sum_{\text{ord}(z)=d} z = (n')^{-\beta} \sum_{d|n'} \mu(d) \frac{\varphi_\beta(d)}{\varphi(d)}.\end{aligned}$$

The type of the factor states $\psi_{n,\beta}$ is computed in Chapter 7. □

We point out that, despite requiring an additional assumption in Theorem 5.14, our argument shows that the measures constructed in Section 5.1 constitute all of the atomic β-subconformal measures for each $0 < \beta \leq 1$. This shows that the extreme points in the (non-closed) atomic simplex are also extreme points in the simplex of β-subconformal measures. Thus, the states $\psi_{n,\beta}$ are extremal KMS$_\beta$ states.

Chapter 6

Quotients and symmetries

This chapter is dedicated to a family of quotients of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, which we have termed *the modular quotients.* These arise from quotients of the monoid $\mathbb{N}^\times \ltimes \mathbb{Z}$, wherein the additive subgroup $\mathbb{Z}$ is replaced with $\mathbb{Z}/m\mathbb{Z}$ for $m \in \mathbb{N}^\times$.

Quotients of right $ax+b$ Toeplitz algebras were the principal focus of [19], particularly the so-called *boundary quotients.* These are quotients obtained by imposing the condition that some, or all, of the generating isometries should be unitary. In [19], the Toeplitz algebra considered was $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{N})$, generated by isometries $V_a, a \in \mathbb{N}^\times$ and S^n, $n \in \mathbb{N}$. This algebra admits three main boundary quotients: the *additive boundary*, with $SS^* = 1$; the *multiplicative boundary*, with $V_aV_a^* = 1, a \in \mathbb{N}^\times$; and the *Crisp-Laca boundary*, with all generating isometries replaced by unitaries. These were shown to be the Toeplitz algebras of various submonoids of $\mathbb{Q}_+^\times \ltimes \mathbb{Q}$: the additive boundary is the Toeplitz algebra of $\mathbb{N}^\times \ltimes \mathbb{Z}$, as we discussed; the multiplicative boundary is the Toeplitz algebra of $\mathbb{Q}_+^\times \ltimes \mathbb{Q}_+$; and the Crisp-Laca boundary is the Toeplitz algebra of $\mathbb{Q}_+^\times \ltimes \mathbb{Q}$, i.e. its group C*-algebra. Each monoid is generated in $\mathbb{Q}_+^\times \ltimes \mathbb{Q}$ by elements of $\mathbb{N}^\times \ltimes \mathbb{N}$ with the appropriate inverses, and may be viewed as enveloping monoids of $\mathbb{N}^\times \ltimes \mathbb{N}$.

There is another type of quotient of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{N})$, arising from quotients of the monoid $\mathbb{N}^\times \ltimes \mathbb{N}$, rather than from enveloping monoids. In Section 6.1, we introduce a type of $ax+b$ monoid, $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ for $m \in \mathbb{N}^\times$. This monoid is left-cancellative, and the quotient $\mathbb{N}^\times \ltimes \mathbb{Z} \to \mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ induces a quotient of the respective Toeplitz algebras. In the spirit of the boundary quotients, we will see that this quotient is obtained by imposing the relation $U^m = 1$ on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$.

The dynamics on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ descend to dynamics on $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$, and the transpose of the quotient map is an inclusion from the KMS_β states of the quotient system into the KMS_β states of the full system. We will show that the KMS_β states which factor through the quotient are precisely the KMS_β states arising from subconformal measures ν on $\mathbb{T}$ which are supported on the set of m^{th} roots of unity. Thus, the characterization of KMS_β states on $(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}), \sigma_t)$ yields a characterization of the KMS_β states on $(\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})), \sigma_t)$ for each $m \in \mathbb{N}^\times$, which may be found in Theorem 6.4.

In Section 6.2, we examine a novel feature of the modular quotient systems: the existence of symmetries. While the algebra $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ does not appear to admit any *-homomorphisms commuting with the dynamics (besides the gauge action), the quotient $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ carries a very natural action of the multiplicative group $(\mathbb{Z}/m\mathbb{Z})^*$. These *-homorphisms permute the KMS_β states when $\beta > 1$ and leave the KMS_β states invariant when $\beta \leq 1$, thus giving a new example of spontaneous symmetry-breaking. These symmetries provide some insight into the formulas for $\nu_{n,\beta}$, which we elaborate at the end of the chapter.

6.1 Modular quotients

For $m \in \mathbb{N}^\times$, consider the monoid $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ with the binary operation

$$(a, [x]) \cdot (b, [y]) = (ab, [bx + y]),$$

where $[x]$ denotes the class of $x \in \mathbb{Z}$ modulo $m\mathbb{Z}$.

Proposition 6.1. *The monoid* $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ *is left-cancellative and right LCM. It is not right-cancellative.*

Proof. For left-cancellativity, assume that $(b, [y]) \cdot (a, [x]) = (b, [y]) \cdot (c, [z])$ for some $a, b, c \in \mathbb{N}^\times$ and $x, y, z \in \mathbb{Z}$. That is, $(ab, [ay + x]) = (cb, [cy + z])$. Then this implies that $ab = cb$, so that $a = c$, by cancellativity of $\mathbb{N}^\times$. Also, we have $[ay + x] = [cy + z]$, which is equivalent to $[x] = [z]$ since $a = c$. Therefore, $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ is left-cancellative.

To see that $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ is not right cancellative, we point out that $(1, [0]) \cdot (m, [0]) = (m, [0])$ and $(1, [1]) \cdot (m, [0]) = (m, [m]) = (m, [0])$, whence the claim follows.

The right-multiples of an element $(a, [x])$ are of the form $(ac, [cx + z])$ for $c \in \mathbb{N}^\times$, $z \in \mathbb{Z}/m\mathbb{Z}$, or equivalently, of the form $(ac, [z])$. We conclude that the common multiples of $(a, [x])$ and $(b, [y])$ are of the form $(\mathrm{lcm}(a, b)c, [z])$, which are the right-multiples of $(\mathrm{lcm}(a, b), [0])$. $\square$

This shows that $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ satisfies the assumptions imposed in Chapter 2, allowing us to consider the Toeplitz algebra $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$. The homomorphism $N(a, [x]) = \ln(a)$ induces an $\mathbb{R}$-action on $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$, as previously discussed.

The isometries u and $v_a, a \in \mathbb{N}^\times$ on $\mathbb{B}(\ell^2(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})))$ defined by

$$u\delta_{(b,[x])} = \delta_{(b,[x+b])} \quad \text{and} \quad v_a\delta_{(b,[x])} = \delta_{(ab,[x])} \quad a \in \mathbb{N}^\times$$

satisfy relations (0)–(3) of Section 2.4. Thus, the universal property yields a *-homomorphism $q_m : \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) \to \mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$, satisfying $q_m(V_a) = v_a$ and $q_m(U) = u$. It is clear that q_m is equivariant with respect to the $\mathbb{R}$-actions on these algebras.

Proposition 6.2. *The *-homomorphism* q_m *descends to an isomorphism to* $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ *from the quotient of* $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ *obtained by imposing the relation*

(4) $U^m = 1$.

The proof follows what are now standard techniques in the subject. A reference for these techniques is e.g. [2, Theorem 1.2], and we refer there for the history of these techniques as well.

Proof. Let $\mathcal{I}_m$ be the closed two-sided ideal of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ generated by $U^m - 1$. It is clear that q_m descends to a *-homomorphism $\overline{q}_m$ between $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})/\mathcal{I}_m$ and $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$, since $u^m = 1$. It remains to show that $\overline{q}_m$ is faithful. For this, it suffices to show that $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ is universal for the relations (0)–(4) (c.f. Section 2.4).

We start by introducing a new algebra, $\mathcal{T}_u(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$, the universal C*-algebra generated by a unitary r and isometries w_a, $a \in \mathbb{N}^\times$ that satisfy the conditions:

(M0) $w_a^* w_a = rr^* = r^* r = r^m = 1$,

(M1) $r w_a = w_a r^a$,

(M2) $w_a w_b = w_{ab}$,

(M3) $w_a w_b^* = w_b^* w_a$ when $\gcd(a,b) = 1$.

Relations (M0)–(M2) say that w_a and r form a representation of $\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})$ by isometries; relation (M3) implies that this representation is Nica-covariant. It is clear that relations (M0)–(M3) are equivalent to relations (0)–(4) of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, by the assignment $r = U$, $w_a = V_a$, so by universality of these two algebras, we obtain an isomorphism $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})/\mathcal{I}_m \cong \mathcal{T}_u(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$.

The algebra $\mathcal{T}_u(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ carries a natural gauge action, as relations (M0)–(M3) are invariant under the transformations $r \mapsto r$, $w_a \mapsto \chi(a) w_a$, $\chi \in \widehat{\mathbb{Q}}_+^*$. As was discussed in Section 3.1, the gauge action induces a faithful conditional expectation $\Phi_u^m : \mathcal{T}_u(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})) \to \mathfrak{D}_u^m$, where the C*-subalgebra $\mathfrak{D}_u^m$ is the closed linear span of monomials of the $w_a r^k w_a^*$, $a \in \mathbb{N}^\times$, $0 \leq k < m$. The algebra $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ also carries a gauge action of $\widehat{\mathbb{Q}}_+^*$, following the same spatial construction appearing in Section 3.1. This again induces a conditional expectation $\Phi^m : \mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})) \to \mathfrak{D}^m$ to the C*-subalgebra $\mathfrak{D}^m$ which is the closed linear span of monomials of the form $v_a u^k v_a^*$, $a \in \mathbb{N}^\times$, $0 \leq k < m$. Our formulas yield a commuting diagram,

$$\begin{tikzcd}
\mathcal{T}_u(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})) \arrow[r, "\overline{q}_m"] \arrow[d, "\Phi_u^m"] & \mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})) \arrow[d, "\Phi^m"] \\
\mathfrak{D}_u^m \arrow[r, "\overline{q}_m|_{\mathfrak{D}_u^m}"] & \mathfrak{D}^m
\end{tikzcd}$$

A standard argument shows that $\overline{q}_m$ is faithful if and only if $\overline{q}_m|_{\mathfrak{D}_u^m}$ is faithful. Indeed, if $b \geq 0$ and $\overline{q}_m(b) = 0$, then $0 = \Phi^m(\overline{q}_m(b)) = \overline{q}_m|_{\mathfrak{D}_u^m}(\Phi_u^m(b))$ implies that $b = 0$, since Φ_u^m is faithful.

By the same arguments appearing in Chapter 3, $\mathfrak{D}_u^m$ and $\mathfrak{D}^m$ are the inductive limits of the subalgebras

$$(\mathfrak{D}_u^m)_a = \operatorname{span}\{w_b r^k w_b^* : b|a,\ 0 \leq k < m\}, \qquad \mathfrak{D}_a^m = \operatorname{span}\{v_b u^k v_b^* : b|a,\ 0 \leq k < m\}.$$

The restriction of $\overline{q}_m$ to $(\mathfrak{D}_u^m)_a$ is clearly an isomorphism onto $\mathfrak{D}_a^m$. The restrictions $\overline{q}_m|_{(\mathfrak{D}_u^m)_a}$ induce an isomorphism between the inductive limits, $\mathfrak{D}_u^m \to \mathfrak{D}^m$, which agrees with $\overline{q}_m$ on the dense subspace spanned by $(\mathfrak{D}_u^m)_a$, $a \in \mathbb{N}^\times$. Therefore, $\overline{q}_m$ is an isomorphism. $\square$

Before the next lemma, we point out that the kernel of q_m in $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ is equal to the closed linear span of elements of the form $V_a(U^n - U^k)V_b^*$, where $n = k + m\ell$ for some $\ell \in \mathbb{Z}$. It is clear that these elements belong to the kernel. Conversely, the kernel is generated as an ideal by $U^m - 1$, so is spanned by elements of the form $x(U^m - 1)y$. Using the monomials which densely-span $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, we obtain a dense subspace of the ideal spanned by elements of the form

$$V_a U^n V_b^* (U^m - 1) V_c U^k V_d^* = V_{ac'}(U^{nc' + b'k + mcb} - U^{nc' + b'k})V_{b'd}^*,$$

where $b' = \frac{b}{\gcd(b,c)}$ and $c' = \frac{c}{\gcd(b,c)}$. This proves the claim.

Using relation (4), we obtain the following factorization of KMS$_\beta$ states, similar to [1, Lemma 6.2]:

Lemma 6.3. *Let $n \in \mathbb{N}^\times$, $\beta > 0$, and let ν be a probability measure supported on Z_m satisfying* (4.4). *Then the* KMS$_\beta$ *state $\phi_{\nu,\beta} \circ \Phi$ of Theorem 4.2 factors through q_m. Conversely, if ν is a probability measure satisfying* (4.4) *such that $\phi_{\nu,\beta} \circ \Phi$ factors through q_m, then ν is supported on Z_m.*

Proof. For the first claim, it suffices to show that $\phi_{\nu,\beta} \circ \Phi$ vanishes on the kernel of q_m. Let $a, b \in \mathbb{N}^\times$ and $n, k \in \mathbb{Z}$, such that $n = k + m\ell$ for some $\ell \in \mathbb{Z}$. If $a \neq b$, then $\Phi(V_a(U^n - U^k)V_b^*) = 0$, so suppose that $a = b$. We then have

$$\phi_{\nu,\beta}(V_a(U^n - U^k)V_a^*) = a^{-\beta} \int_{\mathbb{T}} \mathfrak{z}^k(\mathfrak{z}^{m\ell} - 1)d\nu.$$

Then $\mathfrak{z}^{m\ell} - 1$ is equal to 0 on $Z_m \supseteq \operatorname{supp}(\nu)$, and the claim follows.

For the converse, suppose that ν is not supported on Z_{2m}. Let A be a connected-component of $\mathbb{T} \backslash Z_{2m}$ such that $\nu(A) > 0$. One has that $\Im(\mathfrak{z}^m)$ is either strictly positive or strictly negative on A (assume positive, WLOG). For any positive function $f \in C(\mathbb{T})$

supported on A, one then has

$$\Im\left(\int_{\mathbb{T}} f\cdot \mathfrak{z}^m d\nu\right)=\int_A f\cdot\Im(\mathfrak{z}^m)d\nu\geq 0,$$

with strict inequality for some choices of f. On the other hand,

$$\Im\left(\int_{\mathbb{T}} f d\nu\right)=0,$$

so $\phi_{\nu,\beta}\circ\Phi$ does not vanish on the kernel, which contains $\mathfrak{z}^m-1$. A similar argument applies when ν is supported on Z_{2m} but not on Z_m. □

Combining this lemma with the classification of KMS_β states on $\mathcal{T}(\mathbb{N}^\times\ltimes\mathbb{Z})$ leads to a classification of the KMS_β states on $\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z}))$.

Theorem 6.4. *Let $\sigma:\mathbb{R}\to\operatorname{Aut}\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z}))$ be the dynamics on $\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z}))$ determined by $\sigma_t(u)=u$ and $\sigma_t(v_a)=a^{it}v_a$.*

(1) For each m^{th} root of unity z and each $\beta>1$, there is a unique extremal KMS_β state of $\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z}))$ such that

$$\psi^{(m)}_{z,\beta}(v_a u^k v_b^*)=\delta_{a,b}\frac{a^{-\beta}}{\zeta(\beta)}\sum_{c\in\mathbb{N}^\times}c^{-\beta}z^{ck}. \tag{6.1}$$

The map $z\mapsto\psi^{(m)}_{z,\beta}$ extends to an affine w-homeomorphism of the simplex of probability measures on Z_m onto the simplex of KMS_β states of $(\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z})),\sigma)$.*

(2) For each $n\in\Delta_m:=\{d\in\mathbb{N}^\times : d|m\}$ and each $\beta\leq 1$, there is a unique extremal KMS_β state of $\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z}))$ such that

$$\psi^{(m)}_{n,\beta}(v_a u^k v_b^*)=\delta_{a,b}a^{-\beta}\left(\frac{n}{\gcd(n,k)}\right)^{-\beta}\sum_{d|\frac{n}{\gcd(n,k)}}\mu(d)\frac{\varphi_\beta(d)}{\varphi(d)}. \tag{6.2}$$

The map $n\mapsto\psi^{(m)}_{n,\beta}$ extends to an affine w-homeomorphism of the simplex of probability measures on Δ_m onto the simplex of KMS_β states of $(\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z})),\sigma)$. The state $\psi^{(m)}_{m,\beta}$ is faithful on $\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z}))$.*

Proof. Since q_m is a surjective equivariant *-homomorphism, the map $\psi\mapsto\psi\circ q_m$ is an affine injective continuous map from the simplex of KMS_β states on $\mathcal{T}(\mathbb{N}^\times\ltimes(\mathbb{Z}/m\mathbb{Z}))$ to the KMS_β states in $\mathcal{T}(\mathbb{N}^\times\ltimes\mathbb{Z})$. By Lemma 6.3 and Theorem 4.2, the range of this affine map is the finite-dimensional simplex of states of the form $\phi_{\nu,\beta}\circ\Phi$, for ν a measure on Z_m satisfying (4.4). Formula (6.1) for the extreme points of this simplex when $\beta>1$ follows from the formulas in [19, Theorem 8.1], and formula (6.2) for the extreme points when $\beta\leq 1$ follows from Theorem 1.1. □

6.2 Symmetries

For $\beta \le 1$, the KMS_β states on $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ do not appear to admit symmetries, that is, non-trivial automorphisms of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ that commute with σ and leave the state invariant. However, the quotient $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ does admit symmetries, corresponding to units in the ring $\mathbb{Z}/m\mathbb{Z}$.

For each integer k relatively prime to m, the isometries v_a and the unitary u^k satisfy relations (0)–(4), and the universal property yields a *-homomorphism:

$$\theta_k : \mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})) \to \mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})), \qquad \theta_k(v_a u^x v_b^*) = v_a u^{kx} v_b^*.$$

This is in fact an isomorphism, with inverse θ_ℓ when $k\ell \equiv 1 \pmod{m}$, using (4).

The following proposition is a simple consequence of Theorem 6.4.

Proposition 6.5. *The representation* $\theta : (\mathbb{Z}/m\mathbb{Z})^* \to \operatorname{Aut}\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ *commutes with the* $\mathbb{R}$*-action* σ *on* $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$.

(1) For $\beta > 1$*, one has* $\psi^{(m)}_{z,\beta} \circ \theta_k = \psi^{(m)}_{z^k,\beta}$*. Thus, for each* $n|m$*, the transpose action* $\theta_k^*(\psi) = \psi \circ \theta_k$ *is transitive on the set of extremal* KMS_β *states of the form* $\psi^{(m)}_{z,\beta}$*,* $z \in Z_n^*$*, and factors through the group homomorphism* $(\mathbb{Z}/m\mathbb{Z})^* \to (\mathbb{Z}/n\mathbb{Z})^*$*.*

(2) For $\beta \le 1$*, one has* $\psi^{(m)}_{n,\beta} \circ \theta_k = \psi^{(m)}_{n,\beta}$*.*

This shows that the KMS_β states of $(\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z})), \sigma)$ exhibits spontaneous symmetry breaking. This result is of particular interest, as previous instances of spontaneous symmetry breaking appearing in quantum statistical systems of number theory relied on the existence of a unique KMS_β state at the critical temperature. This is the first instance of phase transition at the critical temperature wherein the extremal KMS_β states exhibit spontaneous symmetry breaking, besides trivial examples such as direct sums of C*-dynamical systems.

In the next chapter, we will examine how these symmetries relate to the GNS representation of the KMS_β states. If $\psi \circ q_m$ is a KMS_β state, then the representation π appearing in the GNS triple (H, π, Ω) of $\psi \circ q_m$ also factors through q_m, since

$$\langle \pi(x)\Omega, \pi(ay)\Omega \rangle = \psi(q_m(x^* a y)) = 0 \qquad \forall x, y \in \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}),\ a \in \ker(q_m).$$

The symmetries of $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ may be used to define unitaries on H that leave the space of operators $\pi(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))$ invariant under conjugation. This is perhaps a more natural notion of symmetry in our context, because the automorphisms of $\mathcal{T}(\mathbb{N}^\times \ltimes (\mathbb{Z}/m\mathbb{Z}))$ do not extend to automorphisms of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$.

We conclude this chapter with a comment on our formulas for KMS_β states. The formulas (6.1) and (6.2) appear quite different, and it is worth asking how they may be related to each other; Proposition 6.5 provides us with one explanation.

For $\beta > 1$, consider the states $\psi^{(m)}_{z,\beta}$ for $z \in Z^*_n$, $n|m$. The dual action θ^* may be integrated against the normalized Haar measure on $(\mathbb{Z}/m\mathbb{Z})^*$, and yields

$$\int_{(\mathbb{Z}/m\mathbb{Z})^*} \theta^*_k(\psi^{(m)}_{z,\beta})dk = \frac{1}{\varphi(m)} \sum_{k\in(\mathbb{Z}/m\mathbb{Z})^*} \psi^{(m)}_{z^k,\beta} = \frac{1}{\varphi(m)}\frac{\varphi(m)}{\varphi(n)} \sum_{w\in Z^*_n} \psi^{(m)}_{w,\beta}$$
$$= \frac{1}{\varphi(n)} \sum_{w\in Z^*_n} \psi^{(m)}_{w,\beta},$$

for each $z \in Z^*_n$. Under the affine isomorphism of Theorem 6.4, this is the KMS$_\beta$ state corresponding to the uniform measure on Z^*_n.

Evaluating this state on u^k with k relatively prime to n yields

$$\begin{aligned}\frac{1}{\varphi(n)} \sum_{w\in Z^*_n} \psi^{(m)}_{w,\beta}(u^k) &= \frac{1}{\varphi(n)\zeta(\beta)} \sum_{w\in Z^*_n} \sum_{c\in\mathbb{N}^\times} c^{-\beta} w^{ck} \\ &= \frac{1}{\varphi(n)\zeta(\beta)} \sum_{c\in\mathbb{N}^\times} c^{-\beta} \frac{\varphi(n)}{\varphi(n/\gcd(n,c))} \sum_{w\in Z^*_{n/\gcd(n,c)}} w \\ &= \frac{1}{\zeta(\beta)} \sum_{c\in\mathbb{N}^\times} \frac{c^{-\beta}}{\varphi(n/\gcd(n,c))}\mu\left(\frac{n}{\gcd(n,c)}\right) \\ &= \frac{1}{\zeta(\beta)} \sum_{d|n} \mu(d)\frac{1}{\varphi(d)}\left(\frac{n}{d}\right)^{-\beta} \sum_{\substack{c\in\mathbb{N}^\times \\ \gcd(d,c)=1}} c^{-\beta},\end{aligned}$$

where we have substituted $\frac{n}{\gcd(n,c)}$ with d, both generic divisors of n. The infinite series (including the zeta function) may be substituted with products over primes,

$$\begin{aligned}& n^{-\beta}\sum_{d|n} \mu(d)\frac{d^\beta}{\varphi(d)}\left(\prod_p 1 - p^{-\beta}\right)\left(\prod_{p\nmid d} \frac{1}{1-p^{-\beta}}\right) \\ &= n^{-\beta}\sum_{d|n} \mu(d)\frac{d^\beta}{\varphi(d)}\left(\prod_{p|d} 1 - p^{-\beta}\right) \\ &= n^{-\beta}\sum_{d|n} \mu(d)\frac{\varphi_\beta(d)}{\varphi(d)}.\end{aligned}$$

This is the same expression as the right-hand side of (6.2), the only difference being the range of β for which the formula is valid. A similar argument in the case where k is not relatively prime to n shows that $\int_{(\mathbb{Z}/m\mathbb{Z})^*} \theta^*_k(\psi^{(m)}_{z,\beta})dk = \psi^{(m)}_{n,\beta}$ in general, if one allows for $\beta > 1$ in the formula for $\psi^{(m)}_{n,\beta}$.

Chapter 7
Spatial representations

For an extremal KMS$_\beta$ state ψ, the von Neumann algebra $\pi_\psi(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))''$ generated by the GNS representation of ψ is a factor. This factor has type I if $\beta > 1$. Our aim in this section is to show that $\pi_\psi(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))''$ has type III$_1$ when $\psi = \psi_{n,\beta}$ for $n \in \mathbb{N}^\times$ and $\beta \leq 1$.

Our formulas for the measures $\nu_{n,\beta}$ and their extensions to the spectrum of $\mathfrak{D}$ allow for a detailed description of the GNS representations and their relation to the symmetries θ.

7.1 GNS representations of supercritical equilibrium

Our main theorem in this chapter is the following:

Theorem 7.1. *For $n \in \mathbb{N}^\times$ and $\beta \leq 1$, let (H, π, Ω) be the GNS representation of $\psi_{n,\beta}$. The von Neumann algebra $\pi(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))''$ is a type* III$_1$ *factor.*

The only KMS$_\beta$ states excluded from this theorem are $\psi_{\infty,\beta}$, $\beta \leq 1$. We expect these von Neumann algebras will also be type III$_1$, but do not currently know of a proof.

We will write capital letters (F, G, etc.) to denote elements of $\mathfrak{D}$, which we view as functions on Spec($\mathfrak{D}$), in order to avoid confusion with functions on $\mathbb{T}$, which we continue to denote with lowercase letters (f, g, etc.).

Recall from [19, Proposition 8.3] that $\mathbb{N}^\times$ acts on $\mathfrak{D}$ by injective *-endomorphisms, given by $\upsilon_a(F) = V_a F V_a^*$, whose ranges are ideals of $\mathfrak{D}$. The action admits left-inverses given by $\upsilon_a^\dagger(F) = V_a^* F V_a$ and respects the lattice structure in the sense of [27, Definition 3], meaning that $\upsilon_a(1)\upsilon_b(1) = \upsilon_{\mathrm{lcm}(a,b)}(1)$. (We have slightly changed notation from [19], using υ_a instead of α_a to avoid confusion with Chapter 3.) The crossed product $\mathfrak{D} \rtimes^\upsilon \mathbb{N}^\times$ is the universal C*-algebra generated by elements $F \in \mathfrak{D}$ and isometries W_a, $a \in \mathbb{N}^\times$ satisfying $W_a F W_a^* = \upsilon_a(F)$. There is a surjective map from

$\mathfrak{D} \rtimes^{\upsilon} \mathbb{N}^\times$ to $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$, defined on the generators by

$$F \mapsto F,\ F \in \mathfrak{D}, \qquad W_a \mapsto V_a,\ a \in \mathbb{N}^\times,$$

since $V_a F V_a^* = \upsilon_a(F)$, by definition. This map is in fact an isomorphism; we refer to [19, Proposition 8.3] for the details.

Proposition 7.2. *Let $\widetilde{\nu}_{n,\beta}$ be the measure on $X = \operatorname{Spec}\mathfrak{D}$ such that*

$$\psi_{n,\beta}(G) = \int_X G \; d\widetilde{\nu}_{n,\beta} \qquad G \in C(X) \cong \mathfrak{D}.$$

Let H be the Hilbert space $L^2(X, \widetilde{\nu}_{n,\beta}) \otimes \ell^2(\mathbb{Q}_+^\times)$, let $\rho_{n,\beta} : \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}) \to B(H)$ be the representation

$$\rho_{n,\beta}(F)(G \otimes \delta_b) = FG \otimes \delta_b, \qquad \rho_{n,\beta}(V_a)(G \otimes \delta_b) = a^{\beta/2}\upsilon_a(G) \otimes \delta_{ab},$$

and let $\Omega = 1_X \otimes \delta_1$. Let $M = \overline{\rho_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))\Omega}$ and $\rho^0_{n,\beta}$ the corner-representation of $\rho_{n,\beta}$ on the invariant subspace M. Then the triple $(M, \rho^0_{n,\beta}, \Omega)$ is the GNS representation for the state $\psi_{n,\beta}$.

Proof. We start by confirming that $\rho_{n,\beta}$ is a representation of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. The operator $\rho_{n,\beta}(V_a)$ is an isometry, since for any $G \otimes \delta_x$ we have

$$\begin{aligned}\langle \rho_{n,\beta}(V_a)G \otimes \delta_x, \rho_{n,\beta}(V_a)G \otimes \delta_x\rangle &= a^\beta \langle \upsilon_a(G) \otimes \delta_{ax}, \upsilon_a(G) \otimes \delta_{ax}\rangle \\ &= a^\beta \psi_{n,\beta}(\upsilon_a(|G|^2)) = \psi_{n,\beta}(|G|^2) = \langle G \otimes \delta_x, G \otimes \delta_x\rangle .\end{aligned}$$

The adjoint of $\rho_{n,\beta}(V_a)$ is given by $\rho_{n,\beta}(V_a)^* G \otimes \delta_x = a^{-\beta/2}\upsilon_a^\dagger(G) \otimes \delta_{x/a}$ when $G \in L^2(\upsilon_a^*(X), \widetilde{\nu}_{n,\beta})$ and 0 when $G \in L^2(X - \upsilon_{a*}(X), \widetilde{\nu}_{n,\beta})$. Equivalently,

$$\rho_{n,\beta}(V_a)^* G \otimes \delta_x = a^{-\beta/2}\upsilon_a^\dagger(\upsilon_a(1_X)G) \otimes \delta_{x/a}.$$

It is easily verified that $a \mapsto \rho_{n,\beta}(V_a)$ is a representation of $\mathbb{N}^\times$ by isometries, and these isometries are covariant with respect to the action of $\mathbb{N}^\times$ on $\mathfrak{D}$:

$$\begin{aligned}\rho_{n,\beta}(V_a)\rho_{n,\beta}(F)\rho_{n,\beta}(V_a^*)G \otimes \delta_x &= \rho_{n,\beta}(V_a)\rho_{n,\beta}(F)\Big(\upsilon_a^\dagger(\upsilon_a(1_X)G) \otimes \delta_{x/a}\Big) \\ &= \upsilon_a(F\upsilon_a^\dagger(\upsilon_a(1_X)G)) \otimes \delta_x = \upsilon_a(F)\upsilon_a(1_X)G \otimes \delta_x = \rho_{n,\beta}(\upsilon_a(F))G \otimes \delta_x,\end{aligned}$$

where $\upsilon_a \circ \upsilon_a^\dagger(\upsilon_a(1_X)G) = V_aV_a^*(V_aV_a^*G)V_aV_a^* = \upsilon_a(1_X)G$.

The vector Ω is cyclic for M and $\rho_{n,\beta}$ by definition, so it only remains to verify that $\psi_{n,\beta}(x) = \left\langle \rho^0_{n,\beta}(x)\Omega, \Omega\right\rangle$ on the generators of $\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$. For $f \in C(\mathbb{T}) \cong \mathfrak{D}_1$, note that $\int_X f d\widetilde{\nu}_{n,\beta} = \int_{\mathbb{T}} f d\nu_{n,\beta}$. For $a, b \in \mathbb{N}^\times$, we find that

$$\begin{aligned}\langle \rho_{n,\beta}(V_b f V_a^*)\Omega, \Omega\rangle &= \left\langle a^{-\beta/2} f \otimes \delta_{1/a}, b^{-\beta/2} 1_X \otimes \delta_{1/b}\right\rangle \\ &= \delta_{a,b} a^{-\beta} \int_{\mathbb{T}} f d\nu_{n,\beta} = \psi_{n,\beta}(V_b f V_a^*). \qquad \square\end{aligned}$$

Remark 7.3. The Hilbert space M may be identified with the space

$$\bigoplus_{p/q \in \mathbb{Q}_+^\times} \upsilon_p(1_X) L^2(X, \widetilde{\nu}_{n,\beta}) = \bigoplus_{p/q \in \mathbb{Q}_+^\times} L^2(\upsilon_p^*(X), \widetilde{\nu}_{n,\beta}),$$

where the fraction $\frac{p}{q}$ is reduced. This identification takes a function G in the $\frac{p}{q}$-summand to the vector $G \otimes \delta_{\frac{p}{q}}$ in H.

To see that the range of this map is M, consider p and q relatively prime, so that $\upsilon_{pq}(1_X) = \upsilon_p(1_X)\upsilon_q(1_X)$. The $\frac{p}{q}$-direct summand contains a dense subspace of elements of the form $\upsilon_p(G) \otimes \delta_{p/q}$ for $G \in \mathfrak{D}$, which can be expressed as

$$\upsilon_p(G) \otimes \delta_{p/q} = \upsilon_p(G)\upsilon_q^\dagger(\upsilon_{pq}(1_X)) \otimes \delta_{p/q} = \rho_{n,\beta}(V_q^* \upsilon_{pq}(G) V_p)\Omega.$$

Conversely, for any $V_q^* G V_p \in \mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z})$ with p, q relatively prime (WLOG), we have

$$\rho_{n,\beta}(V_q^* G V_p)\Omega = \rho_{n,\beta}(V_q^*)(G\upsilon_p(1_X) \otimes \delta_p) = \upsilon_q^\dagger(G\upsilon_{pq}(1_X)) \otimes \delta_{p/q},$$

where the function $\upsilon_q^\dagger(G\upsilon_{pq}(1_X))$ belongs to the ideal $\upsilon_p(\mathfrak{D}) \cong C(\upsilon_p^*(X))$.

For each $k \in \mathbb{N}^\times$, consider the map $\widetilde{\omega}_k^* : \mathfrak{D} \to \mathfrak{D}$ defined by

$$\widetilde{\omega}_k^*(V_a f V_a^*) = V_a(f \circ \omega_k)V_a^*,$$

for $f \in C(\mathbb{T})$, $a \in \mathbb{N}^\times$. This is an injective unital *-homomorphism, and is therefore dual to a surjective map $\widetilde{\omega}_k : X \to X$ such that $\widetilde{\omega}_k^*(F) = F \circ \widetilde{\omega}_k$. This map is not injective, in general, but behaves nicely with respect to $\widetilde{\nu}_{n,\beta}$ when k is relatively prime to n.

Proposition 7.4. *When k is relatively prime to n, the map*

$$w_k \in \mathbb{B}(L^2(X, \widetilde{\nu}_{n,\beta})), \quad w_k(G) = G \circ \widetilde{\omega}_k$$

is a unitary which depends only on the residue of k modulo n.

Proof. We start by showing that w_k is an isometry. Recall from Lemma 5.4 that $\omega_{k*}\nu_{n,\beta} = \nu_{n,\beta}$ when k is relatively prime to n. For any $f, g \in C(\mathbb{T})$, $a, b \in \mathbb{N}^\times$, and

$a' = \frac{a}{\gcd(a,b)}$ and $b' = \frac{b}{\gcd(a,b)}$, we have

$$\begin{aligned}\langle w_k(V_a f V_a^*), w_k(V_b g V_b^*)\rangle &= \psi_{n,\beta}(V_{ab'}(f\circ\omega_{kb'})(\overline{g}\circ\omega_{ka'})V_{a'b}) \\ &= (a'b)^{-\beta}\int_{\mathbb{T}}(f\circ\omega_{b'})(\overline{g}\circ\omega_{a'})\;d\omega_{k*}\nu_{n,\beta} \\ &= \langle V_a f V_a^*, V_b g V_b^*\rangle\,.\end{aligned}$$

Since elements of this form span a dense subspace of $L^2(X,\widetilde{\nu}_{n,\beta})$, it follows that w_k is an isometry.

Next we show that w_k is periodic ($w_k^m = 1$ for some $m \in \mathbb{N}^\times$), and is therefore a unitary. The map $g \mapsto a^{\beta/2}V_a g V_a^*$ (defined for continuous g and extended continuously) is an isometry from $L^2(\mathbb{T},\nu_{n,\beta})$ to $L^2(X,\widetilde{\nu}_{n,\beta})$, since

$$\int_X |a^{\beta/2}V_a g V_a^*|^2 d\widetilde{\nu}_{n,\beta} = \int_X a^\beta V_a|g|^2V_a^* d\widetilde{\nu}_{n,\beta} = a^\beta\psi_{n,\beta}(V_a|g^2|V_a^*) = \int_{\mathbb{T}}|g|^2 d\nu_{n,\beta}.$$

Under this isometry, w_k is conjugate to the map $g \mapsto g\circ\omega_k$. This map is periodic with period $m = \mathrm{ord}_{(\mathbb{Z}/n\mathbb{Z})^*}(k)$ (i.e. m the smallest positive integer such that $k^m \equiv 1$ modulo n), since $\nu_{n,\beta}$ is supported on the n^{th} roots of unity. Since the linear span of the elements $V_a g V_a^*$ is dense in $L^2(X,\widetilde{\nu}_{n,\beta})$, it follows that w_k is also m-periodic.

If $k \equiv k'$ modulo n, then clearly $w_k(V_a g V_a^*) = w_{k'}(V_a g V_a^*)$, and it follows that $w_k = w_{k'}$, again by density. □

Corollary 7.5. *The map*

$$W : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{B}(H), \quad W(k) = W_k := w_k \otimes 1$$

is a unitary representation of $(\mathbb{Z}/n\mathbb{Z})^*$ *such that* $W_k\rho_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))W_k^* = \rho_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))$ *and* $W_k\Omega = \Omega$. *In particular,* W_k *restricts to a unitary on* M *which leaves the image of* $\rho^0_{n,\beta}$ *and the vector state* $\langle\rho_{n,\beta}(\cdot)\Omega,\Omega\rangle$ *invariant under conjugation.*

Proof. It is clear from the previous proposition that W is a representation and $W_k\Omega = \Omega$. The remaining claim follows from the fact that $\widetilde{\omega}_k$ is continuous on X and commutes with υ_{a*} for all $a \in \mathbb{N}^\times$. □

Our next aim is to compute the fixed-point subalgebra for this action. This will require a few lemmas relating the measure $\widetilde{\nu}_{n,\beta}$ to the primitive n^{th} roots of unity.

For each n^{th} root of unity $z \in \mathbb{T}$, let $\widetilde{\delta}_z \in C(\mathbb{T})$ be a positive continuous function which is 1 at z and 0 at all other n^{th} roots of unity. The exact choice of $\widetilde{\delta}_z$ will not matter for our purposes, so long as it satisfies these two properties.

Lemma 7.6. *The functions* $\widetilde{\delta}_z$ *and* $V_p\widetilde{\delta}_z\circ\omega_p V_p^*$ *are equal* $\widetilde{\nu}_{n,\beta}$*-almost everywhere if and only if* $p\cdot\mathrm{ord}(z)|n$.

Proof. For any prime p, the function $(1-V_pV_p^*)\widetilde{\delta}_z = \widetilde{\delta}_z - V_p(\widetilde{\delta}_z \circ \omega_p)V_p^*$ is positive, and hence

$$\psi_{n,\beta}(\widetilde{\delta}_z - V_p\widetilde{\delta}_z \circ \omega_p V_p^*) = \int_X \widetilde{\delta}_z - V_p(\widetilde{\delta}_z \circ \omega_p)V_p^* d\widetilde{\nu}_{n,\beta} = 0$$

if and only if $\widetilde{\delta}_z = V_p(\widetilde{\delta}_z \circ \omega_p)V_p^*$ almost everywhere. Using the KMS condition, the left-hand expression can also be written

$$\psi_{n,\beta}(\widetilde{\delta}_z - p^{-\beta}\widetilde{\delta}_z \circ \omega_p) = \nu_{n,\beta}(\{z\}) - p^{-\beta}\nu_{n,\beta}(\omega_p^{-1}(z)). \tag{7.1}$$

By Lemma 5.4, $\nu_{n,\beta}(\omega_p^{-1}(\{z\})) = \omega_{p*}\nu_{n,\beta}(\{z\}) = \nu_{n',\beta}(\{z\})$, where $n' = \frac{n}{\gcd(p,n)}$. We consider 3 cases:

1. $p|n$ and $\operatorname{ord}(z)|\frac{n}{p}$, so $p \cdot \operatorname{ord}(z)|n$. One has

$$\begin{aligned}(7.1) &= \nu_{n,\beta}(\{z\}) - p^{-\beta}\nu_{n/p,\beta}(\{z\}) \\ &= n^{-\beta}\frac{\varphi_\beta(\operatorname{ord}(z))}{\varphi(\operatorname{ord}(z))} - p^{-\beta}\left(\frac{n}{p}\right)^{-\beta}\frac{\varphi_\beta(\operatorname{ord}(z))}{\varphi(\operatorname{ord}(z))} = 0.\end{aligned}$$

2. $p|n$ and $\operatorname{ord}(z) \nmid \frac{n}{p}$, so $p \cdot \operatorname{ord}(z) \nmid n$.. In this case, one has

$$(7.1) = \nu_{n,\beta}(\{z\}) - p^{-\beta}\nu_{n/p,\beta}(\{z\}) = n^{-\beta}\frac{\varphi_\beta(\operatorname{ord}(z))}{\varphi(\operatorname{ord}(z))} > 0.$$

3. $p \nmid n$, so $p \cdot \operatorname{ord}(z) \nmid n$. One has

$$(7.1) = \nu_{n,\beta}(\{z\}) - p^{\beta}\nu_{n,\beta}(\{z\}) = (1-p^{-\beta})n^{-\beta}\frac{\varphi_\beta(\operatorname{ord}(z))}{\varphi(\operatorname{ord}(z))} > 0. \quad \square$$

Lemma 7.7. *The functions $V_a\widetilde{\delta}_zV_a^*$, $a \in \mathbb{N}^\times$, $\operatorname{ord}(z) = n$ span a dense subspace of $L^2(X, \widetilde{\nu}_{n,\beta})$.*

Proof. We know that the functions $V_afV_a^*$, $f \in C(\mathbb{T})$ span a dense subspace of $L^2(X, \widetilde{\nu}_{n,\beta})$, and clearly $V_afV_a^* = \sum_{z^n=1} f(z)V_a\widetilde{\delta}_zV_a^*$. It follows immediately from Lemma 7.6 that $\widetilde{\delta}_z = V_{d_0}\widetilde{\delta}_z \circ \omega_{d_0}V_{d_0}^*$, where $d_0 = \frac{n}{\operatorname{ord}(z)}$. Our technique is to expand $\widetilde{\delta}_z \circ \omega_{d_0}$ as a sum of functions $\widetilde{\delta}_w$ with $w^{d_0} = z$, and to iterate this procedure to obtain an infinite series of functions of the form $V_b\widetilde{\delta}_wV_b^*$ with $\operatorname{ord}(w) = n$. We will show that this series converges absolutely in the L^2-norm, and converges point-wise $\widetilde{\nu}_{n,\beta}$-a.e. to $V_a\widetilde{\delta}_zV_a^*$.

For our recursive construction, we let $e_0 = \operatorname{ord}(z)$, n_1 the largest divisor of n whose prime factors divide e_0, define $c_1 = \frac{n_1}{e_0}$, and $d_1 = \frac{n}{e_0}$. As was suggested above, we start by expanding $\widetilde{\delta}_z \circ \omega_{d_1}$ as a sum over the set $\omega_{d_1}^{-1}(z)$. Note that since every prime divisor of c_1 divides $\operatorname{ord}(z)$, it follows that every element of $\omega_{c_1}^{-1}(z)$ has order $c_1 \cdot \operatorname{ord}(z)$.

The elements of $\omega_{d_1}^{-1}(z) = \omega_{\frac{n}{n_1}}^{-1}(\omega_{c_1}^{-1}(z))$ may have any order divisible by $c_1 \cdot \mathrm{ord}(z)$ and dividing n, since $\frac{n}{n_1}$ is relatively prime to $\mathrm{ord}(z)$. This allows us to write

$$\widetilde{\delta}_z = V_{d_1}\widetilde{\delta}_z \circ \omega_{d_1} V_{d_1}^* = V_{d_1}\left(\sum_{e_1|\frac{n}{n_1}} \sum_{\substack{w_1^{d_1}=z \\ \frac{\mathrm{ord}(w_1)}{c_1 \mathrm{ord}(z)}=e_1}} \widetilde{\delta}_{w_1}\right) V_{d_1}^*.$$

Whenever e_1 is not equal to $\frac{n}{n_1}$, we may replace $\widetilde{\delta}_{w_1}$ with $V_{d_2}\widetilde{\delta}_{w_1} \circ \omega_{d_2} V_{d_2}$, where $d_2 = \frac{n}{\mathrm{ord}(w_1)}$, which may be rewritten $d_2 = \frac{d_1}{e_1 c_1}$. More generally, given n_k, e_k, c_k, and d_k, we will define n_{k+1} to be the largest divisor of n whose prime divisors also divide $\prod_{i=0}^{k} c_i e_i$, define $c_{k+1} = n_{k+1}\left(\prod_{i=0}^{k} c_i e_i\right)^{-1}$, and $d_{k+1} = \frac{d_k}{e_k c_k}$; we let e_{k+1} be any divisor of $\frac{n}{n_{k+1}}$. Recursively substituting $\widetilde{\delta}_{w_k}$ (terms appearing in the kth iteration of our recursion) with $V_{d_{k+1}}\widetilde{\delta}_{w_k} V_{d_{k+1}}^*$ when $\mathrm{ord}(w) \neq n$ and expanding, we obtain an infinite series whose summands involve only $\widetilde{\delta}_w$ for $\mathrm{ord}(w) = n$, as desired. The terms in this series which appear in the kth step of our recursion are of the form $V_{d_1 d_2 \cdots d_k}\widetilde{\delta}_w V_{d_1 d_2 \cdots d_k}^*$; note that d_k does not just depend on k, but on the previous terms in the sequence $(c_i, d_i, e_i)_{i=1}^{k-1}$.

To see that this series converges pointwise a.e., consider a generic point x in X. Recall from Proposition 3.13 that there is a surjective map $\pi : X \to \overline{\mathbb{N}}$, and the pullback of the characteristic function of $C_k = \{N \in \overline{\mathbb{N}} : k|N\}$ is $V_k V_k^*$. We claim that the set

$$\bigcup_{p \in \mathcal{P}} (\alpha^p \circ \pi)^{-1}(\infty) = \{x \in \overline{\mathbb{N}} : \alpha^p(\pi(x)) = \infty \text{ for some } p \in \mathcal{P}\}$$

has measure 0 under $\widetilde{\nu}_{n,\beta}$, where $\alpha^p(x)$ is the largest power k such that $p^k|x$ and $\alpha^p(x) = \infty$ if no such power exists, c.f. 3.3. Indeed, from Proposition 3.13, one has

$$X = \bigsqcup_{n \in \mathbb{N} \cup \{\infty\}} (\alpha_p \circ \pi)^{-1}(n).$$

For $n < \infty$, the characteristic function of $(\alpha_p \circ \pi)^{-1}(n)$ is $V_{p^n} V_{p^n}^* - V_{p^{n+1}} V_{p^{n+1}}^*$, which has $\widetilde{\nu}_{n,\beta}$-measure $p^{-\beta n} - p^{-\beta(n+1)}$. Thus, one has

$$\widetilde{\nu}_{n,\beta}((\alpha_p \circ \pi)^{-1}(\infty)) = 1 - \left(\sum_{n=0}^{\infty} p^{-\beta n} - p^{-\beta(n+1)}\right).$$

The series telescopes to 1, so that $\widetilde{\nu}_{n,\beta}((\alpha^p \circ \pi)^{-1}(\infty)) = 0$, whence the claim follows by countable subadditivity of $\widetilde{\nu}_{n,\beta}$. We may therefore assume that $\alpha^p(\pi(x)) < \infty$ for every $p \in \mathcal{P}$.

For each $p|n$, it follows from the recursive process that there are only finitely many summands such that $\alpha^p(d_1 d_2 \cdots d_k) \leq \alpha^p(x)$. (This will be verified in our argument that the series converges in L^2.) Evaluating the series at x then results in a finite sum; indeed, if $\alpha^p(d_1 d_2 \cdots d_k) > \alpha^p(x)$, then one has

$$0 \leq V_{d_1 d_2 \cdots d_k} \widetilde{\delta}_z V^*_{d_1 d_2 \cdots d_k}(x) \leq V_{d_1 d_2 \cdots d_k} V_{d_1 d_2 \cdots d_k}(x) = \chi_{C_{d_1 d_2 \cdots d_k}}(\pi(x)) = 0.$$

Therefore, the series converges pointwise a.e. and, by its construction, the pointwise limit is $\widetilde{\delta}_z$.

To see that the series is absolutely convergent in L^2, we start with a few observations. First, the summands are all projections of the form $V_{d_1 d_2 \cdots d_k} \widetilde{\delta}_{w_{k+1}} V_{d_1 d_2 \cdots d_k}$ with $\operatorname{ord}(w_{k+1}) = n$, and thus has L^2-norm

$$\| V_{d_1 d_2 \cdots d_k} \widetilde{\delta}_{w_{k+1}} V^*_{d_1 d_2 \cdots d_k} \| = (d_1 d_2 \cdots d_k)^{-\beta/2} \cdot n^{-\beta/2} \sqrt{\frac{\varphi_\beta(n)}{\varphi(n)}}.$$

Thus, we may factor out a constant $n^{-\beta/2}\sqrt{\frac{\varphi_\beta(n)}{\varphi(n)}}$ from the series of norms.

Second, the summand $\widetilde{\delta}_{w_k}$ (ignoring the isometries for the moment) appears in a sum indexed by the set of $w \in \mathbb{T}$ satisfying $w^{d_k} = w_{k-1}$ and $\frac{\operatorname{ord}(w)}{c_k \cdot \operatorname{ord}(w_{k-1})} = e_k$. We showed that the norm of this summand does not depend on the choice of w_k from this set, so we may replace the sum of norms with the number of summands multiplied by the (constant) norm. The function $w \mapsto w^{d_k}$ maps $Z^*_{\operatorname{ord}(w_k)}$ surjectively to $Z^*_{\operatorname{ord}(w_{k-1})}$ such that the pre-images all have the same cardinality; it follows that, regardless of w_{k-1}, the number of summands is given by

$$\frac{\varphi(\operatorname{ord}(w_k))}{\varphi(\operatorname{ord}(w_{k-1}))} = \frac{\varphi(c_k e_k \operatorname{ord}(w_{k-1}))}{\varphi(\operatorname{ord}(w_{k-1}))}.$$

By our construction, e_k is relatively prime to $c_k \operatorname{ord}(w_{k-1})$, so $\varphi(c_k e_k \operatorname{ord}(w_{k-1})) = \varphi(e_k)\varphi(c_k \operatorname{ord}(w_{k-1}))$. Also, we have chosen c_k to share prime factors with $\operatorname{ord}(w_{k-1})$, so the product formula for φ yields $\varphi(c_k \operatorname{ord}(w_{k-1})) = c_k \varphi(\operatorname{ord}(w_{k-1}))$. Therefore, the number of summands is $c_k \varphi(e_k)$. Repeating this argument for w_{k-1}, w_{k-2}, etc. yields the following sum of norms:

$$\sum_{k=1}^{\infty} \sum_{(c_i, d_i, e_i)_{i=1}^k} \prod_{i=1}^{k} c_i \varphi(e_i) d_i^{-\beta/2},$$

where the second sum is indexed over tuples (c_i, d_i, e_i) obtained from our recursive process, which terminates when $\operatorname{ord}(w_k) = \operatorname{ord}(z) \prod_{i=1}^k c_i e_i = n$. This tuple ultimately depends only on $(e_i)_{i=1}^k$, since c_i and d_i may be recovered from these.

We start by computing $\prod_{i=1}^k c_i \varphi(e_i)$. Note that, by our construction, e_i and e_{i+1} do not share any prime factors, and every prime which divides n but does not divide

$\mathrm{ord}(z)$ (i.e. divides $\frac{n}{n_1}$), must divide some e_i. It follows that

$$\prod_{i=1}^{k} c_i\varphi(e_i) = \left(\prod_{i=1}^{k} c_i e_i\right)\left(\prod_{p|\frac{n}{n_1}} 1 - p^{-1}\right) = \frac{n}{\mathrm{ord}(z)} \prod_{p|\frac{n}{n_1}} 1 - p^{-1},$$

regardless of the sequence $(c_i, d_i, e_i)_{i=1}^k$.

Next we consider $\prod_{i=1}^k d_i$. We will use the identity $d_k = \frac{n}{n_{k-1}e_{k-1}}$ when $k > 1$, where $n_k = c_k \mathrm{ord}(w_k) = c_k \mathrm{ord}(z) \prod_{i=1}^{k-1} c_i e_i$. It follows that

$$\prod_{i=1}^{k} d_i = \frac{n^k}{\mathrm{ord}(z)\prod_{i=1}^{k-1} n_i e_i} = \frac{n}{\mathrm{ord}(z)} \cdot \frac{n^{k-1}}{\prod_{i=1}^{k-1} n_i e_i}.$$

The last step is to reindex the double series. Recall that the terms e_i are relatively prime, or in particular, that if $p|e_i$, then $p \nmid \frac{n}{n_j}$ for all $j > i$. The sequence $(e_i)_{i=1}^k$ is therefore determined by the integers k_p and b_p for each $p|\frac{n}{n_1}$, where k_p is the index where $p|e_{k_p}$ and $b_p = \alpha^p\left(\frac{n}{e_{k_p}}\right)$; conversely, any pairs of integers $k_p \geq 1$ and $\alpha^p(n) > b_p \geq 0$ for each $p|\frac{n}{n_1}$ gives rise to a sequence $(e_i)_{i=1}^k$. This can be related to the product of d_i by noticing that

$$\alpha^p\left(\prod_{i=2}^{k} d_i\right) = (k_p - 1)\alpha^p(n) + b_p,$$

whence it follows that

$$\prod_{i=1}^{k} d_i = \frac{n}{\mathrm{ord}(z)} \cdot \prod_{p|\frac{n}{n_1}} p^{(k_p-1)\alpha^p(n)+b_p}.$$

Since $k_p - 1$ can be any integer greater than 0 and b_p can be any integer between 0 and $\alpha^p(n) - 1$ inclusive, we conclude that

$$\sum_{k=1}^{\infty} \sum_{(c_i,d_i,e_i)_{i=1}^k} \prod_{i=1}^{k} c_i\varphi(e_i) d_i^{-\beta/2} = \left(\frac{n}{\mathrm{ord}(z)}\right)^{\beta/2} \cdot \left(\prod_{p|\frac{n}{n_1}} 1 - p^{-1}\right) \sum_{d\in\mathbb{N}_{n/n_1}^{\times}} d^{-\beta/2},$$

where $\mathbb{N}_d^{\times}$ denotes the set of positive integers whose prime factors also divide d. This series converges absolutely for $\beta > 0$, as desired. □

For each divisor $e|n$, let

$$u_e = \sum_{\mathrm{ord}(w)=e} \widetilde{\delta}_w.$$

Letting n_1 be the largest divisor of n whose prime factors divide e and $d_1 = \frac{n}{e}$, one has

$$\begin{aligned} u_e = \sum_{\operatorname{ord}(w)=e} \widetilde{\delta}_w &= V_{d_1} \left(\sum_{\operatorname{ord}(w)=e} \widetilde{\delta}_w \circ \omega_{\frac{n}{e}} \right) V_{d_1}^* \\ &= V_{d_1} \left(\sum_{e_1 | \frac{n}{n_1}} u_{n_1 \cdot e_1} \right) V_{d_1}^* \end{aligned}$$

and we expand this to an infinite series, recursively. The same algebraic manipulations as in the previous proof imply that the resulting series is

$$u_e = V_{\frac{n}{e}} \left(\sum_{d \in \mathbb{N}^\times_{\frac{n}{n_1}}} V_d u_n V_d^* \right) V_{\frac{n}{e}}^*,$$

which converges $\widetilde{\nu}_{n,\beta}$-a.e.

Corollary 7.8. *For any divisor $c|n$, the series*

$$\sum_{d \in \mathbb{N}_c^\times} \rho_{n,\beta} \left(V_d u_n V_d^* \right)$$

converges in the strong-operator topology.

Proof. By Lemma 7.7, it suffices to show that

$$\left\| \sum_{d \in \mathbb{N}_c^\times \setminus S} \rho_{n,\beta} \left(V_d u_n V_d^* \right) \left(V_b \widetilde{\delta}_z V_b^* \otimes \delta_x \right) \right\|$$

tends to 0 as $S \to \mathbb{N}_c^\times$, for any $z \in Z_n^*$ and $x \in \mathbb{Q}_+^\times$. Factoring $b = b'a$ where $a = \gcd(b, n)$, one finds that

$$\rho_{n,\beta}(V_d u_n V_d^*)(V_b \delta_z V_b^* \otimes \delta_x) = V_{b'} \left(V_{da'} (u_n \circ \omega_{a'}) (\widetilde{\delta}_z \circ \omega_{d'}) V_{d'a} \right) V_{b'} \otimes \delta_x,$$

where $a' = \frac{a}{\gcd(d,a)}$ and $d' = \frac{d}{\gcd(d,a)}$. Both a' and d' are divisors of $\operatorname{ord}(z) = n$; if either one is not equal to 1, then the product is 0. Therefore, we have

$$\left\| \sum_{d \in \mathbb{N}_c^\times \setminus S} \rho_{n,\beta} \left(V_d u_n V_d^* \right) \left(V_b \widetilde{\delta}_z V_b^* \otimes \delta_x \right) \right\| = \begin{cases} \| V_b \delta_z V_b^* \otimes \delta_x \| & \text{if } \gcd(n, b) \in \mathbb{N}_c^\times \setminus S, \\ 0 & \text{otherwise,} \end{cases}$$

whence the result follows. ☐

Lemma 7.9. *For any* $a \in \mathbb{N}^\times$, *one has*

$$\rho_{n,\beta}(V_a u_n V_a^*) = \sum_{d|n} \mu(d) \rho_{n,\beta}(V_{ad} V_{ad}^*) \tag{7.2}$$

Proof. By factoring $a = a' \gcd(a, n)$ and factoring $\rho_{n,\beta}(V_{a'})$, it suffices to consider $a \in \mathbb{N}_n^\times$.

We will start by computing formulas for $\rho_{n,\beta}(V_a V_a^*)$ and $\rho_{n,\beta}(V_a u_n V_a^*)$. By Lemma 7.7, it suffices to consider their actions on vectors of the form $V_b \delta_z V_b^* \otimes \delta_x$ for z a primitive n^{th} root of unity (the tensor with δ_x is unimportant for the representation of $\mathfrak{D}$, so we will omit it in this proof). Since $a \in \mathbb{N}_n^\times$, the prime divisors of a are also divisors of n; in particular, for any divisor $d|a$, we have that $\gcd(d, n) = 1$ if and only if $d = 1$. It follows that

$$\begin{aligned}\rho_{n,\beta}(V_a V_a^*) V_b \widetilde{\delta}_z V_b^* = V_{a'b} \widetilde{\delta}_z \circ \omega_{a'} V_{a'b}^* &= \begin{cases} V_{a'b} \widetilde{\delta}_z V_{a'b} & \text{if } a' = 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} V_b \widetilde{\delta}_z V_b & \text{if } a|b \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where $a' = \frac{a}{\gcd(a,b)}$. Similarly, we have

$$\begin{aligned}\rho_{n,\beta}(V_a u_n V_a^*) V_b \delta_z V_b^* &= V_{\operatorname{lcm}(a,b)} (u_n \circ \omega_{b'})(\delta_z \circ \omega_{a'}) V_{\operatorname{lcm}(a,b)} \\ &= \begin{cases} V_b u_n \circ \omega_{b'} \delta_z V_b & \text{if } a|b \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} V_b \delta_z V_b & \text{if } a|b \text{ and } \gcd\left(\frac{b}{a}, n\right) = 1 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

where $a'b = ab' = \operatorname{lcm}(a, b)$, simplifiying to $b' = \frac{b}{a}$ when $a|b$. We finally check the right-hand side of (7.2) and confirm that it agrees with $\rho_{n,\beta}(V_a u_n V_a^*)$:

$$\begin{aligned}\sum_{d|n} \mu(d) \rho_{n,\beta}(V_{ad} V_{ad}^*) V_b \delta_z V_b^* &= \begin{cases} \sum_{d|\gcd\left(\frac{b}{a},n\right)} \mu(d) V_b \delta_z V_b^* & \text{if } a|b \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} V_b \delta_z V_b^* & \text{if } a|b \text{ and } \gcd\left(\frac{b}{a}, n\right) = 1 \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

using the well-known identity $\sum_{d|a} \mu(d) = 1$ if $a = 1$, and $\sum_{d|a} \mu(d) = 0$ otherwise. ☐

Lemma 7.10. *The fixed-point algebra* $\rho_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))^W$ *is strong-operator dense in*

$$\mathcal{W}^*(\mathbb{N}^\times) := \left(\rho_{n,\beta}(\mathbb{N}^\times) \cup \rho_{n,\beta}(\mathbb{N}^\times)^*\right)'' = \{\rho_{n,\beta}(V_a), \rho_{n,\beta}(V_a)^* : a \in \mathbb{N}^\times\}''.$$

Proof. The elements of $\rho_{n,\beta}(\mathbb{N}^\times)$ are fixed under the action of W, so it suffices to show that $\rho_{n,\beta}(\mathfrak{D})^W \subseteq \mathcal{W}^*(\mathbb{N}^\times)$. The elements of $\rho_{n,\beta}(\mathfrak{D})^W$ are of the form

$$E_W(F) = \frac{1}{\varphi(n)} \sum_k W_k \rho_{n,\beta}(F) W_k^*$$

for $F \in \mathfrak{D}$. If $F = V_a g V_a^*$ for some $g \in C(\mathbb{T})$, then we have that

$$\frac{1}{\varphi(n)} \sum_k W_k \rho_{n,\beta}(F) W_k^* = \rho_{n,\beta}\left(V_a \frac{1}{\varphi(n)} \left(\sum_k g \circ \omega_k\right) V_a^*\right).$$

The function $\sum_k g \circ \omega_k$ is constant on the set of d^{th} roots of unity for each $d|n$, and thus can be written in the form

$$\sum_k g \circ \omega_k = \sum_{d|n} \lambda_d u_d.$$

By Lemma 7.7 and Lemma 7.9, this is equal almost everywhere to a S.O.-convergent series of projections from $\mathcal{W}^*(\mathbb{N}^\times)$, and hence $\rho_{n,\beta}(\mathfrak{D})^W \subseteq \mathcal{W}^*(\mathbb{N}^\times)$. □

We are now prepared to prove Theorem 7.1. Our proof makes use of a standard technique in the theory of von Neumann algebras. Recall that if M is a von Neumann factor and N is a subfactor of M, a conditional expectation $E : M \to N$ has *finite index* if there exists some $K \geq 1$ such that $K \cdot E - \mathrm{id}_M$ is positive on M. When N is type III, the existence of a conditional expectation $E : M \to N$ implies that M is also type III; if E has finite index, then M is type III$_1$ if and only if N is type III$_1$. We refer to [17] for the details.

Given an action γ of a finite group G on a von Neumann factor M, the conditional expectation on M defined by

$$E_G(x) = \frac{1}{|G|} \sum_{g \in G} \gamma_g(x)$$

is finite index. Indeed, if let $K = |G|$, then it follows that, for each $x \geq 0$,

$$|G| \cdot E(G)(x) - x = \sum_{g \in G \setminus \{e\}} \gamma_g(x) \geq 0.$$

Proof of Theorem 7.1. Proposition 7.2 says that $(M, \rho^0_{n,\beta}, \Omega)$ is the GNS representation of $\psi_{n,\beta}$, and the algebra $\rho^0_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))''$ is a factor since $\psi_{n,\beta}$ is an extremal KMS$_\beta$ state. By Lemma 7.10, we have

$$(\rho^0_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))'')^W = (\rho^0_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))^W)'' = \mathcal{W}^*(\mathbb{N}^\times)''.$$

The vector state $\langle \rho^0_{n,\beta}(x)\Omega, \Omega\rangle$ on $\mathcal{W}^*(\mathbb{N}^\times)$ is KMS$_\beta$ for the action $\sigma_t(V_a) = a^{it}V_a$, hence $W^*(\mathbb{N}^\times)$ is isomorphic to the von Neumann algebra generated by the GNS representation of the unique KMS$_\beta$ state on $\mathcal{T}(\mathbb{N}^\times)$, which is type III$_1$, [5, Proposition 8]. Since W is an action of a finite group, we conclude that $\rho^0_{n,\beta}(\mathcal{T}(\mathbb{N}^\times \ltimes \mathbb{Z}))''$ is also type III$_1$. $\square$

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