

A NOTE ON THE CONSTRUCTION OF THE PROXIMATE
TYPE OF AN ENTIRE DIRICHLET SERIES WITH
INDEX-PAIR (p,q)

By

H.M. SRIVASTAVA and H.S. KASANA

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1. INTRODUCTION AND DEFINITIONS

Let $f(s)$ be an entire function defined by a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \quad (1.1)$$

$$(s = \sigma + it; 0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_n \uparrow \infty),$$

which is absolutely convergent for all s . Denote by $M(\sigma)$ the maximum modulus of $f(s)$ so that

$$M(\sigma) \equiv M(\sigma, f) = \sup_{-\infty < t < \infty} |f(\sigma + it)|. \quad (1.2)$$

We begin by recalling the concept of the index-pair (p, q) , the (p, q) -order, and the (p, q) -type of an entire Dirichlet series $f(s)$ for integers p and q such that

$$p \geq q + 1 \geq 1 \quad (1.3)$$

(see [2] and [3]).

DEFINITION 1. An entire function $f(s)$, defined by (1.1), is said to be of (p, q) -order ρ if

$$\rho \equiv \rho(p, q) = \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log^{[p]} M(\sigma)}{\log^{[q]} \sigma} \right\}, \quad (1.4)$$

where $M(\sigma)$ is given by (1.2), and (for convenience)

$$\log^{[m]} x = \exp^{[-m]} x = \log(\log^{[m-1]} x) = \exp(\exp^{[-m-1]} x) \quad (1.5)$$

$$(m = 0, \pm 1, \pm 2, \dots)$$

provided that

$$0 < \log^{[m-1]}_x < \infty \quad (m = 1, 2, 3, \dots) \quad (1.6)$$

with, of course,

$$\log^{[0]}_x = \exp^{[0]}_x = x. \quad (1.7)$$

DEFINITION 2. An entire function $f(s)$, defined by (1.1) and having the (p, q) -order ρ ($b < \rho < \infty$), is said to be of (p, q) -type τ if

$$\tau \equiv \tau(p, q) = \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log^{[p-1]}_{M(\sigma)}}{(\log^{[q-1]}_{\sigma})^{\rho}} \right\}, \quad (1.8)$$

where $b = 0$ if $p > q + 1$, and $b = 1$ if $p = q + 1$.

DEFINITION 3. For ρ and b given by Definitions 1 and 2, respectively, an entire Dirichlet series (1.1) is said to be of index-pair (p, q) for integers p and q constrained by (1.3) if

$$b < \rho(p, q) < \infty$$

and $\rho(p-1, q-1)$ is not a non-zero finite number.

The notion of Lindelöf's proximate order of an entire function (cf., e.g., [6, pp. 64-67]) has been extended, in a natural way, to entire Dirichlet series. As a matter of fact, Azpeitia [1, p. 495] has obtained some interesting propositions of existence of linear proximate orders $R(\sigma)$ and linear lower proximate orders $L(\sigma)$ for $f(s)$ defined by (1.1), and Srivastava and Singh [5, p. 5, Theorem 1] have proved the corresponding existence theorem for the proximate type $T(\sigma)$ of the entire Dirichlet series (1.1). Recently, Kasana [4, p. 447, Theorem 1]

and

$$(iv) \quad \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log [p-2]_{M(\sigma)}}{\exp\{(\log [q-1]_{\sigma})^{\rho} T(\sigma)\}} \right\} = a,$$

where, for convenience,

$$\Lambda_{[m]}(x) = \prod_{j=0}^m \log [j]_x. \quad (2.1)$$

REMARK. For entire Dirichlet series of finite positive (p,q)-type, our assertion (iii) is equivalent to the corresponding assertion due to Kasana [4, p. 447, Equation (1.3)].

Our demonstration of this general theorem, given in Section 3 below, is rather simple and is markedly different from the proof presented earlier by Kasana [4, pp. 447-449]. Moreover, it includes a much wider class of entire Dirichlet series with index-pair (p,q).

3. CONSTRUCTION OF THE (p,q)-PROXIMATE TYPE $T(\sigma)$

We shall show that the properties asserted by the theorem are possessed by $T(\sigma)$ in each of the following possible alternatives.

If there exists a positive number ℓ such that

$$\log [p]_{M(\sigma)} < \rho \log [q]_{\sigma} \quad \text{when } \sigma > \ell, \quad (3.1)$$

then, for a given number a ($0 < a < \infty$), we define

$$\phi_{[q]}^p(\sigma) = \begin{cases} \sup_{\ell \leq t \leq \sigma} \left\{ \frac{\log [3] (\log [p-3]_{M(t)})^{1/a}}{\log [q]_t} \right\} - \rho & (\sigma > \ell) \\ \phi_{[q]}^p(\ell) & (0 < \sigma \leq \ell). \end{cases} \quad (3.2a)$$

$$(0 < \sigma \leq \ell). \quad (3.2b)$$

Otherwise we define

$$\phi_q^p(\sigma) = \sup_{t \geq \sigma} \left\{ \frac{\log^{[3]} (\log^{[p-3]} M(t))^{1/a}}{\log^{[q]} t} \right\} - \rho. \quad (3.2c)$$

The following cases arise naturally:

CASE 1. Let

$$\limsup_{\sigma \rightarrow \infty} \left\{ \phi_q^p(\sigma) \log^{[q]} \sigma \right\} = -\infty, \quad (3.3)$$

where $\phi_q^p(\sigma)$ is defined by (3.2a). In this case we set

$$T(\sigma) = \sup_{t \geq \sigma} \left\{ \exp \left\{ \phi_q^p(t) \log^{[q]} t \right\} \right\}. \quad (3.4)$$

CASE 2. Let

$$\limsup_{\sigma \rightarrow \infty} \left\{ \phi_q^p(\sigma) \log^{[q]} \sigma \right\} = \infty. \quad (3.5)$$

In this case we set

$$T(\sigma) = \sup_{t \leq \sigma} \left\{ \exp \left\{ \phi_q^p(t) \log^{[q]} t \right\} \right\}. \quad (3.6)$$

CASE 3. Let

$$\limsup_{\sigma \rightarrow \infty} \left\{ \phi_q^p(\sigma) \log^{[q]} \sigma \right\} = \log \delta \quad (0 < \delta < \infty). \quad (3.7)$$

In this case we set

$$T(\sigma) = \delta + \sup_{l_1 \leq t \leq \sigma} \left\{ \frac{\log \left\{ \sup_{x \geq t} \left\{ \frac{a^{-1} \log [p-2]_M(x)}{\exp(\delta (\log [q-1]_x)^\rho)} \right\} \right\}}{(\log [q-1]_t)^\rho} \right\}, \text{ if } \gamma = 0, \quad (3.8)$$

where l_1 is such that

$$\log [p-2]_M(\sigma) < a \exp(\delta (\log [q-1]_\sigma)^\rho) \quad (3.9)$$

for all $\sigma > l_1$;

$$T(\sigma) = \delta + \frac{\log \gamma}{\log [q-1]_\sigma}, \text{ if } 0 < \gamma < \infty; \quad (3.10)$$

and

$$T(\sigma) = \delta + \sup_{t \geq \sigma} \left\{ \frac{\log \left\{ \sup_{x \leq t} \left\{ \frac{a^{-1} \log [p-2]_M(x)}{\exp(\delta (\log [q-1]_x)^\rho)} \right\} \right\}}{(\log [q-1]_t)^\rho} \right\}, \text{ if } \gamma = \infty, \quad (3.11)$$

where, for convenience,

$$\gamma = \limsup_{\sigma \rightarrow \infty} \left\{ \frac{\log [p-2]_M(\sigma)}{\exp(\delta (\log [q-1]_\sigma)^\rho)} \right\}. \quad (3.12)$$

We note here that, if $f(s)$ is of finite positive (p,q) -type, then $\delta = \tau$.

We now illustrate our general approach by establishing the theorem for the case corresponding to (3.8). For this case, if we let

$$\mu(\sigma) = \sup_{\ell \geq t \geq \sigma} \left\{ \frac{a^{-1} \log^{[p-2]} M(t)}{\exp(\delta (\log^{[q-1]} t)^\rho)} \right\} \quad (\ell > \exp^{[q-2]} 1), \quad (3.13)$$

then it is easily verified that $\mu(\sigma)$ is continuous and non-increasing, and that

$$\mu(\sigma) \rightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty.$$

Here ℓ is fixed so that $\log \mu(\sigma)$ is always negative. Let \mathcal{E} be the set of $\sigma \in [\ell, \infty)$ for which $\mu(\sigma) = \mu(t)$ for some $t > \sigma$; then \mathcal{E} is a set of bounded half-open intervals $\{[\alpha_n, \beta_n)\}$ and may be empty, bounded, or unbounded. In the complement of the set of open intervals $\{(\alpha_n, \beta_n)\}$, we have

$$\mu(\sigma) = \frac{a^{-1} \log^{[p-2]} M(\sigma)}{\exp(\delta (\log^{[q-1]} \sigma)^\rho)}. \quad (3.14)$$

If $\sigma \in (\alpha_n, \beta_n)$, it follows that

$$\begin{aligned} 0 > \frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^\rho} &= \frac{\log^{[p-1]} M(\beta_n) - \delta (\log^{[q-1]} \beta_n)^\rho - \log a}{(\log^{[q-1]} \sigma)^\rho} \\ &\geq \frac{\log \left(a^{-1} \log^{[p-2]} M(\beta_n) \right)}{(\log^{[q-1]} \beta_n)^\rho} - \delta, \end{aligned} \quad (3.15)$$

and, if σ lies outside these intervals, we have

$$0 > \frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^\rho} = \frac{\log \left(a^{-1} \log^{[p-2]} M(\sigma) \right)}{(\log^{[q-1]} \sigma)^\rho} - \delta. \quad (3.16)$$

Hence, by virtue of the definition of δ , we conclude that

$$\frac{\log \mu(\sigma)}{(\log^{[q-1]} \sigma)^\rho} \rightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty.$$

Thus

$$v(\sigma) = \sup_{\ell \leq t \leq \sigma} \left\{ \frac{\log \mu(t)}{(\log [q-1]_t)^\rho} \right\} \quad (\sigma > \ell) \quad (3.17)$$

is a negative and non-decreasing function, and

$$v(\sigma) \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty.$$

If, for some $\sigma_n \in (\alpha_n, \beta_n)$, we have

$$v(\sigma_n) = \frac{\log \mu(\sigma_n)}{(\log [q-1]_{\sigma_n})^\rho} = \frac{\log \mu(\beta_n)}{(\log [q-1]_{\sigma_n})^\rho}, \quad (3.18)$$

we must also have

$$v(\sigma) = \frac{\log \mu(\beta_n)}{(\log [q-1]_\sigma)^\rho} \quad \text{for } \sigma_n \leq \sigma \leq \beta_n. \quad (3.19)$$

Equation (3.19) holds true, in particular, for $\sigma = \beta_n$. Now the set of σ for which

$$v(\sigma) = \frac{\log \mu(\sigma)}{(\log [q-1]_\sigma)^\rho} \quad (3.20)$$

is necessarily unbounded, and we have shown that this set cannot be a subset of the set of open intervals $\{(\alpha_n, \beta_n)\}$. Hence

$$v(\sigma) = \frac{\log \mu(\sigma)}{(\log [q-1]_\sigma)^\rho} = \frac{\log(a^{-1} \log [p-2]_{M(\sigma)})}{(\log [q-1]_\sigma)^\rho} - \delta \quad (3.21)$$

for an unbounded set of σ .

For the case under discussion,

$$T(\sigma) = v(\sigma) + \delta, \quad (3.22)$$

and hence we have

$$T(\sigma) = \frac{\log(a^{-1} \log^{[p-2]}_{M(\sigma)})}{(\log^{[q-1]}_{\sigma})^{\rho}} \quad (3.23)$$

for an unbounded set of σ . Equation (3.23) readily implies the assertions (i) and (ii) of the theorem. Since, by definition,

$$T(\sigma) \cong \frac{\log(a^{-1} \log^{[p-2]}_{M(\sigma)})}{(\log^{[q-1]}_{\sigma})^{\rho}} \quad \text{for all } \sigma > \ell, \quad (3.24)$$

the assertion (iv) of the theorem also follows at once.

The set $\{\sigma \mid \ell < \sigma < \infty\}$ can be divided into a sequence of intervals in which

$$v(\sigma) = \text{a constant}, \quad (3.25)$$

$$v(\sigma) (\log^{[q-1]}_{\sigma})^{\rho} = \text{a constant}, \quad (3.26)$$

or

$$v(\sigma) = \frac{\log(a^{-1} \log^{[p-2]}_{M(\sigma)})}{(\log^{[q-1]}_{\sigma})^{\rho}} - \delta, \quad (3.27)$$

and since $M(\sigma)$ is differentiable in adjacent intervals, the same is obviously true for $v(\sigma)$ and hence also for $T(\sigma)$. It follows from (3.25), (3.26), and (3.27) [or (3.21)] that

$$v'(\sigma) = 0, \quad (3.28)$$

$$v'(\sigma) = - \frac{\rho v(\sigma)}{\Lambda_{[q-1]}(\sigma)}, \quad (3.29)$$

or

$$v'(\sigma) = - \frac{\rho v(\sigma)}{\Lambda_{[q-1]}(\sigma)} + \frac{\mu'(\sigma)}{\mu(\sigma) (\log^{[q-1]}_{\sigma})^{\rho}}, \quad (3.30)$$

where $\Lambda_{[m]}(x)$ is defined by (2.1). Since

$$v(\sigma) = \sup_{\ell \leq t \leq \sigma} \left\{ \frac{\log \mu(t)}{(\log [q-1]_t)^\rho} \right\} \quad (\sigma > \ell) \quad (3.17)$$

is a negative and non-decreasing function, and

$$v(\sigma) \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty.$$

If, for some $\sigma_n \in (\alpha_n, \beta_n)$, we have

$$v(\sigma_n) = \frac{\log \mu(\sigma_n)}{(\log [q-1]_{\sigma_n})^\rho} = \frac{\log \mu(\beta_n)}{(\log [q-1]_{\sigma_n})^\rho}, \quad (3.18)$$

we must also have

$$v(\sigma) = \frac{\log \mu(\beta_n)}{(\log [q-1]_\sigma)^\rho} \quad \text{for } \sigma_n \leq \sigma \leq \beta_n. \quad (3.19)$$

Equation (3.19) holds true, in particular, for $\sigma = \beta_n$. Now the set of σ for which

$$v(\sigma) = \frac{\log \mu(\sigma)}{(\log [q-1]_\sigma)^\rho} \quad (3.20)$$

is necessarily unbounded, and we have shown that this set cannot be a subset of the set of open intervals $\{(\alpha_n, \beta_n)\}$. Hence

$$v(\sigma) = \frac{\log \mu(\sigma)}{(\log [q-1]_\sigma)^\rho} = \frac{\log(a^{-1} \log [p-2]_{M(\sigma)})}{(\log [q-1]_\sigma)^\rho} - \delta \quad (3.21)$$

for an unbounded set of σ .

For the case under discussion,

$$T(\sigma) = v(\sigma) + \delta, \quad (3.22)$$

and hence we have

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or

$$v(\sigma) = \frac{\log(a^{-1} \log^{[p-2]} M(\sigma))}{(\log^{[q-1]} \sigma)^\rho} - \delta, \quad (3.27)$$

and since $M(\sigma)$ is differentiable in adjacent intervals, the same is obviously true for $v(\sigma)$ and hence also for $T(\sigma)$. It follows from (3.25), (3.26), and (3.27) [or (3.21)] that

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or

$$v'(\sigma) = - \frac{\rho v(\sigma)}{\Lambda_{[q-1]}(\sigma)} + \frac{\mu'(\sigma)}{\mu(\sigma) (\log^{[q-1]} \sigma)^\rho}, \quad (3.30)$$

where $\Lambda_{[m]}(x)$ is defined by (2.1). Since

$$T'(\sigma) \equiv v'(\sigma), \quad (3.31)$$

in view of (3.22), the assertion (iii) of the theorem is easily proved for (3.25) and (3.28), and for (3.26) and (3.29) in which case

$$\frac{\Lambda_{[q-1]}(\sigma)T'(\sigma)}{T(\sigma)} = -\frac{\rho v(\sigma)}{T(\sigma)} \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty, \quad (3.32)$$

by the definition (3.26). The assertion (iii) of the theorem holds true also for (3.27) [or (3.21)] and (3.30), since (3.22) and (3.30) imply that

$$0 \leq \frac{\Lambda_{[q-1]}(\sigma)T'(\sigma)}{T(\sigma)} \leq -\frac{\rho v(\sigma)}{v(\sigma) + \delta} \rightarrow 0 \quad \text{as } \sigma \rightarrow \infty, \quad (3.33)$$

where we have also used the facts that $\mu'(\sigma) \leq 0$ and $v'(\sigma) \geq 0$, and that $v(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$.

The assertion (iii) of the theorem can similarly be established for $T'(\sigma-0)$ or $T'(\sigma+0)$ when these derivatives are unequal.

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