ACCEPTED
FACULTY OF GRADUATE STUDIES
[redacted]
DEAN
DATE May 26, 86

ROTATION NUMBERS FOR UNIONS OF STARS

by

GARY MACGILLIVRAY
B.Sc. University of Victoria, 1985

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the department
of
Mathematics

We accept this thesis as comforming
to the required standard

[redacted]

E.J. Cockayne

[redacted]

D.J. Miller

[redacted]

H.P. Smith

[redacted]

J.A. Ellis

[redacted]

F. Ruskey

Supervisor: E J Cockayne

ABSTRACT

Let G be a p-vertex graph which is rooted at x. Informally, the *rotation number* $h(G,x)$ is the smallest number of edges in a p-vertex graph F such that, for every vertex y of F, there is a copy of G in F with x at y.

Firstly, we characterize all rooted graphs (G,x) such that $h(G,x) = q(G)$, the number of edges in G. We also establish necessary and sufficient conditions under which $h(G,x) = \binom{p}{2}$. Secondly, rotation numbers are calculated for the graph S which is a union of two stars $K_{1,n}$ and $K_{1,m}$ where $n \geq m$. Each of the four possibilities for the root x is considered. Thirdly, we extend the idea of rotation numbers to graphs rooted at an edge by defining the *edge rotation number* $h'(G,e)$. For these numbers we characterize all edge rooted graphs (G,e) such that $h'(G,e) = q(G)$ and establish necessary and sufficient conditions under which $h'(G,e) = \binom{p}{2}$. Edge rotation numbers are calculated for the graph S mentioned above and the graph which is a union of two complete graphs. In both cases, both possibilities for the root are considered. Finally, we examine some properties of rooted subgraphs and describe a backtracking algorithm which, given two rooted graphs (G,x) and (F,y), decides if (G,x) is a rooted subgraph of (F,y).

[redacted]

E J Cockayne

[redacted] [redacted]

[redacted] H P Smith

[redacted] [redacted]

I A Ellis F Ruskey

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

A *rooted graph* is an ordered pair (G, x) where G is a graph and x is a vertex of G called the *root*. Let F be a graph and (G, x) be a rooted graph. We say, informally, that (G, x) is a *homogeneous rooted subgraph* of F if for every $y \in V(F)$ there is a copy of G in F with x at y. This property is denoted by $(G, x) < F$.

The *rotation number* $h(G, x)$ of the rooted graph (G, x) with p vertices is the smallest number of edges in a p-vertex graph F such that $(G, x) < F$. A graph for which this minimum is achieved is called an *extremal graph*.

Rotation numbers were first defined by Cockayne and Lorimer [5], who were motivated by [7]. Since their introduction, rotation numbers and extremal graphs have been calculated for complete bipartite graphs [1, 5, 9], unions of circuits [2, 4] and lollipops [10]. It is the purpose of this thesis to contribute to the study of rotation numbers.

In Chapter 2 a summary of the basic graph theory used in this thesis is presented. Rotation numbers are formally defined, as is the analagous concept of edge rotation numbers. The latter is an extension of the former to graphs rooted at an edge.

Chapter 3 presents a survey of previous work on rotation numbers, namely the calculation of the extremal graphs and numbers for complete bipartite graphs, unions of circuits and lollipops. The connection between rotation numbers and minimum broadcast graphs is discussed. We show that minimum broadcast graphs are also related to edge rotation numbers.

All graphs G such that G is an extremal graph for (G, x) are characterized in Chapter 4, as are all rooted graphs (G, x) for which K_p is an extremal graph. The corresponding results for edge rooted graphs are also proven.

In Chapter 5 we study rotation numbers for unions of two stars (i.e. $G = K_{1,m} \bigcup K_{1,n}$ where $n \geq m$). When G is not rooted at the center of K_m, we are able to calculate all such numbers and give extremal graphs. For the case when the root x is the center of K_m, we are able to calculate rotation numbers and extremal graphs when n=m and $n \geq 3m$. Some extensions of these results are also mentioned.

In Chapter 6 we direct our attention to edge rotation numbers. These numbers and extremal graphs are calculated for unions of two stars and unions of two complete graphs.

Finally, in Appendix A we discuss the problem (G, x) < F. Some general theorems are proven, the complexity of the problem is examined and an algorithm is presented which, given (G, x) and F, decides if (G, x) < F.

CHAPTER 2

PRELIMINARIES

It is the purpose of this chapter to present a summary of the basic graph theory used in this thesis. For a more detailed discussion of this material, the reader is referred to Bondy and Murty [3] or Harary [8].

A *graph* is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is a finite set of *vertices* and $E(G)$ is a set of unordered pairs of distinct vertices. The elements of $E(G)$ are called *edges*. Throughout this chapter, let G and H be graphs.

The number of vertices of G is denoted by $p(G)$ and the number of edges of G is denoted by $q(G)$. When no confusion can result, we sometimes write p and q for $p(G)$ and $q(G)$, respectively.

A *rooted graph* is an ordered pair (G, x) where G is a graph and x is a vertex of G called the *root*. Similarly, an *edge-rooted graph* is an ordered pair (G, e) where G is a graph and e is an edge of G called the *root*.

An edge $[u, v] \in E(G)$ is said to be *incident* with the vertices u and v, which are referred to as the ends of e; the vertices u and v are also said to be *incident* with the edge e. Two vertices are *adjacent* if they are incident with a common edge, as are two edges incident with a common vertex. Let $S \subseteq V(G)$ and $v \in V(G)$. We say that v is *adjacent* to S if v is adjacent to every vertex of $S - \{v\}$. Two subsets S_1 and S_2 of $V(G)$ are *adjacent* if each vertex in S_1 is adjacent to S_2.

For subsets S_1 and S_2 of $V(G)$, we denote by $[S_1, S_2]$ the set of all edges of G with one end in S_1 and the other end in S_2. A subset X of $V(G)$ is called an *independent set* if $[X, X] = \emptyset$ (i.e. no edge of G has both ends in X).

The *line graph* $L(G)$ of a graph G has vertex set $E(G)$ and edge set

$$E(L(G)) = \{ [e_1, e_2] \mid e_1 \text{ and } e_2 \text{ are adjacent in } G \}$$

Two graphs G and H are *isomorphic* if there exists a 1-1 onto function $f: V(G) \rightarrow V(H)$ such that $[u, v] \in E(G)$ if and only if $[f(u), f(v)] \in E(H)$. Such a function f is called an *isomorphism*. An *automorphism* of G is an isomorphism from G to itself. The set of all automorphisms of G, together with the operation of function composition, is a group, called the *automorphism group* of G and denoted by $\Gamma(G)$. A graph G is *vertex-transitive* if for any two vertices $u, v \in V(G)$, there is an automorphism $\alpha \in \Gamma(G)$ such that $\alpha(u) = v$. G is *edge-transitive* if, for any two edges $[u, v]$ and $[x, y]$ of G, there is an automorphism $\beta \in \Gamma(G)$ such that $\beta(\{u, v\}) = \{x, y\}$.

If $[u, v] \in E(G)$ for every pair of distinct vertices $u, v \in V(G)$, G is called a *complete graph*. We write K_n to denote the unique (up to isomorphism) complete graph on n vertices. If $V(G)$ can be partitioned into nonempty subsets X and Y such that each edge of G has one end in X and the other in Y, G is called a *bipartite graph*. Such a partition (X, Y) is called a *bipartition of G* (note that X and Y are independent sets). When X and Y are adjacent, G is called a *complete bipartite graph*. We denote the unique (up to isomorphism) complete bipartite graph with $|X| = m$ and $|Y| = n$ by $K_{m,n}$. A *circulant graph* is a graph G such that the cyclic group $Z_{p(G)}$ is a subgroup of $\Gamma(G)$.

A graph H is a *subgraph* of G if there exists a 1-1 function $f: V(H) \rightarrow V(G)$ such that $[u, v] \in E(H)$ implies $[f(u), f(v)] \in E(G)$. If H is a subgraph of G, we call G a *supergraph* of H. Let (G, x) and (F, y) be rooted graphs. We say that (G, x) is a *rooted subgraph* of (F, y) if G is a subgraph of F and, additionally, $f(x) = y$. We denote this property by $(G, x) < (F, y)$. When we want to emphasize that this is accomplished under the function f, we write $(G, x) <^f (F, y)$. If $(G, x) < (F, y)$ for every $y \in V(F)$ then we say (G, x) is a *homogeneous rooted subgraph* of (F, y) and write $(G, x) < F$. Similarly, if $(G, [u, v])$ and $(F, [x, y])$ are edge rooted graphs we say $(G, [u, v])$ is an *edge rooted subgraph* of $(F, [x, y])$ if G is a subgraph of F and additionally, $f(\{u, v\}) = \{x, y\}$. This property is denoted by $(G, [u, v]) << (F, [x, y])$. When we want to emphasize that this is accomplished under the function f, we write $(G, [u, v]) <<^f (F, [x, y])$. If F is a supergraph of G and $(G, e) << (F, c)$ for every $c \in E(F)$, we say (G, e) is a *homogeneous edge-rooted subgraph* of (F, c) and write $(G, c) << F$.

Let $S \subseteq V(G)$. The *subgraph of* G *induced by* S is the subgraph of G whose vertex set is S and whose edge set os the set of all edges in E(G) which have both ends in S; it is denoted G[S]. Similarly, if $D \subseteq E(G)$, the *subgraph of* G *induced by* D is the subgraph of G with edge set D and whose vertex set is the set of all vertices of G incident with at least one edge in D; it is denoted G[D]. Let H be a subgraph of G. We write G-H to denote the subgraph of G with vertex set V(G) and edge set E(G)-E(H).

Let G_1 and G_2 be subgraphs of G. We say that G_1 and G_2 are *vertex disjoint* if $V(G_1) \bigcap V(G_2) = \emptyset$ and *edge disjoint* if $E(G_1) \bigcap E(G_2) = \emptyset$.

Suppose that G and H have disjoint vertex sets (and hence disjoint edge sets). The *union* of G and H, denoted $G \bigcup H$, has vertex set $V(G) \bigcup V(H)$ and edge set $E(G) \bigcup E(H)$. We sometimes abbreviate $G \bigcup G$ to 2G etc. The *join* of G and H is denoted G+H and is formed from $G \bigcup H$ by adding all possible edges with one end in V(G) and the other in V(H).

The *degree* $d_G(v)$ *of a vertex* $v \in V(G)$ is the number of vertices of G to which v is adjacent. We sometimes refer to a vertex of degree k as a *k-vertex*. The ***degree*** $d_G(e)$ *of an edge* $e \in E(G)$ is the number of edges of G with which e is adjacent.

The following result is sometimes called the handshaking lemma - because when a number of people shake hands, the total number of hands shaken is always even.

Remark 2.1. $\sum_{v \in V(G)} d_G(v) = 2q(G)$

Corollary 2.2. In any graph, the number of vertices of odd degree is even.

The *average degree* of G equals $\dfrac{2q(G)}{p(G)}$.

A graph G is *k-regular* if every vertex of G has degree exactly k, and *almost k-regular* if p(G)-1 vertices have degree k and the remaining vertex has degree k+1. If G is k-regular (almost k-regular) for some k, we say that G is *regular (almost regular)*.

A *path* in G is a finite nonempty sequence of distinct vertices $P = v_0, v_1, \ldots, v_n$ such that $[v_{i-1}, v_i] \in E(G)$, $1 \leqslant i \leqslant n$. The vertices v_0 and v_n are called *the origin* and *the terminus* of P, respectively. The integer n is called the *length* of P.

Two vertices u and v of G are *connected* if there is a path in G with origin u and terminus v. Connection is an equivalence relation and thus partitions V(G) into nonempty subsets $V_1, V_2, \ldots, V_m$ such that two vertices u and v are in V_i if and only if they are connected. The subgraphs $G[V_1], G[V_2], \ldots, G[V_m]$ are called the *components* (or *connected components*) of G. We say that G is *connected* if it has exactly one component, otherwise we say G is *disconnected*.

A *circuit* or *cycle* in G is a finite sequence of vertices $C = v_0, v_1, \ldots, v_n, v_0$ such that the v_i are distinct and $[v_i, v_{i+1}] \in E(G)$, $0 \leqslant i \leqslant n$ (where addition is modulo n+1). The integer n is the *length* of C and the circuit C is sometimes called an *n-circuit* and denoted by C_n. A p(G)-circuit is called a *hamilton circuit* and a graph which has a hamilton circuit is referred to as a *hamiltonian graph*. A *tree* is a connected graph which contains no circuits.

The following result characterizes those values of n for which K_n is in some sense composed entirely of hamilton circuits.

Theorem 2.3. K_n is an edge disjoint union of hamilton circuits if and only if n is odd.

The next theorem provides a useful characterization of bipartite graphs.

Theorem 2.4. A graph G is bipartite if and only if it contains no circuits of odd length.

Let $v \in V(G)$. The *open neighbourhood* of v, denoted $N_G(v)$, is the set of all vertices adjacent to v. The *closed neighbourhood* of v is $N_G[v] = N_G(v) \bigcup \{v\}$.

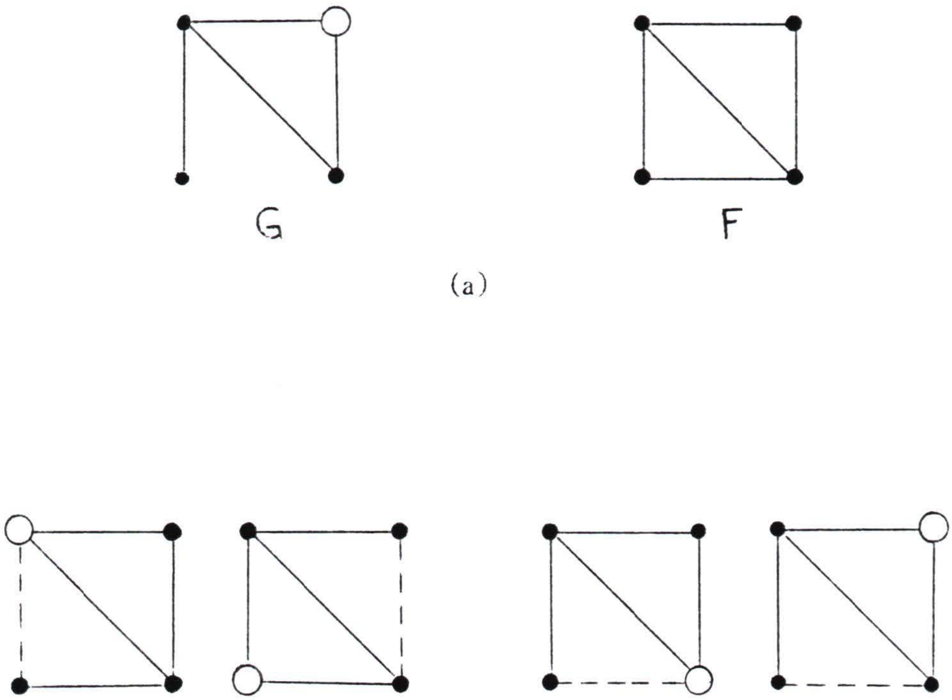
A *matching* is a subset M of E(G) such that no two of its elements are adjacent in G; the two ends of an edge in M are said to be *matched*. If $V(G[M]) = V(G)$, we say that M is a *perfect matching*. A perfect matching is also called a *1-factor*. A graph G is *1-factorable* if there are edge disjoint 1-factors $F_1, F_2, \ldots, F_n$ such that $G = F_1 \bigcup F_2 \bigcup \cdots \bigcup F_n$. $F_1, F_2, \ldots, F_n$ is called a *1-factorization of G*. The following theorem tells when K_n is 1-factorable.

Theorem 2.5. K_n is 1-factorable if and only if n is even.

The *vertex rotation number* (or *rotation number*) $h(G, x)$ of the p-vertex rooted graph (G, x) is defined to be the smallest number of edges in a p-vertex graph F such that $(G, x) < F$. Such a graph F is called an *extremal* (G, x) *graph*.

In figure 2.1 (a) we show a rooted graph (G, x) and a graph F such that $(G, x) < F$. Figure 2.1 (b) shows the copies of (G, x) rooted at each vertex of F. It is easy to show that $h(G, x) = 5$.

The above concepts have an analog for edge rooted graphs. The *edge rotation number* $h(G, e)$ of the p-vertex edge rooted graph (G, e) is defined to be the smallest number of edges in a p-vertex graph F such that $(G, e) << F$. Such a graph F is called



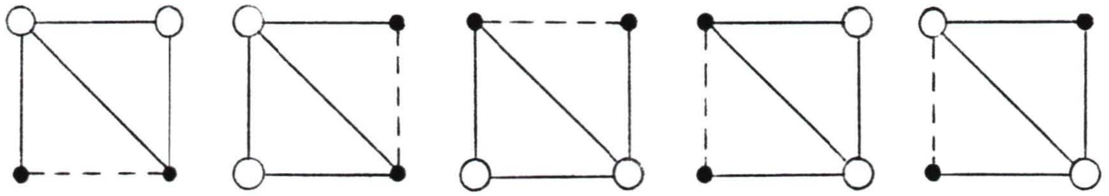
(b)

Figure 2.1 An example of $(G, x) < F$.

an *extremal* (G, e) *graph*.

In figure 2.2 (a) we show an edge rooted graph (G, e) and a graph F such that $(G, e) \ll F$. Figure 2.2(b) shows the copies of (G, e) rooted at each edge of F. It is easy to show that $h'(G, e) = 5$.

(a)



(b)

Figure 2.2. An example of $(G, e) \ll F$

CHAPTER 3

PREVIOUS WORK

It is the purpose of this chapter to survey previous work on rotation numbers. These numbers were first defined by Cockayne and Lorimer [5], whose motivation came from a special case of the problem studied by Farley et al. [7]. Since their introduction, rotation numbers and extremal graphs have been calculated for complete bipartite graphs [1, 4, 9], unions of circuits [2, 4] and lollipops [10]. We survey the results for each of these classes of graphs and the connection between rotation numbers and minimum broadcast graphs. In the latter section, we show that minimum broadcast graphs are also related to edge rotation numbers. No proofs which appear elsewhere are included in this chapter.

3.1. Minimum Broadcast Graphs

Consider the problem of information dissemination in a communication network (graph) whereby a single message known initially to one member (*the source*) is communicated to all members (vertices). This is to be accomplished by a sequence of phone calls over the edges of the network, subject to:

(i) Each phone call requires one unit of time.

(ii) A member may participate in at most one phone call per time unit.

(iii) A member may only call an adjacent member.

We refer to this process as broadcasting.

For a p-vertex graph, at least $\lceil \log_2 p \rceil$ time units are required to broadcast from any source. A *minimum broadcast graph* is a p-vertex graph with the smallest number of edges such that broadcasting from any source takes $\lceil \log_2 p \rceil$ time units.

Broadcasting from source u defines a rooted tree (T, u). When $p = 2^k$ there is only one such tree, (T_k, u), which is defined recursively as follows: $(T_1, u) = (K_2, u)$. For $k > 1$, let (R_{k-1}, r) and (S_{k-1}, s) be two disjoint copies of (T_{k-1}, u). Then

$$(T_k, u) = (((R_{k-1}, r) \bigcup (S_{k-1}, s)) + [r, s], r)$$

Note that, in order to broadcast in time k the first call must use the edge $[r, s]$.

If G has 2^k vertices, broadcasting from source x is possible in time k if and only if $(T_k, u) < (G, x)$. It follows that broadcasting in time k is possible from any source if and only if $(T_k, u) < G$ and the number of edges in a minimum broadcast graph is $h(T_k, u)$.

Theorem 3.1.1 [Farley, Hedetniemi, Mitchell and Proskurowski, 1979].
$h(T_k, u) = k2^{k-1}$

In the proof of theorem 3.1.1, a family $\{G_k\}$ of minimum broadcast graphs such that $q(G_k) = k2^{k-1}$ and $(T_k, u) < G_k$ is constructed. A slightly specialized version of their construction is the following: $G_1 = K_2$. For $k > 1$, $G_k = 2G_{k-1}$, together with a set of edges which forms a perfect matching between corresponding vertices in the two copies of G_{k-1} (i.e. the 1-1 onto mapping induced by these edges is an isomorphism). The rooted tree (T_3, u) and the graph G_3 are shown in figure 3.1.1.

If we root T_k at $[u, v]$ (see figure 3.1.1) then $(T_k, [u, v]) << G_k$ and G_k has the minimum number of edges with respect to this property. In other words,

Theorem 3.1.2. $h(T_k, [u, v]) = k2^{k-1}$, and G_k is an extremal $(T_k, [u, v])$ graph.

Proof. Let F be an extremal $(T_k, [u, v])$ graph. It suffices to show that $k2^{k-1}$ is the smallest number of edges required so that broadcasting from any source is possible in time k, regardless of over which edge the initial call is placed.

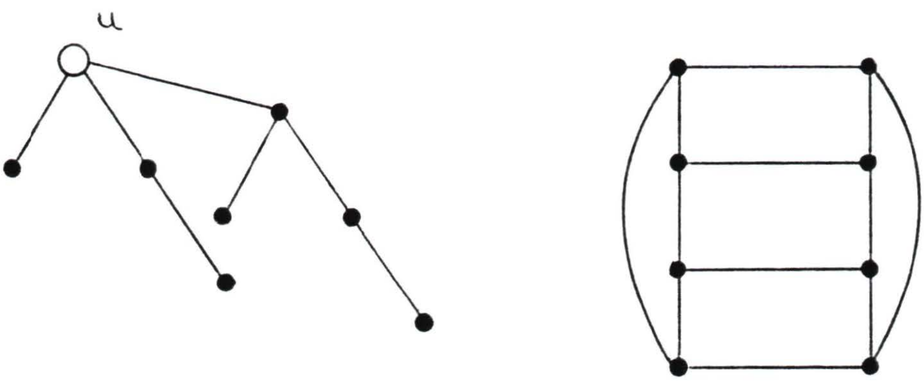


Figure 3.1.1 The graphs (T_3, u) and G_3

For broadcasting to be possible in time k, the number of "informed" vertices must double at each time unit. Therefore, each vertex must have degree at least k. This establishes $k2^{k-1}$ as a lower bound for $q(F)$.

The proof will be complete if we can show that $(T_k, [u, v]) \ll G_k$. We do this by induction on k.

Basis. If $k = 1$, $G_1 = K_2$ and the statement is true.

Induction. Assume that $(T_{k-1}, [u, s]) \ll G_{k-1}$. Consider $(T_k, [u, v]) \ll^{f} (G_k, e)$. If e is an edge of the perfect matching between corresponding vertices in the two copies of G_{k-1}, our problem is reduced to that of broadcasting in time k-1 from a vertex in each copy and this is possible by theorem 3.1.1. Otherwise, e joins a pair of vertices in the same copy of G_{k-1}. At time 2, let each of these vertices call the corresponding vertex in the other copy. Note that the two called vertices are adjacent by construction. The problem is now equivalent to broadcasting in time k-1 in each copy, where the first calls are placed over e and its corresponding edge. But this is possible by the induction hypothesis.

The result now follows. ■

Minimum broadcast graphs provided the motivation for the study of rotation numbers. Extremal (G, x) graphs are to general rooted graphs as minimum broadcast graphs on 2^k vertices are to the rooted trees (T_k, u).

3.2 Complete Bipartite Graphs

Let $G = K_{n,p-n}$ $(p>n)$ and let x be a vertex of the defining independent set of size n.

If F is a p-vertex graph such that $(G, x) < F$, every $v \in V(F)$ is in an n-subset $B_v \subseteq V(F)$ such that the bipartite subgraph of F induced by $[B_v, \bar{B}_v]$ is complete.

Let **X** be a class of n-subsets of $S = \{1, 2, \ldots, p\}$ such that $\bigcup \mathbf{X} = S$. We define the graph $F(\mathbf{X})$ to have vertex set S and

$$E(F(\mathbf{X})) = \{[a, b] \mid a\epsilon B, b\epsilon\bar{B} \text{ for some } B\epsilon\mathbf{X}\}$$

An extremal (G, x) graph is isomorphic to a graph $F(\mathbf{X})$ with the smallest number of edges, where the minimum is over all classes **X**.

An edge $[a, b] \in E(\overline{F(\mathbf{X})})$ if and only if for each $B \in \mathbf{X}$ we have $a \in B$ iff $b \in B$. This induces an equivalence relation on S and it follows that $\overline{F(\mathbf{X})}$ is a disjoint union of complete subgraphs. It is therefore convenient to tackle the problem of maximizing $q(\overline{F(\mathbf{X})})$ instead of the equivalent problem of minimizing $q(F(\mathbf{X}))$.

Rotation numbers for complete bipartite graphs were first calculated by Cockayne and Lorimer [5].

Theorem 3.2.1 [Cockayne and Lorimer, 1980]. Let $p = nq+r$ where $r = 0, 1$ or $n-1$. Then the complement of $(q-1)K_n \bigcup 2K_r \bigcup K_{n-r}$ is the unique extremal graph.

Corollary 3.2.2 [Cockayne and Lorimer, 1980]. The conclusion of theorem 3.2.1 holds for $n = 1, 2, 3$.

The only cases in which $n = 4$ or 5 not covered by theorem 3.2.1 are $p = 4q+2$, $p = 5q+2$ and $p = 5q+3$. The extremal graphs for these cases are given in the next two results.

Theorem 3.2.3 [Cockayne and Lorimer, 1980]. Let $p = 4q+2$.

(a) If $q = 1$ $(G = K_{4,2})$ there are exactly two extremal graphs, namely K_6-3K_2 and

K_6-K_3.

(b) If $q = 2$ $(G = K_{4,6})$ there are three extremal graphs, namely the complements of

$3K_2 \bigcup K_4$, $K_4 \bigcup K_3 \bigcup 3K_1$ and $3K_3 \bigcup K_1$.

(c) If $q > 2$ then the complement of $(q-2)K_4$ together with any of the three graphs in

(b) is an extremal graph.

Theorem 3.2.4 [Cockayne and Lorimer, 1980].

(a) If $p = 5q+2$ then the unique extremal graph is the complement of

$(q-1)K_5 \bigcup K_4 \bigcup K_1$.

(b) If $p = 5q+3$ then the unique extremal graph is the complement of

$$\begin{array}{ll} 2K_3 \bigcup K_2 & q = 1 \\ (q-2)K_5 \bigcup 3K_4 \bigcup K_1 & q \geqslant 2 \end{array}$$

The above work was greatly extended by Bollobas and Cockayne [1], who obtained the following results.

Theorem 3.2.5 [Bollobas and Cockayne, 1982]. If $n \leqslant p \leqslant 2n$ then the extremal graphs are the complements of

$$K_{n-1} \bigcup (r+1)K_1 \qquad r \leqslant \frac{2}{3}(n-1)$$

$$2K_r \bigcup K_{n-r} \qquad r \geqslant \frac{2}{3}(n-1)$$

Let $p = nq+r$ where $0 \leqslant r < n$. All cases in which $q = 1$ are covered by theorem 3.2.5, so assume $q \geqslant 2$. We define the graphs

$$G_1 = (q-1)K_n \bigcup K_{n-1} \bigcup (r+1)K_1$$

$$G_2 = \begin{cases} (q-n+r)K_n \bigcup (n-r+1)K_{n-1} \bigcup K_1 & n-r \leqslant q \\ (q+1)K_{n-u_0} \bigcup K_{u_0} & n-r > q \end{cases}$$

where $u_0 = \left\lfloor \dfrac{n-r}{q} \right\rfloor$. Define the functions

$$\lambda(u) = (r+2u)(n-u)$$

$$\mu(u) = (n-r+2u)(n-u)$$

Theorem 3.2.6 [Bollobás and Cockayne, 1982]. For $q \geqslant 2$, $n \geqslant 6$

(a) If $\lambda(1) < \mu(u_0)$ then $\overline{G}_1$ is the unique extremal graph.

(b) If $\dfrac{n-r}{q}$ is integral and $\lambda(1) > \mu(u_0)$ then $\overline{G}_2$ is the unique extremal graph.

(c) If $\dfrac{n-r}{q}$ is integral and $\lambda(1) = \mu(u_0)$ then $\overline{G}_1$ and $\overline{G}_2$ are the only two extremal graphs.

Hell and Haagvist [9] have calculated the rotation numbers and extremal graphs for all cases not covered by the above results.

3.3. Unions of Circuits

Let $G = C_k \bigcup C_{p-k}$ where $k < \frac{p}{2}$ and the root x is a vertex of C_k (the case where x is on C_{p-k} is trivial). If $(G, x) < F$, every $v \in V(F)$ is in a k-circuit whose deletion from F leaves a hamiltonian graph. An extremal (G, x) graph is a p-vertex graph with the smallest number of edges and the above property

The case where $k = 3$ has been settled by Cockayne and Fang [4] who proved the following theorem

Theorem 3.3.1 [Cockayne and Fang, 1983] Let $k=3$ and $p>9$

(a) If $p \equiv 0 \pmod 3$ then $h(G, x) = \left\lfloor \frac{9p+1}{2} \right\rfloor$

(b) If $p \equiv 1 \pmod 3$ then $h(G, x) = \left\lfloor \frac{9p+8}{2} \right\rfloor$

(c) If $p \equiv 2 \pmod 3$ then $h(G, x) = \left\lfloor \frac{9p+6}{2} \right\rfloor$

(d) $h(C_3 \bigcup C_5, x) = 14$ and $h(C_3 \bigcup C_5, x) = 15$

Extremal graphs for the case where 3 divides p are shown in figure 3.3.1. The reader is referred to the paper by Cockayne and Fang for the others.

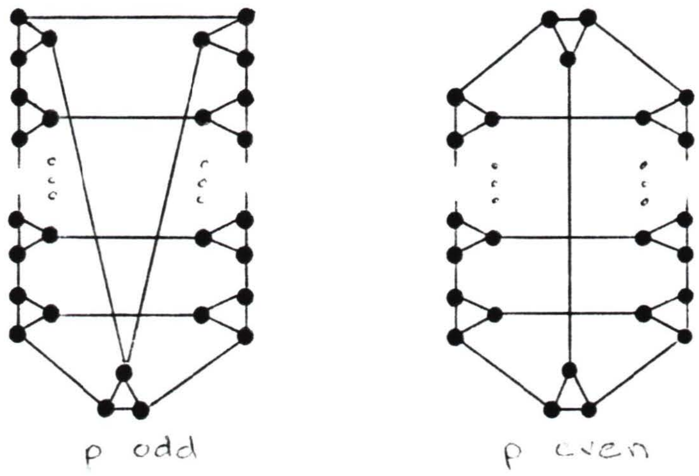


Figure 3.3.1. Extremal $(C_3 \bigcup C_{p-3}, x)$ graphs for the case when $3|p$

The above work has been extended by Bollobas, Cockayne and Fang [2]. They consider the case $G = C_k \bigcup C_r$ where $4 \leqslant k < r$ and the root x is on C_k. A lower bound is established for the rotation number and an infinite class of graphs which achieve the bound is exhibited. The bound is used to establish rotation numbers for the cases $k = 4$, $p \not\equiv 3$ or 4 (mod 7).

Theorem 3.3.2 [Bollobas, Cockayne and Fang, 1984]. Let $p \geqslant 2k+1$ ($k \geqslant 4$) and $(G, x) < F$. Then the average degree of F is at least $2 + \dfrac{6}{2k-1}$.

If $(2k-1) \mid p$, the graph F depicted in figure 3.3.2 satisfies $(G, x) < F$ and achieves the lower bound of theorem 3.3.2. This is sufficient to prove

Theorem 3.3.3 [Bollobas, Cockayne and Fang, 1984]. If $k \geqslant 4$ and $(2k-1) \mid p$ then

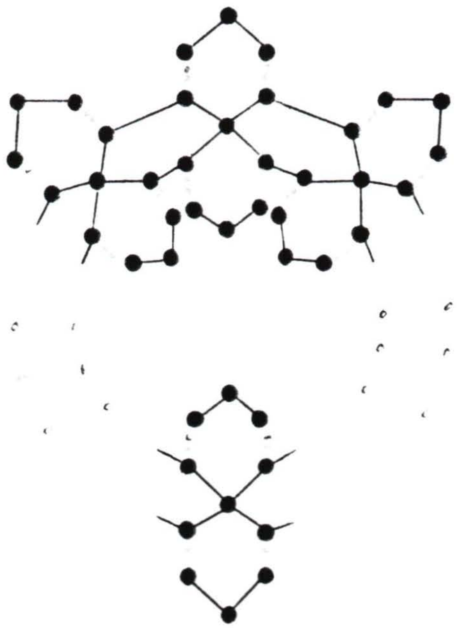


Figure 3.3.2. An extremal $(C_k \bigcup C_{p-k}, x)$ graph for the case when $(2k-1) \mid p$

$$h(C_k \bigcup C_{p-k}, x) = p\,(1 + \frac{3}{2k-1})$$

Bollobás, Cockayne and Fang conjecture that, if $(2k-1)$ does not divide p, the extremal graphs consist of about $\frac{p}{2k-1}$ pairs of k-circuits joined at a single vertex together with some extra edges to compensate for the indivisibility. This is the case when $k = 4$.

Theorem 3.3.4 [Bollobas, Cockayne and Fang, 1984]. Let $q \geqslant 2$. Then

(a) $p = 7q+1$ implies $h(C_4 \bigcup C_{p-4}, x) = 10q+2$

(b) $p = 7q+2$ implies $h(C_4 \bigcup C_{p-4}, x) = 10q+4$

(c) $p = 7q+5$ implies $h(C_4 \bigcup C_{p-4}, x) = 10q+8$

(d) $p = 7q+6$ implies $h(C_4 \bigcup C_{p-4}, x) = 10q+9$

In the case where $p \equiv 3$ or $4 \pmod 7$, lower bounds for $h(C_4 \bigcup C_{p-4}, x)$ have been calculated but no extremal graphs have been found. However, Bollobas, Cockayne and Fang report that they have constructed graphs F with $(G, x) < F$ and only one more edge than the bound specifies.

3.4. Lollipops

Let L_n be the n-vertex lollipop graph, i.e. a vertex a of degree one joined to an (n-1)-cycle $b_0, b_1, \ldots, b_{n-2}, b_0$.

Holyer and Cockayne [10] have calculated rotation numbers for (L_n, a), (L_n, b_0) and (L_n, b_1).

Theorem 3.4.1 [Holyer and Cockayne, 1984]. $h(L_n, b_0) = h(L_n, a) = \left\lceil \frac{3}{2} n \right\rceil$

It turns out that in both cases each vertex must have degree at least three. Extremal graphs, depending on the parity of n, are shown in figure 3.4.1. When L_n is

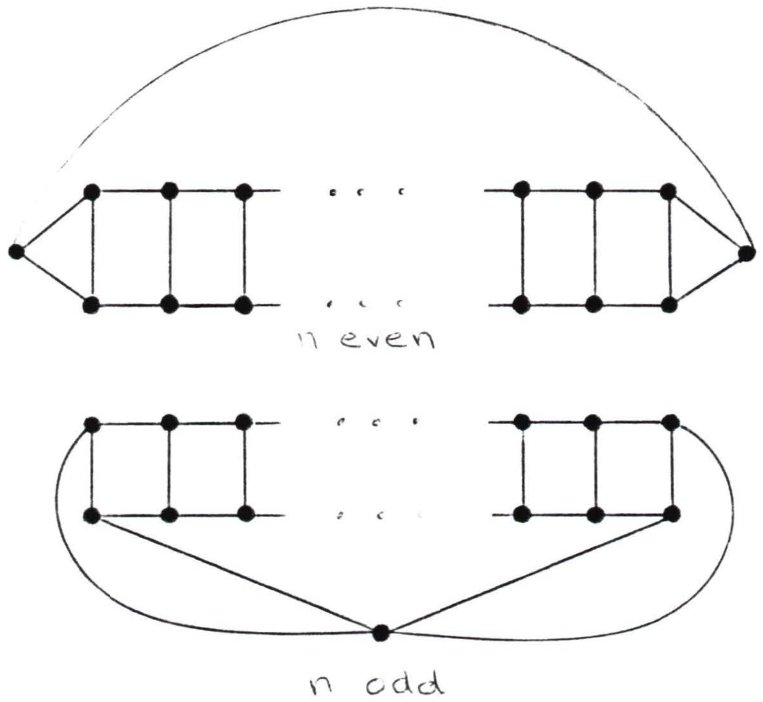


Figure 3 4 1 Extremal (L_n, b_0) and (L_n, a) graphs

rooted at b_1 the problem is much more challenging In this case Holyer and Cockayne prove that the average degree of F must be at least $\frac{18}{7}$ This result arises from a careful study of the number and arrangement of the vertices of degree two

Theorem 3 4 2 [Holyer and Cockayne 1984] Let $n \geqslant 5$

(a) If $n \not\equiv 3 \pmod 7$ then $h(L_n, b_1) = \left\lceil \frac{9}{7} n \right\rceil$

(b) If $n \equiv 3 \pmod 7$ then $\left\lceil \frac{9}{7} n \right\rceil \leqslant h(L_n, b_1) \leqslant \left\lceil \frac{9}{7} n \right\rceil + 1$

Extremal graphs for the cases $n \equiv 0, 3$ or $6 \pmod 7$ are shown in figure 3.4.2.

3.5. Conclusion.

In this chapter we have surveyed the work done to date on rotation numbers. Most of this work has focussed on calculating the numbers and extremal graphs for rooted graphs where vertices have two orbits (if there is only one orbit then G is vertex transitive and $(G, x) < G$). This is not the case for lollipop graphs, but there the orbits have order one or two.

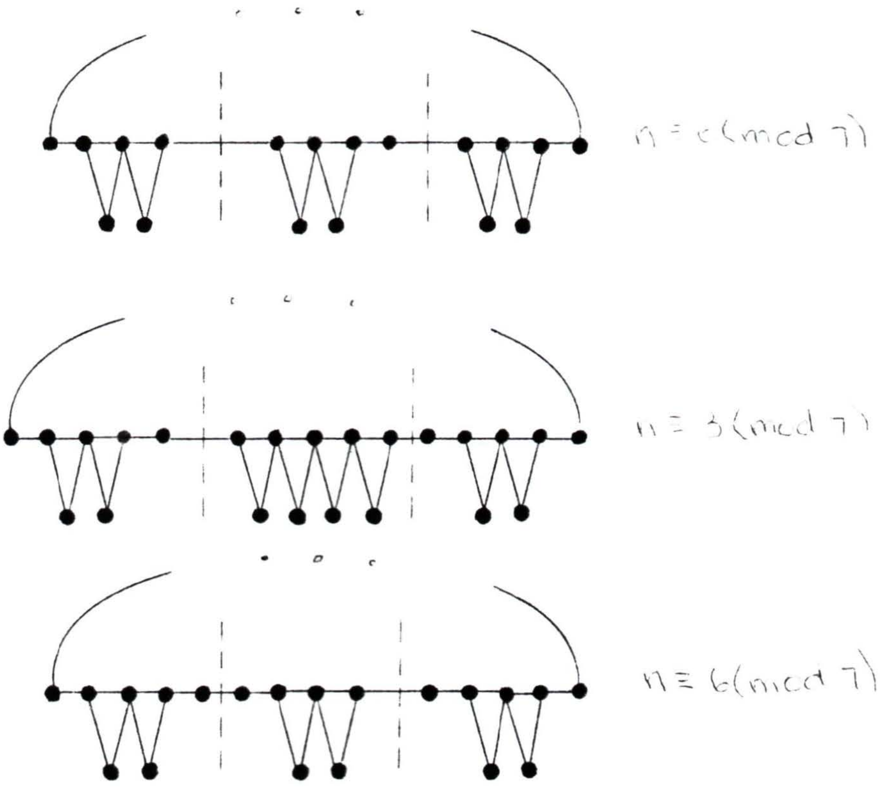


Figure 3.4.2. Some extremal (L_n, b_1) graphs.

Even for graphs with this simple structure, the calculation of rotation numbers appears to be a very difficult problem. Determining these numbers for each class of graphs seems to require a detailed examination of the properties particular to that class. This makes it difficult to believe that many results of general applicability will ever be proven. We conjecture that, given an arbitrary rooted graph (G, x), the problem of determining $h(G, x)$ is NP-complete (in appendix A we show that, given (G, x) and F, the problem of deciding if $(G, x) < F$ is NP-complete).

CHAPTER 4

COMPLETE EXTREMAL GRAPHS TRANSITIVITY

It is the purpose of this section to prove four results concerning vertex and edge rotation numbers All graphs such that G is an extremal (G x) graph are characterized as are all graphs such that K_p is an extremal (G x) graph The corresponding results for edge rooted graphs are also proven

It is clear that if G is vertex transitive $h(G\ x) = q(G)$ for any $x \in V(G)$ The converse is also true as we now show

Theorem 4 1 $h(G\ x) = q(G)$ if and only if G is vertex transitive

Proof Suppose first that $h(g\ x) = q(G)$ Then $(G\ x) < G$ Hence $(G\ x) <^f (G\ y)$ implies that the mapping on E(G) induced by f is 1-1 and onto since each edge of G is mapped 1-1 to an edge of G Hence $[f(u)\ f(v)] \in E(G)$ implies $[u\ v] \in E(G)$ Combining this with the definition of f we deduce that f is an automorphism

Let $a\ b \in V(G)$ We must find an automorphism g such that $g(a) = b$ Let $(G\ x) <^{\alpha} (G\ a)$ and $(G\ x) <^{\beta} (G\ b)$ Let g be the automorphism $\beta o \alpha^{-1}$ Then

$$g(a) = \beta(\alpha^{-1}(a)) = \beta(x) = b$$

as required Therefore G is vertex transitive

On the other hand if G is vertex transitive $(G\ x) < G$ by definition ■

Theorem 4 2 Let $p > 2$ K_p is an extremal (G x) graph if and only if either $d_G(x) = p-1$ or G-x is isomorphic to $\bar{K}_{p-1}$

Proof Suppose that $d_G(x) < p-1$ and G-x is not isomorphic to K_{p-1}. It follows that there is a vertex y nonadjacent to x and distinct vertices a and b such that $[a, b] \notin E[G]$. Let $F = K_p - [u, v]$. For each $z \in V(F)$, we will find f such that $(G, x) <^f (F, z)$. There are two cases to consider.

CASE 1 $(G, x) <^f (F, v)$

Let $f(y) = u$ and map the remaining vertices in any 1-1 fashion. Since the only pair of nonadjacent vertices in F is $\{u, v\}$, it follows that $(G, x) <^f (F, v)$.

CASE 2 $(G, x) <^f (F, w)$, $w \neq u, v$

Let $f(a) = u$, $f(b) = v$ and map the remaining vertices in any 1-1 fashion. Since the only pair of nonadjacent vetices in F is $\{u, v\}$, the result follows as above.

Both cases have been considered and the result now follows.

On the other hand, if $d_G(x) = p-1$ the conclusion is obvious, so assume G-x is isomorphic to K_{p-1}. Let F be an extremal (G, x) graph and $s, t \in V(F)$. Since $p > 2$, there is a vertex $w \in V(F)$, $w \neq s, t$. Consider $(G, x) <^f (F, w)$. Then $[s, t] \in E(F)$ because f(G-x) is complete. Hence all pairs of vertices in F are joined by an edge, i.e. F is complete. ■

The graph $\overline{K}_2$ shows that the condition $p > 2$ cannot be omitted (since $\overline{K}_2$-x is isomorphic to K_1 but $h(\overline{K}_2, x) = 0$).

We now prove the corresponding results for edge rooted graphs. It is clear that $h'(G, e) = q(G)$ for any edge e whenever G is edge transitive. The converse is also true.

Theorem 4.3 $h'(G, e) = q(G)$ if and only if G is edge transitive.

Proof Suppose first that $h'(G, e) = q(G)$. Then $(G, e) << G$. Therefore the mapping on E(G) induced by f is 1-1 and onto, since each edge of G is mapped 1-1 to an edge of G Hence $(G, e) <<^f (G, z)$ implies that f is a 1-1 onto function on E(G) and so $[f(u), f(v)] \in E(G)$ implies $[u, v] \in E(G)$ Combining this with the definition of f we deduce that f is an automorphism

Let $s, t \in E(G)$ We want to find an automorphism g such that $g(s) = t$ Let $(G, e) <<^{\alpha} (G, s)$ and $(G, e) <<^{\beta} (G, t)$ Consider the automorphism $g = \beta o \alpha^{-1}$ Then

$$g(s) = \beta(\alpha^{-1}(s)) = \beta(e) = t$$

as required Therefore G is edge transitive

On the other hand it follows directly from the definition of an edge transitive graph G that transitive graph that $h(G, e) = q(G)$ ∎

We conclude this section with a characterization of all edge rooted graphs (G, e) such that K_p is an extremal (G, e) graph

Theorem 4 4 Let $p > 4$ K_p is an extremal (G, e) graph if and only if either $d_G(e) = 2p-2$ or G-u-v is isomorphic to K_{p-2} where $e = [u, v]$

Proof Suppose that $d_G(e) < 2p-2$ and G-u-v is nonisomorphic to K_{p-2} It follows that there is a vertex $w \in V(G)$ which is nonadjacent to u, $w \neq v$ There are also vertices s and t in G-u-v such that $[s, t] \notin E(G)$ For each edge $e \in E(K_p - [x, y])$ we will find f such that $(G, e) <<^f (K_p - [x, y])$ thus proving that K_p is not an extremal graph There are two cases to consider

CASE 1 $(G, e) <<^f (K_p, [x, a])$

Let $f(u) = x$, $f(v) = a$ and $f(w) = y$ Map all remaining vertices in an arbitrary 1-1 fashion Since all edges except $[x, y]$ are present this case is OK

CASE 2 $(G, e) <<^f (K_p, [a, b])$, $a, b \neq x, y$

Let $f(u) = a$, $f(v) = b$, $f(s) = x$ and $f(t) = y$ Map all remaining vertices in an arbitrary 1-1 fashion Then as above we are done

Both cases have been considered and the result now follows.

On the other hand, if $d_G(e) = 2p-2$ the result is obvious, so suppose G-u-v is isomorphic to K_{p-2}. Let F be an extremal (G, e) graph. No generality is lost by assuming that the vertices of F are labelled identically to those of G. We must show that an arbitrary vertex w is adjacent to both u and v. Since $p > 4$ and F-u-v is complete, there are vertices s and t, distinct from u, v, and w, such that $[s, t] \in E(G)$. Consider $(G, e) <<^f (F, [s, t])$. Since f(G)-f(u)-f(v) is complete, and since $f(u) = s$ and $f(v) = t$, we deduce w is adjacent to both u and v. Therefore, F is complete. ■

The graph $2K_2$ shows that the hypothesis $p > 4$ cannot be omitted (since $2K_2$-e is isomorphic to K_2 but $h(2K_2, e) = 2 \neq 6$).

CHAPTER 5

ROTATION NUMBERS FOR UNIONS OF STARS

In this chapter we calculate rotation numbers for the graph $S = K_{1\,n} \bigcup K_{1\,m}$ $n \geqslant m$ This graph is illustrated in figure 5 0 1 and the vertex labels shown there are used throughout the chapter Each of the four possibilities for the root is considered Note that $p(S) = n+m+2$ Throughout this chapter F denotes a graph which has a minimal number of edges with respect to the constraints imposed upon it

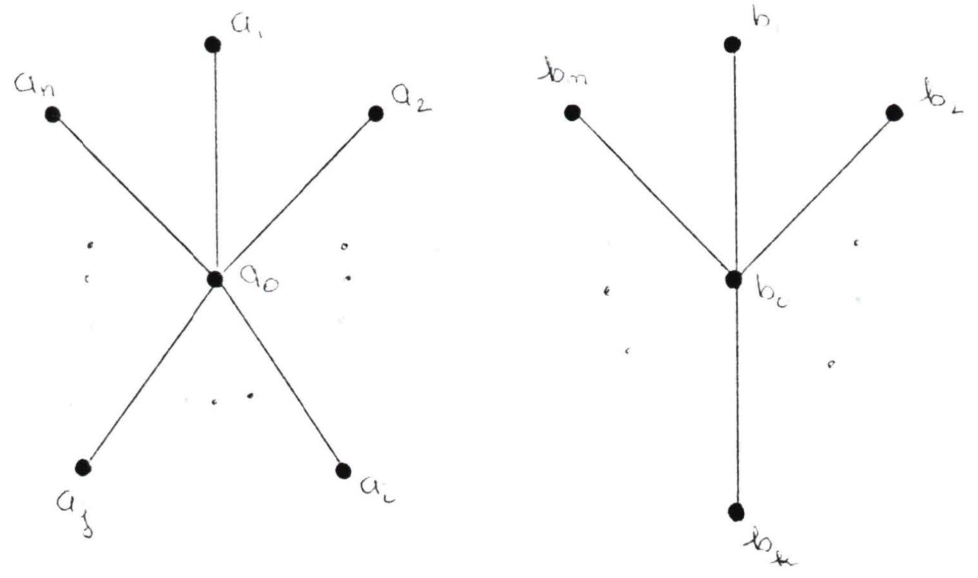


Figure 5 0 1 The graph $S = K_{1\,n} \bigcup K_{1\,m}$

5.1 Calculation of $h(S, a_0)$

Theorem 5.1.1. $h(S, a_0) = \left\lceil \frac{n(n+m+2)}{2} \right\rceil$

Proof. Since $(S, a_0) < (F, v)$ for every $v \in V(F)$, each vertex of F must have degree at least n. Hence $2q(F) \geq n(n+m+2)$ or $q(F) \geq \left\lceil \frac{n(n+m+2)}{2} \right\rceil$. To complete the proof of the theorem, we construct graphs for which this lower bound is achieved. There are three cases to consider.

CASE 1. n is even.

Let F be the n-regular circulant graph with vertex set $\{0, 1, \ldots, n+m+1\}$. Vertices i and j are joined if $i-\frac{n}{2} \leqslant j \leqslant i+\frac{n}{2}$ (where addition is taken modulo n+m+2). For any $i \in V(F)$ we must find a vertex $j \in V(F)$ such that i and j are not adjacent and j is adjacent to all vertices nonadjacent to i. But since $n \geqslant m$, the vertex $j = i + \left\lceil \frac{n+m+2}{2} \right\rceil$ has this property.

CASE 2. n and m are both odd.

Let $n+m+2 = 2r$. Take two disjoint copies of K_r, call them C_0 and C_1. Let $V(C_i) = \{v_{i,1}, v_{i,2}, \ldots, v_{i,r}\}$, $i = 0, 1$. For $j = 1, 2, \ldots, r$ join $v_{0,j}$ to $v_{1,j+k}$, $k = 1, 2, \ldots, n-r+1$ (where addition is modulo r). Let F be the resulting graph. Since F is n-regular, it suffices to find a vertex which is adjacent to all vertices nonadjacent to a vertex $v_{i,j}$ and is itself nonadjacent to $v_{i,j}$. Note that all vertices nonadjacent to $v_{i,j}$ are in C_{1-i} and $[v_{0,j}, v_{1,j}] \notin V(F)$. Therefore $v_{1-i,j}$ satisfies all conditions.

CASE 3. n is odd and m is even.

Let $m+n+2 = 2r+1$. Take a copy of $K_{r,r}$ and a disjoint vertex z. Let $K_{r,r}$ have bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_r\}$. Let $V(F) = X \cup Y \cup \{z\}$. $E(F)$ is constructed from $E(K_{r,r})$ as follows. Delete $[x_i, y_i]$, $1 \leqslant i \leqslant r$. Join z to x_i, $1 \leqslant i \leqslant r$. Join z to y_j, $1 \leqslant j \leqslant k$ where $k = n+1-r$. Since $n+1$ is even, so is $|\{y_{k+1}, y_{k+2}, \ldots, y_r\}|$. Join y_{k+1} to y_{k+2}, y_{k+3} to y_{k+4}, $\ldots$, y_{r-1} to y_r. In the graph constructed so far, z has degree $n+1$ and all other vertices have degree r. There are now two subcases to consider.

SUBCASE 3.1. r is even.

Let $F_1, F_2, \ldots, F_{r-1}$ be a 1-factorization of K_r where, without loss of generality, $F_{r-1} = \{[1, 2], [3, 4], \ldots, [r-1, r]\}$. Since $m \geqslant 2$, $n-d(v) \leqslant r-1$ for all vertices $v \neq z$. Also $n-r \leqslant r-3$. Add the edges corresponding to $F_1, F_2, \ldots, F_{n-r}$ to the subgraphs induced by X and Y. In the resulting graph, z has degree $n+1$ and all other vertices have degree n. Hence

$$q(F) = \left\lfloor \frac{n+1+n(n+m+1)}{2} \right\rfloor = \left\lfloor \frac{n(n+m+2)}{2} \right\rfloor$$

SUBCASE 3.2. r is odd.

If r is odd, $n-r$ is even. Let $C_1, C_2, \ldots, C_{\frac{r-1}{2}}$ be a factorization of K_r into hamilton circuits where, without loss of generality, $C_{\frac{r-1}{2}} = 1, 2, \ldots, r, 1$. Since $m \geqslant 2$, $n-d(v) \leqslant r-2$ for all vertices $v \neq z$. Also $\frac{n-2}{2} \leqslant \frac{r-3}{2}$. Add the edges corresponding to $C_1, C_2, \ldots, C_{\frac{n-r}{2}}$ to the subgraphs induced by X and Y. In the resulting graph z has degree $n+1$ and all other vertices have degree n. Hence

$$q(F) = \left\lfloor \frac{n(n+m+2)}{2} \right\rfloor$$

In either subcase, the desired graph has been constructed.

Note that $[x_i, y_i] \notin E(F)$ and at least one of x_i and y_i is adjacent to z, $1 \leqslant i \leqslant r$. Consider $(S, a_0) <^f (F, v)$. If v is z, x_1 is adjacent to all vertices nonadjacent to v. Otherwise, x_1

(y_i) is nonadjacent to y_i (x_i) and is adjacent to all vertices nonadjacent to y_i (x_i).

The result now follows. ■

5.2. Calculation of $h(S, a_1)$

In this section we calculate rotation numbers for the rooted graph (S, a_1). We first dispense with the trivial case $m = 1$.

Theorem 5.2.1. If $n=m=1$ then $h(S, a_1) = 2$.

Proof. When $n=m=1$, S is isomorphic to $K_2 \bigcup K_2$ and hence is vertex transitive. ■

S is not vertex transitive unless $m=n=1$. In what follows, this case should be understood to be excluded.

Lemma 5.2.2. Let F be an extremal (S, a_1) graph. Then $\delta_F \geqslant 2$. F has at least two vertices of degree at least n and either another vertex of degree at least n or at least one vertex of degree at least $m+1$.

Proof. Since $(S, a_1) < (F, v)$ for every $v \epsilon V(F)$, every vertex is adjacent to a vertex of degree at least n. It follows that F has at least two vertices of degree at least n. Let x and y be two such vertices.

Suppose that x and y are the only two vertices of degree at least n. Consider $(S, a_1) <^f (F, x)$. It follows that $f(a_0) = y$. The vertex $w = f(b_0)$ is adjacent to at least m vertices distinct from x and y, but must also be adjacent to at least one of x and y. Therefore, if there is not a third vertex of degree at least n, there is a vertex of degree at least $m+1$.

Let $z \epsilon V(F)$, $d_F(z) < m+1$ (note $m+1 \geqslant 2$) and let v be a vertex of degree at least n with $[z, v] \epsilon E(F)$. We claim that $d_F(z) \geqslant 2$. Consider $(S, a_1) <^g (F, v)$. Then either

$z \in g(K_{1,n})$ or $z \in g(K_{1,m})$. In either case, z is adjacent to a vertex diferent from v. Therefore $d_F(z) \geqslant 2$. Since $m+1 \geqslant 2$ it now follows that $\delta_F \geqslant 2$.

The result now follows. ■

Theorem 5.2.3. If m=1 and n>m=1 then $h(S, a_1) = 2(m+n)-1 = 2n+1$.

Proof. By lemma 5.2.2, $\delta_F \geqslant 2$ and there are at least two vertices of degree at least n. Hence

$$2q(F) \geqslant 2n+2(m+n) = 4(m+n)-2$$

i.e. $q(F) \geqslant 2(m+n)-1$. This bound is achieved by a 5-circuit when n=2 and, when n>2, by the graph formed from $K_2 + \overline{(K_n+e)}$ by joining a disjoint copy of K_2 to one vertex of the K_2 (i.e. so that a triangle is formed). ■

For the remainder of this section, we consider only those cases where $m \geqslant 2$.

Suppose that F has only three vertices a, b, c of degree at least m and let $A = \{a, b, c\}$. The minimality of q(F) ensures that no two vertices of $V(F) - A$ are adjacent (because neither may be $f(b_0)$ or $f(a_0)$). Therefore each of the m+n-1 vertices of $V(F) - A$ is adjacent to two (lemma 5.2.2) vertices of A and a further two edges are required to ensure that each vertex of A is adjacent to a vertex of degree at least m. We conclude

$$q(F) \geqslant 2(m+n-1)+1+1 = 2(m+n) \qquad (2.1)$$

and therefore an extremal graph F with fewer than 2(m+n) edges has at least four vertices of degree at least m.

Theorem 5.2.4. If $n=m \geqslant 2$ then $h(S, a_1) = 2(m+n)-2 = 4n-2$.

Proof. Suppose there exists an extremal graph F with fewer than 2(m+n) edges. F must have at least four vertices of degree at least m=n. Hence, if $k \geqslant 4$ vertices have degree at least m

$$2q(F) \geqslant kn+2(m+n+2-k) \geqslant 4n+2(m+n-2) = 4m+4n-4$$

from which it follows that $q(F) \geqslant 2(m+n)-2$. The graph $2(K_2 + \overline{K}_{n-1})$ achieves the minimum. ■

Rotation numbers for the remaining cases are given in the following theorem.

Theorem 5.2.5. If $n > m \geqslant 2$ then $h(S, a_1) = 2(m+n)-1$.

Proof. By lemma 5.2.2 and (2.1) we need only consider the cases where four or more vertices have degree at least m. Since $m \geqslant 2$

$$2q(F) \geqslant 2n+(m+1)+m+2(n+m-2) = 4n+4m-3$$

from which it follows that $q(F) \geqslant 2(m+n)-1$. Extremal (S, a_1) graphs which achieve this bound is shown in figure 5.2.1. ■

5.3. Calculation of $h(S, b_1)$

In this section we calculate rotation numbers for the rooted graph (S, b_1). The case where m=n is covered in the previous section.

The following lemma is the key result in calculating rotation numbers for the remaining cases.

Lemma 5.3.1. Let $n>m$ and F be an extremal (S, b_1) graph. Then there are three vertices x, y and z such that $d_F(x) \geqslant n$, $d_F(y) \geqslant n$ and $d_F(z) \geqslant m$. Furthermore, if $m>1$ then $\delta_F \geqslant 2$.

Proof. It is clear that at least two vertices, x and y, must have degree at least n. It follows from considering $(S, b_1) < (F, x)$ that there is another vertex z of degree at least m.

Let $m>1$, $v \in V(F)$ with $d_F(v) < m$. Then v is adjacent to some vertex w of degree at least m. Consider $(S, b_1) <^f (F, w)$. Since $w \neq f(b_0)$, we deduce that v is

$$2q(F) \geqslant kn+2(m+n+2-k) \geqslant 4n+2(m+n-2) = 4m+4n-4$$

from which it follows that $q(F) \geqslant 2(m+n)-2$. The graph $2(K_2 + \overline{K}_{n-1})$ achieves the minimum. ■

Rotation numbers for the remaining cases are given in the following theorem.

Theorem 5.2.5. If $n > m \geqslant 2$ then $h(S, a_1) = 2(m+n)-1$.

Proof. By lemma 5.2.2 and (2.1) we need only consider the cases where four or more vertices have degree at least m. Since $m \geqslant 2$

$$2q(F) \geqslant 2n+(m+1)+m+2(n+m-2) = 4n+4m-3$$

from which it follows that $q(F) \geqslant 2(m+n)-1$. Extremal (S, a_1) graphs which achieve this bound is shown in figure 5.2.1. ■

5.3. Calculation of $h(S, b_1)$

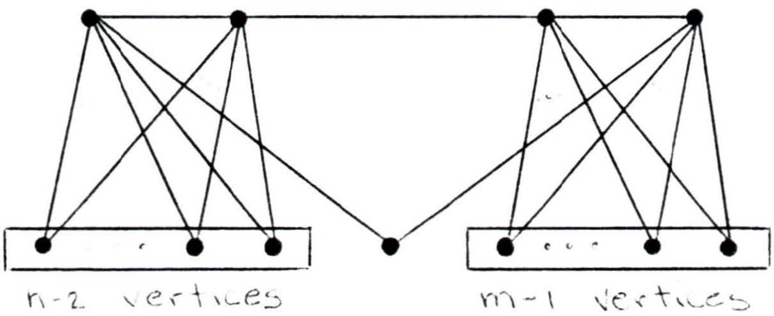
In this section we calculate rotation numbers for the rooted graph (S, b_1). The case where m=n is covered in the previous section.

The following lemma is the key result in calculating rotation numbers for the remaining cases.

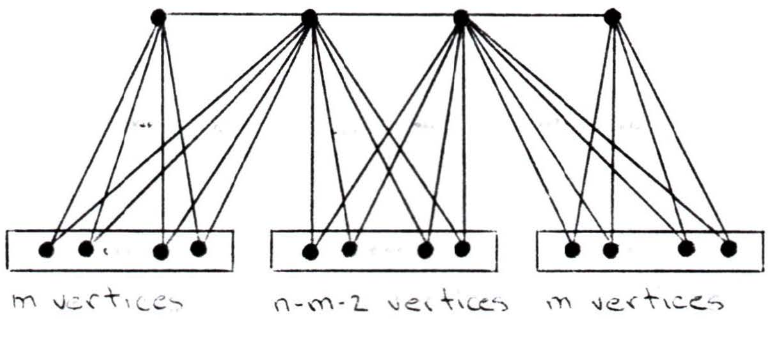
Lemma 5.3.1. Let $n>m$ and F be an extremal (S, b_1) graph. Then there are three vertices x, y and z such that $d_F(x) \geqslant n$, $d_F(y) \geqslant n$ and $d_F(z) \geqslant m$. Furthermore, if $m>1$ then $\delta_F \geqslant 2$.

Proof. It is clear that at least two vertices, x and y, must have degree at least n. It follows from considering $(S, b_1) < (F, x)$ that there is another vertex z of degree at least m.

Let $m>1$, $v \in V(F)$ with $d_F(v) < m$. Then v is adjacent to some vertex w of degree at least m. Consider $(S, b_1) <^f (F, w)$. Since $w \neq f(\bar{b}_0)$, we deduce that v is



n = m+1



n > m+1

Figure 5.2.1 Extremal (S, a_1) graphs when $n > m \geqslant 2$

adjacent to at least one other vertex, so $d_F(v) \geqslant 2$. Since $m \geqslant 2$ we conclude that $\delta_F \geqslant 2$. ■

Theorem 5.3.2. If $m=1$ and $n>m$ then $h(S, b_1) = 2(m+n)-1 = 2n+1$.

Proof. It is easy to see that five edges are required when $n=2$ and seven edges are required when $n=3$. Therefore assume $n \geqslant 4$. By lemma 5.3.1 there are at least two vertices x and y of degree at least n. If there is a third vertex of degree at least n, then $q(F) \geqslant 3n-3 \geqslant 2n+1$.

Consider now the case where x and y are the only two vertices of degree at least n in F. Suppose first that $[x, y] \in E(F)$. Let $(S, b_1) <^f (F, x)$. Then $f(a_0)=y$, so $d_F(y) \geqslant n+1$. A similar argument shows $d_F(x) \geqslant n+1$. Thus

$$q(F) \geqslant 2(n+1)-1 = 2n+1 = 2(m+n)-1$$

On the other hand suppose $[x, y] \notin E(F)$. If $d_F(x)+d_F(y) \geqslant 2n+1$ then so is $q(F)$. Therefore assume $d_F(x)=d_F(y)=n$. Since F has $n+1$ other vertices, x and y must be mutually adjacent to $n-1$ of them. Thus at least $n-1 > 0$ vertices in $V(F) - \{x, y\}$ have degree at least two. Let $v \in N(x) \bigcap N(y)$ and consider $(S, b_1) <^g (F, v)$. If $g(b_0) = x$, we must have $g(a_0) = y$. But since y is adjacent to v, this implies that $d_F(y) \geqslant n+1$ contrary to our assumption. The case where $g(b_0) = y$ similarly leads to a contradiction. We therefore conclude that $d_F(v) \geqslant 3$. In addition, each of the two remaining vertices must have degree at least one (since we must have $(S, b_1) < (F, u)$ for every $u \in V(F)$). Hence

$$2q(F) \geqslant 2n+3(n-1)+2 = 5n-1 \geqslant 4n+1$$

from which it follows that $q(F) \geqslant 2n+1 = 2(n+m)-1$.

Thus $2(m+n)-1$ has been established as a lower bound for $q(F)$. This bound is achieved by the graph formed from K_{2n} by joining a vertex of degree one to one of the vertices of degree three. ■

Rotation numbers for the remaining cases are calculated in the following theorem.

Theorem 5.3.3. Let $n > m \geqslant 2$. Then $h(S, b_1) = 2(m+n)-1$.

Proof. We will first show that a graph F with $(S, b_1) < F$ and fewer than $2(m+n)-1$ edges must have exactly two vertices of degree at least n (by lemma 5.3.1, F must have at least two such vertices). We will then prove that $2(m+n)-1$ is a lower bound on the number of edges of graphs in this class.

Suppose that $m=2$ (the case $m>2$ will be handled separately). Recall that $\delta_F \geqslant 2$ (lemma 5.3.1). If $k>2$ vertices have degree at least n,

$$2q(F) \geqslant kn+2(m+n+2-k) \geqslant 2n+k+2(n-m) \geqslant 4n+4m-3.$$

We conclude that $q(F) \geqslant 2(m+n)-1$ in this case.

Now suppose that $m>2$. Suppose further that F has only three vertices a, b, c of degree at least m and let $A = \{a, b, c\}$. The minimality of F ensures that no two vertices of $V(F)-A$ are adjacent (because neither may be $f(a_0)$ or $f(b_0)$). Therefore each of the $m+n-1$ vertices of $V(F)-A$ is adjacent to two (lemma 5.3.1) vertices of A and a further two edges are required to ensure that each vertex of A is adjacent to a vertex of degree at least m. We conclude $q(F) \geqslant 2(m+n-1)+2 = 2(m+n)-1$ in this case. On the other hand, suppose that three or more vertices have degree at least n and at least one other vertex has degree at least m. Since $\delta_F \geqslant 2$, we deduce

$$2q(F) \geqslant 3n+m+2(m+n-2) \geqslant 2n+2m+1+2(n+m-2) = 4n+4m-3.$$

It now follows that $2(m+n)-1$ is a lower bound for $q(F)$ in this case.

Therefore, an extremal (S, b_1) graph with fewer than $2(m+n)-1$ edges must have exactly two vertices x and y with degree at lest n. By lemma 5.3.1 at least one other vertex has degree at least m and $\delta_F \geqslant 2$.

We claim that $d_F(x)+d_F(y) \geqslant 2n+1$. Suppose first that $[x, y] \in E(F)$ and consider $(S, b_1) <^f (F, x)$. It follows that $f(a_0) = y$ and, since $x \in f(K_{1,m})$, $d_F(y) \geqslant n+1$. Similarly $d_F(x) \geqslant n+1$, so $d_F(x)+d_F(y) \geqslant 2n+2$. On the other hand assume $[x, y] \notin E(F)$

Since $n>m$ there is a vertex $v \in N(x) \bigcap N(y)$ Consider $(S, b_1) <^g (F, v)$ then one of x or y must be $g(a_0)$ Since $v \in g(K_{1,m})$ $d_F(g(a_0)) \geqslant n+1$ Therefore $d_F(x)+d_F(y) \geqslant 2n+1$

If $m>2$ and exactly one other vertex has degree m or more then as above we deduce $q(F) \geqslant 2(m+n-1)+1+1$

Oterwise $k>1$ vertices distinct from x and y have degree at least m and

$$2q(F) \geqslant 2n+1 + km + 2(m+n-k)$$
$$\geqslant 4n + 4m - 3$$

since $m \geqslant 2$ This establishes $2(m+n)-1$ as a lower bound for $h(S, b_1)$ An extremal (S, b_1) graph which achieves this bound is shown in figure 5 3 1 ■

5.4 Calculation of $h(S, b_0)$

In this section rotation numbers are calculated for the case where S is rooted at b_0 We give the exact solution when $n=m$ and $n \geqslant 3m$ Some results concerning the cases where $m<n<3m$ are mentioned

The case $n=m$ is covered by theorem 5 1 1 so throughout this section assume $m>n$ We now dispense with the trivial cases $m=0, 1$

Theorem 5 4 1 If $m=0$ and $n>m$ $n \neq 2$ then $h(S, b_0) = 2n+1 = 2(m+n)+1$ If $n=2$ then $h(S, b_0) = 4$

Proof If $n=1$ then $h(S, b_0) = 3 = 2n+1$ by Theorem 4 2 If $n=2$ then it is easy to see that C_4 is an extremal (S, b_0) graph So assume $n>2$ Let the vertices of F be labelled identically to those of S (i e so that $[a_0, a_1] \in E(F)$ and so on) Consider $(S, b_0) <^f (F, a_0)$ If $d_F(a_0) = n+1$ then $q(F) \geqslant 2n+1$ since $f(a_0)$ is adjacent to n vertices different from a_0 Otherwise assume $d_F(a_0) = n$ There are now two cases to consider

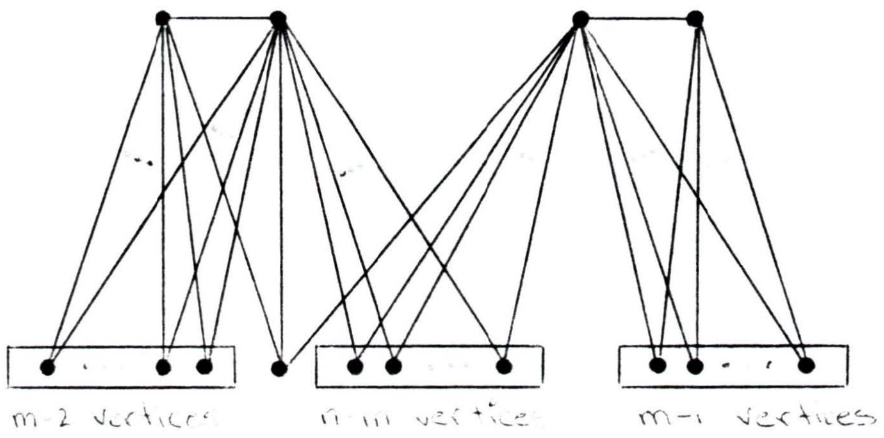


Figure 5.3.1. An extremal (S, b_1) graph when $m \geqslant 2$

CASE 1. $f(a_0) = a_i$ for some i, $1 \leqslant i \leqslant n$.

If $d_F(b_0) = 1$ then $(S, b_0) < (F, a_i)$ implies that b_0 is adjacent to another vertex, a contradiction. Otherwise $d_F(b_0) \geqslant 2$. But then $q(F) \geqslant n+n+1$, since b_0 is adjacent to a vertex different from a_i.

CASE 2. $f(a_0) = b_0$.

Note that $[b_0, a_1] \in E(F)$. Consider $(S, b_0) <^g (F, a_1)$. Since $[a_0, b_0] \notin E(F)$, $g(a_0) \neq a_0$.

Therefore some other vertex is also adjacent to a_1 whence $q(F) \geqslant n+n+1$

Both cases have been considered and $2n+1$ has been established as a lower bound for $q(F)$. This bound is achieved by the graph $K_2 + \overline{K}_n$. ■

Since K_2 is vertex transitive, the case $m=1$ and $n>m$ is the same as when S was rooted at b_1. We therefore have the following result the proof of which may be found in Section 5.3.

Theorem 5.4.2 If $m=1$ and $n>m$ then $h(S, b_0) = 2(m+n)-1$

From this point onward, assume $m \geqslant 2$. Suppose that $(S, b_0) < F$. We define a vertex of degree at least n in F to be an *N-vertex*. A vertex v with $m<d_F(v)<n$ will be referred to as a *B-vertex* (think Between). We observe that $\delta_F \geqslant m$ and that F must have at least two N-vertices.

The majority of the work in calculating rotation numbers for (S, b_0) involves a careful study of the number, arrangement and connection between N-vertices and m-vertices (i.e. vertices of degree m in F).

Lemma 5.4.3 No m-vertex can be adjacent to all N-vertices.

Proof Let F be an extremal graph and suppose that v is an m-vertex which is adjacent to every N-vertex in F. Consider $(S, b_0) <^f (F, v)$. Since $d_F(v) = m$, $f(K_{1,m}) = N[v]$. But then all vertices in $\overline{f(K_{1,m})}$ have degree less than n, so it is impossible to find an image of $K_{1,n}$ (because none of these vertices may be $f(a_0)$). This contradicts our choice of F as an extremal graph and the result now follows. ■

Lemma 5.4.4 If $d_F(v)<n$ then v is adjacent to an N-vertex.

Proof Since $d_F(v)<n$, there is a vertex $u \in V(F)$ such that $[u, v] \notin E(F)$. Consider $(S, b_0) <^f (F, u)$. Since $[u, v] \notin E(F)$, $v \in f(K_{1,n})$. Now, $d_F(v)<n$ implies that

$v \neq f(a_0)$ therefore v is adjacent to an N-vertex (namely $f(a_0)$) ■

Lemma 5.4.5 Let $d_F(u) = d_F(v) = m$. If the set of N-vertices adjacent to u contains the set of N-vertices adjacent to v, then $[u, v] \in E(F)$.

Proof Consider $(S, b_0) <^f (F, u)$. Since $d_F(u)=m$, $f(K_{1,m}) = N[u]$. Therefore no N-vertex adjacent to v can be $f(a_0)$. This implies that $v \in f(K_{1,m})$, which implies that $[u, v] \in E(F)$. ■

By Lemma 5.4.5, if two m-vertices are not adjacent we can conclude that each is adjacent to an N-vertex which is nonadjacent to the other.

An upper bound on $h(S, b_0)$ is provided by the graph G, which is constructed as follows. Let H be an (m-1)-regular graph with n vertices (if n(m-1) is odd, let H be almost (m-1)-regular). Let $I = H + K_1$. G is formed by joining each vertex in $V(H) \bigcap V(I)$ to a distinguished vertex of K_{m+1}. Note that

$$q(G) = \left\lfloor \frac{(m+3)n+m(m+1)}{2} \right\rfloor$$

It turns out that, for any given $m \geqslant 2$, G is an extremal (S, b_0) graph for all but finitely many values of n.

Suppose that G is not an extremal graph. We will derive an upper bound for the number of N-vertices in an extremal (S, b_0) graph F. If F has k N-vertices and $q(F) < q(G)$ then

$$kn+(m+n+2-k)m < (m+3)n+m(m+1)$$

So

$$k < 2 + \frac{n+m}{n-m} \tag{4.1}$$

Notice that for fixed m the right hand side of (4.1) is a decreasing function of n. In particular, for n=3m we get

$$k < 2 + \frac{4m}{2m} = 4$$

Therefore, when $n \geqslant 3m$, an extremal graph F with fewer edges than G must have either exactly two or exactly three N-vertices. We will show that no such graph can exist in lemmas 5.4.6 to 5.4.9 and hence establish G as an extremal graph for this case.

Lemma 5.4.6 If F has exactly two N-vertices and $(S, b_0) < F$, then $q(F) \geqslant q(G)$.

Proof Let x and y be the two N-vertices. If F has no m-vertices,

$$2q(F) \geqslant 2n+(m+n)(m+1) = (m+3)n+m(m+1)$$

from which it follows that $q(F) \geqslant q(G)$. We may therefore assume that F has at least one m-vertex z which, without loss of generality, is adjacent to x (note that by lemma 5.4.3, z cannot be adjacent to y).

We claim that x is adjacent to every vertex except possibly y. Define

$$C_1 = \{u \mid [u, y] \notin E(F), u \neq x\}$$
$$C_2 = \{v \mid [v, x] \notin E(F), v \neq y\}$$
$$C_3 = V(F) - C_1 - C_2$$

Note that $z \in C_1$ and that all vertices in C_3 are B-vertices. Let $w \in V(F) - \{x, y\}$ and consider $(S, b_0) <^f (F, w)$. If $x = f(a_0)$, then $C_2 \subseteq f(K_{1,m})$ and so w is adjacent to C_2. Similarly, $y = f(a_0)$ implies that w is adjacent to C_1. Hence every vertex is adjacent to C_1 or C_2 or both. Since C_1 contains an m-vertex, at most m+1 vertices can be adjacent to C_1. Therefore, at least n+1 vertices are adjacent to C_2. It follows that each vertex in C_2 has degree at least n. But x and y are the only two N-vertices in F, so this is possible only if $C_2 = \varnothing$. Therefore, x is adjacent to every vertex except possibly y, and the claim is proved.

It follows that all m-vertices are adjacent to x and hence by lemma 5.4.5 these m-vertices are adjacent. Hence there are at most m of these and at least n B-vertices. Now

$$2q(F) \geqslant n+(n+m)+m^2+(m+1)n \geqslant (m+3)n+m(m+1)$$

from which it follows that $q(F) \geqslant q(G)$ ■

By Lemma 5 4 6 when $n \geqslant 3m$ we need only consider the case where F has exactly three N-vertices

Lemma 5 4 7 Let $n>m+1$ If F has 3 or more N-vertices and some m-vertex is adjacent to only one of them then $q(F) \geqslant q(G)$

Proof Let x be an N-vertex and w be an m-vertex such that x is the only N-vertex to which w is adjacent By lemma 5 4 5 w is adjacent to every other m-vertex adjacent to x Thus at most m m-vertices are adjacent to x

Let u be nonadjacent to w and consider $(S\ b_0) <^f (F\ u)$ Then $w \in f(K_{1\ n})$ so $x = f(a_0)$ Therefore for every vertex r which is nonadjacent to x we have $r \in f(K_{1\ m})$ This implies that $[r\ u] \in E(F)$ for every vertex u which is nonadjacent to w Since there are n+1 such vertices u $d_F(r) \geqslant n$ We conclude that x is adjacent to all vertices with degree less than n

If F has $k \geqslant 3$ N-vertices

$$\begin{aligned} 2q(F) &\geqslant (n+m+2-k)+(k-1)n+(n+m+2-k)m+(n+2-k) \\ &= (m+3)n+m(m+1)+(k-2)(n-m-2) \\ &\geqslant (m+3)n+m(m+1) \end{aligned}$$

from which it follows that $q(F) \geqslant q(G)$ ■

Thus a graph F with $(S\ b_0) < F$ and $q(F)<q(G)$ must have exactly three N-vertices and every m-vertex must be adjacent to exactly two of them (by lemma 5 4 3 no m-vertex can be adjacent to all three N-vertices) Moreover lemma 5 4 5 says that any pair of m-vertices adjacent to the same two N-vertices are adjacent to each other

Lemma 5 4 8 Let $n \geqslant 2m-1$ Suppose that F has exactly three N-vertices and every m-vertex is adjacent to exactly two of them. Then F has at most 2m-2 m-vertices.

Proof. Let x, y and z be the N-vertices and C_{xy}, C_{xz} and C_{yz} be the sets of m-vertices adjacent to the pairs of N-vertices {x, y}, {x, z} and {y, z} respectively. Note that by lemma 5.4.5 each of these sets induces a complete subgraph and has cardinality at most m-1. If one of C_{xy}, C_{xz}, C_{yz} is empty, there is nothing to prove, so assume that all of them are non-empty. Let $C_0 = V(F) - C_{xy} - C_{xz} - C_{yz}$. We note $\{x, y, z\} \subseteq C_0$.

Let $v \in C_0$ and $(S, b_0) <^f (F, v)$. We partition $C_0 = A_x \bigcup A_y \bigcup A_z$ according to which vertex from {x, y, z} is used for $f(a_0)$. If $v \in A_x$ then $C_{yz} \subseteq f(K_{1,m})$ and hence v is adjacent to C_{yz}. Similarly $v \in A_y$ (A_z) implies that v is adjacent to C_{xz} (C_{yz}).

We now fix v_{xy}, v_{xz}, v_{yz} in C_{xy}, C_{xz}, C_{yz} respectively and note that, by the above, v_{xy}, v_{xz}, v_{yz} are adjacent to A_x, A_y, A_z respectively. These adjacencies include only one of the two edges from each of v_{xy}, v_{xz}, v_{yz} to the N-vertices and we conclude the number of edges from each of v_{xy}, V_{xz}, v_{yz} is at least

$$|A_x| + |A_y| + |A_z| + 3 = |C_0| + 3$$

Since v_{xy} is adjacent to $|C_{xy}| - 1$ vertices in C_{xy} etc., we have

$$\begin{aligned} 3m &\geqslant (|C_{xy}|-1)+(|C_{xz}|-1)+(|C_{yz}|-1)+|C_0|+3 \\ &= |C_{xy}|+|C_{xz}|+|C_{yz}|+|C_0| = m+n+2 \end{aligned}$$

Therefore $2m-2 \geqslant n$, a contradiction.

We conclude that at least one of C_{xy}, C_{xz} or C_{yz} is empty. It now follows that there are at most 2(m-1) m-vertices. ■

Lemma 5.4.9. Let n>2m-1. If F satisfies the hypothesis of lemma 5.4.8 then $q(F) \geqslant q(G)$.

Proof. Let x, y and z be the N-vertices. It follows from the proof of lemma 5.4.8 that some pair of N-vertices are not mutually adjacent to an m-vertex. Therefore some N-vertex, x say, is adjacent to every m-vertex.

If x is also adjacent to all B-vertices then

$$2q(F) \geqslant 2n+(m+n-1)+(m+n-1)(m+1)-(2m-2)$$
$$= (m+3)n+m(m+1)+n-2m$$
$$\geqslant (m+3)n+m(m+1)$$

from which it follows that $q(F)\geqslant q(G)$.

We may therefore assume that some B-vertex is nonadjacent to x. Let C_x, C_y and C_z be the sets of vertices, excluding x, y and z, which are nonadjacent to x, y and z respectively. Let $C_0 = V(F) - C_x - C_y - C_z - \{x, y, z\}$. Note that C_x contains no m-vertices. If, in addition, C_y (C_z) also contains no m-vertices, then no m-vertices are non-adjacent to y (z). Therefore all m-vertices are adjacent to x and y (x and z). Since by lemma 5.4.5 all such m-vertices are adjacent, it follows that there are at most m-1 m-vertices. Hence

$$2q(F) \geqslant 3n+(m-1)m+(m+n-1-(m-1))(m+1)$$
$$= (m+3)n+m(m+1)+n-2m$$
$$\geqslant (m+3)n+m(m+1)$$

from which it follows that $q(F)\geqslant q(G)$.

We may therefore assume further that both of C_y and C_z contain an m-vertex. Thus F contains at most 2(m-1) m-vertices (because there are at most m-1 m-vertices adjacent to x and y (x and z)). Let $v \in W = V(F) - \{x, y, z\}$ and consider $(S, b_0) <^f (F, v)$. If $f(a_0)=x$ then v is adjacent to C_x. Similarly for y and C_y, and for z and C_z. Since both C_y and C_z contain m-vertices, at most m-1 vertices of W can be adjacent to each of them. Thus at least n+m-1-2(m-1) = n-m+1 vertices of W are adjacent to C_x, so every vertex in C_x has degree at least n-m. All vertices in $V(F) - C_y - \{x, y, z\}$ which are not m-vertices must be B-vertices, hence

$$
\begin{aligned}
2q(F) &\geqslant 2n+(n+m-1-|C_x|)+(n+m-1-|C_x|)(m+1)-(2m-2)+(n-m)|C_x| \\
&= (m+3)n+m(m+1)+(n-2m-2)|C_x|+n-2m \\
&\geqslant (m+3)n+m(m+1)+2 > 2q(G)
\end{aligned}
$$

The result now follows ■

Combining Lemma's 5.4.4 through 5.4.9 we have proved

Theorem 5.4.10 If $n \geqslant 3m$ ($m \geqslant 2$) then

$$h(S, b_0) = \left\lceil \frac{(m+3)n+m(m+1)}{2} \right\rceil$$

and G is an extremal (S, b_0) graph

The methods used to prove theorem 5.4.10 can also be used to prove the following results which we state without proof

Result 5.4.11 Let $3m > n > 3m-6$. Then $q(G) \geqslant h(S, b_0) \geqslant q(G)-1$

Result 5.4.12 Let $3m-6 \geqslant n \geqslant 2m$, $n \neq 3m-7$. Then $h(S, b_0) = q(G)-1$

Result 5.4.13 Let $2m-1 > n \geqslant \frac{3}{2}m$. Then $h(S, b_0) = \left\lceil \frac{(m+2)n+m(m+1)}{2} \right\rceil$

In addition, we conjecture that G is an extremal graph for $3m > n > 3m-6$ and $n = 3m-7$. We also believe that the rotation number given in result 5.4.13 covers all cases $m < n < 2m-1$

Beyond the results mentioned above, it seems very difficult to extend the methods of this section to calculate the rotation numbers for the unresolved cases. This is because, as n approaches m, the number of X-vertices in a cadidate extremal graph grows as a function of m. It was the fact that this number is bounded for $n \geqslant \frac{3}{2}m$ which enabled our approach to work

CHAPTER 6

CALCULATION OF SOME EDGE ROTATION NUMBERS

In this chapter we calculate edge rotation numbers and extremal graphs for the graphs $S = K_{1,m} \bigcup K_{1,n}$ $(m \leqslant n)$ and $K = K_m \bigcup K_n$ $(m \leqslant n)$. In each case both possibilities for the root are considered.

6.1. Edge Rotation Numbers for Unions of Stars.

Let $S = K_{1,m} \bigcup K_{1,n}$ $(n \geqslant m)$ the graph pictured in figure 5.0.1. In this section we calculate $h'(S, [a_0, a_1])$ and $h'(S, [b_0, b_1])$ and exhibit the associated extremal graphs.

If n=1 or n=m then S is edge transitive, so

$$h'(S, [a_0, a_1]) = h'(S, b_0, b_1]) = n+m$$

The two following theorems cover all remaining cases.

Theorem 6.1.1. Let $n > m \geqslant 1$. Then $h'(S,[a_0, a_1]) = 2n$.

Proof. It is convenient to think of F as being labelled so that a subgraph isomorphic to S is evident (i.e. $[a_0, a_1] \in E(F)$ and so on). Consider $(S, [a_0, a_1]) << (F, [b_0, b_1])$. Then either b_0 or b_1 has degree at least n; let x denote this vertex. If $[x, a_0] \in E(F)$ then $(S, [a_0, a_1]) << (F, [x, a_0])$ implies the existence of a vertex y which is incident with m edges not incident with x or a_0. Hence $q(F) \geqslant 2n-1+m \geqslant 2n$ in this case. If $[x, a_0] \notin E(F)$ then clearly $q(F) \geqslant d_F(a_0)+d_F(x) \geqslant 2n$. A graph which achieves this bound is constructed from $K_{1,n} \bigcup K_{1,m}$ by joining the "center" of the m-star to n-m vertices of degree one on the n-star. ■

Theorem 6.1.2 Let $n > m \geqslant 1$. Then $h(S,[b_0, b_1]) = 2n+1$.

Proof. As above, let the labels assigned to $V(F)$ make explicit a subgraph isomorphic to S. Consider $(S,[b_0, b_1]) <<^f (F,[a_0, a_1])$. This implies that there is another vertex of degree at least n $(=f(a_0))$. There are two cases to consider.

CASE 1. $f(a_0) = a_k$, $1 < k \leqslant n$.

Therefore a_k is adjacent to n vertices different from a_0. Hence $q(F) \geqslant n+n+m \geqslant 2n+1$, where m is the number of edges $[b_0, b_i]$, $1 \leqslant i \leqslant m$.

CASE 2. $f(a_0) = b_j$, $0 \leqslant j \leqslant m$.

Therefore b_j is adjacent to n vertices from $V(F) - \{a_0\}$. Let $v \in N(b_j) \bigcap N(a_0)$ and consider $(S,[b_0, b_1]) <<^g (F,[v, a_0])$. If $b_j = g(a_0)$, then $d_F(b_j) \geqslant n+1$ and $q(F) = n+(n+1)$. Otherwise, there is another vertex of degree at least n which is adjacent to n vertices from $V(F) - \{a_0\}$. Since $n \geqslant 2$, this means that at least one edge not incident with a_0 or b_j has been found. Therefore $q(F) \geqslant n+n+1$.

In both cases $2n+1$ has been established as a lower bound for $q(F)$. A graph which achieves this bound is constructed from $K_{1,n} \bigcup K_{1,m}$ by joining the "center" of the m-star to n-m+1 vertices of degree one in the n-star. ■

6.2 Edge Rotation Numbers for Unions of Complete Graphs

In this section we calculate edge rotation numbers for the graph $K = K_m \bigcup K_n$, where $1 < m < n$. The cases $m = 1$ and $m = n$ are trivial.

We will consider first the case where K is rooted at e_m, an edge of K_m.

Lemma 6.2.1 If $(K, e_m) << F$ then $\bar{F}$ is bipartite.

Proof Suppose that $\bar{F}$ contains a cycle C and consider $(K, e_m) <<^f (F, e)$. Since $f(K_m)$ and $f(K_n)$ are complete, adjacent vertices of C must be alternately in $f(K_m)$ and $f(K_n)$. It follows that the length of C is even. Therefore $\bar{F}$ contains no odd circuits and hence is bipartite ■

In order to minimize $q(F)$, it suffices to maximize $q(\bar{F})$. Let H be the subgraph of $\bar{F}$ induced by the set of vertices which are not isolated in $\bar{F}$. Suppose that H has k connected components with bipartitions (U_i, V_i), $1 \leqslant i \leqslant k$, respectively. Without loss of generality we can assume $|U_i| \leqslant |V_i|$, $1 \leqslant i \leqslant k$.

Lemma 6.2.2 If $(K, e_m) <<^f (F, e)$ then $f(K_m)$ and $f(K_n)$ each contain exactly one of U_i or V_i, $1 \leqslant i \leqslant k$.

Proof Without loss of generality suppose that U_1 is contained partly in $f(K_n)$ and partly in $f(K_m)$. Let $v \in V_1$ be in $f(K_n)$, the case where $v \in f(K_m)$ is similar. Then, since $f(K_n)$ and $f(K_m)$ are complete, v is adjacent in H only to those vertices in U_1 which are in $f(K_m)$. Similarly, a vertex $u \in U_1$ in $f(K_n)$ is adjacent in H only to those vertices in V_1 which are in $f(K_m)$. But no pair of vertices in $f(K_m)$ may be adjacent in H; therefore there is no path in H from u to v, contradicting the connectedness of $H[U_1 \bigcup V_1]$. ■

Lemma 6.2.3 If $k = 1$ (i.e. H has only one component) then $q(H) \leqslant (m-1)(m-2)$

Proof An application of lemma 6.2.2 with e joining a pair of vertices in V_1 gives $|V_1| \leqslant m$ and $|U_1| \leqslant m$. Since $m < n$, there is a vertex $x \in V(\bar{F})-U_1-V_1$. Another application of lemma 6.2.2, this time with $e = [x, v]$, $v \in V_1$ yields $|V_1| \leqslant m-1$. This implies that there exists a vertex $y \neq x \in V(\bar{F})-V_1-U_1$. One more application of lemma 6.2.2, this time using $e = [x, y]$, yields $|U_1| \leqslant m-2$. Therefore

$$q(H) \leqslant |U_1|\,|V_1| \leqslant (m-1)(m-2)$$

as required. ■

Lemma 6.2.4. If $k > 1$ then $q(H) \leqslant (m-1)(m-2)$.

Proof. An application of lemma 6.2.2 with $e = [u, v]$ where $u \in U_j$ and $v \in V_i$, $i \neq j$ gives

$$|V_i| + \sum_{i \neq j} |U_j| \leqslant m \qquad i=1, 2, \ldots, k \tag{6.1}$$

since $v \in V_i$ must be in $f(K_m)$ and therefore all of V_i must be contained in $f(K_m)$. It now follows that

$$\sum_i |U_i| \leqslant m \qquad \text{and} \qquad |V_j| \leqslant m-k \quad j=1, 2, \ldots, k \tag{6.2}$$

By (6.2), if $|U_i| = |V_i|$, $1 \leqslant i \leqslant k$, then $m < n$ implies there is a vertex $x \in V(\bar{F})-V(H)$. Using lemma 6.2.2 with $e = [x, u]$, where $u \in U_1$ we find

$$\sum_{i=1}^{k} |U_i| \leqslant m-1 \tag{6.3}$$

Otherwise suppose $|U_j| < |V_j|$ for some j. Then (6.1) implies (6.3).

If $|V_1| = m-1$ then it follows from (6.1) that $k=2$ and $|U_2|=1$. Applying lemma 6.2.2 with $e=[v_1, v_2]$, $v_i \in V_i$, $i=1, 2$ gives $|V_2|=1$. Now $m < n$ implies there exists $b \in V(\bar{F})-V(H)$. Another application of lemma 6.2.2, this time with $e=[b, u]$, $u \in U_1$ yields $|U_1| + |U_2| \leqslant m-1$. We deduce that there is a vertex $c \neq b \in V(\bar{F})-V(H)$. Yet another application of lemma 6.2.2 ($e = [b, c]$) gives $|U_1| + |U_2| \leqslant m-2$. Thus

$$\begin{aligned} q(H) &\leqslant (m-1)(|U_1| + |U_2|) \\ &\leqslant (m-1)(m-2) \end{aligned}$$

The case where $|V_2| = m-1$ is similar.

Otherwise $|V_i| \leqslant m-2$, $1 \leqslant i \leqslant k$. By (6.3) we have

$$q(H) = \sum_{i=1}^{k} |U_i| \; |V_i|$$

$$\leqslant (m-2) \sum_{i} |U_i|$$

$$\leqslant (m-2)(m-1)$$

The result now follows. ■

We can now prove the main result of this chapter.

Theorem 6.2.5. $K_{m+n} - K_{m-1\,m-2}$ is an extremal (K, e_m) graph and

$$h(K, e_m) = \binom{m+n}{2} - (m-1).$$

Proof. This is just a straightforward verification that $(K, e_m) << K_{m+n} - K_{m-1\,m-2}$ and is left for the reader. ■

We now consider the case where K is rooted at e_n, an edge of K_n.

Theorem 6.2.6. $h'(K, e_n) = \dfrac{n(n-1)}{2} + \dfrac{m(m-1)}{2} + m(n-m)$

Proof. Without loss of generality assume that $V(F)$ is labelled to make a subgraph isomorphic to K explicit. These are the K_n and K_m mentioned below. Let e be an edge of K_m and consider $(K, e_n) << (F, e)$. This implies that each end of e (and therefore each vertex of K_m) is adjacent to at least n-m vertices of K_n. Thus $q(F) \geqslant \dfrac{n(n-1)}{2} + \dfrac{m(m-1)}{2} + m(n-m)$. A graph which achieves this bound may be constructed from $K_n \bigcup K_m$ by joining each vertex of K_m to n-m vertices of K_n. ■

REFERENCES

[1] Bollobas B and E J Cockayne More Rotation Numbers for Complete Bipartite Graphs Journal of Graph Theory 6 (1982) 403-411

[2] Bollobas B E J Cockayne and Z Y Fang Rotation Numbers for Unions of Circuits Journal of Graph Theory 8 (1984) 69-81

[3] Bondy J A and U S R Murty *Graph Theory with Applications* Elsevier Publishing New York New York 1982

[4] Cockayne E J and Z Y Fang Rotation Numbers for Unions of Triangles and Circuits Annals of Discrete Mathematics 17 (1983) 203-209

[5] Cockayne E J and P J Lorimer On Rotation Numbers for Complete Bipartite Graphs Congressus Numeratium 28 (1980) 281-291

[6] Cockayne E J D J Miller and Z Y Fang Rotation Numbers for Rooted Graphs A Survey University of Victoria Department of Mathematics Report DM-231-IR (1981)

[7] Farley A M S T Hedetniemi S Mitchell and A Proskurowski Minimum Broadcast Graphs (Note) Discrete Mathematics 25 (1979) 189-193

[8] Harary F *Graph Theory* Addison Wesley Reading Mass 1972

[9] Hell P and R Haagvist Rotation Numbers for Complete Bipartite Graphs (in preparation)

[10] Holyer I and E J Cockayne Rotation Numbers for Lollipop Graphs University of Victoria Department of Mathematics Report DM-321-IR (1984)

[11] Papadimitriou C H and K Steiglitz *Combinatorial Optimization, Algorithms and Complexity* Prentice-Hall Englewood Cliffs New Jersey 1982

APPENDIX A

PROPERTIES OF ROOTED SUBGRAPHS AND AN ALGORITHM

To calculate the rotation number $h(G, x)$ one must find the minimum number of edges in a graph F such that $(G, x) < F$. The problem of deciding if a rooted graph (G, x) is a homogeneous rooted subgraph of a graph F is therefore closely related to the problem of determining $h(G, x)$. In this appendix we examine the problems $(G, x) < (F, y)$ and $(G, x) < F$. A backtracking algorithm which solves the former problem (and therefore also the latter) is presented.

We first present some miscellaneous results.

Theorem A.1. Let $p(G) = p(F)$. If $(G, x) < (F, y)$ then $(\bar{F}, y) < (\bar{G}, x)$.

Proof. Let $(G, x) <^f (F, y)$. Then f is a 1-1 mapping of V(G) onto V(F) and $[a, b] \in E(G)$ implies $[f(a), f(b)] \in E(F)$. Therefore, if $[c, d] \notin E(F)$, $[f^{-1}(c), f^{-1}(d)] \notin E(G)$, or in other words $[c, d] \in E(\bar{F})$ implies $[f^{-1}(c), f^{-1}(d)] \in E(\bar{G})$. Since f^{-1} is a 1-1 mapping of V(F) onto V(G), it follows that $(\bar{F}, y) <^{f^{-1}} (\bar{G}, x)$. ■

Theorem A.2. If $(G, a) << (F, b)$ then $(L(G), a) < (L(F), b)$, where L(X) denotes the line graph of X.

Proof. Let $(G, a) <<^f (F, b)$. Then f is a 1-1 mapping of V(G) into V(F) and $[u, v] \in E(G)$ implies $[f(u), f(v)] \in V(F)$. Thus f induces a 1-1 mapping from E(G) into E(F). This is equivalent to a 1-1 mapping from V(L(G)) into V(L(F)).

Suppose x and y are adjacent edges in G. Then f(x) and f(y) are adjacent edges in F or, in other words, $[x, y] \in E(L(G))$ implies $[f(x), f(y)] \in E(L(F))$. Therefore $(L(G), a) <^f (L(F), b)$. ■

Corollary A.3. If $(G, e) << F$ then $(L(G), e) < L(F)$.

Corollary A.3 provides an equivalent definition of the edge rotation number $h'(G, e)$ is the smallest number of vertices in a line graph L such that $(L(G), e) < L$.

Given two arbitrary rooted graphs (G, x) and (F, y), the problem of deciding if $(G, x) < (F, y)$ is evidently equivalent to the subgraph isomorphism problem and, therefore, is NP-complete. One would expect that the problem of deciding if $(G, x) < F$ for an arbitrary rooted graph (G, x) and an arbitrary graph F is at least as hard as the previous problem. The following theorem shows that this is the case.

Theorem A.4. Given an arbitrary rooted graph (G, x) and and arbitrary graph F, the problem of deciding if $(G, x) < F$ is NP-complete.

Proof. The problem is easily seen to be in NP. A nondeterministic Turing machine need only guess p 1-1 onto functions $f_i: V(G) \rightarrow V(F)$, $1 \leqslant i \leqslant p$ and then check to see that all of the required properties hold.

The problem $(C_p, x) < F$ is exactly the hamilton circuit problem, so the result follows by restricting G to be a circuit with $p(F)$ vertices. ■

We now describe a backtracking algorithm which, given two p-vertex rooted graphs, (G, x) and (F, y), decides if $(G, x) < (F, y)$. This algorithm may be used to decide if $(G, x) < F$ by applying it to each $y \in V(F)$.

Backtracking algorithms are based on the idea of intelligently enumerating the solution space (in our case, the set of all 1-1 mappings from $V(G)$ into $V(F)$) by partitioning it into mutually exclusive and exhaustive subsets. It is hoped that this can be done in such a way that only a small portion of the solution space must be examined before a solution is found or we have determined that no solution exists.

In order to specify our algorithm, we must describe strategies for

(i) Partitioning the solution space into mutually exclusive and exhaustive subsets.

(ii) Eliminating nodes that do not lead to solutions.

(iii) Deciding which node to expand and how this should be done.

Each of these issues is tackled in turn.

(i) Partitioning. If there is a function f with the required properties, it may be found by examining all 1-1 mappings from V(G) into V(F).

Let us define a *partial embedding* of (G, x) to be an ordered triple (g, S, L) where $S \subseteq V(G)$, g is a 1-1 map from S into V(F) with the property that $[u, v] \in E(G)$ implies $[g(u), g(v)] \in E(F)$ and L is a family of subsets of $\overline{g(S)}$ which is indexed by $\bar{S}$ and gives for each $v \in \bar{S}$ the set of possible images of v under an extension of g. We shall say that a partial embedding f is *obtainable* from g if and only if the restriction of f to S is g.

Using the above definitions, we can describe our approach to the problem and consequently the partitioning rule. Each node corresponds to a partial embedding. When a node (g, S, L) is expanded, its children are $\{(g_i, S \bigcup \{v\}, L-M_v)\}$, where $v \in \bar{S}$, $1 \leqslant i \leqslant k = |M_v|$, g_i is the extension of g to $S \bigcup \{u_i\}$, $u_i \in M_v$ and M_v is the set of possible images of v. The criteria for choosing (g, S, L) and v are described in section (iii). Note that every partial embedding obtainable from (g, S, L) is obtainable from exactly one of its children. Hence the partitioning rule divides the space of partial embeddings obtainable from (g, S, L) into mutually exclusive and exhaustive subsets.

We require an initialization. The choice is obvious:

$$((x, y), \{x\}, \{M_v \mid v \neq x \in V(G)\})$$

where $u \in V(F)$ is in M_v if and only if $d_F(u) \geqslant d_G(v)$ and $[x, v] \in E(G)$ implies

$[y, u] \in E(F)$. It follows from the definition of the partitioning rule that, using this initialization, all nodes generated are distinct and that all partial embeddings in which $x \rightarrow y$ may be generated.

(ii) Elimination. It is a waste of time to examine the children of a node which cannot be extended into an isomorphism of (G, x) into (F, y).

A new piece of information is gained each time a child is generated. Suppose that in generating the child node, v is mapped to u. Therefore, whenever $[v, w] \in E(G)$ we must also have $[u, z] \in E(F)$, for all $z \in M_w$. Thus we can scan the sets M_w and eliminate possible mappings using the above rule (u can also be eliminated as it is now used). If $M_\alpha = \emptyset$ for some α and $S \neq V(G)$ we can conclude that the child cannot be extended to produce a solution.

The above elimination rule can be made much stronger. The key is the following theorem of Hall.

Theorem [Hall]. Let G be a bipartite graph with bipartition (X, Y), where $|X| = |Y|$. Then G has a perfect matching if and only if

$$|N(S)| \geqslant |S| \quad \text{for all } S \subseteq X$$

In order to apply Hall's theorem to our problem, we define a bipartite graph B with vertex set $\bar{S} \bigcup \overline{g(S)}$ and edge set

$$E(B) = \{[a, b] \mid a \in \bar{S}, b \in \overline{g(S)} \bigcap M_a\}$$

An extension of (g, S, L) into an isomorphism of (G, x) into (F, y) corresponds to a perfect matching in B. Unfortunately, not every perfect matching in B corresponds to an isomorphism. However, Hall's theorem enables us to deduce that if no perfect matching exists, the given node can not be extended to produce a solution. Note that B can be constructed in polynomial time and that we can determine in polynomial time whether B has a perfect matching [11] This elimination rule covers all of the cases handled by its predecessor and many more

(iii) Expansion We require two rules one for deciding which node to expand and another for deciding the vertex within that node to expand upon These rules must be chosen with a view toward minimizing the total number of nodes to be examined

It would be nice if our node selection criteria had two properties

(1) A node likely to lead to a solution will be expanded "soon"

(2) A node unlikely to lead to a solution will either be expanded and eliminated "soon" or will be ignored while "more promising" nodes are considered

One would expect that a node "near" a solution would have few potential descendants (i e $\sum_{v \in \bar{S}} |M_v|$ is small) a node which could be eliminated soon would have $|M_v|$ small for at least one v a node requiring lots of processing would have lots of potential descendants (i e $\sum_{v \in \bar{S}} |M_v|$ is large) This suggests three possible expansion rules

Expand the node with

(1) smallest $\sum |M_v|$

(2) smallest $|\bar{S}|^{-1} \sum |M_v|$

(3) smallest $|M_v|$ for some $v \in \bar{S}$

Each of these rules gives rise to a variation of the algorithm We have implemented (1) and (2) For the graphs in this thesis (and a few others) fewer nodes were generated when rule (1) was used (rather than rule (2))

The vertex expansion rule should complement the node expansion rule In view of the fact that it is not efficient in terms of space (memory) to generate a lot of nodes it makes sense to expand on a vertex v with small $|M_v|$ and possibly $d_G(v)$ large

Thus we could expand on the vertex v with

(1) smallest $|M_v|$

(2) largest $d_G(v)$

(3) smallest $\frac{|M_v|}{d_G(v)}$

We have implemented (1) and (2) and have obtained mixed results Sometimes (1) is better than (2) sometimes the opposite is true Rule (2) seems to be better whenever G has a few vertices whose degrees are large compared to the rest of the vertices in G and if F has the same property

The above discussion leads to the algorithm shown in figure A 1 This algorithm has been useful in finding many of the extremal graphs in this thesis A straightforward extension enables the algorithm to handles the cases where F has more vertices than G

```
INPUT (G, x) and (F, y)
queue := empty;
initialize;
while queue not empty do
  (g, S, L) := queue_remove;
  if S=V(G) then
    OUTPUT "(G, x) < (F, y)";
    stop;
  else
    Let v∈S̄ have smallest |M_v|;
    For all u∈M_v do
      generate the child corresponding to u;
      construct B for this child;
      if B has no perfect matching then
        eliminate this node
      else
        insert this child into the queue
      end if;
    next u;
  end if;
end while;
OUTPUT "F has no subgraph isomorphic to (G, x) rooted at y";
```

The queue is kept ordered according to the node expansion criteria.

Figure A.1. An algorithm that decides (G, x) < (F, y).

VITA

Surname MACGILLIVRAY *Given Name* GARY

Place of Birth *GATESHEAD UPON TYNE ENGLAND*
Date of Birth DECEMBER 9 1960

Educational Institutions Attended with Dates of Entering and Leaving

UNIVERSITY OF VICTORIA B C 1980 to 1984
UNIVERSITY OF VICTORIA B C 1985 to 1986

Degrees, Diplomas, Etc , Awarded, with Dates and Names of Institutions

B SC (HONOURS) 1985 UNIVERSITY OF VICTORIA B C

Honours and Awards

NSERC POSTGRADUATE SCHOLARSHIP 1985 and JAN-APR 1986
ROBERT S EVANS MEMORIAL SCHOLARSHIP 1984/85
UNIVERSITY OF VICTORIA PRESIDENTS SCHOLARSHIP
1981/82 1983/84 and 1984/85
NSERC UNDERGRADUATE RESEARCH AWARD (INDUSTRY)
JAN-APR 1983 and MAY-AUG 1984

PARTIAL COPYRIGHT LICENSE

I hereby grant the right to lend my thesis or dissertation (the title of which is shown below) to users of the University of Victoria Library and to make single copies only for such users or in response to a request from the Library of any other University or similar institution on its behalf or for one of its users. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by me or a member of the University designated by me. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis/Dissertation

ROTATION NUMBERS FOR UNIONS OF STARS

Author [redacted]

Gary MacGillivray

April 17 1986