

**QUASI-CONVOLUTION PROPERTIES OF CERTAIN  
SUBCLASSES OF ANALYTIC FUNCTIONS**

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# Quasi-Convolution Properties of Certain Subclasses of Analytic Functions

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## Abstract

Two subclasses  $\mathcal{A}(p, n, \alpha)$  and  $\mathcal{B}(p, n, \alpha)$  of analytic functions in the open unit disk are introduced. The object of the present paper is to give a number of quasi-convolution properties of functions belonging to each of the classes  $\mathcal{A}(p, n, \alpha)$  and  $\mathcal{B}(p, n, \alpha)$ .

## 1. Introduction and Definitions

Let  $\mathcal{T}(p, n)$  be the class of functions  $f(z)$  of the form:

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0; \quad p, n \in \mathbb{N} := \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let  $\mathcal{T}(p, n, \alpha)$  denote the subclass of  $\mathcal{T}(p, n)$  consisting of functions  $f(z)$  which also satisfy the inequality:

$$\Re \left\{ \frac{f(z)}{z f'(z)} \right\} > \alpha \quad \left( z \in \mathcal{U}; \quad 0 \leq \alpha < \frac{1}{p} \right) \quad (1.2)$$

for some  $\alpha$  ( $0 \leq \alpha < 1/p$ ). We note that a function  $f(z)$  belonging to the class  $\mathcal{T}(p, n, \alpha)$  is  $p$ -valently starlike in  $\mathcal{U}$ .

For the class  $\mathcal{T}(p, n, \alpha)$ , Yamakawa [6] has shown that, if  $f(z) \in \mathcal{T}(p, n)$  satisfies the inequality:

$$\sum_{k=p+n}^{\infty} (2k - \alpha pk - p) a_k \leq p(1 - \alpha p) \quad \left(0 \leq \alpha < \frac{1}{p}\right), \quad (1.3)$$

then  $f(z) \in \mathcal{T}(p, n, \alpha)$ . Applying this fact, Yamakawa [6] gave the following equivalence relations:

$$f(z) \in \mathcal{A}(p, n, \alpha) \Leftrightarrow f(z) \in \mathcal{T}(p, n) \quad \text{and} \quad \sum_{k=p+n}^{\infty} (2k - \alpha pk - p) a_k \leq p(1 - \alpha p) \quad \left(0 \leq \alpha < \frac{1}{p}\right); \quad (1.4)$$

$$f(z) \in \mathcal{B}(p, n, \alpha) \Leftrightarrow f(z) \in \mathcal{T}(p, n) \quad \text{and} \quad \sum_{k=p+n}^{\infty} k(2k - \alpha pk - p) a_k \leq p^2(1 - \alpha p) \quad \left(0 \leq \alpha < \frac{1}{p}\right). \quad (1.5)$$

Let the functions  $f_j(z)$  given by

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k \quad (j = 1, 2) \quad (1.6)$$

be in the class  $\mathcal{T}(p, n)$ . Then the quasi-convolution (or *modified* Hadamard product)  $(f_1 * f_2)(z)$  of the functions  $f_1(z)$  and  $f_2(z)$  is defined here (and in what follows) by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,1} a_{k,2} z^k. \quad (1.7)$$

The quasi-convolution (1.7) was introduced and studied earlier by Owa ([1] and [2]), and by Schild and Silverman [3] for  $p = 1$ . [See also Srivastava *et al.* ([4] and [5]).] In the present paper we aim at giving several quasi-convolution properties of functions in the subclasses  $\mathcal{A}(p, n, \alpha)$  and  $\mathcal{B}(p, n, \alpha)$  characterized by (1.4) and (1.5), respectively.

## 2. A Set of Lemmas

We begin by recalling the following lemmas due to Yamakawa [6], which will be needed in proving our main results (Theorem 1 and Theorem 2 below).

**Lemma 1** (Yamakawa [6]). *If  $f_j(z) \in \mathcal{A}(p, n, \alpha_j)$  ( $j = 1, 2$ ), then  $(f_1 * f_2)(z) \in \mathcal{A}(p, n, \beta)$ , where*

$$\beta = \frac{[p + 2n - \alpha_1 p(p + n)][p + 2n - \alpha_2 p(p + n)] - p(1 - \alpha_1 p)(1 - \alpha_2 p)(p + 2n)}{p \{ [p + 2n - \alpha_1 p(p + n)][p + 2n - \alpha_2 p(p + n)] - p(1 - \alpha_1 p)(1 - \alpha_2 p)(p + n) \}}. \quad (2.1)$$

The result is sharp for the functions  $f_j(z)$  given by

$$f_j(z) = z^p - \frac{p(1 - \alpha_j p)}{p + 2n - \alpha_j p(p + n)} z^{p+n} \quad (j = 1, 2). \quad (2.2)$$

**Lemma 2** (Yamakawa [6]). *If  $f_j(z) \in \mathcal{B}(p, n, \alpha_j)$  ( $j = 1, 2$ ), then  $(f_1 * f_2)(z) \in \mathcal{B}(p, n, \beta)$ , where*

$$\beta = \frac{[p + 2n - \alpha_1 p(p + n)][p + 2n - \alpha_2 p(p + n)](p + n) - p^2(1 - \alpha_1 p)(1 - \alpha_2 p)(p + 2n)}{p \{ [p + 2n - \alpha_1 p(p + n)][p + 2n - \alpha_2 p(p + n)](p + n) - p^2(1 - \alpha_1 p)(1 - \alpha_2 p)(p + n) \}}. \quad (2.3)$$

The result is sharp for the functions  $f_j(z)$  given by

$$f_j(z) = z^p - \frac{p^2(1 - \alpha_j p)}{(p + n)[p + 2n - \alpha_j p(p + n)]} z^{p+n} \quad (j = 1, 2). \quad (2.4)$$

### §3. Main Results and Their Consequences

One of our main results is contained in

**Theorem 1.** *If  $f_j(z) \in \mathcal{A}(p, n, \alpha_j)$  for each  $j = 1, \dots, m$ , then*

$$(f_1 * f_2 * \dots * f_m)(z) \in \mathcal{A}(p, n, \beta),$$

where

$$\beta = \frac{1}{p} - \frac{n p^{m-2} \prod_{j=1}^m (1 - \alpha_j p)}{\prod_{j=1}^m [p + 2n - \alpha_j p(p + n)] - p^{m-1}(p + n) \prod_{j=1}^m (1 - \alpha_j p)}. \quad (3.1)$$

The result is sharp for the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) given by

$$f_j(z) = z^p - \frac{p(1 - \alpha_j p)}{p + 2n - \alpha_j p(p + n)} z^{p+n} \quad (j = 1, \dots, m). \quad (3.2)$$

*Proof.* Our proof of Theorem 1 is by induction on  $m$ . Indeed the assertion of Theorem 1 holds true when  $m = 1$ . For  $m = 2$ , we find from Lemma 1 that  $(f_1 * f_2)(z) \in \mathcal{A}(p, n, \beta)$  with

$$\begin{aligned} \beta &= \frac{[p + 2n - \alpha_1 p(p + n)][p + 2n - \alpha_2 p(p + n)] - p(1 - \alpha_1 p)(1 - \alpha_2 p)(p + 2n)}{p \{ [p + 2n - \alpha_1 p(p + n)][p + 2n - \alpha_2 p(p + n)] - p(1 - \alpha_1 p)(1 - \alpha_2 p)(p + n) \}} \\ &= \frac{1}{p} - \frac{n(1 - \alpha_1 p)(1 - \alpha_2 p)}{[p + 2n - \alpha_1 p(p + n)][p + 2n - \alpha_2 p(p + n)] - p(1 - \alpha_1 p)(1 - \alpha_2 p)(p + n)}. \end{aligned} \quad (3.3)$$

Therefore, Theorem 1 is true also for  $m = 2$ .

Next we suppose that Theorem 1 is true for a fixed natural number  $m$ . Then, applying Lemma 1 once again, we see that

$$(f_1 * f_2 * \cdots * f_{m+1})(z) = (f_1 * f_2 * \cdots * f_m)(z) * f_{m+1}(z) \in \mathcal{A}(p, n, \gamma), \quad (3.4)$$

where

$$\begin{aligned} \gamma &= \frac{1}{p} - \frac{n(1 - \alpha_{m+1}p)(1 - \beta p)}{[p + 2n - \alpha_{m+1}p(p + n)][p + 2n - \beta p(p + n)] - p(1 - \alpha_{m+1}p)(1 - \beta p)(p + n)} \\ &= \frac{1}{p} - \frac{n p^{m-1} \prod_{j=1}^{m+1} (1 - \alpha_j p)}{\prod_{j=1}^{m+1} [p + 2n - \alpha_j p(p + n)] - p^m(p + n) \prod_{j=1}^{m+1} (1 - \alpha_j p)}. \end{aligned} \quad (3.5)$$

Therefore, Theorem 1 is true also for  $m + 1$ . Thus, by mathematical induction, we conclude that Theorem 1 is true for any natural number  $m$ .

Finally, if we take the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) given by (3.2), then we have

$$\begin{aligned} (f_1 * f_2 * \cdots * f_m)(z) &= z^p - p^m \frac{\prod_{j=1}^m (1 - \alpha_j p)}{\prod_{j=1}^m [p + 2n - \alpha_j p(p + n)]} z^{p+n} \\ &= z^p - \Omega z^{p+n} \end{aligned} \quad (3.6)$$

where, for convenience,

$$\Omega = p^m \frac{\prod_{j=1}^m (1 - \alpha_j p)}{\prod_{j=1}^m [p + 2n - \alpha_j p(p + n)]}. \quad (3.7)$$

Therefore, in view of (1.4), we obtain

$$\begin{aligned} &\sum_{k=p+n}^{\infty} \frac{2k - \beta p k - p}{p(1 - \beta p)} \Omega \\ &= \frac{2(p + n) - \beta p(p + n) - p}{p(1 - \beta p)} \cdot \frac{p^m \prod_{j=1}^m (1 - \alpha_j p)}{\prod_{j=1}^m [p + 2n - \alpha_j p(p + n)]} \\ &= 1, \end{aligned}$$

which shows that the assertion of Theorem 1 is sharp for the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) given by (3.2).

Setting  $\alpha_j = \alpha$  ( $j = 1, \dots, m$ ) in Theorem 1, we readily obtain

**Corollary 1.** *If  $f_j(z) \in \mathcal{A}(p, n, \alpha)$  for all  $j = 1, \dots, m$ , then*

$$(f_1 * f_2 * \dots * f_m)(z) \in \mathcal{A}(p, n, \beta),$$

where

$$\beta = \frac{1}{p} - \frac{n p^{m-2} (1 - \alpha p)^m}{[p + 2n - \alpha p(p + n)]^m - p^{m-1} (p + n) (1 - \alpha p)^m}. \quad (3.8)$$

The result is sharp for the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) given by

$$f_j(z) = z^p - \frac{p(1 - \alpha p)}{p + 2n - \alpha p(p + n)} z^{p+n} \quad (j = 1, \dots, m). \quad (3.9)$$

The proof of Theorem 1 can be applied *mutatis mutandis* in order to derive

**Theorem 2.** *If  $f_j(z) \in \mathcal{B}(p, n, \alpha_j)$  for each  $j = 1, \dots, m$ , then*

$$(f_1 * f_2 * \dots * f_m)(z) \in \mathcal{B}(p, n, \beta),$$

where

$$\beta = \frac{1}{p} - \frac{n p^{m-1} \prod_{j=1}^m (1 - \alpha_j p)}{(p + n) \left\{ (p + n)^{m-2} \prod_{j=1}^m [p + 2n - p(p + n)\alpha_j] - p^m \prod_{j=1}^m (1 - \alpha_j p) \right\}}. \quad (3.10)$$

The result is sharp for the functions  $f_j(z)$  given by

$$f_j(z) = z^p - \frac{p^2(1 - \alpha_j p)}{(p + n)[p + 2n - p(p + n)\alpha_j]} z^{p+n} \quad (j = 1, \dots, m). \quad (3.11)$$

For  $\alpha_j = \alpha$  ( $j = 1, \dots, m$ ), Theorem 2 immediately yields

**Corollary 2.** *If  $f_j(z) \in \mathcal{B}(p, n, \alpha)$  for all  $j = 1, \dots, m$ , then*

$$(f_1 * f_2 * \dots * f_m)(z) \in \mathcal{B}(p, n, \beta),$$

where

$$\beta = \frac{1}{p} - \frac{n p^{m-1} (1 - \alpha p)^m}{(p + n) \{ (p + n)^{m-2} [p + 2n - p(p + n)\alpha]^m - p^m (1 - \alpha p)^m \}}. \quad (3.12)$$

The result is sharp for the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) given by

$$f_j(z) = z^p - \frac{p^2(1 - \alpha p)}{(p + n)[p + 2n - p(p + n)\alpha]} z^{p+n} \quad (j = 1, \dots, m). \quad (3.13)$$

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