

***CERTAIN FAMILIES OF ANALYTIC FUNCTIONS  
WITH NEGATIVE COEFFICIENTS***

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## Abstract

Let  $\mathcal{F}_\lambda(\alpha, A, B)$  denote the class of functions  $f$  of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$

which are analytic and univalent in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and satisfy the condition:

$$\left| \frac{[(1-\lambda)z^{-1}f(z) + \lambda f'(z)] - 1}{B[(1-\lambda)z^{-1}f(z) + \lambda f'(z)] - [B + (A-B)(1-\alpha)]} \right| < 1$$

( $\lambda \geq 0$ ;  $0 \leq \alpha < 1$ ;  $-1 \leq A < B \leq 1$ ;  $0 < B \leq 1$ ;  $z \in \mathcal{U}$ ).

Also let  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$  if and only if  $zf'(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ . In this paper we give basic properties of the function classes  $\mathcal{F}_\lambda(\alpha, A, B)$  and  $\mathcal{R}(\lambda, \alpha, A, B)$  and obtain numerous sharp results including (for example) coefficient estimates, distortion theorems, closure theorems, linear combinations, and modified Hadamard products of several functions belonging to the classes  $\mathcal{F}_\lambda(\alpha, A, B)$  and  $\mathcal{R}(\lambda, \alpha, A, B)$ . We also obtain radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class  $\mathcal{F}_\lambda(\alpha, A, B)$  and consider integral operators

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associated with functions belonging to the class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Finally, we determine the support points of these classes of analytic and univalent functions with negative coefficients, and extend some of the aforementioned distortion theorems to hold true for certain operators of fractional calculus (that is, fractional integral and fractional derivative).

## 1. Introduction

Let  $\mathcal{S}$  denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic and univalent in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let  $\mathcal{T}$  denote the subclass of  $\mathcal{S}$  whose elements can be expressed in the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \quad (1.2)$$

A function  $f(z) \in \mathcal{T}$  is said to be in the class  $\mathcal{F}_\lambda(\alpha)$  if it satisfies the inequality:

$$\Re \left\{ (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \right\} > \alpha \quad (1.3)$$

for some  $\lambda$  ( $\lambda \geq 0$ ) and  $\alpha$  ( $0 \leq \alpha < 1$ ), and for all  $z \in \mathcal{U}$ . Further, a function  $f(z)$  belonging to  $\mathcal{T}$  is said to be in the class  $\mathcal{R}(\lambda, \alpha)$  if it satisfies the inequality:

$$\Re \{ f'(z) + \lambda z f''(z) \} > \alpha \quad (1.4)$$

for some  $\lambda$  ( $\lambda \geq 0$ ) and  $\alpha$  ( $0 \leq \alpha < 1$ ), and for all  $z \in \mathcal{U}$ . It is clear that

$$f(z) \in \mathcal{R}(\lambda, \alpha) \iff z f'(z) \in \mathcal{F}_\lambda(\alpha). \quad (1.5)$$

A function  $f(z)$  belonging to  $\mathcal{T}$  is said to be in the class

$$\mathcal{F}_\lambda(\alpha, A, B) \quad (\lambda \geq 0; \quad 0 \leq \alpha < 1; \quad -1 \leq A < B \leq 1; \quad 0 < B \leq 1)$$

if and only if there exists a function  $w(z)$  analytic in  $\mathcal{U}$ , and satisfying  $w(0) = 0$  and  $|w(z)| < 1$  for  $z \in \mathcal{U}$ , such that

$$\begin{aligned} (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) &= (1 - \alpha) \frac{1 + Aw(z)}{1 + Bw(z)} + \alpha \\ &= \frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}. \end{aligned} \quad (1.6)$$

It is easy to see that the above condition is equivalent to

$$\left| \frac{[(1 - \lambda) z^{-1} f(z) + \lambda f'(z)] - 1}{B[(1 - \lambda) z^{-1} f(z) + \lambda f'(z)] - [B + (A - B)(1 - \alpha)]} \right| < 1 \quad (z \in \mathcal{U}). \quad (1.7)$$

Furthermore, a function  $f(z) \in \mathcal{T}$  is said to belong to the class  $\mathcal{R}(\lambda, \alpha, A, B)$  if and only if

$$zf'(z) \in \mathcal{F}_\lambda(\alpha, A, B).$$

We note that, by specializing the parameters  $\lambda$ ,  $\alpha$ ,  $A$ , and  $B$ , we obtain the following interesting subclasses including those that were studied by various earlier authors.

- (i)  $\mathcal{F}_\lambda(\alpha, -1, 1) = \mathcal{F}_\lambda(\alpha)$  (Bhoosnurmath and Swamy [5]);
- (ii)  $\mathcal{R}(\lambda, \alpha, -1, 1) = \mathcal{R}(\lambda, \alpha)$  (Altıntaş [1]);
- (iii)  $\mathcal{F}_0(\alpha, -1, 1)$  (Sarangi and Uralegaddi [17]);
- (iv)  $\mathcal{F}_1(\alpha, -1, 1) = \mathcal{R}(0, \alpha, -1, 1)$  (Sarangi and Uralegaddi [17] and Al-Amiri [2]);
- (v)  $\mathcal{F}_1(\alpha, -\beta, \beta) = \mathcal{R}(0, \alpha, -\beta, \beta) = \mathcal{P}^*(\alpha, \beta)$  ( $0 < \beta \leq 1$ ) (Gupta and Jain [10]);
- (vi)  $\mathcal{R}(1, \alpha, -\beta, \beta)$  ( $0 < \beta \leq 1$ ) denotes the class of functions  $f(z) \in \mathcal{T}$  such that  $zf'(z) \in \mathcal{P}^*(\alpha, \beta)$ ;
- (vii)  $\mathcal{F}_1(0, A, B) = \mathcal{R}(0, 0, A, B) = \mathcal{P}^*(1, A, B)$  (Shukla and Dashrath [19] and Owa and Srivastava [14]);
- (viii)  $\mathcal{R}(1, 0, A, B)$  denotes the class of functions  $f(z) \in \mathcal{T}$  such that  $zf'(z) \in \mathcal{P}^*(1, A, B)$  (Owa and Srivastava [14]);
- (ix)  $\mathcal{F}_1(\alpha, A, B) = \mathcal{R}(0, \alpha, A, B) = \mathcal{P}(1, A, B, \alpha)$  (Aouf [3]);
- (x)  $\mathcal{R}(1, \alpha, A, B)$  denotes the class of functions  $f(z) \in \mathcal{T}$  such that

$$zf'(z) \in \mathcal{P}(1, A, B, \alpha).$$

## 2. A Theorem on Coefficient Estimates

**Theorem 1.** *Let the function  $f(z)$  be defined by (1.2). Then  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  if and only if*

$$\sum_{n=2}^{\infty} (1+B)[1+(n-1)\lambda] a_n \leq (B-A)(1-\alpha). \quad (2.1)$$

*The result is sharp.*

**Proof.** Assume that the inequality (2.1) holds true and let  $|z| = 1$ . Then we obtain

$$\begin{aligned} & |[(1-\lambda)f(z) + \lambda z f'(z)] - z| - |B[(1-\lambda)f(z) + \lambda z f'(z)] \\ & \quad - [B + (A-B)(1-\alpha)]z| = \left| - \sum_{n=2}^{\infty} [1+(n-1)\lambda] a_n z^n \right| \\ & \quad - \left| (B-A)(1-\alpha)z - B \sum_{n=2}^{\infty} [1+(n-1)\lambda] a_n z^n \right| \\ & \leq \sum_{n=2}^{\infty} (1+B)[1+(n-1)\lambda] a_n - (B-A)(1-\alpha) \\ & \leq 0, \end{aligned}$$

by hypothesis. Hence, by the maximum modulus theorem, we have  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ .

To prove the converse, assume that  $f(z)$  is defined by (1.2) and is in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ , so that the condition (1.7) readily yields

$$\begin{aligned} & \left| \frac{[(1-\lambda)f(z) + \lambda z f'(z)] - z}{B[(1-\lambda)f(z) + \lambda z f'(z)] - [B + (A-B)(1-\alpha)]z} \right| \\ & = \left| \frac{- \sum_{n=2}^{\infty} [1+(n-1)\lambda] a_n z^n}{(B-A)(1-\alpha)z - B \sum_{n=2}^{\infty} [1+(n-1)\lambda] a_n z^n} \right| < 1 \quad (z \in \mathcal{U}). \end{aligned} \quad (2.2)$$

Since  $|\Re(z)| \leq |z|$  for all  $z$ , we find from (2.2) that

$$\Re \left\{ \frac{\sum_{n=2}^{\infty} [1+(n-1)\lambda] a_n z^n}{(B-A)(1-\alpha)z - B \sum_{n=2}^{\infty} [1+(n-1)\lambda] a_n z^n} \right\} < 1. \quad (2.3)$$

Choose values of  $z$  on the real axis so that  $[(1-\lambda)z^{-1}f(z) + \lambda f'(z)]$  is real. Upon clearing the denominator in (2.3) and letting  $z \rightarrow 1-$  through *real* values, we have

$$\sum_{n=2}^{\infty} [1 + (n-1)\lambda] a_n \leq (B-A)(1-\alpha) - B \sum_{n=2}^{\infty} [1 + (n-1)\lambda] a_n, \quad (2.4)$$

which obviously is the required assertion (2.1).

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z) = z - \frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} z^n \quad (n \geq 2). \quad (2.5)$$

**Corollary 1.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then*

$$a_n \leq \frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} \quad (n \geq 2). \quad (2.6)$$

*The equality in (2.6) is attained for the function  $f(z)$  given by (2.5).*

**Corollary 2.** *Let the function  $f(z)$  be defined by (1.2). Then  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$  if and only if*

$$\sum_{n=2}^{\infty} n(1+B)[1+(n-1)\lambda] a_n \leq (B-A)(1-\alpha). \quad (2.7)$$

*The result is sharp for the function  $f(z)$  given by*

$$f(z) = z - \frac{(B-A)(1-\alpha)}{n(1+B)[1+(n-1)\lambda]} z^n \quad (n \geq 2). \quad (2.8)$$

**Proof.** Note that  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$  if and only if  $zf'(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ . Therefore, Corollary 2 follows easily from Theorem 1.

**Corollary 3.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then*

$$a_n \leq \frac{(B-A)(1-\alpha)}{n(1+B)[1+(n-1)\lambda]} \quad (n \geq 2). \quad (2.9)$$

*The equality in (2.9) is attained for the function  $f(z)$  given by (2.8).*

### 3. Basic Properties of the Classes $\mathcal{F}_\lambda(\alpha, A, B)$ and $\mathcal{R}(\lambda, \alpha, A, B)$

We begin by recalling the following result:

**Lemma 1** (Aouf [3, p. 222, Theorem 1]). *A function  $f(z)$  defined by (1.2) is in the class  $\mathcal{P}(1, A, B, \alpha)$  if and only if*

$$\sum_{n=2}^{\infty} n(1+B)a_n \leq (B-A)(1-\alpha). \quad (3.1)$$

*The result is sharp.*

Making use of Lemma 1, we now prove

**Theorem 2.** *If the function  $f(z)$  defined by (1.2) is in the class  $\mathcal{P}(1, A, B, \alpha)$  ( $0 \leq \alpha < 1$ ), then  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  provided that  $0 \leq \lambda \leq 1$ .*

**Proof.** It follows from the hypothesis  $0 \leq \lambda \leq 1$  that

$$(1+B)\{n - [1 + (n-1)\lambda]\} = (1+B)(n-1)(1-\lambda) \geq 0 \quad (n \geq 2; 0 < B \leq 1). \quad (3.2)$$

Thus, by means of Lemma 1, we have

$$\begin{aligned} \sum_{n=2}^{\infty} (1+B)[1 + (n-1)\lambda] a_n &\leq \sum_{n=2}^{\infty} n(1+B)a_n \\ &\leq (B-A)(1-\alpha), \end{aligned}$$

which (in view of Theorem 1) proves the assertion that  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ .

**Theorem 3.** *Let  $0 \leq \lambda_1 \leq \lambda_2$ ,  $0 \leq \alpha < 1$ ,  $-1 \leq A < B \leq 1$ , and  $0 < B \leq 1$ . Then*

$$\mathcal{F}_{\lambda_2}(\alpha, A, B) \subseteq \mathcal{F}_{\lambda_1}(\alpha, A, B).$$

**Proof.** It follows from Theorem 1 that

$$\begin{aligned} \sum_{n=2}^{\infty} (1+B)[1 + (n-1)\lambda_1] a_n &\leq \sum_{n=2}^{\infty} (1+B)[1 + (n-1)\lambda_2] a_n \\ &\leq (B-A)(1-\alpha) \end{aligned} \quad (3.3)$$

for  $f(z) \in \mathcal{F}_{\lambda_2}(\alpha, A, B)$ .

**Corollary 4.** *Let  $0 \leq \lambda_1 \leq \lambda_2$ ,  $0 \leq \alpha < 1$ ,  $-1 \leq A < B \leq 1$ , and  $0 < B \leq 1$ . Then*

$$\mathcal{R}(\lambda_2, \alpha, A, B) \subseteq \mathcal{R}(\lambda_1, \alpha, A, B).$$

**Theorem 4.** *Let  $\lambda \geq 0$ ,  $0 \leq \alpha_1 \leq \alpha_2 < 1$ ,  $-1 \leq A < B \leq 1$ , and  $0 < B \leq 1$ . Then*

$$\mathcal{F}_\lambda(\alpha_2, A, B) \subseteq \mathcal{F}_\lambda(\alpha_1, A, B).$$

The proof of Theorem 4 uses Theorem 1 in a straightforward manner. The details may be omitted.

**Corollary 5.** *Let  $\lambda \geq 0$ ,  $0 \leq \alpha_1 \leq \alpha_2 < 1$ ,  $-1 \leq A < B \leq 1$ , and  $0 < B \leq 1$ . Then*

$$\mathcal{R}(\lambda, \alpha_2, A, B) \subseteq \mathcal{R}(\lambda, \alpha_1, A, B).$$

**Theorem 5.** *Let  $\lambda \geq 0$ ,  $0 \leq \alpha < 1$ ,  $-1 \leq A_1 \leq A_2 < 1$ ,  $0 < B_1 \leq B_2 \leq 1$ , and  $A_2 < B_1$ . Then*

$$\mathcal{F}_\lambda(\alpha, A_2, B_1) \subseteq \mathcal{F}_\lambda(\alpha, A_1, B_2).$$

Theorem 5 follows easily from the definition of the class  $\mathcal{F}_\lambda(\alpha, A, B)$ .

**Theorem 6.** *Let  $\lambda \geq 0$ ,  $0 \leq \alpha < 1$ ,  $-1 \leq A_1 \leq A_2 < 1$ ,  $0 < B_1 \leq B_2 \leq 1$ , and  $A_2 < B_1$ . Then*

$$\mathcal{R}(\lambda, \alpha, A_2, B_1) \subseteq \mathcal{R}(\lambda, \alpha, A_1, B_2).$$

**Proof.** Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda, \alpha, A_2, B_1)$  and let  $B_2 = B_1 + \epsilon$ . Then, by using Corollary 2, we have

$$\sum_{n=2}^{\infty} n(1 + B_1) [1 + (n-1)\lambda] a_n \leq (B_1 - A_2)(1 - \alpha). \quad (3.4)$$

Hence

$$\begin{aligned}
& \sum_{n=2}^{\infty} n(1+B_2) [1+(n-1)\lambda] a_n \\
&= \sum_{n=2}^{\infty} n(1+B_1+\epsilon) [1+(n-1)\lambda] a_n \\
&= \sum_{n=2}^{\infty} n(1+B_1) [1+(n-1)\lambda] a_n + \epsilon \sum_{n=2}^{\infty} n [1+(n-1)\lambda] a_n \\
&\leq (B_1 - A_2)(1 - \alpha) + \epsilon \frac{(B_1 - A_2)(1 - \alpha)}{1 + B_1} \\
&\leq (B_1 - A_2)(1 - \alpha) + \epsilon(B_1 - A_2)(1 - \alpha) \\
&\leq (B_2 - A_1)(1 - \alpha),
\end{aligned}$$

which exhibits the fact that  $f(z) \in \mathcal{R}(\lambda, \alpha, A_1, B_2)$ .

**Theorem 7.** *Let  $0 \leq \lambda_1 \leq \lambda_2$ ,  $0 \leq \alpha_1 \leq \alpha_2 < 1$ ,  $-1 \leq A_1 \leq A_2 < B_1 \leq B_2 \leq 1$ , and  $0 < B_1 \leq B_2 \leq 1$ . Then*

$$\mathcal{F}_{\lambda_2}(\alpha_2, A_2, B_1) \subseteq \mathcal{F}_{\lambda_1}(\alpha_1, A_1, B_2).$$

Theorem 7 is an immediate consequence of the definition of the class  $\mathcal{F}_{\lambda}(\alpha, A, B)$ .

**Theorem 8.** *Let  $0 \leq \lambda_1 \leq \lambda_2$ ,  $0 \leq \alpha_1 \leq \alpha_2 < 1$ ,  $-1 \leq A_1 \leq A_2 < B_1 \leq B_2 \leq 1$ , and  $0 < B_1 \leq B_2 \leq 1$ . Then*

$$\mathcal{R}(\lambda_2, \alpha_2, A_2, B_1) \subseteq \mathcal{R}(\lambda_1, \alpha_1, A_1, B_2).$$

**Proof.** Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda_2, \alpha_2, A_2, B_1)$  and let  $B_2 = B_1 + \epsilon$ . Then, by using Corollary 2, we get

$$\sum_{n=2}^{\infty} n(1+B_1) [1+(n-1)\lambda_2] a_n \leq (B_1 - A_2)(1 - \alpha_2). \quad (3.5)$$

Hence

$$\begin{aligned}
& \sum_{n=2}^{\infty} n(1+B_2) [1+(n-1)\lambda_1] a_n \\
&= \sum_{n=2}^{\infty} n(1+B_1+\epsilon) [1+(n-1)\lambda_1] a_n \\
&= \sum_{n=2}^{\infty} n(1+B_1) [1+(n-1)\lambda_1] a_n + \epsilon \sum_{n=2}^{\infty} n [1+(n-1)\lambda_1] a_n \\
&\leq \sum_{n=2}^{\infty} n(1+B_1) [1+(n-1)\lambda_2] a_n + \epsilon \sum_{n=2}^{\infty} n [1+(n-1)\lambda_2] a_n \\
&\leq (B_1 - A_2)(1 - \alpha_2) + \epsilon \frac{(B_1 - A_2)(1 - \alpha_2)}{1 + B_1} \\
&\leq (B_1 - A_2)(1 - \alpha_2) + \epsilon(B_1 - A_2)(1 - \alpha_2) \\
&\leq (B_2 - A_1)(1 - \alpha_1),
\end{aligned}$$

which implies that  $f(z) \in \mathcal{R}(\lambda_1, \alpha_1, A_1, B_2)$ .

**Theorem 9.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then  $f(z)$  also belongs to the following classes:*

(i)  $\mathcal{F}_\lambda \left( \frac{1+\alpha}{2}, A, B \right)$ , that is,

$$\mathcal{R}(\lambda, \alpha, A, B) \subseteq \mathcal{F}_\lambda \left( \frac{1+\alpha}{2}, A, B \right);$$

(ii)  $\mathcal{F}_\lambda \left( \alpha, \frac{B+A}{2}, B \right)$ , that is,

$$\mathcal{R}(\lambda, \alpha, A, B) \subseteq \mathcal{F}_\lambda \left( \alpha, \frac{B+A}{2}, B \right).$$

**Proof.** (i) Since  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ , we have

$$\begin{aligned}
\sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_n &\leq \frac{(B-A)(1-\alpha)}{2} \\
&= (B-A) \left( 1 - \frac{1+\alpha}{2} \right)
\end{aligned}$$

with the aid of Corollary 2. Furthermore, since

$$0 < \frac{1 + \alpha}{2} < 1$$

for  $0 \leq \alpha < 1$ , we have Part (i) of Theorem 9.

(ii) Similarly, since  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ , we have

$$\begin{aligned} \sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_n &\leq \frac{(B-A)(1-\alpha)}{2} \\ &= \left( B - \frac{B+A}{2} \right) (1-\alpha). \end{aligned}$$

Note also that

$$-1 \leq \frac{B+A}{2} < B \quad \text{for} \quad -1 \leq A < B \leq 1 \quad \text{and} \quad 0 < B \leq 1.$$

Thus Part (ii) of Theorem 9 follows at once.

**Theorem 10.** *Let the function  $f(z)$  defined by (1.2) and the function  $g(z)$  defined by*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0) \tag{3.6}$$

*be in the classes  $\mathcal{F}_\lambda(\alpha, A, B)$  and  $\mathcal{R}(\lambda, \alpha, A, B)$ , respectively. Then the function  $k(z)$  defined by*

$$k(z) = z - \frac{2}{3} \sum_{n=2}^{\infty} (a_n + b_n) z^n \quad (a_n \geq 0; \quad b_n \geq 0) \tag{3.7}$$

*is in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ .*

**Proof.** Since  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  and  $g(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ , by Theorem 1 and Corollary 2, we get

$$\sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_n \leq (B-A)(1-\alpha)$$

and

$$\sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] b_n \leq \frac{(B-A)(1-\alpha)}{2},$$

respectively. Therefore, we have

$$\frac{2}{3} \sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] (a_n + b_n) \leq (B-A)(1-\alpha),$$

which implies that  $k(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ , and the proof of Theorem 10 is thus completed.

#### 4. Distortion Theorems

**Theorem 11.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ .*

*Then, for  $|z| = r < 1$ ,*

$$r - \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r^2 \leq |f(z)| \leq r + \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r^2. \quad (4.1)$$

*Furthermore, for  $\lambda \geq 1$ ,*

$$1 - \frac{2(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r \leq |f'(z)| \leq 1 + \frac{2(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r. \quad (4.2)$$

*The estimates (4.1) and (4.2) are sharp.*

**Proof.** Since  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ , in view of Theorem 1, we have

$$\begin{aligned} (1+B)(1+\lambda) \sum_{n=2}^{\infty} a_n &\leq \sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_n \\ &\leq (B-A)(1-\alpha), \end{aligned} \quad (4.3)$$

which immediately yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)}. \quad (4.4)$$

Consequently, for  $|z| = r < 1$ , we obtain

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} a_n r^n \\ &\leq r + r^2 \sum_{n=2}^{\infty} a_n \\ &\leq r + \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r^2 \end{aligned} \quad (4.5)$$

and

$$\begin{aligned}
|f(z)| &\geq r - \sum_{n=2}^{\infty} a_n r^n \\
&\geq r - r^2 \sum_{n=2}^{\infty} a_n \\
&\geq r - \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r^2,
\end{aligned} \tag{4.6}$$

which prove the assertion (4.1) of Theorem 11.

From (4.4) and Theorem 1, it follows also that

$$\begin{aligned}
\sum_{n=2}^{\infty} n a_n &\leq \frac{(B-A)(1-\alpha)}{\lambda(1+B)} + \frac{\lambda-1}{\lambda} \sum_{n=2}^{\infty} a_n \\
&\leq \frac{2(B-A)(1-\alpha)}{(1+B)(1+\lambda)}.
\end{aligned} \tag{4.7}$$

Consequently, for  $|z| = r < 1$ , we have

$$\begin{aligned}
|f'(z)| &\leq 1 + \sum_{n=2}^{\infty} n a_n r^{n-1} \\
&\leq 1 + r \sum_{n=2}^{\infty} n a_n \\
&\leq 1 + \frac{2(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
|f'(z)| &\geq 1 - \sum_{n=2}^{\infty} n a_n r^{n-1} \\
&\geq 1 - r \sum_{n=2}^{\infty} n a_n \\
&\geq 1 - \frac{2(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r,
\end{aligned} \tag{4.9}$$

which prove the assertion (4.2) of Theorem 11.

The bounds in (4.1) and (4.2) are attained for the function  $f(z)$  given by

$$f(z) = z - \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} z^2 \quad (z = \pm r). \tag{4.10}$$

**Corollary 6.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then the disk  $|z| < 1$  is mapped onto a domain that contains the disk:*

$$|w| < \frac{(1+B)(1+\lambda) - (B-A)(1-\alpha)}{(1+B)(1+\lambda)}. \quad (4.11)$$

*The result is sharp, the extremal function  $f(z)$  being given by (4.10).*

Corollary 6 follows upon letting  $r \rightarrow 1-$  in the left-hand side of (4.1).

**Corollary 7.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then, for  $|z| = r < 1$ ,*

$$r - \frac{(B-A)(1-\alpha)}{2(1+B)(1+\lambda)} r^2 \leq |f(z)| \leq r + \frac{(B-A)(1-\alpha)}{2(1+B)(1+\lambda)} r^2, \quad (4.12)$$

and

$$1 - \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r \leq |f'(z)| \leq 1 + \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} r. \quad (4.13)$$

*Equalities in (4.12) and (4.13) hold true for the function  $f(z)$  given by*

$$f(z) = z - \frac{(B-A)(1-\alpha)}{2(1+B)(1+\lambda)} z^2 \quad (z = \pm r). \quad (4.14)$$

**Corollary 8.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then the disk  $|z| < 1$  is mapped onto a domain that contains the disk:*

$$|w| < \frac{2(1+B)(1+\lambda) - (B-A)(1-\alpha)}{2(1+B)(1+\lambda)}. \quad (4.15)$$

*The result is sharp, the extremal function  $f(z)$  being given by (4.14).*

Corollary 8 follows upon letting  $r \rightarrow 1-$  in the left-hand side of (4.12).

## 5. Radii of Close-to-Convexity, Starlikeness, and Convexity

**Theorem 12.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then  $f(z)$  is close-to-convex of order  $\rho$  ( $0 \leq \rho < 1$ ) in  $|z| < r_1(\lambda, \alpha, \rho, A, B)$ , where*

$$r_1(\lambda, \alpha, \rho, A, B) = \inf_n \left[ \frac{(1-\rho)(1+B)[1+(n-1)\lambda]}{n(B-A)(1-\alpha)} \right]^{1/(n-1)} \quad (n \geq 2). \quad (5.1)$$

The result is sharp, the extremal function  $f(z)$  being given by (2.5).

**Proof.** It is sufficient to show that

$$|f'(z) - 1| \leq 1 - \rho \quad \text{for } |z| < r_1(\lambda, \alpha, \rho, A, B).$$

Indeed we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus

$$|f'(z) - 1| \leq 1 - \rho \quad \text{if} \quad \sum_{n=2}^{\infty} \left( \frac{n}{1 - \rho} \right) a_n |z|^{n-1} \leq 1. \quad (5.2)$$

But, from Theorem 1, we obtain

$$\sum_{n=2}^{\infty} \frac{(1 + B) [1 + (n - 1)\lambda]}{(B - A)(1 - \alpha)} a_n \leq 1. \quad (5.3)$$

Hence (5.2) will be true if

$$\left( \frac{n}{1 - \rho} \right) |z|^{n-1} \leq \frac{(1 + B) [1 + (n - 1)\lambda]}{(B - A)(1 - \alpha)} \quad (n \geq 2). \quad (5.4)$$

Solving (5.4) for  $|z|$ , we obtain

$$|z| \leq \left[ \frac{(1 - \rho)(1 + B) [1 + (n - 1)\lambda]}{n(B - A)(1 - \alpha)} \right]^{1/(n-1)} \quad (n \geq 2). \quad (5.5)$$

Theorem 12 follows easily from (5.5).

**Theorem 13.** Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then  $f(z)$  is starlike of order  $\rho$  ( $0 \leq \rho < 1$ ) in  $|z| < r_2(\lambda, \alpha, \rho, A, B)$ , where

$$r_2(\lambda, \alpha, \rho, A, B) = \inf_n \left[ \frac{(1 - \rho)(1 + B) [1 + (n - 1)\lambda]}{(n - \rho)(B - A)(1 - \alpha)} \right]^{1/(n-1)} \quad (n \geq 2). \quad (5.6)$$

The result is sharp, the extremal function  $f(z)$  being given by (2.5).

**Proof.** We must show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad \text{for } |z| < r_2(\lambda, \alpha, \rho, A, B).$$

Indeed we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad \text{if} \quad \sum_{n=2}^{\infty} \frac{(n-\rho) a_n |z|^{n-1}}{1-\rho} \leq 1. \quad (5.7)$$

Hence, by using (5.3), (5.7) will be true if

$$\frac{(n-\rho) |z|^{n-1}}{1-\rho} \leq \frac{(1+B) [1+(n-1)\lambda]}{(B-A)(1-\alpha)} \quad (n \geq 2), \quad (5.8)$$

that is, if

$$|z| \leq \left[ \frac{(1-\rho)(1+B) [1+(n-1)\lambda]}{(n-\rho)(B-A)(1-\alpha)} \right]^{1/(n-1)} \quad (n \geq 2). \quad (5.9)$$

Theorem 13 follows easily from (5.9).

**Corollary 9.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then  $f(z)$  is convex of order  $\rho$  ( $0 \leq \rho < 1$ ) in  $|z| < r_3(\lambda, \alpha, \rho, A, B)$ , where*

$$r_3(\lambda, \alpha, \rho, A, B) = \inf_n \left[ \frac{(1-\rho)(1+B) [1+(n-1)\lambda]}{n(n-\rho)(B-A)(1-\alpha)} \right]^{1/(n-1)} \quad (n \geq 2). \quad (5.10)$$

*The result is sharp, the extremal function  $f(z)$  being given by (2.5).*

## 6. A Family of Integral Operators

**Theorem 14.** *Let  $c$  be a real number such that  $c > -1$ . If  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ , then the function  $F(z)$  defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \quad (6.1)$$

*also belongs to the class  $\mathcal{R}(\lambda, \alpha, A, B)$ .*

**Proof.** Let the function  $f(z)$  be defined by (1.2). Then, from the representation (6.1) of  $F(z)$ , it follows that

$$F(z) = z - \sum_{n=2}^{\infty} b_n z^n,$$

where

$$b_n = \left( \frac{c+1}{c+n} \right) a_n.$$

Therefore, we have

$$\begin{aligned} \sum_{n=2}^{\infty} n(1+B) [1+(n-1)\lambda] b_n &= \sum_{n=2}^{\infty} n(1+B) [1+(n-1)\lambda] \left( \frac{c+1}{c+n} \right) a_n \\ &\leq \sum_{n=2}^{\infty} n(1+B) [1+(n-1)\lambda] a_n \\ &\leq (B-A)(1-\alpha), \end{aligned}$$

since  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ . Hence, by Corollary 2,  $F(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ .

**Theorem 15.** *Let  $c$  be a real number such that  $c > -1$ . If  $F(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ , then the function  $f(z)$  given by (6.1) is univalent for  $|z| < R^*$ , where*

$$R^* = \inf_n \left[ \frac{(c+1)(1+B) [1+(n-1)\lambda]}{(c+n)(B-A)(1-\alpha)} \right]^{1/(n-1)} \quad (n \geq 2). \quad (6.2)$$

*The result is sharp.*

**Proof.** Let

$$F(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \quad (6.3)$$

It follows then from (6.1) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}}{c+1} \frac{d}{dz} \{z^c F(z)\} \\ &= z - \sum_{n=2}^{\infty} \left( \frac{c+n}{c+1} \right) a_n z^n. \end{aligned}$$

To prove the assertion of Theorem 15, it suffices to show that

$$|f'(z) - 1| < 1 \quad \text{for } |z| < R^*,$$

where  $R^*$  is defined by (6.2). Now

$$\begin{aligned} |f'(z) - 1| &= \left| - \sum_{n=2}^{\infty} \frac{n(c+n)}{c+1} a_n z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} n \left( \frac{c+n}{c+1} \right) a_n |z|^{n-1}. \end{aligned}$$

Thus

$$|f'(z) - 1| < 1 \quad \text{if} \quad \sum_{n=2}^{\infty} n \left( \frac{c+n}{c+1} \right) a_n |z|^{n-1} < 1. \quad (6.4)$$

But Corollary 2 confirms that

$$\sum_{n=2}^{\infty} \frac{n(1+B) [1+(n-1)\lambda]}{(B-A)(1-\alpha)} a_n \leq 1.$$

Thus (6.3) will be satisfied if

$$\left( \frac{c+n}{c+1} \right) |z|^{n-1} \leq \frac{(1+B) [1+(n-1)\lambda]}{(B-A)(1-\alpha)} \quad (n \geq 2),$$

that is, if

$$|z| \leq \left[ \frac{(c+1)(1+B) [1+(n-1)\lambda]}{(c+n)(B-A)(1-\alpha)} \right]^{1/(n-1)} \quad (n \geq 2). \quad (6.5)$$

The required result follows now from (6.4).

Finally, the result is sharp for the function  $f(z)$  given by

$$f(z) = z - \frac{(c+n)(B-A)(1-\alpha)}{n(c+1)(1+B) [1+(n-1)\lambda]} z^n \quad (n \geq 2). \quad (6.6)$$

## 7. Closure Theorems

Let the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \quad (a_{n,j} \geq 0; \quad j = 1, \dots, m; \quad z \in \mathcal{U}). \quad (7.1)$$

**Theorem 16.** *Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then the function  $h(z)$  defined by*

$$h(z) = z - \frac{1}{2} \sum_{n=2}^{\infty} (a_{n,1} + a_{n,2}) z^n \quad (a_{n,j} \geq 0; \quad j = 1, 2) \quad (7.2)$$

*is also in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ .*

**Proof.** Since  $f_1(z)$  and  $f_2(z)$  are in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ , by using Theorem 1 we have

$$\sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_{n,1} \leq (B-A)(1-\alpha) \quad (7.3)$$

and

$$\sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_{n,2} \leq (B-A)(1-\alpha). \quad (7.4)$$

From (7.3) and (7.4), we obtain

$$\frac{1}{2} \sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] (a_{n,1} + a_{n,2}) \leq (B-A)(1-\alpha). \quad (7.5)$$

Hence, by Theorem 1,  $h(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ , which completes the proof of Theorem 16.

**Corollary 10.** *Let the functions  $f_j(z)$  defined by (7.1) be in the classes  $\mathcal{F}_\lambda(\alpha_j, A_j, B_j)$  for each  $j = 1, \dots, m$ . Then the function  $h(z)$  defined by*

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left( \sum_{j=1}^m a_{n,j} \right) z^n \quad (7.6)$$

*is in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ , where*

$$\alpha = \min_{1 \leq j \leq m} \{\alpha_j\}, \quad A = \min_{1 \leq j \leq m} \{A_j\} \quad \text{and} \quad B = \max_{1 \leq j \leq m} \{B_j\}. \quad (7.7)$$

**Proof.** Since  $f_j(z) \in \mathcal{F}_\lambda(\alpha_j, A_j, B_j)$  for each  $j = 1, \dots, m$ , we observe that

$$\sum_{n=2}^{\infty} (1+B_j) [1+(n-1)\lambda] a_{n,j} \leq (B_j - A_j)(1-\alpha_j) \quad (7.8)$$

with the aid of Theorem 1. Therefore, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} [1+(n-1)\lambda] \left( \frac{1}{m} \sum_{j=1}^m a_{n,j} \right) \\ &= \frac{1}{m} \sum_{j=1}^m \left\{ \sum_{n=2}^{\infty} [1+(n-1)\lambda] a_{n,j} \right\} \\ &\leq \frac{1}{m} \sum_{j=1}^m \left\{ \frac{(B_j - A_j)(1-\alpha_j)}{1+B_j} \right\} \\ &\leq \frac{(B-A)(1-\alpha)}{1+B}. \end{aligned}$$

Thus

$$\sum_{n=2}^{\infty} (1+B)[1+(n-1)\lambda] \left( \frac{1}{m} \sum_{j=1}^m a_{n,j} \right) \leq (B-A)(1-\alpha), \quad (7.9)$$

which shows that  $h(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ , completing the proof of Theorem 17.

**Corollary 11.** *Let the functions  $f_j(z)$  defined by (7.1) be in the classes  $\mathcal{R}(\lambda, \alpha_j, A_j, B_j)$  for each  $j = 1, \dots, m$ . Then the function  $h(z)$  defined by (7.6) is also in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ , where  $\alpha$ ,  $A$ , and  $B$  are defined by (7.7).*

## 8. Theorems Depicting Extremal Properties

**Theorem 18.** *The class  $\mathcal{F}_\lambda(\alpha, A, B)$  is a convex set.*

**Proof.** Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then it is sufficient to prove that the function  $h(z)$  given by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \leq \mu \leq 1) \quad (8.1)$$

is also in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Since, for  $0 \leq \mu \leq 1$ ,

$$h(z) = z - \sum_{n=2}^{\infty} \{ \mu a_{n,1} + (1-\mu) a_{n,2} \} z^n, \quad (8.2)$$

with the aid of Theorem 1, we have

$$\sum_{n=2}^{\infty} (1+B)[1+(n-1)\lambda] \{ \mu a_{n,1} + (1-\mu) a_{n,2} \} \leq (B-A)(1-\alpha), \quad (8.3)$$

which implies that  $h(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ . Hence the class  $\mathcal{F}_\lambda(\alpha, A, B)$  is convex.

As a consequence of Theorem 18, there exist the extreme points of the class  $\mathcal{F}_\lambda(\alpha, A, B)$ , which are given by

**Theorem 19.** *Let  $f_1(z) = z$  and*

$$f_n(z) = z - \frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} z^n \quad (n \geq 2). \quad (8.4)$$

Then  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  if and only if it can be expressed in the form:

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

where

$$\mu_n \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \mu_n = 1.$$

**Proof.** Suppose that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \mu_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} \mu_n z^n. \end{aligned}$$

Then we get

$$\sum_{n=2}^{\infty} \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} \frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} \mu_n = \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1.$$

By virtue of Theorem 1, this shows that  $f(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ .

On the other hand, suppose that the function  $f(z)$  defined by (1.2) is in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Again, by using Theorem 1, we can show that

$$a_n \leq \frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} \quad (n \geq 2).$$

Setting

$$\mu_n = \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} a_n \quad (n \geq 2)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n,$$

we have

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

which completes the proof of Theorem 19.

**Corollary 12.** *The class  $\mathcal{R}(\lambda, \alpha, A, B)$  is a convex set.*

**Corollary 13.** Let  $f_1(z) = z$  and

$$f_n(z) = z - \frac{(B-A)(1-\alpha)}{n(1+B)[1+(n-1)\lambda]} z^n \quad (n \geq 2). \quad (8.5)$$

Then  $f(z) \in \mathcal{R}(\lambda, \alpha, A, B)$  if and only if it can be expressed in the form:

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z),$$

where

$$\mu_n \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \mu_n = 1.$$

## 9. Linear Combination of Functions in the Classes

$\mathcal{F}_\lambda(\alpha, A, B)$  and  $\mathcal{R}(\lambda, \alpha, A, B)$

**Theorem 20.** Let each of the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) defined by (7.1) be in the same class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then the function  $h(z)$  given by

$$h(z) = \frac{1}{m} \sum_{j=1}^m f_j(z) \quad (9.1)$$

is also in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ .

**Proof.** By the definition (9.1) of  $h(z)$ , we have the expansion:

$$h(z) = z - \sum_{n=2}^{\infty} \left( \frac{1}{m} \sum_{j=1}^m a_{n,j} \right) z^n. \quad (9.2)$$

Since  $f_j(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  for every  $j = 1, \dots, m$ , by using Theorem 1 we obtain

$$\sum_{n=2}^{\infty} (1+B)[1+(n-1)\lambda] \left( \frac{1}{m} \sum_{j=1}^m a_{n,j} \right) \leq (B-A)(1-\alpha), \quad (9.3)$$

which, in view of Theorem 1, yields Theorem 20.

**Corollary 14.** Let each of the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) defined by (7.1) be in the same class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then the function  $h(z)$  given by (9.1) is also in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ .

## 10. Theorems Involving Support Points

A function  $f(z)$  in the class  $\mathcal{F}_\lambda(\alpha, A, B)$  is said to be a support point of  $\mathcal{F}_\lambda(\alpha, A, B)$  if there exists a continuous linear functional  $J$  on  $\mathcal{T}$  such that  $\Re(J(f)) \geq \Re(J(g))$  for all  $g(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  and  $\Re(J)$  is nonconstant on  $\mathcal{F}_\lambda(\alpha, A, B)$ . We denote by  $\text{Supp} \{\mathcal{F}_\lambda(\alpha, A, B)\}$  the set of support points of  $\mathcal{F}_\lambda(\alpha, A, B)$ , and by  $\text{Ext} \{\mathcal{F}_\lambda(\alpha, A, B)\}$  the set of extreme points of  $\mathcal{F}_\lambda(\alpha, A, B)$ .

Let  $\mathcal{F}$  be a subfamily of univalent functions in the unit disk  $\mathcal{U}$  whose set of extreme points is countable, let  $f_0(z)$  be a support point of  $\mathcal{F}$ , and let  $J$  be a corresponding continuous linear functional. Let

$$\mathcal{G}_J = \{f \in \mathcal{F} : \Re(J(f)) = \Re(J(f_0))\}. \quad (10.1)$$

We begin by recalling the following known result:

**Lemma 2** (Deeb [7]). *Let  $\mathcal{G}_J$  be defined by (10.1). Then  $\mathcal{G}_J$  is convex,*

$$\text{Ext}\{\mathcal{G}_J\} \subset \text{Ext}\{\mathcal{F}\},$$

and

$$\mathcal{G}_J = \left\{ f \in \mathcal{F} : f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \left( \mu_n \geq 0; \right. \right. \\ \left. \left. \sum_{n=1}^{\infty} \mu_n = 1; f_n(z) \in \text{Ext}\{\mathcal{G}_J\} \right) \right\}. \quad (10.2)$$

Let  $\mathcal{A}_1$  be the class of functions of the form:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

which are analytic in the unit disk  $\mathcal{U}$ . Brickman *et al.* [6] proved the following result.

**Lemma 3.** *Let  $\{b_n\}_{n=0}^{\infty}$  be a sequence of complex numbers such that*

$$\limsup_{n \rightarrow \infty} |b_n|^{\frac{1}{n}} < 1,$$

and set

$$J(f) = \sum_{n=0}^{\infty} a_n b_n$$

for  $f(z) \in \mathcal{A}_1$ . Then  $J$  is a continuous linear functional on  $\mathcal{A}_1$ .

Conversely, any continuous linear functional on  $\mathcal{A}_1$  is given by the sequence  $\{b_n\}_{n=0}^{\infty}$  defined above.

Making use of Lemma 3, we now prove

**Theorem 21.** *The set  $\text{Supp } \{\mathcal{F}_\lambda(\alpha, A, B)\}$  of support points of  $\mathcal{F}_\lambda(\alpha, A, B)$  is given by*

$$\text{Supp } \{\mathcal{F}_\lambda(\alpha, A, B)\} = \left\{ f \in \mathcal{F}_\lambda(\alpha, A, B) : f(z) = z - \sum_{n=2}^{\infty} \frac{(B-A)(1-\alpha)\mu_n}{(1+B)[1+(n-1)\lambda]} z^n \right. \\ \left. \left( \mu_n \geq 0; \sum_{n=2}^{\infty} \mu_n \leq 1; \mu_j = 0 \text{ for some } j \geq 2 \right) \right\}.$$

**Proof.** Let

$$f_0(z) = z - \sum_{n=2}^{\infty} \frac{(B-A)(1-\alpha)\mu_n}{(1+B)[1+(n-1)\lambda]} z^n \quad (10.3)$$

be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ , where

$$\mu_n \geq 0, \quad \sum_{n=2}^{\infty} \mu_n \leq 1, \quad \text{and} \quad \mu_j = 0 \quad \text{for some } j \geq 2.$$

If  $b_n = 0$  ( $n \geq 2; n \neq j$ ) and  $b_1 = b_j = 1$ , then

$$\limsup_{n \rightarrow \infty} |b_n|^{\frac{1}{n}} < 1.$$

Therefore, by Lemma 3, we define the continuous linear functional  $J$  given by the sequence  $\{b_n\}_{n=0}^{\infty}$ . It follows that

$$J(f_0) = 1 \quad \text{and} \quad J(f) = 1 - a_j \leq 1 \quad \text{for } f(z) \in \mathcal{F}_\lambda(\alpha, A, B).$$

This implies that

$$\Re\{J(f_0)\} \geq \Re\{J(f)\} \quad \text{for all } f(z) \in \mathcal{F}_\lambda(\alpha, A, B).$$

Hence  $f_0(z)$  is a support point of the class  $\mathcal{F}_\lambda(\alpha, A, B)$ .

Conversely, suppose that  $f_0(z)$  is a support point of  $\mathcal{F}_\lambda(\alpha, A, B)$ , and that its continuous linear functional  $J$  is given by the sequence  $\{b_n\}_{n=0}^\infty$ . Then  $\Re(J)$  is also continuous and linear on  $\mathcal{F}_\lambda(\alpha, A, B)$ . Therefore, by the Krein-Milman theorem (see, *e.g.*, Duren [8, p. 281]), there exists an extreme point  $f_n(z)$  of  $\mathcal{F}_\lambda(\alpha, A, B)$  such that

$$\Re\{J(f_0)\} = \max\{\Re\{J(f)\} : f \in \mathcal{F}_\lambda(\alpha, A, B)\} = \Re\{J(f_n)\}. \quad (10.4)$$

Let

$$\mathcal{G}_J = \{f_n : \Re\{J(f_0)\} = \Re\{J(f_n)\}, \quad f_n \in \text{Ext} \{\mathcal{F}_\lambda(\alpha, A, B)\}\}.$$

Then we note that  $\text{Ext} \{\mathcal{F}_\lambda(\alpha, A, B)\}$  is countable by means of Theorem 19. If  $\mathcal{G}_J = \text{Ext} \{\mathcal{F}_\lambda(\alpha, A, B)\}$ , then  $\Re(J)$  must be constant on  $\mathcal{F}_\lambda(\alpha, A, B)$ . This contradicts the hypothesis that  $f_0(z)$  is a support point of  $\mathcal{F}_\lambda(\alpha, A, B)$ . Consequently, there exists a  $j$  such that

$\Re\{J(f_0)\} > \Re\{J(f_j)\}$ . It follows from this fact that

$$\text{Ext} \{\mathcal{G}_J\} \subset \{f_n : f_n \in \text{Ext}\{\mathcal{F}_\lambda(\alpha, A, B)\} \quad (n \geq 2; \quad n \neq j)\}. \quad (10.5)$$

Consequently, with the help of Lemma 2, we have

$$f_0(z) = \sum_{n=2}^{\infty} \mu_n f_n(z), \quad (10.6)$$

where

$$\mu_n \geq 0, \quad \sum_{n=2}^{\infty} \mu_n = 1, \quad \text{and} \quad f_n(z) \in \text{Ext} \{\mathcal{G}_J\} \quad (n \geq 2; \quad n \neq j).$$

Further, by using Theorem 19, we obtain

$$f_0(z) = z - \sum_{\substack{n=2 \\ n \neq j}}^{\infty} \frac{(B-A)(1-\alpha)\mu_n}{(1+B)[1+(n-1)\lambda]} z^n, \quad (10.7)$$

which completes the proof of Theorem 21.

By using the same technique as in the proof of Theorem 21, and with the aid of Corollary 2, we have

**Theorem 22.** *The set  $\text{Supp } \{\mathcal{R}(\lambda, \alpha, A, B)\}$  of support points of  $\mathcal{R}(\lambda, \alpha, A, B)$  is given by*

$$\begin{aligned} \text{Supp } \{\mathcal{R}(\lambda, \alpha, A, B)\} = & \left\{ f \in \mathcal{R}(\lambda, \alpha, A, B) : f(z) = z \right. \\ & \left. - \sum_{n=2}^{\infty} \frac{(B-A)(1-\alpha)\mu_n}{n(1+B)[1+(n-1)\lambda]} z^n \quad (\mu_n \geq 0; \right. \\ & \left. \sum_{n=2}^{\infty} \mu_n \leq 1; \quad \mu_j = 0 \text{ for some } j \geq 2) \right\}. \end{aligned} \quad (10.8)$$

## 11. Definitions and Applications of Fractional Calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (*cf.*, *e.g.*, [4], [9, Chapter 13], [11], [13], [15], [16], [20, p. 21 *et seq.*], and [22]). We find it to be convenient to recall here the following definitions which were used earlier by Owa [12] (and by Srivastava and Owa [21]).

**Definition 1.** The *fractional integral of order  $k$*  is defined, for a function  $f(z)$ , by

$$D_z^{-k} f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-k}} d\zeta \quad (k > 0), \quad (11.1)$$

where  $f(z)$  is an analytic function in a simply-connected region of the  $z$ -plane containing the origin, and the multiplicity of  $(z-\zeta)^{k-1}$  is removed by requiring  $\log(z-\zeta)$  to be real when  $z-\zeta > 0$ .

**Definition 2.** The *fractional derivative of order  $k$*  is defined, for a function  $f(z)$ , by

$$D_z^k f(z) = \frac{1}{\Gamma(1-k)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^k} d\zeta \quad (0 \leq k < 1), \quad (11.2)$$

where  $f(z)$  is constrained, and the multiplicity of  $(z-\zeta)^{-k}$  is removed, as in Definition 1.

**Definition 3.** Under the hypotheses of Definition 2, the *fractional derivative of order  $n+k$*  is defined by

$$D_z^{n+k} f(z) = \frac{d^n}{dz^n} D_z^k f(z) \quad (0 \leq k < 1; \quad n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}). \quad (11.3)$$

**Theorem 23.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then*

$$|D_z^{-k} f(z)| \geq \frac{|z|^{1+k}}{\Gamma(2+k)} \left\{ 1 - \frac{2(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} |z| \right\} \quad (11.4)$$

and

$$|D_z^{-k} f(z)| \leq \frac{|z|^{1+k}}{\Gamma(2+k)} \left\{ 1 + \frac{2(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} |z| \right\} \quad (11.5)$$

for  $k > 0$  and  $z \in \mathcal{U}$ . The result is sharp.

**Proof.** Let

$$\begin{aligned} F(z) &= \Gamma(2+k) z^{-k} D_z^{-k} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+k)}{\Gamma(n+1+k)} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \Theta(n) a_n z^n, \end{aligned} \quad (11.6)$$

where

$$\Theta(n) = \frac{\Gamma(n+1)\Gamma(2+k)}{\Gamma(n+1+k)} \quad (k > 0; n \geq 2). \quad (11.7)$$

It is easily seen that

$$0 < \Theta(n) \leq \Theta(2) = \frac{2}{2+k} \quad (k > 0; n \geq 2). \quad (11.8)$$

Therefore, by using Theorem 1 and (11.8), we find that

$$\begin{aligned} |F(z)| &\geq |z| - \Theta(2) |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{2(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} |z|^2 \end{aligned} \quad (11.9)$$

and

$$\begin{aligned} |F(z)| &\leq |z| + \Theta(2) |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{2(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} |z|^2, \end{aligned} \quad (11.10)$$

which prove the inequalities (11.4) and (11.5) of Theorem 23. Furthermore, the equalities in (11.4) and (11.5) are attained for the function  $f(z)$  defined by

$$D_z^{-k} f(z) = \frac{z^{1+k}}{\Gamma(2+k)} \left\{ 1 - \frac{2(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} z \right\} \quad (11.11)$$

or, equivalently, by

$$f(z) = z - \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} z^2. \quad (11.12)$$

**Corollary 15.** *Under the hypotheses of Theorem 23,  $D_z^{-k} f(z)$  is included in the disk with center at the origin and radius*

$$\frac{1}{\Gamma(2+k)} \left\{ 1 + \frac{2(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} \right\}.$$

Next, by using Lemma 2, we similarly have

**Theorem 24.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ .*

*Then*

$$|D_z^{-k} f(z)| \geq \frac{|z|^{1+k}}{\Gamma(2+k)} \left\{ 1 - \frac{(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} |z| \right\} \quad (11.13)$$

*and*

$$|D_z^{-k} f(z)| \leq \frac{|z|^{1+k}}{\Gamma(2+k)} \left\{ 1 + \frac{(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} |z| \right\} \quad (11.14)$$

*for  $k > 0$  and  $z \in \mathcal{U}$ . The result is sharp for the function  $f(z)$  given by*

$$f(z) = z - \frac{(B-A)(1-\alpha)}{2(1+B)(1+\lambda)} z^2. \quad (11.15)$$

**Corollary 16.** *Under the hypotheses of Theorem 24,  $D_z^{-k} f(z)$  is included in the disk with center at the origin and radius*

$$\frac{1}{\Gamma(2+k)} \left\{ 1 + \frac{(B-A)(1-\alpha)}{(2+k)(1+B)(1+\lambda)} \right\}.$$

**Theorem 25.** *Let the function  $f(z)$  defined by (1.2) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ .*

*Then*

$$|D_z^k f(z)| \geq \frac{|z|^{1-k}}{\Gamma(2-k)} \left\{ 1 - \frac{(B-A)(1-\alpha)}{(2-k)(1+B)(1+\lambda)} |z| \right\} \quad (11.16)$$

*and*

$$|D_z^k f(z)| \leq \frac{|z|^{1-k}}{\Gamma(2-k)} \left\{ 1 + \frac{(B-A)(1-\alpha)}{(2-k)(1+B)(1+\lambda)} |z| \right\} \quad (11.17)$$

*for  $0 \leq k < 1$  and  $z \in \mathcal{U}$ . The result is sharp.*

**Proof.** Let

$$\begin{aligned}
G(z) &= \Gamma(2-k) z^k D_z^k f(z) \\
&= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(2-k)}{\Gamma(n+1-k)} a_n z^n \\
&= z - \sum_{n=2}^{\infty} n \Phi(n) a_n z^n,
\end{aligned} \tag{11.18}$$

where

$$\Phi(n) = \frac{\Gamma(n) \Gamma(2-k)}{\Gamma(n+1-k)} \quad (0 \leq k < 1; n \geq 2). \tag{11.19}$$

Noting that

$$0 < \Phi(n) \leq \Phi(2) = \frac{1}{2-k} \quad (0 \leq k < 1; n \geq 2), \tag{11.20}$$

and with the aid of Corollary 2, we have

$$\begin{aligned}
|G(z)| &\geq |z| - \Phi(2) |z|^2 \sum_{n=2}^{\infty} n a_n \\
&\geq |z| - \frac{(B-A)(1-\alpha)}{(2-k)(1+B)(1+\lambda)} |z|^2
\end{aligned} \tag{11.21}$$

and

$$\begin{aligned}
|G(z)| &\leq |z| + \Phi(2) |z|^2 \sum_{n=2}^{\infty} n a_n \\
&\leq |z| + \frac{(B-A)(1-\alpha)}{(2-k)(1+B)(1+\lambda)} |z|^2
\end{aligned} \tag{11.22}$$

which prove the inequalities (11.16) and (11.17) of Theorem 25. Since the equalities in (11.16) and (11.17) are attained for the function  $f(z)$  defined by

$$D_z^k f(z) = \frac{z^{1-k}}{\Gamma(2-k)} \left\{ 1 - \frac{(B-A)(1-\alpha)}{(2-k)(1+B)(1+\lambda)} z \right\} \tag{11.23}$$

or, equivalently, by

$$f(z) = z - \frac{(B-A)(1-\alpha)}{2(1+B)(1+\lambda)} z^2, \tag{11.24}$$

we thus complete the proof of Theorem 25.

**Corollary 17.** *Under the hypotheses of Theorem 25,  $D_z^k f(z)$  is included in the disk with center at the origin and radius*

$$\frac{1}{\Gamma(2-k)} \left\{ 1 + \frac{(B-A)(1-\alpha)}{(2-k)(1+B)(1+\lambda)} \right\}.$$

## 12. A Generalized Fractional Integral Operator

We need the following definition of a generalized fractional integral operator given by Srivastava *et al.* [24].

**Definition 4.** For real numbers  $k > 0$ ,  $\gamma$ , and  $\delta$ , the *generalized fractional integral operator*  $I_{0,z}^{k,\gamma,\delta}$  is defined by

$$I_{0,z}^{k,\gamma,\delta} f(z) = \frac{z^{-k-\gamma}}{\Gamma(k)} \int_0^z (z-\zeta)^{k-1} F\left(k+\gamma, -\delta; k; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta, \quad (12.1)$$

where  $f(z)$  is an analytic function in a simply-connected region of the  $z$ -plane containing the origin, with the order

$$f(z) = O(|z|^\epsilon) \quad (z \rightarrow 0),$$

where  $\epsilon > \max\{0, \gamma - \delta\} - 1$ ,  $F(a, b; c; z)$  is the Gauss hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (12.2)$$

and  $(a)_n$  is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0) \\ a(a+1) \cdots (a+n-1) & (n \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}), \end{cases} \quad (12.3)$$

and the multiplicity of  $(z-\zeta)^{k-1}$  is removed by requiring  $\log(z-\zeta)$  to be real when  $z-\zeta > 0$ .

**Remark 1.** Clearly, in the special case when  $\gamma = -k$ , we have

$$I_{0,z}^{k,-k,\delta} f(z) = D_z^{-k} f(z),$$

where  $D_z^{-k}$  is the fractional integral operator given by Definition 1.

In order to prove our results for the generalized fractional integral operator  $I_{0,z}^{k,\gamma,\delta}$ , we recall here the following lemma:

**Lemma 4** (Srivastava *et al.* [24]). *If  $k > 0$  and  $\nu > \gamma - \delta - 1$ , then*

$$I_{0,z}^{k,\gamma,\delta} z^\nu = \frac{\Gamma(\nu+1) \Gamma(\nu-\gamma+\delta+1)}{\Gamma(\nu-\gamma+1) \Gamma(\nu+k+\delta+1)} z^{\nu-\gamma}. \quad (12.4)$$

With the aid of Lemma 4, we now prove

**Theorem 26.** *Let  $k > 0$ ,  $\gamma < 2$ ,  $k + \delta > -2$ ,  $\gamma - \delta < 2$ , and  $\gamma(k + \delta) \leq 3k$ . If the function  $f(z)$  defined by (1.2) is in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ , then*

$$\begin{aligned} \left| I_{0,z}^{k,\gamma,\delta} f(z) \right| &\geq \frac{\Gamma(2 - \gamma + \delta) |z|^{1-\gamma}}{\Gamma(2 - \gamma) \Gamma(2 + k + \delta)} \\ &\cdot \left\{ 1 - \frac{2(B - A)(1 - \alpha)(2 - \gamma + \delta)}{(1 + B)(1 + \lambda)(2 - \gamma)(2 + k + \delta)} |z| \right\} \end{aligned} \quad (12.5)$$

and

$$\begin{aligned} \left| I_{0,z}^{k,\gamma,\delta} f(z) \right| &\leq \frac{\Gamma(2 - \gamma + \delta) |z|^{1-\gamma}}{\Gamma(2 - \gamma) \Gamma(2 + k + \delta)} \\ &\cdot \left\{ 1 + \frac{2(B - A)(1 - \alpha)(2 - \gamma + \delta)}{(1 + B)(1 + \lambda)(2 - \gamma)(2 + k + \delta)} |z| \right\} \end{aligned} \quad (12.6)$$

for  $z \in \mathcal{U}_0$ , where

$$\mathcal{U}_0 = \begin{cases} \mathcal{U} & (\gamma \leq 1) \\ \mathcal{U} \setminus \{0\} & (\gamma > 1). \end{cases}$$

The equalities in (12.5) and (12.6) are attained for the function  $f(z)$  given by (11.12).

**Proof.** By using Lemma 4, we have

$$\begin{aligned} I_{0,z}^{k,\gamma,\delta} f(z) &= \frac{\Gamma(2 - \gamma + \delta)}{\Gamma(2 - \gamma) \Gamma(2 + k + \delta)} z^{1-\gamma} \\ &- \sum_{n=2}^{\infty} \frac{\Gamma(n + 1) \Gamma(n - \gamma + \delta + 1)}{\Gamma(n - \gamma + 1) \Gamma(n + k + \delta + 1)} a_n z^{n-\gamma}. \end{aligned} \quad (12.7)$$

Letting

$$\begin{aligned} H(z) &= \frac{\Gamma(2 - \gamma) \Gamma(2 + k + \delta)}{\Gamma(2 - \gamma + \delta)} z^\gamma I_{0,z}^{k,\gamma,\delta} f(z) \\ &= z - \sum_{n=2}^{\infty} \Psi(n) a_n z^n, \end{aligned} \quad (12.8)$$

where

$$\Psi(n) = \frac{(2 - \gamma + \delta)_{n-1} n!}{(2 - \gamma)_{n-1} (2 + k + \delta)_{n-1}} \quad (n \geq 2), \quad (12.9)$$

we can see that  $\Psi(n)$  is non-increasing for integers  $n \geq 2$ , and we have

$$0 < \Psi(n) \leq \Psi(2) = \frac{2(2 - \gamma + \delta)}{(2 - \gamma)(2 + k + \delta)} \quad (n \geq 2). \quad (12.10)$$

Therefore, by using Theorem 1 and (12.10), we have

$$\begin{aligned}
|H(z)| &\geq |z| - \Psi(2) |z|^2 \sum_{n=2}^{\infty} a_n \\
&\geq |z| - \frac{2(B-A)(1-\alpha)(2-\gamma+\delta)}{(1+B)(1+\lambda)(2-\gamma)(2+k+\delta)} |z|^2
\end{aligned} \tag{12.11}$$

and

$$\begin{aligned}
|H(z)| &\leq |z| + \Psi(2) |z|^2 \sum_{n=2}^{\infty} a_n \\
&\leq |z| + \frac{2(B-A)(1-\alpha)(2-\gamma+\delta)}{(1+B)(1+\lambda)(2-\gamma)(2+k+\delta)} |z|^2.
\end{aligned} \tag{12.12}$$

This completes the proof of Theorem 26.

**Remark 2.** In view of Remark 1, taking  $\gamma = -k$  in Theorem 26, we readily obtain Theorem 23 as a special case.

Finally, by employing the same technique as in the proof of Theorem 26, and applying Corollary 2, we obtain

**Theorem 27.** *Let  $k > 0$ ,  $\gamma < 2$ ,  $k + \delta > -2$ ,  $\gamma - \delta < 2$ , and  $\gamma(k + \delta) \leq 3k$ . If the function  $f(z)$  defined by (1.2) is in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ , then*

$$\begin{aligned}
\left| I_{0,z}^{k,\gamma,\delta} f(z) \right| &\geq \frac{\Gamma(2-\gamma+\delta) |z|^{1-\gamma}}{\Gamma(2-\gamma) \Gamma(2+k+\delta)} \\
&\cdot \left\{ 1 - \frac{(B-A)(1-\alpha)(2-\gamma+\delta)}{(1+B)(1+\lambda)(2-\gamma)(2+k+\delta)} |z| \right\}
\end{aligned} \tag{12.13}$$

and

$$\begin{aligned}
\left| I_{0,z}^{k,\gamma,\delta} f(z) \right| &\leq \frac{\Gamma(2-\gamma+\delta) |z|^{1-\gamma}}{\Gamma(2-\gamma) \Gamma(2+k+\delta)} \\
&\cdot \left\{ 1 + \frac{(B-A)(1-\alpha)(2-\gamma+\delta)}{(1+B)(1+\lambda)(2-\gamma)(2+k+\delta)} |z| \right\}
\end{aligned} \tag{12.14}$$

for  $z \in \mathcal{U}_0$ , where  $\mathcal{U}_0$  is defined as in Theorem 26. The equalities in (12.13) and (12.14) are attained for the function  $f(z)$  given by (11.24).

### 13. Inclusion Properties Involving Modified Hadamard Products

Let the functions  $f_j(z)$  ( $j = 1, \dots, m$ ) be defined, as before, by (7.1). Then the *modified* Hadamard product of  $f_1(z)$  and  $f_2(z)$  is defined here by

$$f_1 \star f_2(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \quad (13.1)$$

**Theorem 28.** *Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the classes  $\mathcal{F}_\lambda(\alpha_j, A_j, B_j)$  ( $j = 1, 2$ ), respectively. Then the modified Hadamard product  $f_1 \star f_2(z)$  defined by (13.1) belongs to the class  $\mathcal{F}_\lambda(\alpha(2 - \alpha), A(2B - A), B^2)$ , where*

$$\alpha = \min\{\alpha_1, \alpha_2\}, \quad A = \min\{A_1, A_2\}, \quad \text{and} \quad B = \max\{B_1, B_2\}. \quad (13.2)$$

**Proof.** Since  $f_1(z) \in \mathcal{F}_\lambda(\alpha_1, A_1, B_1)$  and  $f_2(z) \in \mathcal{F}_\lambda(\alpha_2, A_2, B_2)$ , by using Theorem 1, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} (1 + B^2) [1 + (n - 1)\lambda] a_{n,1} a_{n,2} \\ & \leq \sum_{n=2}^{\infty} (1 + B) [1 + (n - 1)\lambda] a_{n,1} a_{n,2} \\ & \leq (B - A)(1 - \alpha) \frac{(B_0 - A)(1 - \alpha)}{(1 + B_0)(1 + \lambda)} \\ & \leq (B - A)^2(1 - \alpha)^2 = \{B^2 - A(2B - A)\} \{1 - \alpha(2 - \alpha)\}, \end{aligned} \quad (13.3)$$

where  $\alpha$ ,  $A$ , and  $B$  are given by (13.2), and (for convenience)  $B_0 = \min\{B_1, B_2\}$ . Now observe that

$$-1 \leq A(2B - A) < B^2 \leq 1 \quad \text{and} \quad 0 < B^2 \leq 1$$

for

$$-1 \leq A < B \leq 1 \quad \text{and} \quad 0 < B \leq 1,$$

and also that

$$0 \leq \alpha(2 - \alpha) < 1 \quad \text{for} \quad 0 \leq \alpha < 1.$$

Hence, by Theorem 1, the modified Hadamard product  $f_1 \star f_2(z)$  is in the class

$$\mathcal{F}_\lambda(\alpha(2 - \alpha), A(2B - A), B^2).$$

In a similar manner, with the aid of Corollary 2, we can prove

**Theorem 29.** *Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the classes  $\mathcal{R}(\lambda, \alpha_j, A_j, B_j)$  ( $j = 1, 2$ ), respectively. Then the modified Hadamard product  $f_1 \star f_2(z)$  defined by (13.1) belongs to the class  $\mathcal{R}(\lambda, \alpha(2 - \alpha), A(2B - A), B^2)$ , where  $\alpha, A$ , and  $B$  are given by (13.2).*

Employing the technique used earlier by Schild and Silverman [18], we next prove

**Theorem 30.** *Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the same class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then  $f_1 \star f_2(z) \in \mathcal{F}_\lambda(\beta(\lambda, \alpha, A, B), A, B)$ , where*

$$\beta(\lambda, \alpha, A, B) = 1 - \frac{(B - A)(1 - \alpha)^2}{(1 + B)(1 + \lambda)}. \quad (13.4)$$

*The result is sharp.*

**Proof.** Under the hypotheses of Theorem 30, we find from Theorem 1 that

$$\sum_{n=2}^{\infty} \frac{(1 + B)[1 + (n - 1)\lambda]}{(B - A)(1 - \alpha)} a_{n,1} \leq 1 \quad (13.5)$$

and

$$\sum_{n=2}^{\infty} \frac{(1 + B)[1 + (n - 1)\lambda]}{(B - A)(1 - \alpha)} a_{n,2} \leq 1. \quad (13.6)$$

We only need to find the largest  $\beta = \beta(\lambda, \alpha, A, B)$  such that

$$\sum_{n=2}^{\infty} \frac{(1 + B)[1 + (n - 1)\lambda]}{(B - A)(1 - \beta)} a_{n,1} a_{n,2} \leq 1. \quad (13.7)$$

From (13.5) and (13.6) we get, by means of the Cauchy-Schwarz inequality,

$$\sum_{n=2}^{\infty} \frac{(1 + B)[1 + (n - 1)\lambda]}{(B - A)(1 - \alpha)} \sqrt{a_{n,1} a_{n,2}} \leq 1. \quad (13.8)$$

Thus it is sufficient to show that

$$\begin{aligned} & \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\beta)} a_{n,1} a_{n,2} \\ & \leq \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} \sqrt{a_{n,1} a_{n,2}} \quad (n \geq 2), \end{aligned} \quad (13.9)$$

that is, that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{1-\beta}{1-\alpha} \quad (n \geq 2). \quad (13.10)$$

Note that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} \quad (n \geq 2). \quad (13.11)$$

Consequently, we need only to prove that

$$\frac{(B-A)(1-\alpha)}{(1+B)[1+(n-1)\lambda]} \leq \frac{1-\beta}{1-\alpha} \quad (n \geq 2), \quad (13.12)$$

or, equivalently, that

$$\beta \leq 1 - \frac{(B-A)(1-\alpha)^2}{(1+B)[1+(n-1)\lambda]} \quad (n \geq 2). \quad (13.13)$$

Since  $\Xi(n)$  defined by

$$\Xi(n) = 1 - \frac{(B-A)(1-\alpha)^2}{(1+B)[1+(n-1)\lambda]} \quad (13.14)$$

is an increasing function of  $n$  ( $n \geq 2$ ), letting  $n = 2$  in (13.14) we obtain

$$\beta \leq \Xi(2) = 1 - \frac{(B-A)(1-\alpha)^2}{(1+B)(1+\lambda)}, \quad (13.15)$$

which proves the main assertion of Theorem 30.

Finally, by taking the functions  $f_j(z)$  ( $j = 1, 2$ ) given by

$$f_j(z) = z - \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} z^2 \quad (j = 1, 2), \quad (13.16)$$

we can see that the result is sharp.

**Corollary 18.** *Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the same class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then  $f_1 \star f_2(z) \in \mathcal{R}(\lambda, \gamma(\lambda, \alpha, A, B), A, B)$ , where*

$$\gamma(\lambda, \alpha, A, B) = 1 - \frac{(B-A)(1-\alpha)^2}{2(1+B)(1+\lambda)}. \quad (13.17)$$

The result is sharp for the functions  $f_j(z)$  ( $j = 1, 2$ ) given by

$$f_j(z) = z - \frac{(B - A)(1 - \alpha)}{2(1 + B)(1 + \lambda)} z^2 \quad (j = 1, 2). \quad (13.18)$$

**Theorem 31.** *Let the function  $f_1(z)$  defined by (7.1) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ , and let the function  $f_2(z)$  defined by (7.1) be in the class  $\mathcal{F}_\lambda(\nu, A, B)$ . Then*

$$f_1 \star f_2(z) \in \mathcal{F}_\lambda(\xi(\lambda, \alpha, \nu, A, B)),$$

where

$$\xi(\lambda, \alpha, \nu, A, B) = 1 - \frac{(B - A)(1 - \alpha)(1 - \nu)}{(1 + B)(1 + \lambda)}. \quad (13.19)$$

The result is sharp.

**Proof.** Proceeding as in the proof of Theorem 30, we get

$$\xi \leq \Lambda(n) = 1 - \frac{(B - A)(1 - \alpha)(1 - \nu)}{(1 + B)[1 + (n - 1)\lambda]} \quad (n \geq 2). \quad (13.20)$$

Since the function  $\Lambda(n)$  is an increasing function of  $n$  ( $n \geq 2$ ), letting  $n = 2$  in (13.20) we obtain

$$\xi \leq \Lambda(2) = 1 - \frac{(B - A)(1 - \alpha)(1 - \nu)}{(1 + B)(1 + \lambda)}, \quad (13.21)$$

which evidently proves the main assertion of Theorem 31.

Furthermore, taking

$$f_1(z) = z - \frac{(B - A)(1 - \alpha)}{(1 + B)(1 + \lambda)} z^2 \quad (13.22)$$

and

$$f_2(z) = z - \frac{(B - A)(1 - \nu)}{(1 + B)(1 + \lambda)} z^2, \quad (13.23)$$

we can show that the result of Theorem 31 is sharp.

**Corollary 19.** *Let the function  $f_1(z)$  defined by (7.1) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$  and let the function  $f_2(z)$  defined by (7.1) be in the class  $\mathcal{R}(\lambda, \nu, A, B)$ . Then*

$$f_1 \star f_2(z) \in \mathcal{R}(\lambda, \eta(\lambda, \alpha, \nu, A, B), A, B),$$

where

$$\eta(\lambda, \alpha, \nu, A, B) = 1 - \frac{(B - A)(1 - \alpha)(1 - \nu)}{2(1 + B)(1 + \lambda)}. \quad (13.24)$$

The result is sharp for the functions  $f_1(z)$  and  $f_2(z)$  given by

$$f_1(z) = z - \frac{(B - A)(1 - \alpha)}{2(1 + B)(1 + \lambda)} z^2$$

and

$$f_2(z) = z - \frac{(B - A)(1 - \nu)}{2(1 + B)(1 + \lambda)} z^2.$$

**Theorem 32.** *Let the functions  $f_j(z)$  ( $j = 1, 2, 3$ ) defined by (7.1) be in the same class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then*

$$f_1 \star f_2 \star f_3(z) \in \mathcal{F}_\lambda(\zeta(\lambda, \alpha, A, B), A, B),$$

where

$$\zeta(\lambda, \alpha, A, B) = 1 - \frac{(B - A)^2 (1 - \alpha)^3}{(1 + B)^2 (1 + \lambda)^2}. \quad (13.25)$$

The result is the best possible for the functions  $f_j(z)$  ( $j = 1, 2, 3$ ) given by

$$f_j(z) = z - \frac{(B - A)(1 - \alpha)}{(1 + B)(1 + \lambda)} z^2 \quad (j = 1, 2, 3). \quad (13.26)$$

**Proof.** From Theorem 30, we have

$$f_1 \star f_2(z) \in \mathcal{F}_\lambda(\beta(\lambda, \alpha, A, B), A, B),$$

where  $\beta$  is given by (13.4). Also, by using Theorem 31, we get

$$f_1 \star f_2 \star f_3(z) \in \mathcal{F}_\lambda(\zeta(\lambda, \alpha, A, B), A, B),$$

where

$$\begin{aligned} \zeta(\lambda, \alpha, A, B) &= 1 - \frac{(B - A)(1 - \alpha)(1 - \beta)}{(1 + B)(1 + \lambda)}, \\ &= 1 - \frac{(B - A)^2 (1 - \alpha)^3}{(1 + B)^2 (1 + \lambda)^2}. \end{aligned}$$

This completes the proof of Theorem 32.

**Corollary 20.** *Let the functions  $f_j(z)$  ( $j = 1, 2, 3$ ) defined by (7.1) be in the same class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then*

$$f_1 \star f_2 \star f_3(z) \in \mathcal{R}(\lambda, \kappa(\lambda, \alpha, A, B), A, B),$$

where

$$\kappa(\lambda, \alpha, A, B) = 1 - \frac{(B - A)^2 (1 - \alpha)^3}{4(1 + B)^2 (1 + \lambda)^2}. \quad (13.27)$$

The result is the best possible for the functions  $f_j(z)$  ( $j = 1, 2, 3$ ) given by

$$f_j(z) = z - \frac{(B - A)(1 - \alpha)}{2(1 + B)(1 + \lambda)} z^2 \quad (j = 1, 2, 3). \quad (13.28)$$

**Theorem 33.** *Let  $f_1(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  and  $f_2(z) \in \mathcal{F}_\lambda(\nu, A, B)$ . Then*

$$f_1 \star f_2(z) \in \mathcal{R}(\lambda, \mu(\lambda, \alpha, \nu, A, B), A, B),$$

where

$$\mu(\lambda, \alpha, \nu, A, B) = 1 - \frac{2(B - A)(1 - \alpha)(1 - \nu)}{(1 + B)(1 + \lambda)}. \quad (13.29)$$

The result is sharp.

**Proof.** Since  $f_1(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  and  $f_2(z) \in \mathcal{F}_\lambda(\nu, A, B)$ , we have

$$\sum_{n=2}^{\infty} (1 + B) [1 + (n - 1)\lambda] a_{n,1} \leq (B - A)(1 - \alpha) \quad (13.30)$$

and

$$\sum_{n=2}^{\infty} (1 + B) [1 + (n - 1)\lambda] a_{n,2} \leq (B - A)(1 - \nu). \quad (13.31)$$

It follows from (13.30) and (13.31) that

$$\sum_{n=2}^{\infty} (1 + B)^2 [1 + (n - 1)\lambda]^2 a_{n,1} a_{n,2} \leq (B - A)^2 (1 - \alpha)(1 - \nu). \quad (13.32)$$

We want to find the largest  $\mu = \mu(\lambda, \alpha, \nu, A, B)$  such that

$$\sum_{n=2}^{\infty} n(1 + B) [1 + (n - 1)\lambda] a_{n,1} a_{n,2} \leq (B - A)(1 - \mu).$$

This will certainly be satisfied if

$$\frac{n(1+B)[1+(n-1)\lambda]}{(B-A)(1-\mu)} \leq \frac{(1+B)^2[1+(n-1)\lambda]^2}{(B-A)^2(1-\alpha)(1-\nu)} \quad (n \geq 2), \quad (13.33)$$

that is, if

$$\mu \leq 1 - \frac{n(B-A)(1-\alpha)(1-\nu)}{(1+B)[1+(n-1)\lambda]} \quad (n \geq 2). \quad (13.34)$$

Since  $\Omega(n)$  defined by

$$\Omega(n) = 1 - \frac{n(B-A)(1-\alpha)(1-\nu)}{(1+B)[1+(n-1)\lambda]} \quad (13.35)$$

is an increasing function of  $n$  ( $n \geq 2$ ), letting  $n = 2$  in (13.35) we obtain

$$\mu \leq \Omega(2) = 1 - \frac{2(B-A)(1-\alpha)(1-\nu)}{(1+B)(1+\lambda)}, \quad (13.36)$$

which proves the main assertion of Theorem 33. Finally, the result is sharp for the functions

$$f_1(z) = z - \frac{(B-A)(1-\alpha)}{(1+B)(1+\lambda)} z^2 \in \mathcal{F}_\lambda(\alpha, A, B)$$

and

$$f_2(z) = z - \frac{(B-A)(1-\nu)}{(1+B)(1+\lambda)} z^2 \in \mathcal{F}_\lambda(\nu, A, B).$$

**Theorem 34.** *Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the same class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Then the function  $h(z)$  defined by*

$$h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \quad (13.37)$$

belongs to the class  $\mathcal{F}_\lambda(\delta(\lambda, \alpha, A, B), A, B)$ , where

$$\delta(\lambda, \alpha, A, B) = 1 - \frac{2(B-A)(1-\alpha)^2}{(1+B)(1+\lambda)}. \quad (13.38)$$

The result is sharp for the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (13.16).

**Proof.** By virtue of Theorem 1, we obtain

$$\begin{aligned} & \sum_{n=2}^{\infty} \left\{ \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} \right\}^2 a_{n,1}^2 \\ & \leq \left\{ \sum_{n=2}^{\infty} \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} a_{n,1} \right\}^2 \\ & \leq 1 \end{aligned} \quad (13.39)$$

and

$$\begin{aligned} & \sum_{n=2}^{\infty} \left\{ \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} \right\}^2 a_{n,2}^2 \\ & \leq \left\{ \sum_{n=2}^{\infty} \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} a_{n,2} \right\}^2 \\ & \leq 1. \end{aligned} \quad (13.40)$$

It follows from (13.39) and (13.40) that

$$\sum_{n=2}^{\infty} \frac{1}{2} \left\{ \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} \right\}^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \quad (13.41)$$

Therefore, we need to find the largest  $\delta$  such that

$$\frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\delta)} \leq \frac{1}{2} \left\{ \frac{(1+B)[1+(n-1)\lambda]}{(B-A)(1-\alpha)} \right\}^2 \quad (n \geq 2), \quad (13.42)$$

that is,

$$\delta \leq 1 - \frac{2(B-A)(1-\alpha)^2}{(1+B)[1+(n-1)\lambda]} \quad (n \geq 2). \quad (13.43)$$

Since  $\Pi(n)$  defined by

$$\Pi(n) = 1 - \frac{2(B-A)(1-\alpha)^2}{(1+B)[1+(n-1)\lambda]} \quad (13.44)$$

is an increasing function of  $n$  ( $n \geq 2$ ), we readily have

$$\delta \leq \Pi(2) = 1 - \frac{2(B-A)(1-\alpha)^2}{(1+B)(1+\lambda)}, \quad (13.45)$$

which completes the proof of Theorem 34.

**Corollary 21.** *Let the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (7.1) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Then the function  $h(z)$  defined by (13.37) belongs to the class  $\mathcal{R}(\lambda, \omega(\lambda, \alpha, A, B), A, B)$ , where*

$$\omega(\lambda, \alpha, A, B) = 1 - \frac{(B-A)(1-\alpha)^2}{(1+B)(1+\lambda)}. \quad (13.46)$$

*The result is sharp for the functions  $f_j(z)$  ( $j = 1, 2$ ) defined by (13.18).*

**Theorem 35.** Let the function  $f_1(z)$  defined by (7.1) be in the class  $\mathcal{F}_\lambda(\alpha, A, B)$ . Also let the function  $f_2(z)$  be defined by

$$f_2(z) = z - \sum_{n=2}^{\infty} a_{n,2} z^n \quad (|a_{n,2}| \leq 1; \quad n = 2, 3, 4, \dots). \quad (13.47)$$

Then  $f_1 \star f_2(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ .

**Proof.** Since

$$\begin{aligned} & \sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] |a_{n,1} a_{n,2}| \\ &= \sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_{n,1} |a_{n,2}| \\ &\leq \sum_{n=2}^{\infty} (1+B) [1+(n-1)\lambda] a_{n,1} \\ &\leq (B-A)(1-\alpha), \end{aligned}$$

by Theorem 1, it follows that  $f_1 \star f_2(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ .

**Corollary 22.** If  $f_1(z) \in \mathcal{F}_\lambda(\alpha, A, B)$  and

$$f_2(z) = z - \sum_{n=2}^{\infty} a_{n,2} z^n \quad (0 \leq a_{n,2} \leq 1; \quad n = 2, 3, 4, \dots), \quad (13.48)$$

then  $f_1 \star f_2(z) \in \mathcal{F}_\lambda(\alpha, A, B)$ .

**Corollary 23.** Let the function  $f_1(z)$  defined by (7.1) be in the class  $\mathcal{R}(\lambda, \alpha, A, B)$ . Also let the function  $f_2(z)$  be defined by

$$f_2(z) = z - \sum_{n=2}^{\infty} a_{n,2} z^n \quad (|a_{n,2}| \leq 1; \quad n = 2, 3, 4, \dots). \quad (13.49)$$

Then  $f_1 \star f_2(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ .

**Corollary 24.** If  $f_1(z) \in \mathcal{R}(\lambda, \alpha, A, B)$  and

$$f_2(z) = z - \sum_{n=2}^{\infty} a_{n,2} z^n \quad (0 \leq a_{n,2} \leq 1; \quad n = 2, 3, 4, \dots), \quad (13.50)$$

then  $f_1 \star f_2(z) \in \mathcal{R}(\lambda, \alpha, A, B)$ .

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