# CERTAIN SUBCLASSES OF FUNCTIONS OF POSITIVE REAL PART WITH NEGATIVE COEFFICIENTS

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# CERTAIN SUBCLASSES OF FUNCTIONS OF POSITIVE REAL PART WITH NEGATIVE COEFFICIENTS*

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## ABSTRACT

The object of the present paper is to derive several useful properties of the class $\mathcal{R}(\alpha)$ which is related to the class $\mathcal{Q}(\alpha)$ defined earlier by H. Silverman and M. Ziegler [Houston J. Math. 4(1978), 269-275]. Relationships between $\mathcal{R}(\alpha)$ and various other classes including $\mathcal{Q}(\alpha)$, and some results for a modified convolution product of functions belonging to the class $\mathcal{R}(\alpha)$ are presented. Finally, a certain functional $\mathcal{J}(q)$ of functions $q(z)$ in $\mathcal{R}(\alpha)$ is considered.


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## 1. INTRODUCTION

Let $\mathcal{Q}(\alpha)$ denote the class of functions of the form

$$q(z) = 1 - \sum_{n=1}^{\infty} b_n z^n \qquad (b_n \geqq 0) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$ satisfying the condition

$$|q(z) - 1| \leqq 1 - \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some $\alpha$ $(0 \leqq \alpha < 1)$. The class $\mathcal{Q}(\alpha)$ was introduced by Silverman and Ziegler [9]. A function $q(z)$ of the form (1.1) is said to be in the class $\mathcal{R}(\alpha)$ if and only if

$$zq'(z) + 1 \in \mathcal{Q}(\alpha).$$

In the present paper, we prove several interesting results for functions belonging to the class $\mathcal{R}(\alpha)$.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.3}$$

which are analytic in the unit disk $\mathcal{U}$. Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of analytic and univalent functions in the unit disk $\mathcal{U}$. Then a function $f(z)$ in $\mathcal{S}$ is said to be starlike of order $\alpha$ $(0 \leqq \alpha < 1)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{1.4}$$

for some $\alpha$ $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions of order $\alpha$ in the unit disk $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be convex of order $\alpha$ $(0 \leqq \alpha < 1)$ if and only if

$$\text{(1.5)} \qquad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U})$$

for some $\alpha$ $(0 \leqq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the class of all convex functions of order $\alpha$ in the unit disk $\mathcal{U}$.

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were first introduced by Robertson [4], and were studied subsequently by Schild [5], MacGregor [2], Pinchuk [3], Jack [1], and others.

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions whose nonzero coefficients, from the second one on, are negative. Thus an analytic and univalent function $f(z)$ is in the class $\mathcal{T}$ if it can be expressed as

$$\text{(1.6)} \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0).$$

We denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ with $\mathcal{T}$; that is,

$$\mathcal{T}^*(\alpha) = \mathcal{T} \cap \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{C}(\alpha) = \mathcal{T} \cap \mathcal{K}(\alpha).$$

The classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ were studied by Silverman [8]. Schild [6] considered a subclass of $\mathcal{T}$ consisting of polynomials having $|z| = 1$ as the radius of univalence, Silverman [8] proved coefficient inequalities, distortion theorems, and covering theorems for $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, and Schild and Silverman [7] gave some interesting results for the convolution product of functions in the classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$.

We require the following lemmas due to Silverman [8] in our investigation.

LEMMA 1. *Let the function* $f(z)$ *be defined by* (1.6). *Then* $f(z)$ *is in the class* $\mathcal{T}^*(\alpha)$ *if and only if*

(1.7) $$\sum_{n=2}^{\infty} (n-\alpha)a_n \leqq 1 - \alpha.$$

LEMMA 2. *Let the function* $f(z)$ *be defined by* (1.6). *Then* $f(z)$ *is in the class* $\mathcal{C}(\alpha)$ *if and only if*

(1.8) $$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leqq 1 - \alpha.$$

## 2. PROPERTIES OF THE CLASS $\mathcal{R}(\alpha)$

We begin by recalling here the following lemma due to Silverman and Ziegler [9].

LEMMA 3. *Let the function* $q(z)$ *be defined by* (1.1). *Then* $q(z)$ *is in the class* $\mathcal{Q}(\alpha)$ *if and only if*

(2.1) $$\sum_{n=1}^{\infty} b_n \leqq 1 - \alpha.$$

Making use of Lemma 3, we shall prove

THEOREM 1. *Let the function* $q(z)$ *be defined by* (1.1). *Then* $q(z)$ *is in the class* $\mathcal{R}(\alpha)$ *if and only if*

(2.2) $$\sum_{n=1}^{\infty} n\, b_n \leqq 1 - \alpha.$$

*The result* (2.2) *is sharp*.

PROOF. Since

(2.3) $$zq'(z) + 1 = 1 - \sum_{n=1}^{\infty} n\, b_n z^n,$$

we prove the assertion (2.2) by substituting $n\, b_n$ for $b_n$ in Lemma 3. Further, for the function defined by

(2.4) $$q(z) = 1 - \left(\frac{1 - \alpha}{n}\right) z^n \qquad (n \geqq 1),$$

we can easily see that the result (2.2) is sharp.

COROLLARY 1. *Let the function* $q(z)$ *defined by* (1.1) *be in the class* $\mathcal{R}(\alpha)$. *Then*

(2.5) $$b_n \leqq \frac{1 - \alpha}{n} \qquad (n \geqq 1).$$

*Equality holds true for the function* $q(z)$ *given by* (2.4).

COROLLARY 2. *Let* $0 \leqq \alpha < 1$. *Then*

(2.6) $$\mathcal{R}(\alpha) \subset \mathcal{Q}(\alpha).$$

Next, by using Theorem 1, we shall prove

THEOREM 2. The class $\mathcal{R}(\alpha)$ is convex.

PROOF. Let the function $q(z)$ defined by (1.1) and the function $g(z)$ defined by

$$g(z) = 1 - \sum_{n=1}^{\infty} c_n z^n \qquad (c_n \geqq 0) \tag{2.7}$$

be in the class $\mathcal{R}(\alpha)$. Then it suffices to prove that the function

$$h(z) = \lambda\, q(z) + (1-\lambda) g(z) \qquad (0 \leqq \lambda \leqq 1)$$

is also in the class $\mathcal{R}(\alpha)$. We note that

$$h(z) = 1 - \sum_{n=1}^{\infty} \{\lambda\, b_n + (1-\lambda) c_n\} z^n \tag{2.8}$$

and

$$\sum_{n=1}^{\infty} n\{\lambda\, b_n + (1-\lambda)c_n\} = \lambda \sum_{n=1}^{\infty} n\, b_n + (1-\lambda) \sum_{n=1}^{\infty} n\, c_n \tag{2.9}$$
$$\leqq 1 - \alpha,$$

which evidently completes the proof of Theorem 2.

We know from Theorem 2 that there are some extreme points of $\mathcal{R}(\alpha)$.

THEOREM 3. Let

$$q_0(z) = 1 \tag{2.10}$$

and

(2.11) $$q_n(z) = 1 - \left(\frac{1-\alpha}{n}\right) z^n \qquad (n \geqq 1).$$

Then the function $q(z)$ is in the class $\mathcal{R}(\alpha)$ if and only if it can be expressed in the form

(2.12) $$q(z) = \sum_{n=0}^{\infty} \lambda_n q_n(z),$$

where $\lambda_n \geqq 0$ $(n \geqq 0)$ and

(2.13) $$\sum_{n=0}^{\infty} \lambda_n = 1.$$

PROOF. We assume that

(2.14) $$q(z) = \sum_{n=0}^{\infty} \lambda_n q_n(z)$$

$$= 1 - \sum_{n=1}^{\infty} \frac{(1-\alpha)\lambda_n}{n} z^n.$$

Then, by appealing to Theorem 1, we have

(2.15) $$\sum_{n=1}^{\infty} n \cdot \frac{(1-\alpha)\lambda_n}{n} = (1-\alpha)(1-\lambda_0) \leqq 1 - \alpha,$$

which implies that the function $q(z)$ belongs to the class $\mathcal{R}(\alpha)$.

Conversely, let us assume that the function $q(z)$ defined by (1.1) is in the class $\mathcal{R}(\alpha)$. Then, since

$$b_n \leqq \frac{1-\alpha}{n} \quad \text{for} \quad n \geqq 1,$$

we can set

$$\lambda_n = \frac{n\, b_n}{1 - \alpha} \qquad (n \geqq 1) \tag{2.16}$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n . \tag{2.17}$$

Consequently, we have the representation (2.12), and the proof of Theorem 3 is completed.

COROLLARY 3. *The extreme points of* $\mathcal{R}(\alpha)$ *are* $q_n(z)$ $(n \geqq 0)$ *given by* (2.10) *and* (2.11).

## 3. SOME INTERESTING RELATIONSHIPS

Silverman and Ziegler [9] gave a relationship between $\mathcal{Q}(\alpha)$ and $\mathcal{T}^*(\alpha)$. We derive several interesting relationships between $\mathcal{Q}(\alpha)$ and $\mathcal{T}^*(\alpha)$, and between $\mathcal{R}(\alpha)$ and $\mathcal{C}(\alpha)$.

THEOREM 4. *Let the function* $q(z)$ *defined by* (1.1) *be in the class* $\mathcal{Q}(\alpha)$. *Then*

$$\int_0^z q(z)dz$$

*is in the class* $\mathcal{T}^*(\alpha)$.

PROOF. We note that

(3.1) $$\int_0^z q(z)dz = z - \sum_{n=1}^{\infty} \left(\frac{b_n}{n+1}\right) z^{n+1}.$$

Hence, by Lemma 3,

(3.2) $$\sum_{n=1}^{\infty} (n+1-\alpha)\left(\frac{b_n}{n+1}\right) \leqq \sum_{n=1}^{\infty} b_n \leqq 1 - \alpha,$$

which, in view of Lemma 1, implies that

$$\int_0^z q(z)dz \in \mathcal{T}^*(\alpha).$$

COROLLARY 4. Let the function $q(z)$ defined by (1.1) be in the class $\mathcal{R}(\alpha)$. Then

$$\int_0^z \{zq'(z) + 1\}dz$$

is in the class $\mathcal{T}^*(\alpha)$.

PROOF. Since $q(z) \in \mathcal{R}(\alpha)$ if and only if

$$zq'(z) + 1 \in \mathcal{Q}(\alpha),$$

the proof of Corollary 4 is straightforward.

THEOREM 5. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{T}^*(\alpha)$. Then $f'(z)$ is in the class $\mathcal{Q}\left(\frac{\alpha}{2-\alpha}\right)$.

PROOF. Note that Lemma 1 gives

(3.3) $$\sum_{n=1}^{\infty} a_{n+1} \leqq \frac{1-\alpha}{2-\alpha}.$$

Therefore, we have

(3.4) $$\sum_{n=1}^{\infty} (n+1)a_{n+1} \leqq 1-\alpha+\alpha\sum_{n=1}^{\infty} a_{n+1}$$
$$\leqq 1-\frac{\alpha}{2-\alpha},$$

which implies that

$$f'(z) \in \mathcal{Q}\left(\frac{\alpha}{2-\alpha}\right).$$

COROLLARY 5. Let the function $f(z)$ be defined by (1.6). Then $f(z)$ is in the class $\mathcal{T}^*(0)$ if and only if $f'(z)$ is in the class $\mathcal{Q}(0)$.

PROOF. Corollary 5 follows easily upon setting $\alpha = 0$ in Theorem 5.

THEOREM 6. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{C}(\alpha)$. Then $f'(z)$ is in the class

$$\mathcal{R}(\alpha) \cap \mathcal{Q}\left(\frac{1}{2-\alpha}\right).$$

PROOF. To prove that $f'(z) \in \mathcal{R}(\alpha)$, we need only show that

(3.5) $$\sum_{n=1}^{\infty} n(n+1)a_{n+1} \leqq 1-\alpha.$$

In fact, from Lemma 2, we obtain

$$(3.6)\qquad \sum_{n=1}^{\infty} n(n+1)a_{n+1} \leqq \sum_{n=1}^{\infty} (n+1)(n+1-\alpha)a_{n+1} \leqq 1 - \alpha,$$

and

$$(3.7)\qquad \sum_{n=1}^{\infty} (n+1)a_{n+1} \leqq 1 - \frac{1}{2 - \alpha} ,$$

showing that

$$f'(z) \in \mathcal{Q}\left(\frac{1}{2 - \alpha}\right).$$

Thus we have Theorem 6.

THEOREM 7. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.6) <u>be in the class</u> $\mathcal{T}^*(\alpha)$. <u>Then</u>

$$(3.8)\qquad \frac{f(z)}{z} \in \mathcal{R}(\alpha) \cap \mathcal{Q}\left(\frac{1}{2 - \alpha}\right).$$

PROOF. Since

$$(3.9)\qquad \frac{f(z)}{z} = 1 - \sum_{n=1}^{\infty} a_{n+1}z^n,$$

we obtain

$$(3.10)\qquad \sum_{n=1}^{\infty} n\, a_{n+1} = \sum_{n=2}^{\infty} (n-1)a_n \leqq \sum_{n=2}^{\infty} (n-\alpha)a_n \leqq 1 - \alpha$$

and

$$(3.11)\qquad \sum_{n=1}^{\infty} a_{n+1} = \sum_{n=2}^{\infty} a_n \leqq 1 - \frac{1}{2 - \alpha} .$$

Now the assertion (3.8) of Theorem 7 follows at once from (3.10) and (3.11).

Similarly, we have

THEOREM 8. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{C}(\alpha)$. Then

$$\frac{f(z)}{z} \in \mathcal{R}\left(\frac{1+\alpha}{2}\right) \cap \mathcal{Q}\left(\frac{3-\alpha}{2(2-\alpha)}\right). \tag{3.12}$$

## 4. A MODIFIED CONVOLUTION PRODUCT

Let $q_j(z)$ $(j = 1,2)$ be defined by

$$q_j(z) = 1 - \sum_{n=1}^{\infty} b_{n,j} z^n \qquad (b_{n,j} \geqq 0). \tag{4.1}$$

We denote by $q_1 * q_2(z)$ a modified convolution product of two functions $q_1(z)$ and $q_2(z)$, defined by

$$q_1 * q_2(z) = 1 - \sum_{n=1}^{\infty} b_{n,1}\, b_{n,2}\, z^n. \tag{4.2}$$

We now consider the modified convolution products of functions in the classes $\mathcal{Q}(\alpha)$ and $\mathcal{R}(\alpha)$.

THEOREM 9. Let the functions $q_i(z)$ $(i = 1,2)$ be defined by (4.1). Also let $q_1(z) \in \mathcal{R}(\alpha)$ and $q_2(z) \in \mathcal{R}(\beta)$. Then the modified convolution product $q_1 * q_2(z)$ defined by (4.2) is in the class $\mathcal{R}(\alpha + \beta - \alpha\beta)$.

PROOF. We have to find the largest $\gamma = \gamma(\alpha,\beta)$ such that

$$(4.3) \qquad \sum_{n=1}^{\infty} n\, b_{n,1}\, b_{n,2} \leqq 1 - \gamma,$$

or equivalently,

$$(4.4) \qquad \sum_{n=1}^{\infty} \left(\frac{n}{1 - \gamma}\right) b_{n,1}\, b_{n,2} \leqq 1.$$

We note from Theorem 1 that

$$(4.5) \qquad \sum_{n=1}^{\infty} \left(\frac{n}{1 - \alpha}\right) b_{n,1} \leqq 1$$

and

$$(4.6) \qquad \sum_{n=1}^{\infty} \left(\frac{n}{1 - \beta}\right) b_{n,2} \leqq 1.$$

By using the Cauchy-Schwarz inequality, we have

$$(4.7) \qquad \sum_{n=1}^{\infty} \sqrt{\frac{n}{1 - \alpha}}\ \sqrt{\frac{n}{1 - \beta}}\ \sqrt{b_{n,1}\, b_{n,2}} \leqq 1.$$

Hence, if

$$(4.8) \qquad \left(\frac{n}{1 - \gamma}\right) \sqrt{b_{n,1}\, b_{n,2}} \leqq \sqrt{\frac{n}{1 - \alpha}}\ \sqrt{\frac{n}{1 - \beta}}$$

for $n \geqq 1$, we have (4.4).

Further, it is sufficient to prove that

$$(4.9) \qquad \frac{1}{1 - \gamma} \leqq \frac{n}{(1-\alpha)(1-\beta)}$$

for $n \geqq 1$.

It follows from (4.9) that

(4.10) $$\gamma \leqq 1 - \frac{(1-\alpha)(1-\beta)}{n}.$$

Since

(4.11) $$\Phi(n) = 1 - \frac{(1-\alpha)(1-\beta)}{n}$$

is an increasing function of $n$ $(n \geqq 1)$, putting $n = 1$ in (4.11), we obtain

(4.12) $$\gamma \leqq \Phi(1) = \alpha + \beta - \alpha\beta < 1.$$

Thus

(4.13) $$q_1 * q_2(z) \in \mathcal{R}(\alpha + \beta - \alpha\beta),$$

which proves Theorem 7.

In a similar manner, we can prove

THEOREM 10. *Let the functions* $q_i(z)$ $(i = 1,2)$ *be defined by* (4.1). *Also let* $q_1(z) \in \mathcal{Q}(\alpha)$ *and* $q_2(z) \in \mathcal{Q}(\beta)$. *Then the modified convolution product* $q_1 * q_2(z)$ *defined by* (4.2) *is in the class* $\mathcal{Q}(\alpha + \beta - \alpha\beta)$.

THEOREM 11. *Let the functions* $q_i(z)$ $(i = 1,2)$ *be defined by* (4.1). *Also let* $q_1(z) \in \mathcal{Q}(\alpha)$ *and* $q_2(z) \in \mathcal{R}(\beta)$. *Then the modified convolution product* $q_1 * q_2(z)$ *defined by* (4.2) *is in the class* $\mathcal{R}(\alpha + \beta - \alpha\beta)$.

## 5. THE FUNCTIONAL $\mathcal{J}(q)$

We introduce the functional $\mathcal{J}(q)$ defined by

(5.1) $$\mathcal{J}(q) = \frac{1}{z}\int_0^z q(z)dz,$$

and prove

THEOREM 12. Let the function $q(z)$ defined by (1.1) be in the class $\mathcal{Q}(\alpha)$. Then $\mathcal{J}(q)$ is in the class

$$\mathcal{R}(\alpha) \cap \mathcal{Q}\left(\frac{1+\alpha}{2}\right).$$

PROOF. Note that

(5.2) $$\mathcal{J}(q) = 1 - \sum_{n=1}^{\infty}\left(\frac{b_n}{n+1}\right)z^n.$$

Hence

(5.3) $$\sum_{n=1}^{\infty} n\left(\frac{b_n}{n+1}\right) \leqq \sum_{n=1}^{\infty} b_n \leqq 1 - \alpha,$$

which implies that $\mathcal{J}(q) \in \mathcal{R}(\alpha)$. Furthermore,

(5.4) $$\sum_{n=1}^{\infty}\frac{b_n}{n+1} \leqq \frac{1}{2}\sum_{n=1}^{\infty} b_n \leqq 1 - \left(\frac{1+\alpha}{2}\right),$$

which shows that

$$\mathcal{J}(q) \in \mathcal{Q}\left(\frac{1+\alpha}{2}\right).$$

In a similar way, we prove

THEOREM 13. *Let the function* $q(z)$ *defined by* (1.1) *be in the class* $\mathcal{R}(\alpha)$. *Then* $\mathcal{J}(q)$ *is in the class*

$$\mathcal{R}\left(\frac{1+\alpha}{2}\right).$$

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