A SIMPLE REDUCIBLE CASE OF DOUBLE HYPERGEOMETRIC SERIES INVOLVING CATALAN'S CONSTANT AND RIEMANN'S ζ–FUNCTION

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It is shown how simply a certain double hypergeometric series can be expressed in terms of Catalan's constant and Riemann's ζ–function. The exact value of an interesting definite integral emerges as a by–product of the analysis presented here.

1. INTRODUCTION

Making use of the Pochhammer symbol $(\lambda)_n$ given by

$$(\lambda)_n = \begin{cases} 1, & \text{if } n = 0\,, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in \mathbb{N}\,, \end{cases} \tag{1}$$

we define a general double hypergeometric series by (cf., e.g., [2, p. 27, Equation 1.3(28, et seq.])

$$F^{p:r;u}_{q:s;v}\left[\begin{matrix} \alpha_1,\cdots,\alpha_p:\ \gamma_1,\cdots,\gamma_r;\ \lambda_1,\cdots,\lambda_u; \\ \\ \beta_1,\cdots,\beta_q:\ \delta_1,\cdots,\delta_s;\ \mu_1,\cdots,\mu_v; \end{matrix}\ x,\ y\right] \tag{2}$$

$$= \sum_{\ell,m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_{\ell+m} \prod_{j=1}^{r} (\gamma_j)_{\ell} \prod_{j=1}^{u} (\lambda_j)_m}{\prod_{j=1}^{q} (\beta_j)_{\ell+m} \prod_{j=1}^{s} (\delta_j)_{\ell} \prod_{j=1}^{v} (\mu_j)_m} \frac{x^m}{m!} \frac{y^n}{n!},$$

provided that the double series converges.

Srivastava and Karlsson [2, pp. 28–32] have recorded a considerably large number of instances in which the double hypergeometric series (2) reduces to simpler functions including, for example, a hypergeometric function of a single variable (see also [2, p. 299 et seq.]). The object of the present note is to point out a particularly simple reducible case of (2) when

$$p = q = r - 1 = s = u - 1 = v = 1, \quad x = y = -1, \tag{3}$$

and the resulting parameters are appropriately specialized. Indeed, in terms of Catalan's constant:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cong 0.915965594177219015\cdots, \tag{4}$$

and Riemann's ζ–function (cf., e.g., Whittaker and Watson [3, Chapter 13]):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1, \tag{5}$$

we shall show that

$$F^{1:2;2}_{1:1;1}\left[\begin{matrix}1:\ \frac{1}{2}, 1; \frac{1}{2}, 1; \\ \\ 2:\ \ \frac{3}{2};\ \ \frac{3}{2};\end{matrix}\ -1, -1\right] = \pi G - \tfrac{7}{4}\,\zeta(3), \tag{6}$$

where $\zeta(3) \cong 1.20205690315959428540\cdots$.

2. DERIVATION OF THE REDUCTION FORMULA (6)

Denote, for convenience, the first member of the reduction formula (6) by Ω. Since [2, p. 21, Equation 1.2(35)]

$$\arctan z = z\ {}_2F_1\left[\tfrac{1}{2}, 1; \tfrac{3}{2}; -z^2\right] \tag{7}$$

in terms of the Gaussian hypergeometric function, we readily have

$$\Omega = 2\int_0^1 \frac{(\arctan z)^2}{z}\, dz\ , \tag{8}$$

which, upon setting $z = \tan(t/2)$, yields

$$\Omega = \tfrac{1}{2}\int_0^{\pi/2} \frac{t^2}{\sin t}\, dt\ . \tag{9}$$

Integrating by parts, we find from (9) that

$$\Omega = -\int_0^{\pi/2} t\ \ell n \tan(t/2)\, dt. \tag{10}$$

Now make use of the elementary expansion [1, p. 356, Equation 121(1.5)]:

$$(11) \qquad \ln \tan(t/2) = -2 \sum_{n=0}^{\infty} \frac{\cos[(2n+1)t]}{2n+1} \qquad (0 < t < \pi),$$

and invert the order of integration and summation; we thus obtain

$$(12) \qquad \Omega = \sum_{n=0}^{\infty} \frac{2}{2n+1} \int_0^{\pi/2} t \cos[(2n+1)t]\, dt.$$

Evaluating this last integral by parts, we find that

$$(13) \qquad \Omega = \pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} - 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}$$

$$= \pi G - 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3},$$

where we have used the definition (4).

Finally, we apply the special case $s = 3$ of the known result [3, p. 271]:

$$(14) \qquad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = (1 - 2^{-s})\, \zeta(s), \qquad \mathrm{Re}(s) > 1,$$

which incidentally follows at once upon separating the series in (5) into its even and odd terms, and we arrive immediately at the right–hand side of the reduction formula (6).

3. CONCLUDING REMARKS

Our derivation of the reduction formula (6) provides yet another interesting illustration of the familiar method of obtaining transformation and reduction formulas for multiple hypergeometric series by evaluating definite integrals in two different ways. For numerous other illustrations of this rather fruitful method, one may refer to Srivastava and Karlsson [2, pp. 325–331].

We conclude by remarking that, as an interesting by–product of our analysis detailed in the preceding section, we find that

$$\int_0^{\pi/2} \frac{t^2}{\sin t}\, dt = 2\pi G - \frac{7}{2}\,\zeta(3), \tag{15}$$

which evidently follows from (6) and (9).

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