THE PLANAR ISOSCELES PROBLEM
FOR MANEFF'S GRAVITATIONAL LAW

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Abstract. Maneff's gravitational law explains with a very good approximation the perihelion advance of the inner planets as well as the orbit of the Moon. Here we study the invariant set of planar isosceles solutions of the 3-body problem for Maneff's model. We show that every solution leads to a collision singularity and consequently has no periodic orbits. Using McGehee's technique we blow-up the triple collision singularity and regularize binary-collision solutions. The flow on the collision manifold is shown to be non-gradient-like and the set of collision/ejection solutions is described. The center manifold and the block-regularization problems are analysed. The network of homoclinic and heteroclinic orbits is further discussed. Finally we study an anisotropic model having the property that the flow on the collision manifold changes dramatically when the mass parameter is varied, giving rise to a subcritical pitchfork bifurcation of the equilibria.

I. INTRODUCTION

In case of the inverse square force law, special relativity cannot explain the perihelion advance of Mercury, however a very good approximation is obtained within general relativity. Unfortunately, the study of a most natural problem, that of 3 gravitating point masses (the so-called 3-body problem of celestial mechanics) encounters a lot of difficulties in the framework of Einstein's model. Therefore there existed pre- and post-relativistic attempts to modify Newton's inverse square force law without leaving the boundaries of classical mechanics. Most of them were unsuccessful. In general they could explain the perihelion advance of the inner planets but were unable to describe accurately the motion of the Moon (which is very well approximated by Newton's theory). Between 1924 and 1930, Maneff [17, 18, 19, 20] proposed and physically analysed a nonrelativistic gravitational law, able to explain both these issues. The computation of the perihelion advance of Mercury for an attractive force given by the sum of an inverse square and an inverse cubic law is an easy, well-known excercise (see[13], Chapter III, p. 123, Problem 14). In spite of this, Maneff's papers seem to be virtually unknown. As far as the knowledge of this author goes, the validity of Maneff's law was not tested by other physical criteria as light-deflection or time-delay.

In a previous paper [11] we have seen that Maneff's model, given by a quasihomogeneous potential function of the form $W = U + V$, where U and V are homogeneous functions of degree -1 and -2 respectively, is rather pathologic in the class of quasihomogeneous potential functions W, where U and V are homogeneous of degree $-a$ and $-b$, with $1 \leq a < b$. More precisely, for $b \neq 2$, all triple-collision solutions of the corresponding 3-body problem have the nice feature of tending ultimately to a b-central configuration, a property also known for Newton's model. By studying the invariant set of rectilinear soutions and imbedding it into the general problem, we have seen in [11] that this is not true for all solutions if $b = 2$.

The goal of this paper is to continue the dynamical analysis of this law by studying the richer invariant set of planar isosceles orbits. We will first show that every solution of this invariant set encounters a collision (which can take place forwards or backwards in time). This implies the nonexistence of noncollision singularities as well as the nonexistence of periodic orbits within the set. The next step will be to blow up the collision singularity and to paste instead a collision manifold using some McGehee-type transformations [20]. Double collisions are further regularized, so the new equations of motion become free of singularities. The collision manifold is topologically a sphere minus 4 points. The flow on the collision manifold is further studied in order to get information on orbits passing close to a triple collision. We see that it is formed only by periodic orbits with the exception of the homoclinic/heteroclinic orbits connecting the 6 equilibria. It is important to notice that a main difference between the flow in Maneff's model and that of the gravitational laws given by general quasihomogeneous potential functions, is that Maneff's case is the only one which is not gradient-like.

We next get a picture of the flow near the collision manifold and are able to describe the solutions leading to a triple collision. They either approach the collision manifold through a rest point or tend to the equator of the collision manifold. This last class of solutions (which approach the collision manifold without asymptotic phase) occur only in Maneff's model. They get lost for all other quasihomogeneous potentials because of the gradient-like property on the collision manifold. The orbits tending to a rest point contain a lower dimensional manifold.

The nonhyperbolicity of the equilibria gives rise to center manifolds. We show that for every energy level there is a unique 2-dimensional analytic center manifold corresponding to the rest point representing the collinear central configuration. The flow on the center manifold is stable but not asymptotically stable.

The problem of block-regularization is further analysed. We prove that if solutions in the set of equatorial collision orbits are not block-regularizable or if this set has dimension less than 3, then the orbits colliding through a planar, noncollinear configuration, are not block-regularizable.

We then prove that for any negative value of the energy constant, the pair of equilibria corresponding to a collinear central configuration, are connected by a unique heteroclinic orbit. In case $m = 1$ the equilibria corresponding to the equilateral configurations are also connected by heteroclinic orbits. For nonnegative values of the energy constant, there are no heteroclinic orbits connecting solutions of the northern hemisphere of the collision manifold with solutions of the southern hemisphere. Also, orbits leaving the rest points of the collision manifold from the northern hemisphere cannot return to it, consequently there are no homoclinic or heteroclinic orbits connecting equilibria of the northern hemisphere. The same is true for the southern hemisphere. All this analysis offers some information on the dynamics of the invariant set given by the planar isosceles solutions of the 3-body problem with Maneff's gravitational law. As a main difference from the classical Newtonian case (studied by Devaney [6, 7, 8]), we remark the fact that the noncollinear configurations to which triple collision orbits tend are not equilateral triangles, in general. They are isosceles triangles and their shape depends on the value of m. The case $m = 1$ is the only one for which the equilateral triangle is recovered. Therefore the degenerate Lagrangean orbits tending to a triple collision occur only if all masses are equal. This is a rather peculiar property of Maneff's gravitational law.

In the final section we discuss an anisotropic model of Maneff's law for which the flow on the collision manifold changes dramatically when the mass parameter is varied. For $0 < m < 7/4$ the flow is similar to the classical Maneff model. However, due to the anisotropy, if $m = 1$, the corresponding noncollinear triple collision orbit doesn't tend to an equilateral triangle but to an isosceles right triangle. If $m \geq 7/4$ the number of equilibria changes from 6 to 2, giving rise to a subcritical pitchfork bifurcation. Physically, this happens because the noncollinear limit configuration of triple-collision orbits degenerates into a segment. The qualitative behavior of the flow is then rather similar to that of the rectilinear invariant set of Maneff's model [11].

The paper is divided into sections. In Section II we derive the equations of motion of the isosceles problem with Maneff's law and prove the ubiquity of collision orbits for this class of solutions. In Section III McGehee's technique is used to blow up the triple-collision singularity and to regularize binary-collision orbits. Section IV deals with triple-collision solutions and Section V is concerned with the analysis of center manifolds. Section VI contains a regularization criterion while Section VII deals with the problem of existence of connecting orbits in phase space. Finally, Section VIII studies an anisotropic Maneff law and compares it with the classical one.

II. THE UBIQUITY OF COLLISIONS

Consider a system formed by 3 particles of masses $m_i > 0, i = 1,2,3$, moving in the Euclidean plane $\Re^2$. In an absolute frame, the *configuration* of the system is given by $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$, where $\mathbf{q}_i \in \Re^2$ is the position vector of m_i. Denote by $A = diag(m_1, m_1, m_2, m_2, m_3, m_3)$ a 6×6 diagonal matrix. The equations of motion of the 3-body problem with Maneff's gravitational law read

$$\ddot{\mathbf{q}} = A^{-1}\nabla\tilde{W}(\mathbf{q}), \tag{1}$$

where $\tilde{W} = \tilde{U} + \tilde{V}$, with

$$\tilde{U}\colon \Re^6 \setminus \Delta \to \Re_+, \quad \tilde{U}(\mathbf{q}) = G \sum_{1\le i<j\le 3} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|},$$

$$\tilde{V}\colon \Re^6 \setminus \Delta \to \Re_+, \quad \tilde{V}(\mathbf{q}) = \frac{3G^2}{2c^2} \sum_{1\le i<j\le 3} \frac{m_i m_j (m_i + m_j)}{|\mathbf{q}_i - \mathbf{q}_j|^2}.$$

Here $\Delta = \bigcup_{1\le i<j\le 3}\{\mathbf{q} \mid \mathbf{q}_i = \mathbf{q}_j\}$ represents the *collision set*, $|.|$ is the Euclidean norm, G denotes the gravitational constant and c is the speed of light. In order to simplify notation, read $\gamma = \frac{3G^2}{2c^2}$ and take physical units such that $G = 1$. Notice that $\tilde{U}$ and $\tilde{V}$ are homogeneous functions of degree -1 and -2 respectively, thus $\tilde{W}$ will be called *quasihomogeneous*.

Standard results of the differential equations theory ensure for given initial conditions outside the collision set, the existence and uniqueness of an analytic solution $(\mathbf{q}, \dot{\mathbf{q}})$ of the equations (1), defined on a maximal interval (t^-, t^+) containing 0. In case t^- or t^+ is finite, the solution is said to experience a *singularity* in the past or in the future, respectively. Otherwise the solution is called *regular*.

Consider further $m_1 = m_2 = 1$ and $m_3 = m$. Then the set of *isosceles* solutions (i.e. those for which $|\mathbf{q}_3(t) - \mathbf{q}_1(t)| = |\mathbf{q}_2(t) - \mathbf{q}_1(t)|$ for all t where the solution is defined) is invariant for the equations (1). It is true, in fact, that if a solution is isosceles, then the symmetric masses have to be equal (see [1] and for a generalization see [9]). Restricting the equations (1) to this invariant set, one can use Jacobi coordinates [24] to simplify the equations of motion. For this, define the transformations given by the analytic diffeomorphism

$$\mathbf{x} = (1/2)(\mathbf{q}_1 - \mathbf{q}_2), \quad \mathbf{y} = \mathbf{q}_3 - (1/2)(\mathbf{q}_1 + \mathbf{q}_2), \tag{2}$$

see Figure 1. Taking into account the symmetries of isosceles orbits and using the fact that $\mathbf{x} = (x, 0)$, $\mathbf{y} = (0, y)$, the equations (1) restricted to the invariant set of isosceles solutions become

$$\begin{cases} \ddot{x} = -\frac{1}{4x^2} - \frac{mx}{(x^2+y^2)^{3/2}} - \frac{\gamma}{2x^3} - \frac{2\gamma m(1+m)x}{(x^2+y^2)^2} \\ \ddot{y} = -\frac{(2+m)y}{(x^2+y^2)^{3/2}} - \frac{2\gamma m(1+m)(2+m)y}{(x^2+y^2)^2}. \end{cases} \tag{3}$$

So, the equations (3) define the planar isosceles problem for Maneff's model and they are the object of study of our paper. Their form already suggests the following result which is true in the isosceles Newtonian problem, too [10].

Theorem 1. *Any solution of the planar isosceles problem leads either to a binary collision between the symmetric particles or to a triple collision (either of them taking place in the future or in the past).*

Proof. Supposing that for all t, $x(t) > 0$ then, by the equations (3), $\ddot{x} < 0$. However, there are no functions $x: \Re \to \Re$ with this property, therefore there exists a $t_0 \in \Re$ such that $x(t_0) = 0$. This implies collision for symmetric particles. In case $y(t_0) = 0$, then a triple collision occurs. We will see later that it is indeed possible that x and y cancel simultaneously, for a lower dimensional set of initial data. In case $x(t) < 0$ then $\ddot{x}(t) > 0$ and the proof proceeds similarly.

Corollary 2. **(a)** *The only singularities of the planar isosceles problem are collisions.*
(b) *There are no periodic orbits in the planar isosceles problem.*

Remark. Statement (b) is understood in the absence of a regularization of binary collisions (see Sections III and VI for details concerning the regularization process).

III. THE COLLISION MANIFOLD

The previous section proves the ubiquity of collision solutions in the planar isosceles problem. We next concentrate on triple-collision and near-triple-collision orbits. For this observe that we can write equations (3) in the form

$$\begin{cases} \dot{\mathbf{z}} = M^{-1}\zeta \\ \dot{\zeta} = \nabla \hat{W}(\mathbf{z}) \end{cases} \tag{4}$$

where $\mathbf{z} = (x, y), \zeta = (2\dot{x}, \frac{2m}{m+2}\dot{y})$, $M = \begin{pmatrix} 2 & 0 \\ 0 & \frac{2m}{m+2} \end{pmatrix}$, $\hat{W} = \hat{U} + \hat{V}$,

$$\hat{U}(x, y) = \frac{1}{2x} + \frac{2m}{(x^2 + y^2)^{1/2}} ,$$

$$\hat{V}(x, y) = \frac{\gamma}{2x^2} + \frac{2\gamma m(1 + m)}{x^2 + y^2}.$$

Consider McGehee's transformations given by the analytic diffeomorphism

$$\begin{cases} r = (\mathbf{z}^T M \mathbf{z})^{1/2} \\ \mathbf{s} = r^{-1}\mathbf{z} \\ v = r\zeta^T \mathbf{s} \\ \mathbf{u} = r\zeta - vM\mathbf{s}, \end{cases} \tag{5}$$

which have the properties: $\mathbf{s}^T M\mathbf{s} = 1$ and $\mathbf{s}^T\mathbf{u} = 0$. Composing (5) with the time transformation

$$d\tau = r^{-2}dt, \tag{6}$$

the equations of motion (4) become

$$\begin{cases} r' = rv \\ v' = v^2 + \mathbf{u}^T M^{-1}\mathbf{u} - r\hat{U}(\mathbf{s}) - 2\hat{V}(\mathbf{s}) \\ \mathbf{s}' = M^{-1}\mathbf{u} \\ \mathbf{u}' = -(\mathbf{u}^T M^{-1}\mathbf{u})M\mathbf{s} + r(\nabla\hat{U}(\mathbf{s}) + \hat{U}(\mathbf{s})M\mathbf{s}) + \nabla\hat{V}(\mathbf{s}) + 2\hat{V}(\mathbf{s})M\mathbf{s}, \end{cases} \tag{7}$$

and the energy relation reads

$$(1/2)(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) - r\hat{U}(\mathbf{s}) - \hat{V}(\mathbf{s}) = 2r^2 h.$$

Further define the transformations given by the analytic diffeomorphism

$$\begin{cases} \mathbf{s} = (\frac{1}{\sqrt{2}}\cos\theta, \sqrt{\frac{m+2}{2m}}\sin\theta) \\ \mathbf{u} = (-\sqrt{2}u\sin\theta, \sqrt{\frac{2m}{m+2}}u\cos\theta), \end{cases} \tag{8}$$

which are compatible with the above mentioned properties of McGehee's coordinates. Using the new energy relation

$$u^2 + v^2 - 2V(\theta) = 2r(rh + U(\theta)), \tag{9}$$

the equations of motion (7) become

$$\begin{cases} r' = rv \\ v' = 2hr^2 + rU(\theta) \\ \theta' = u \\ u' = r\frac{d}{d\theta}U(\theta) + \frac{d}{d\theta}V(\theta), \end{cases} \tag{10}$$

where

$$U(\theta) = \frac{1}{\sqrt{2}\cos\theta} + \frac{2\sqrt{2}m^{3/2}}{(m + 2\sin^2\theta)^{1/2}},$$

$$V(\theta) = \frac{\gamma}{\cos^2\theta} + \frac{4\gamma m^2(1+m)}{m + 2\sin^2\theta},$$

and by abuse of notation we have denoted the coordinates by the same letters. Prime denotes differentiation with respect to the new fictitious time variable τ. Notice also that since $\{r = 0\}$ is an invariant set, a solution tending to a triple collision needs an infinite amount of fictitious time to reach this set, thus $\tau \to \infty$ under such circumstances.

For future reference, notice that V is an even function defined on the open interval $(-\pi/2, \pi/2), \lim_{\theta\to\pm\frac{\pi}{2}} V(\theta) = \infty$ and V has three critical points: a local maximum at

$\theta = 0$ (independent on the parameter m) and two symmetric minima at θ_m and $-\theta_m$, where

$$\theta_m = \arccos\sqrt{\frac{m+2}{2(m\sqrt{2(m+1)}+1)}}. \tag{11}$$

Notice that equations (10) are still singular when $\theta \to \pm\frac{\pi}{2}$, values which physically correspond to binary collisions between the symmetric particles. Geometrically, in terms of (x, y)-coordinates, the value $\theta = \frac{\pi}{2}$ means $x = 0$ and $y < 0$, while $\theta = -\frac{\pi}{2}$ means $x = 0$ and $y < 0$. In order to eliminate these singularities, denote

$$R(\theta) = V(\theta)\cos^2\theta,$$

consider the transformation given by the analytic diffeomorphism

$$w = R^{-1/2}(\theta)(\cos\theta)^{3/2}u \tag{12}$$

and then compose it with the time transformation

$$d\psi = R(\theta)(\cos\theta)^{-3/2}d\tau.$$

The equations of motion (10) become

$$\begin{cases} r' = \frac{\cos^2\theta}{R(\theta)}rv \\ v' = \frac{\cos^2\theta}{R(\theta)}r(2hr + U(\theta)) \\ \theta' = \frac{w}{R^{1/2}(\theta)} \\ w' = (1-\frac{w}{2})\frac{\cos\theta}{R^{3/2}(\theta)}\frac{d}{d\theta}R(\theta) + \frac{3}{2}\frac{\cos^2\theta\sin\theta}{R^{3/2}(\theta)}v^2 - \frac{\sin\theta}{R^{1/2}(\theta)} + r\frac{\cos^3\theta}{R^{3/2}(\theta)}\frac{d}{d\theta}U(\theta), \end{cases} \tag{13}$$

having the energy relation

$$R(\theta)w^2 + v^2\cos^3\theta - 2R(\theta)\cos\theta = 2r(rh + U(\theta))\cos^3\theta. \tag{14}$$

Again, by abuse, we've maintained the same notations for the coordinates and prime denotes now differentiation with respect to the new time variable ψ. For a solution to reach the invariant set $\{r = 0\}$ it also needs $\psi \to \infty$. The equations (13) extend naturally to the interval $[-\pi/2, \pi/2]$ and are thus free of singularities. All solutions of the equations (13) are therefore globally defined. The set

$$C = \{(r, v, \theta, w) \mid r = 0 \text{ and equation (14) holds}\}$$

is an invariant manifold for the equations (13) and will be called the *collision manifold*. It can be analytically written as

$$\frac{w^2}{2\cos\theta} + \frac{v^2}{2V(\theta)} = 1$$

and due to the form of V discussed above, it represents a sphere with four points deleted, like in Figure 2. C was pasted to the phase space instead of the collision singularity and though fictitious, a study of the flow on it, will give us information on the behavior of orbits tending to and/or passing close to triple collisions.

IV. TRIPLE-COLLISION ORBITS

In this section we study the flow on and outside the collision manifold, emphasizing triple-collision orbits. First observe that in order to have equilibria for the equations (13), one necessarily has to impose the conditions $r = 0$ and $w = 0$. From the last equation we then get

$$\frac{\cos\theta \frac{d}{d\theta} W(\theta)}{W^{3/2}(\theta)} + \frac{3\cos^2\theta \sin\theta}{2W^{3/2}(\theta)} - \frac{\sin\theta}{W^{1/2}(\theta)} = 0,$$

which is equivalent to

$$\frac{d}{d\theta} V(\theta) = 3(V(\theta) - \frac{1}{2}v^2)\tan\theta$$

and has the solution $V(\theta) = (1/2)v^2$. Introducing this back into the equation we see how the condition of being a rest point asks also that $\frac{d}{d\theta}V(\theta) = 0$. Thus, critical points of V lying on the collision manifold in the plane $w = 0$, are rest points of the flow. We have thus obtained

Proposition 3. *The equations* (13) *have* 6 *equilibria*

$$E^{\pm} = (0, \pm\sqrt{2V(0)}, 0, 0),\ R^{\pm} = (0, \pm\sqrt{2V(\theta_m)}, \theta_m, 0),\ L^{\pm} = (0, \pm\sqrt{2V(-\theta_m)}, -\theta_m, 0),$$

where θ_m is given by (11), *and they all belong to the collision manifold C.*

Computing the eigenvalues of the linearized system at the restpoints of the flow, one obtains that all of them take real values, $\lambda_r > 0$ for E^+, R^+, L^+, $\lambda_r < 0$ for E^-, R^-, L^-, $\lambda_v = 0$ for $E^{\pm}, R^{\pm}, L^{\pm}$, $\lambda_\theta \cdot \lambda_w < 0$ for $R^{\pm}, L^{\pm}$ and $\mathrm{Re}\lambda_\theta = \mathrm{Re}\lambda_w = 0$ for $E^{\pm}$. Since $v' = 0$ on C, the flow on the collision manifold looks like in Figure 2. Therefore we summarize:

Proposition 4. *The flow on the collision manifold has 6 rest points $E^{\pm}, R^{\pm}, L^{\pm}$. Each of the equilibria $R^{\pm}$ and $L^{\pm}$ has one homoclinic orbit on C. There are two heteroclinic orbits connecting R^+ with L^+ and other two connecting R^- with L^-, on C. All the other orbits on C are cycles. In phase space, E^+ (E^-, respectively) contains a one-dimensional unstable (stable) manifold, while R^+, L^+ (R^-, L^-, respectively) contain each a two-dimensional unstable (stable) manifold and a one-dimensional stable (unstable) manifold.*

Let us see now what is the asymptotic behavior of solutions tending/ejecting to/from a triple approach. For orbits corresponding to $E^{\pm}$ we have

$$y/x = \frac{[(2+m)/2m]^{1/2}\sin\theta}{\frac{1}{\sqrt{2}}\cos\theta} \to 0 \quad \text{when} \quad \theta \to 0,$$

while for orbits corresponding to $R^{\pm}$ and $L^{\pm}$ we obtain

$$(y/x)^2 = \frac{[(2+m)/2m]\sin^2\theta}{\frac{1}{2}\cos^2\theta} \to 2\sqrt{2(1+m)}-1 \quad \text{when} \quad \theta \to \theta_m.$$

This means that the first type of orbits tend to form a rectilinear configuration near the collision/ejection such that the equal masses are equidistant from the third one. The other type of orbits tend to form an isosceles triangle. The shape of this triangle depends on the value of m. It is equilateral only if $m = 1$, it tends to a triangle of sides proportional to $\sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2}}$ and 2 if $m \to 0$, and it degenerates to a line when $m \to \infty$. This is rather surprising since in the Newtonian case all noncollinear configurations are equilateral.

Let us further prove the existence of solutions reaching the collision manifold otherwise than through equilibria. For making the terminology meaningful we will call the cycle **e** (see Figure 2) corresponding to $r = 0$ and $v = 0$, the *equator* of C. For future reference also call the region $r = 0, v > 0$ as the *northern hemisphere* of C and the region $r = 0, v < 0$ as the *southern hemisphere* of C.

Lemma 5. *There exist solutions of the equations* (13) *tending/ejecting to/from the equator* **e** *of the collision manifold* C. *Moreover,* **e** *is the only cycle on* C *having this property.*

Proof. We prove only the existence of solutions ejecting from **e** since the existence of collision solutions follows by symmetry. From the second equation in (13) we see that, for $\theta \neq \pm\frac{\pi}{2}$ and r sufficiently small, $v' > 0$, and that $v' = 0$ only at $\theta = \pm\frac{\pi}{2}$. This follows indeed since U has a positive inferior limit which exceeds $2rh$ for r sufficiently small, even if $h < 0$. Notice also that the values $\theta = \pm\frac{\pi}{2}$ are taken only at discrete fictitious-time instants along an orbit. The fact that $v' \geq 0$ implies that, for $r > 0$, orbits near C increase with respect to the v-variable. If $v > 0$, then $r' > 0$ (excepting the set of discrete values of ψ where $\theta = \pm\frac{\pi}{2}$, and where $r = 0$). However, as long as $v > 0$ and $r > 0$, an orbit through a point in the plane $v = const. \neq 0$ and outside C, will not eject from the cycle $v = const. \neq 0$ of C, because $v' > 0$ ($= 0$ only for the set formed by isolated points mentioned above). Since this is true for any cycle $v = const. > 0$ of C, the corresponding orbit cannot eject from any such periodic trajectory. Therefore, the only chance for an orbit outside C to reach a cycle $v = const.$ of C, is if $v = 0$. Indeed, then $r' \to 0$ and if also $r \to 0$, it follows that $v' \to 0$. Observe that r goes to 0 faster than v does (because $\lim_{\psi\to\infty}(r(\psi)/v(\psi)) = 0$), so one really expects that the equator **e** is approached backwards in time by orbits ejecting from it. For proving this, notice that one can find a constant $k > 0$ such that $v' \geq kr\frac{\cos^2\theta}{R(\theta)}$. Multiplying this inequality by $v > 0$, we get $vv' \geq krv\frac{\cos^2\theta}{R(\theta)} = kr'$, the equality following from the first equation in (13). Consequently, $r \leq (1/2k)v^2 + c$, where c is a constant of integration. For $c = 0$ and when $v \to 0$ through positive values, we necessarily have $r \to 0$ and the existence of orbits ejecting from the equator **e**, follows. This completes the proof.

Figure 3 tries to suggest in a 3-dimensional picture (where r is taken as the radial direction from C), an orbit ejecting from **e** which in fact belongs to a 4-dimensional phase space. The physical interpretation of these triple collision orbits is as follows. The particles m_1 and m_2 experience elastic binary collisions, while m_3 crosses the x-axis (up and down) after every collision. The amplitude of the oscillations of m_3 decreases after every passage through $x = 0$ and after infinitely many binary collisions, the triple collision occurs (in finite real time). For triple ejections this scenario is similar but with reversed time.

It is hard to determine the size of the set Λ_e of solutions tending/ejecting to/from the equator $\mathbf{e}$, or to decide whether it is a manifold or not. The associated *first return* (Poincaré) map has the form

$$(r,v) \to (r + \frac{4\pi^2 U}{2W^2}r^2 + \frac{2\pi}{W}rv + O_3(r,v),\ v + \frac{2\pi}{W}r + \frac{4\pi h}{W}r^2 + O_3(r,v))$$

and is degenerate at $r = 0, v = 0$. There exist some criteria of showing that, under certain conditions, degenerate fixed points may have manifolds attached [2], [21]. Unfortunately none of them is applicable in our case.

V. CENTER MANIFOLDS

In Proposition 4 we have seen that E^+ (E^-, respectively) contains a one-dimensional unstable (stable) manifold. However, since the equilibrium is not hyperbolic, the only conclusion we can draw regarding the size of the set of initial data leading to collinear triple collisions, is that it's dimension is at least one. In order to give a complete answer, a study of the center manifold is necessary. It is known that center manifolds may not be unique and that they rarely are analytic even for analytic vector fields [23]. Here we prove the following result.

Theorem 6. *For every fixed energy level, the equilibrium E^+ (E^-, respectively) of the equations* (13) *has a one-dimensional unstable (stable) analytic manifold and a unique two-dimensional analytic center manifold. The flow on this center manifold is stable but not asymptotically stable.*

From the proof of Theorem 6, the following consequence will become obvious.

Corollary 7. *For every real value h of the energy constant, the set of analytic solutions leading to a collinear triple collision forms a one-dimensional manifold.*

Proof of Theorem 6. Using the energy relation (14) to eliminate the last equation in (13) and translating the origin of the frame to E^+, the equations of motion (13) become

$$\begin{cases} r' = \frac{\cos^2\theta}{R(\theta)}r[\sqrt{2\gamma}(1+2m)+v] \\ v' = \frac{\cos^2\theta}{R(\theta)}r(2hr+U(\theta)) \\ \theta' = [\frac{2r(rh+U(\theta))\cos^3\theta}{R^2(\theta)} - \frac{[v+\sqrt{\gamma}(1+2m)]^2\cos^2\theta}{R^2(\theta)} + \frac{2\cos\theta}{R(\theta)}]^{1/2}. \end{cases}$$

Using the fact that

$$\frac{\cos^2\theta}{R(\theta)} = \frac{1}{\gamma(1+2m)^2} + \frac{4(2+3m+m^2)}{\gamma(1+2m)^4}\theta^2 + O_3(\theta)$$

and that $\frac{d}{d\theta}U(0) = \frac{d}{d\theta}V(0) = 0$, a long but straightforward computation shows that the above equations can be written as

$$\begin{cases} r' = Ar + \mathcal{R}(r, v, \theta) \\ v' = \mathcal{V}(r, \theta) \\ \theta' = \mathcal{T}(r, v, \theta), \end{cases}$$

where

$$\mathcal{R}(r, v, \theta) = a_1 rv + a_2 r\theta^2 + a_3 rv\theta^2 + r(a_4 + v)O_3(\theta),$$

$$\mathcal{V}(r, \theta) = b_1 r + b_2 r^2 + b_3 r\theta^2 + b_4 r^2\theta^2 + O(r) \cdot O_3(\theta),$$

$$\mathcal{T}(r, v, \theta) = c_1 r + c_2 v + c_3 \theta^2 + c_4 r^2 + c_5 v^2 + O_3(r, v, \theta).$$

The rest point E^+ is not hyperbolic since the eigenvalues corresponding to the v and θ component are 0. Therefore center manifolds have the form

$$W^c(\mathbf{0}) = \{(r, v, \theta) \in [0, \infty) \times \Re \times [-\frac{\pi}{2}, \frac{\pi}{2}] \mid r = g(v, \theta),\ |v|, |\theta| < \delta, g(0, 0) = 0, Dg(0, 0) = 0\}$$

for δ sufficiently small (see [14]). The function g will be determined by the equation

$$Ag(v, \theta) + \mathcal{R}(g(v, \theta), v, \theta) = \frac{\partial}{\partial v} g(v, \theta) \cdot \mathcal{V}(g(v, \theta), \theta) + \frac{\partial}{\partial \theta} g(v, \theta) \cdot \mathcal{T}(g(v, \theta), \theta).$$

Supposing g has the form

$$g(v, \theta) = \sum_{\substack{k=2 \\ i+j=k}}^{\infty} a_{ij} v^i \theta^j$$

and introducing it into the above equation we obtain $a_{ij} = 0$, for all admissible i, j. Therefore the only analytic function verifying the above equation is $g(v, \theta) \equiv 0$, so $r \equiv 0$ is the unique analytic center manifold. Since $v' = 0$ on the center manifold, the flow on it looks like in Figure 4. Therefore the flow on $W^c(\mathbf{0})$ is stable but not asymptotically stable. All the above arguments work for E^-. This completes the proof.

A similar result can be obtained for the other equilibrium solutions, however the computations become more complicated.

VI. REGULARIZATION

The concept of block-regularization was first used in celestial mechanics in the context of binary collisions of the classical 2-body problem [12]. This notion is meant to describe how far a collision-orbit can be extended beyond the singularity such that the property of continuity of the solutions with respect to initial data is maintained around the solution experiencing the collision singularity.

Let $\phi: \mathcal{S} \times \Re \to \mathcal{S}$ be a flow on a manifold $\mathcal{S}$, let B be a submanifold of $\mathcal{S}$ of the same dimension having the boundary b. Define the set of *ingress* and *egress* points of the flow with respect to B as being

$$b^- = \{z \in b \mid \phi(z, (-\epsilon, 0)) \cap B = \emptyset \text{ for some } \epsilon > 0\},$$

$$b^+ = \{z \in b \mid \phi(z, (0, \epsilon)) \cap B = \emptyset \text{ for some } \epsilon > 0\}.$$

Then B is called an *isolating block* if $b^+ \cup b^- = b$. Denote further

$$a^- = \{z \in b^+ \mid \phi(z, \psi) \in B \text{ for all time } \psi \geq 0\},$$

$$a^+ = \{z \in b^- \mid \phi(z, \psi) \in B \text{ for all time } \psi \leq 0\},$$

and define $\Gamma: b^- \setminus a^- \to b^+ \setminus a^+$, $\Gamma(z_-) = z_+$, where z_- and z_+ belong to the same orbit of the flow and z_+ is the point where the orbit first hits $b^+ \setminus a^+$ coming from z_-. For ϕ and B fixed, Γ is unique. It is known that Γ is a diffeomorphism [3]. The pair (ϕ, B) will be called *block-regularizable* if Γ can be homeomorphically extended from b^+ to b^-.

Let further ϕ denote a solution of the equations (13) and consider the following notations

$$S_{h,\epsilon} = \{\phi \mid r \leq \epsilon\} \cap \{\phi \mid \text{relation (14) holds}\},$$

$$G = G_{\mu,h,\epsilon} = \{\phi \in S_{h,\epsilon} \mid |v| \leq \mu\},$$

$$G^+ = G^+_{\mu,h,\epsilon} = \{\phi \in S_{h,\epsilon} \mid v \geq \mu\},\ G^- = G^-_{\mu,h,\epsilon} = \{\phi \in S_{h,\epsilon} \mid v \leq -\mu\},$$

$$\gamma^\pm_\mu = \gamma^\pm_{\mu,h,\epsilon} = \{\phi \in G^\pm \mid r = \epsilon\},\ \sigma^\pm_\mu = \sigma^\pm_{\mu,h,\epsilon} = \{\phi \in S_{h,\epsilon} \mid v = \pm\mu\},$$

where $\epsilon > 0$ and $\mu > 0$ are constants.

For a small enough $\epsilon = \bar{\epsilon}$ and for $h = \bar{h}$ fixed, $S = S_{\bar{h},\bar{\epsilon}}$ is an isolating block, where $b^+ = \gamma^+_{0,\bar{h},\bar{\epsilon}}$, $b^- = \gamma^-_{0,\bar{h},\bar{\epsilon}}$, $a^+ = b^+ \cap (\Lambda^+_e \cup \Lambda_{E^+} \cup \Lambda_{L^+} \cup \Lambda_{R^+})$, $a^- = b^- \cap (\Lambda^-_e \cup \Lambda_{E^-} \cup \Lambda_{L^-} \cup \Lambda_{R^-})$. Here $\Lambda^\pm_e, \Lambda_{E^\pm}, \Lambda_{L^\pm}, \Lambda_{R^\pm}$ represent the sets of orbits ejecting (leading) from (to) $e, E^+, L^+, R^+ (e, E^-, L^-, R^-)$. Notice that $\Lambda_e = \Lambda^+_e \cup \Lambda^-_e$ and the union is disjoint. Also denote $\Lambda_E = \Lambda_{E^+} \cup \Lambda_{E^-}$, $\Lambda_L = \Lambda_{L^+} \cup \Lambda_{L^-}$, $\Lambda_R = \Lambda_{R^+} \cup \Lambda_{R^-}$.

We've seen in Section IV that the structure and size of Λ_e are difficult to estimate. The same happens with the question of regularization of orbits in Λ_e. The next result shows the importance of understanding these properties of Λ_e, for solving the regularization question of orbits in $\Lambda_E \cup \Lambda_L \cup \Lambda_R$. Here we restrict to analyse $\Lambda_L \cup \Lambda_R$, i.e. the noncollinear collision orbits.

Theorem 8. *If solutions in Λ_e are not block-regularizable or if* $\dim \Lambda_e < 3$, *then solutions in $\Lambda_L \cup \Lambda_R$ are not block-regularizable.*

Proof. Let $[\psi_1, \psi_2]$, with $\psi_1 < \psi_2$, be a closed interval of the fictitious time variable ψ. Then $\phi([\psi_1, \psi_2])$ is said to be *maximal* in a closed set K, if $\phi([\psi_1, \psi_2]) \subset K$ but $\phi(I) \not\subset K$, for any I containing $[\psi_1, \psi_2]$ but larger than it. We further prove the following property for the flow ϕ of equations (13).

(*) *For ϵ small, if $\phi([\psi_1, \psi_2])$ is maximal in G^+, then $\phi(\psi_1) \in \sigma_\mu^+$ and $\phi(\psi_2) \in \gamma_\mu^+$.*

If $\phi(\psi) \in \sigma_\mu^+$, then $v = \mu > 0$ and $r \leq \epsilon$. Since $\epsilon < 0$ is small, it follows that $v'(\psi) > 0$ and $r'(\psi) > 0$ for all ψ excepting for a discrete set $\{\psi_k\}_{k \in \mathrm{N}}$ of isolated points where $v'(\psi_k) = r'(\psi_k) = 0$. These imply that points on σ_μ^+ are entring G^+, so $\phi(\psi_2) \in \gamma_\mu^+$. Also, points in γ_μ^+ are leaving G^+, so $\phi(\psi_1) \in \sigma_\mu^+$. This concludes the proof of (*).

Take now $\alpha \in a^- \setminus \Lambda_{E^-}$ and let $\mathcal{U} \subset b^-$ be an open neighborhood of α. Notice that $\mathcal{U}$ is a 3-dimensional set and suppose $\dim \Lambda_e < 3$. Take also $\mu > -\sqrt{2V(\theta_m)}$. Since σ_μ^+ (form the proof of (*)) and b^+ (from its definition) are sections of the flow and since $\dim \Lambda_e < 3$, it follows that some of the orbits will not tend to e. Since for those orbits $v' > 0$ (excepting a discrete set of isolated fictitious time instants), there exist orbits in $\mathcal{U}$ reaching σ_μ^+. Such orbits cross σ_μ^+, enter G^+ and by (*) they can leave G^+ only through γ_μ^+. Thus, for the homeomorphism $\Gamma: b^- \setminus a^- \to b^+ \setminus a^+$, we can find a sequence of points $\{z_\mu\}_{\mu \in \mathrm{N}}, z_\mu \to \alpha$ for $\mu \to \infty$, such that $\Gamma(z_\mu) \in \gamma_\mu^+ \subset b^+$ for every μ. But $\bigcap_{\mu \in \mathrm{N}} \gamma_\mu^+ = \emptyset$ and since Γ is a homeomorphism, it cannot be extended as a homeomorphism to a^+ since it cannot be extended to the point α. This shows that orbits in $\Lambda_L \cup \Lambda_R$ are not block-regularizable in case $\dim \Lambda_e < 3$.

If orbits in Λ_e are not block-regularizable and $\dim \Lambda_e = 3$, and if there are still solutions reaching σ_μ^+, the above procedure can be used in the same way. Otherwise, the fact that orbits in Λ_e are not block-regularizable concludes the proof.

Note. The idea of proving (*) occured first in [22], making use of the specific form of the equations of motion of the rectilinear 3-body problem. It was shown in [11] that to prove a statement similar to (*) one can use only the fact that the flow is gradient-like on the collision manifold. In our case we gave a proof for a situation where the flow on the collision manifold is not gradient-like.

VII. CONNECTING ORBITS

The goal of this section is to analyse the existence of connecting orbits for the rest-points $E^\pm, R^\pm, L^\pm$ and the equatorial cycle $\mathbf{e}$. Let us first prove the following result.

Theorem 9. *An orbit leaving (reaching) the collision manifold from E^+, R^+, L^+ or $\mathbf{e}$ (at E^-, R^-, L^- or $\mathbf{e}$), cannot return (start) to (from) the northern (southern) hemisphere of C.*

Proof. We have seen that since there exists a $K > 0$ such that $U(\theta) > K$ for all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it follows that for any fixed h (including the cases when $h < 0$), $v' \geq 0$ for r sufficiently small, being 0 only for a discrete set of points. So, if $v > 0$ (as it is the case at instants of time that make the corresponding orbits to be infinitesimally close to E^+, R^+, L^+ or $\mathbf{e}$) then necessarily $r' > 0$ at those moments of time. Suppose that $v' < 0$ at some later instant of time (otherwise the statement is proved). Thus, in order to reach the northern hemisphere again, an orbit needs to have an ultimately decreasing r-component, at least on a subsequence of time instants. But it also needs $v' \geq 0$ and $v > 0$ in a neighborhood of the northern hemisphere. This implies $r' > 0$, so r is increasing and the orbit cannot come close to the northern hemisphere again. The proof regarding the southern hemisphere is analogous.

Corollary 10. *In the invariant set $\{r > 0\}$ there are no homoclinic orbits for the equilibria $E^{\pm}, R^{\pm}, L^{\pm}$, there are no orbits connecting $E^{\pm}, R^{\pm}$ or $L^{\pm}$ with* $\mathbf{e}$ *and there are no heteroclinic orbits inside the sets $\{E^{+}, R^{+}, L^{+}\}$ and $\{E^{-}, R^{-}, L^{-}\}$.*

The following result gives an even better understanding of the phase space picture. The method of proving the existence of the homoclinic orbits used below is due to Devaney [6].

Theorem 11. *Restrict the solutions of equations (13) to the invariant set $\{r > 0\}$. Then, for any $h < 0$ there exists a unique heteroclinic orbit from E^{+} to E^{-} in the invariant set $D = \{(r, v, \theta, w) \mid \theta = w = 0\}$. Moreover, for $m = 1$ there also exist heteroclinic orbits from R^{+} to R^{-} and from L^{+} to L^{-} in the invariant sets $D_R = \{(r, v, \theta, w) \mid \theta = \theta_m, w = 0\}$ and $D_L = \{(r, v, \theta, w) \mid \theta = -\theta_m, w = 0\}$, respectively. If $h \geq 0$, there are no heteroclinic orbits connecting elements of the sets $\{E^{+}, L^{+}, R^{+}\}$ and $\{E^{-}, R^{-}, L^{-}\}$, and there are no homoclinic orbits for* $\mathbf{e}$.

Proof. Notice first that $\theta = 0$ is a solution of the equation $\frac{d}{d\theta}V(\theta) = 0$, which represents the condition of a central configuration for the inverse cubic law. This condition gives rise (for $\theta = 0$) to an invariant set for equations (13) (see [22]). However, in order to have compatibility with the third equation in (13), one necessarily needs $w = 0$. Thus $D = \{(r, v, \theta, w) \mid \theta = w = 0\}$ is an invariant set for the equations (13).

In order to prove the existence of the heteroclinic orbit from E^{+} to E^{-}, restrict equations (13) to the invariant set D and get

$$\begin{cases} r' = \frac{1}{\gamma(2m+1)^2} rv \\ v' = \frac{4m+1}{\sqrt{2}\gamma(2m+1)^2} r + \frac{2h}{\gamma(2m+1)^2} r^2, \end{cases} \tag{15}$$

with the energy relation

$$v^2 = 2hr^2 + \sqrt{2}(4m+1)r + 2\gamma(2m+1)^2.$$

Some elementary computations show that the energy relation makes sense for all $v \in \Re$ and $r \geq 0$ and that the energy levels in the invariant set D are as in Figure 5. Thus the first part of the theorem follows. For the case $m = 1$, the invariance of D_R and D_L is a consequence of the fact that $\frac{d}{d\theta}U(\pm\theta_m) = 0$. The equations (13) restricted to D_R take the form

$$\begin{cases} r' = \frac{m+2}{2\gamma(m\sqrt{(2m+1)}+1)^2} rv \\ v' = \frac{(m+2)U(\theta_m)}{2\gamma(m\sqrt{(2m+1)}+1)^2} r + \frac{h(m+2)}{\gamma(m\sqrt{(2m+1)}+1)^2} r^2, \end{cases}$$

with the energy relation

$$v^2 = \frac{1}{2}\left(\frac{h}{\cos^3\theta_m} r^2 + rU(\theta_m) + \frac{R(\theta_m)}{\cos^2\theta_m}\right).$$

A similar argument to that given for equations (15) shows that, qualitatively, the phase-space picture of the above equations is the same. Thus, the existence of the heteroclinic orbits from R^+ to R^- follows. The proof of the statement concerning the heteroclinic orbits from L^+ to L^- is proved in the same way. For the last part of the theorem it is enough to notice that since $h \geq 0$, from the equations (13) we have $v' \geq 0$, the value 0 being reached only for a discrete set of time instants. Thus, orbits ejecting from E^+, R^+, L^+ or $\mathbf{e}$, cannot attain values for v lower than the initial positive ones. This completes the proof.

The physical interpretation of the heteroclinic orbits connecting E^+ and E^- is as follows. The particles eject and move homothetically away from the center of mass, m_1 and m_2 leave the origin equidistantly and in opposite directions on the x-axis, while m_3 remains at the origin. They return then homothetically and the orbit dies in a triple collision. The heteroclinic orbits from R^+ to R^- mean that tha particles eject from a triple approach, move homothetically maintaining all the time the shape of an equilateral triangle. They stop at some instant and then return homothetically, leading to a triple collapse in their common center of mass. The interpretation of the orbits connecting L^+ and L^- is the same like for those connecting R^+ and R^- but having m_1 instead of m_2 and m_2 in the place of m_1.

VIII. AN ANISOTROPIC MANEFF LAW

The equilibria of Maneff's flow are not hyperbolic since $\lambda_v = 0$. Still, the corresponding center manifold doesn't give rise to a bifurcation of the equilibria. We see now that this is possible for a slightly different model. In this section we define an anisotropic Maneff law and compare the results with those obtained in previous sections. Anisotropic models are important for the understanding of certain connections between classical and quantum mechanics. Studies of the anisotropic Kepler problem have been initiated by Gutzwiller [15, 16] and continued by Devaney [4, 5]. The anisotropy is this time thought as depending on the body, acting in the interraction of each of m_1 and m_2 with m_3. Take the scalar product

$$< \mathbf{a}, \mathbf{b} >= a_1 b_1 + a_2 b_2 + (1/2) a_3 b_3,$$

and denote by ∇ the gradient with respect to it. Define again the equations (1) as in Section II but considering the new gradient and taking this time a simpler $\tilde{V}$-potential function, given by

$$\tilde{V}: \Re^6 \setminus \Delta \to \Re_+, \quad \tilde{V}(\mathbf{q}) = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^2}.$$

Using transformations (2), the equations of motion of the isosceles problem become

$$\begin{cases} \ddot{x} = -\frac{1}{4x^2} - \frac{mx}{(x^2+y^2)^{3/2}} - \frac{1}{4x^3} - \frac{2mx}{(x^2+y^2)^2} \\ \ddot{y} = -\frac{(1+m)y}{(x^2+y^2)^{3/2}} - \frac{(2+m)y}{(x^2+y^2)^2}. \end{cases} \tag{16}$$

Then Theorem 1 and Corollary 2 are also valid. Equations (4) have the same form, with the same function $\hat{U}$ but having

$$M = \begin{pmatrix} 2 & 0 \\ 0 & \frac{4m}{2+m} \end{pmatrix} \text{ and } \hat{V}(x,y) = \frac{1}{4x^2} + \frac{2m}{x^2+y^2}.$$

Using transformations (5)-(6), it follows that equations (7) and the energy relation take the same form. Instead of transformations (8) define

$$\begin{cases} \mathbf{s} = (\frac{1}{\sqrt{2}}\cos\theta, \sqrt{\frac{m+2}{4m}}\sin\theta) \\ \mathbf{u} = (-\sqrt{2}u\sin\theta, \sqrt{\frac{4m}{m+2}}u\cos\theta). \end{cases} \tag{17}$$

The energy relation (9) and the equations (10) remain unchanged, with the same U but having

$$V(\theta) = \frac{1}{2\cos^2\theta} + \frac{8m^2}{m+2+(m-2)\cos^2\theta}.$$

The critical points of V are, for $0 < m < 7/4$, at $\theta = 0$ and $\theta = \pm\theta_m$, where

$$\theta_m = \arccos\sqrt{\frac{m+2}{2-m+4m\sqrt{2-m}}},$$

and only at $\theta = 0$ for $m \geq 7/4$. Since equations (13), (14) and the collision manifold are the same, it follows that the behavior of V determines the behavior of the flow on the collision manifold. The corresponding equations (13) have the 6 equilibria $E^\pm, R^\pm$ and $L^\pm$, similar to those given by Proposition 3, if $0 < m < 7/4$, and only the equilibria $E^\pm$ if $m \geq 7/4$. For the equilibria in the northern hemisphere as well as for those in the southern hemisphere, a subcritical pitchfork bifurcation occurs (see Figure 6, where the dotted/continuous line means instability/stability for the flow restricted to the collision manifold).

For $0 < m < 7/4$ the qualitative behavior of the flow on the collision manifold is the same like in the classical planar isosceles problem studied in previous sections. For $m \geq 7/4$ the flow is like in the rectilinear problem [11]. Physically this is explained by computing the asymptotic configurations of orbits approaching the rest points of the flow. Thus, for $E^\pm$ we obtain

$$y/x = \frac{\sqrt{\frac{2+m}{4m}}\sin\theta}{\frac{1}{\sqrt{2}}\cos\theta} \to 0, \text{ for } \theta \to 0,$$

while for $R^\pm$ and $L^\pm$ we compute

$$(y/x)^2 = \frac{\frac{2+m}{4m}\sin^2\theta}{\frac{1}{2}\cos^2\theta} \to 2\sqrt{2-m} - 1, \text{ for } \theta \to \theta_m.$$

This shows that $E^{\pm}$ correspond to collinear configurations while $R^{\pm}$ and $L^{\pm}$ regard isosceles triangles which change their shape with the value of m. For $m \to 7/4$ through values smaller than $7/4$, the shape degenerates from a triangle to a segment. On the collision manifold this means that $R^{\pm}$ and $L^{\pm}$ tend to $E^{\pm}$ (see Figure 7). It is also interesting to note that, due to the anisotropy, the case $m = 1$ doesn't correspond to an equilateral but to a right isosceles triangle.

Note finally that the other results of previous sections: Lemma 5, Theorem 6, Corollary 7, Theorem 8, Theorem 9, Corollary 10, are all true for the anisotropic problem discussed here if $0 < m < 7/4$, and with obvious changes if $m \geq 7/4$.

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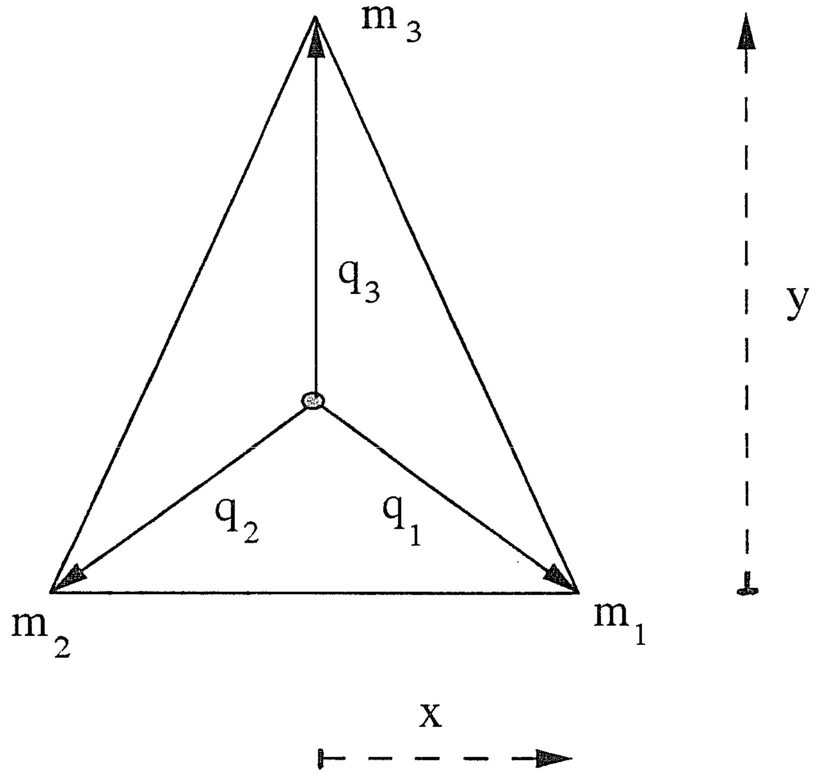


Figure 1

The planar isosceles configuration

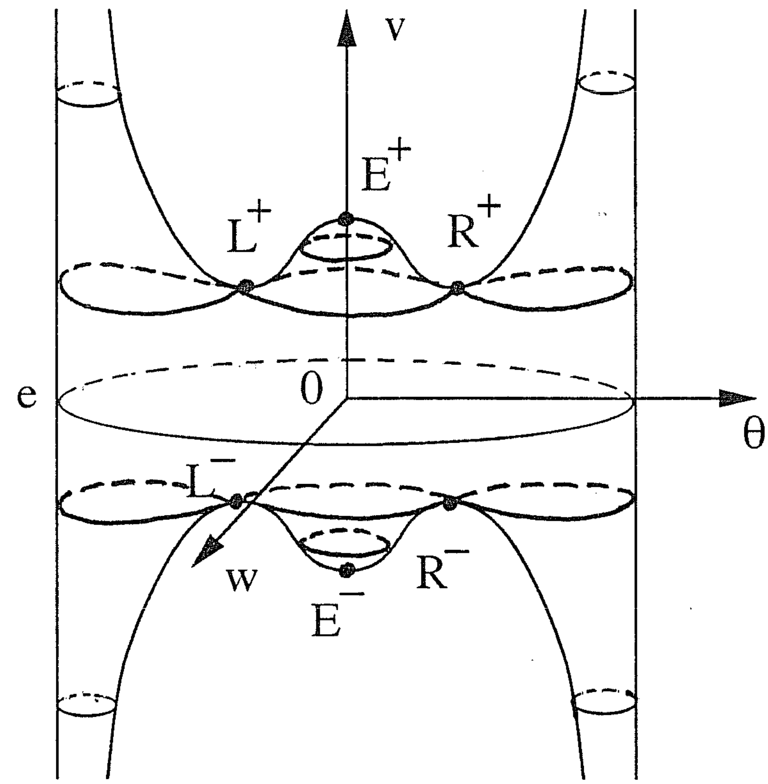


Figure 2

The flow on the collision manifold

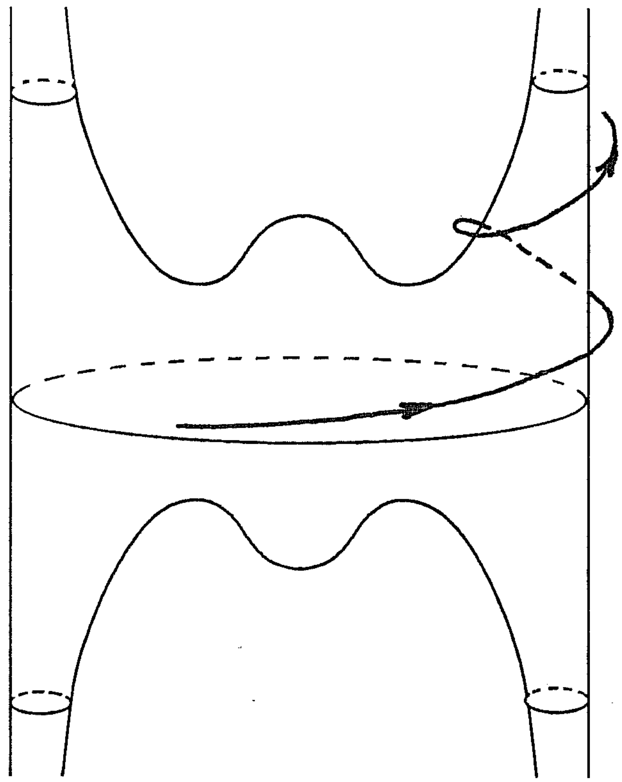


Figure 3

An orbit ejecting from the equator

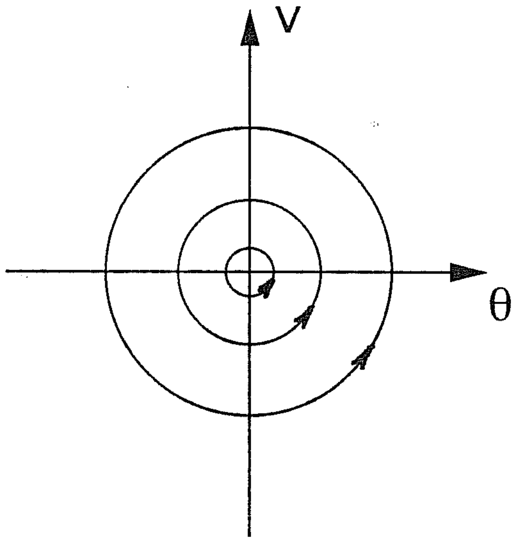


Figure 4

The flow on the analytic center manifold of E^+

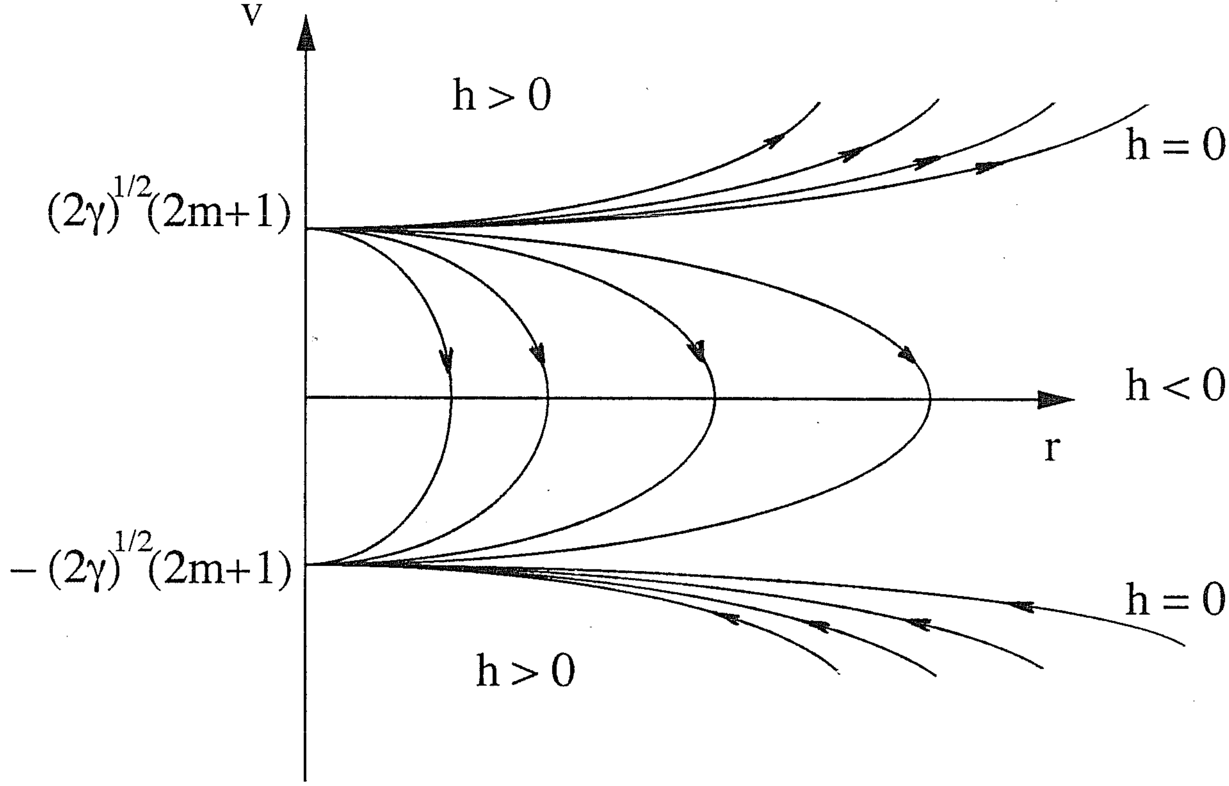


Figure 5

The flow in the invariant set D

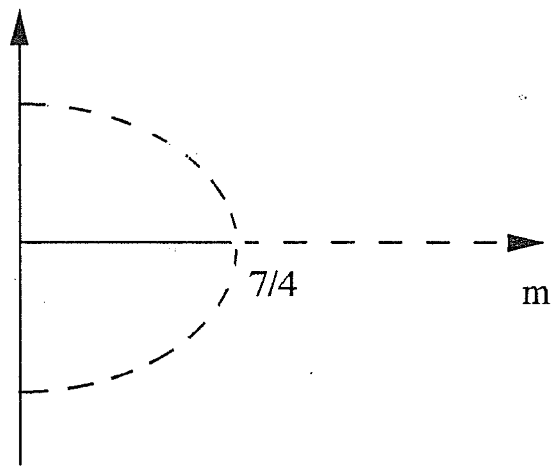
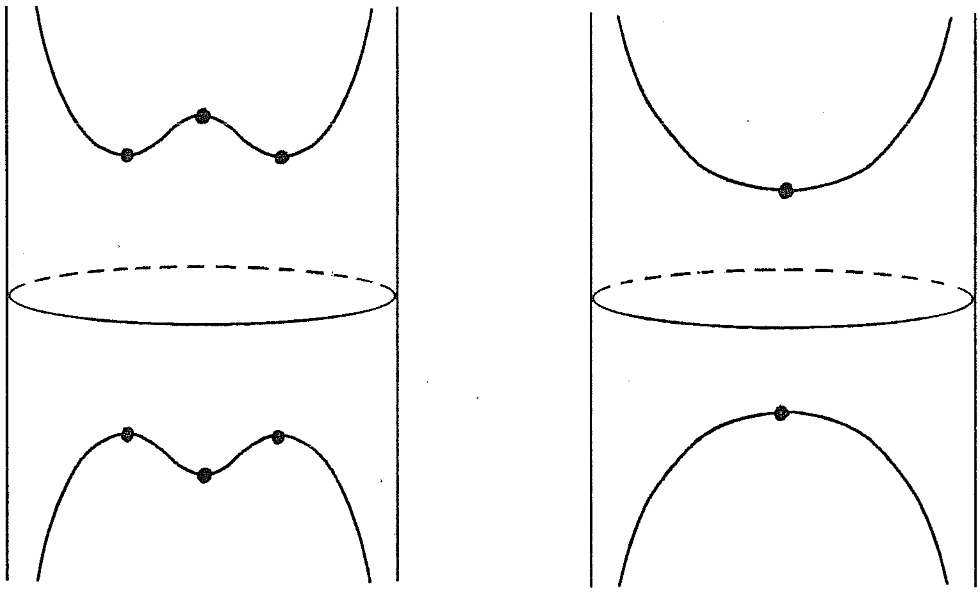


Figure 6

Subcritical pitchfork bifurcation of the equilibria on the collision manifold



$0 < m < 7/4$ $m \geq 7/4$

Figure 7

The qualitative change of the collision manifold for the anisotropic law