

C^* -algebras from actions of congruence monoids

by

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Abstract

We initiate the study of a new class of semigroup C^* -algebras arising from number-theoretic considerations; namely, we generalize the construction of Cuntz, Deninger, and Laca by considering the left regular C^* -algebras of $ax+b$ -semigroups from actions of congruence monoids on rings of algebraic integers in number fields. Our motivation for considering actions of congruence monoids comes from class field theory and work on Bost–Connes type systems. We give two presentations and a groupoid model for these algebras, and establish a faithfulness criterion for their representations. We then explicitly compute the primitive ideal space, give a semigroup crossed product description of the boundary quotient, and prove that the construction is functorial in the appropriate sense. These C^* -algebras carry canonical time evolutions, so that our construction also produces a new class of C^* -dynamical systems. We classify the KMS (equilibrium) states for this canonical time evolution, and show that there are several phase transitions whose complexity depends on properties of a generalized ideal class group. We compute the type of all high temperature KMS states, and consider several related C^* -dynamical systems.

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Introduction

The area of operator algebras was founded in the first half of the 20th century by Murray, von Neumann, Gelfand, and Naimark. Initially, operator algebras were conceived as mathematical models for systems in quantum physics; however, it soon became apparent that the theory was intimately connected with many other areas of mathematics, such as representation theory, measure theory, algebra, topology, and dynamical systems. This has made the theory very robust, and operator algebras is now an active and highly interdisciplinary area of research in mathematics. One of the most important classes of operator algebras is the class of C^* -algebras; these can be realized as operator-norm closed subalgebras of bounded operators on Hilbert space that are closed under taking adjoints. A C^* -algebra equipped with a time evolution is called a C^* -dynamical system; such systems provide a mathematical framework for studying quantum statistical dynamical systems and their equilibrium states in a very general setting.

In the 1990s, Connes discovered several fascinating connections between operator algebras and number theory. This inspired many operator algebraists to consider C^* -algebras of number-theoretic origin, and the study of C^* -algebras from number theory has been an active area of research ever since. One particularly interesting interaction between operator algebras and number theory, pioneered by Cuntz, is by way of C^* -algebras generated by representations of “ $ax + b$ -semigroups” over rings of algebraic integers. The C^* -algebras arising from number theory in this fashion carry canonical time evolutions, and thus give rise to C^* -dynamical systems. This interplay between operator algebras and number theory has benefited the theory of operator algebras by providing new classes of C^* -algebras and C^* -dynamical systems which exhibit interesting phenomena; their analysis has led to new results that can also be applied in much more general contexts.

The content of this thesis comprises two research articles [A] and [B] given in chapters 5 and 6, respectively, in which C^* -algebras of $ax+b$ -semigroups arising from actions of congruence monoids on rings of algebraic integers are introduced and systematically studied. Before getting to these works, I will discuss preliminaries on semigroup C^* -

algebras in general and on the C^* -algebra associated with the full $ax + b$ -semigroup over the ring of integers in an algebraic number field, which was introduced by Cuntz, Deninger, and Laca. In order to explain the motivation for the construction in this thesis, I will also give a brief introduction to the ideal-theoretic formulation of class field theory.

Preliminaries

Semigroup C*-algebras. If P is a left cancellative semigroup, then each element $p \in P$ gives rise to an isometry $\lambda_p \in \mathcal{B}(\ell^2(P))$ such that $\lambda_p \delta_x = \delta_{px}$ where $\{\delta_x : x \in P\}$ is the canonical orthonormal bases for $\ell^2(P)$ consisting of point-mass functions. The *left regular C*-algebra of P* is $C_\lambda^*(P) := C^*(\{\lambda_p : p \in P\})$. This way, one obtains a large class of interesting C*-algebras which contains the reduced group C*-algebras of discrete groups.

Over the last decade, the general theory of semigroup C*-algebras has been significantly advanced. In particular, the work of Li [24, 25, 29] provides general connections between semigroup C*-algebras and groupoid C*-algebras which allows one to import results about the latter class to understand semigroup C*-algebras via a more dynamical approach (see [8, Chapter 5] for a unified treatment of Li's early work).

We shall now briefly explain some of Li's work for the special case of left Ore monoids; this class of semigroups contains the ones arising from actions of congruence monoids in which we are ultimately interested here. Suppose that P is a left Ore monoid with group of left quotients G , that is, P embeds into the group G such $P^{-1}P = G$ where $P^{-1}P := \{q^{-1}p : p, q \in P\}$ is the set of left quotients of P in G . Let \mathcal{J}_P denote the smallest collection of subsets of P such that

- $\emptyset \in \mathcal{J}_P$ and $P \in \mathcal{J}_P$;
- if $X \in \mathcal{J}_P$ and $p \in P$, then $pX \in \mathcal{J}_P$ and $(p^{-1}X) \cap P \in \mathcal{J}_P$;
- if $X, Y \in \mathcal{J}_P$, then $X \cap Y \in \mathcal{J}_P$.

Each $X \in \mathcal{J}_P$ is called a *constructible right ideal of P* ; equipped with the semigroup operation given by set intersection, \mathcal{J}_P becomes a semilattice (that is, a commutative semigroup in which every element is idempotent). The second condition implies that P acts on \mathcal{J}_P by left translation. The semilattice \mathcal{J}_P appears naturally when one studies the C*-algebra $C_\lambda^*(P)$; indeed, if we consider the canonical "diagonal"

subalgebra $D_\lambda(P) := C_\lambda^*(P) \cap \ell^\infty(P)$ of $C_\lambda^*(P)$, then $D_\lambda(P) = \overline{\text{span}}(\{E_X : X \in \mathcal{J}_P\})$ where E_X denotes the orthogonal projection from $\ell^2(P)$ onto $\ell^2(X)$. The semigroup P is said to *satisfy the independence condition* if for $X, X_1, \dots, X \in \mathcal{J}_P$, $X = \bigcup_{i=1}^n X_i \implies X = X_i$ for some i . Let $C_u^*(\mathcal{J}_P)$ be the C^* -algebra of the semilattice \mathcal{J}_P , and let $e_X \in C_u^*(\mathcal{J}_P)$ denote the canonical generating projection corresponding to $X \in \mathcal{J}_P$. Then there is a canonical homomorphism $C_u^*(\mathcal{J}_P) \rightarrow D_\lambda(P)$ such that $e_X \mapsto E_X$ for all $X \in \mathcal{J}_P$. By [25, Corollary 2.7], P satisfies the independence condition if and only if this map is an isomorphism. There is a natural action α of P on $C_u^*(\mathcal{J}_P)$ by endomorphisms such that $\alpha_p(e_X) = e_{pX}$ for all $p \in P$ and $X \in \mathcal{J}_P$. The semigroup crossed product $C_u^*(\mathcal{J}_P) \rtimes_\alpha P$ serves as the *universal C^* -algebra of P* , and there is a canonical surjective homomorphism from $C_u^*(\mathcal{J}_P) \rtimes_\alpha P$ onto $C_\lambda^*(P)$. Satisfying the independence condition is clearly a necessary condition for the canonical homomorphism $C_u^*(\mathcal{J}_P) \rtimes_\alpha P \rightarrow C_\lambda^*(P)$ to be an isomorphism.

The full $ax + b$ -semigroup over a ring of integers. Let K be a number field with ring of integers R , and let $R^\times := R \setminus \{0\}$ be the multiplicative monoid of non-zero elements in R . Then R^\times acts on the additive group of R by multiplication, and one may form the semi-direct product $R \rtimes R^\times$ which is the set $R \times R^\times$ equipped with the product

$$(b, a)(d, c) = (b + ad, ac) \quad \text{for } (b, a), (d, c) \in R \times R^\times.$$

The monoid $R \rtimes R^\times$ is interesting because it encodes both the multiplicative and additive structures of the ring R . The left regular C^* -algebra $C_\lambda^*(R \rtimes R^\times)$ carries a canonical time evolution σ that is determined on the generating isometries by

$$\sigma_t(\lambda_{(b,a)}) = N(a)^{it} \lambda_{(b,a)} \quad \text{for } (b, a) \in R \times R^\times, t \in \mathbb{R},$$

where $N(a) = |R/aR|$ is the norm of the principal ideal aR . Cuntz, Deninger, and Laca [5] initiated the study of $C_\lambda^*(R \rtimes R^\times)$ and of the associated C^* -dynamical system $(C_\lambda^*(R \rtimes R^\times), \sigma)$. They proved that these systems exhibit intriguing properties, including a phase transition involving the ideal class group of K . This and earlier work of Laca and Raeburn [13] led Neshveyev to prove a general result about KMS

states on groupoid C*-algebras [31] which gives an alternative approach to the phase transition theorem from [5].

It is also shown in [5] that the construction of $C_\lambda^*(R \rtimes R^\times)$ from R is functorial for inclusions of rings of algebraic integers. Thus K-theory provides a new invariant of the R (or equivalently of the number field K). Cuntz, Echterhoff, and Li [6, 7] proved a powerful general result on K-theory for a certain class of crossed product C*-algebras that in particular gave a formula for the K-theory of $C_\lambda^*(R \rtimes R^\times)$ in terms of the K-theory of the C*-algebras of certain isotropy groups. However, in general, it is not known how much information about R is contained in the K-theory of $C_\lambda^*(R \rtimes R^\times)$.

Echterhoff and Laca computed the primitive ideal space of $C_\lambda^*(R \rtimes R^\times)$ [10], and Li then studied K-theoretic invariants associated with primitive ideals [26, 27]. Li showed in particular, that if one considers not only the K-theory of $C_\lambda^*(R \rtimes R^\times)$, but also K-theoretic invariants associated with the minimal primitive ideals of $C_\lambda^*(R \rtimes R^\times)$, then one can, under a technical assumption, recover the Dedekind zeta function of the underlying number field (this technical assumption has since been removed, see [3]).

Class field theory. As before, let K be a number field with ring of algebraic integers R . From a number-theoretic point of view, it is natural to consider more general monoids of the form $R \rtimes M$ where M is a submonoid of R^\times defined via congruence conditions. In order to explain this, we must briefly discuss the ideal-theoretic formulation of class field theory.

A finite field extension $L \supseteq K$ is said to be *abelian* if L is Galois over K and the Galois group $\text{Gal}(L/K)$ is abelian. Class field theory is concerned with the study of abelian extensions of number fields (or more generally, of global fields). For the number field \mathbb{Q} , the Kronecker–Weber theorem asserts that every abelian extension is contained in a cyclotomic field, that is, if $L \subseteq \mathbb{C}$ is a finite abelian extension of \mathbb{Q} , then there exists a natural number m such that $L \subseteq \mathbb{Q}(\zeta_m)$, where ζ_m is a primitive m -th root of unity and $\mathbb{Q}(\zeta_m)$ is the smallest subfield of \mathbb{C} containing ζ_m . Abelian extensions of imaginary quadratic fields (or more generally, CM fields) also admit

an explicit description, but in general, it is not known how to explicitly describe all abelian extensions of a given number field. Indeed, Hilbert's 12th problem concerns *explicit class field theory*, that is, the problem of explicitly describing the abelian extensions of a number field in terms of the arithmetic of the field itself. Although Hilbert's 12th problem is still open, it has been known since the early 20th century that given a number field K , there exist finite abelian extensions of K , called ray class fields, which play a role analogous to that played by the cyclotomic fields $\mathbb{Q}(\zeta_m)$ for \mathbb{Q} . Indeed, the ray class fields of a number field K are parameterized by arithmetic data associated with K , and they exhaust all abelian extensions of K in the sense that their union is equal to K^{ab} , the maximal abelian extension of K . However, unlike the cyclotomic fields, ray class fields are typically not given explicitly.

Let \mathcal{P}_K be the set of non-zero prime ideals of R . Then every fractional ideal \mathfrak{a} of K can be written uniquely as $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathcal{P}_K} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$ where $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$ is zero for all but finitely many \mathfrak{p} , that is, the group \mathcal{I} of fractional ideals of K is freely generated by the elements of \mathcal{P}_K . As usual, for each $x \in K^* = K \setminus \{0\}$, we shall write $v_{\mathfrak{p}}(x)$ instead of $v_{\mathfrak{p}}(xR)$, where xR is the principal fractional ideal generated by x . Let $i: K^* \rightarrow \mathcal{I}$ by $i(x) := xR$. Then $\text{Cl}_K := \mathcal{I}/i(K^*)$ is the *ideal class group* of K . This group is an important invariant of K ; it is known to be a finite group, and Cl_K is trivial if and only if R is a unique factorization domain.

Let $V_{K, \mathbb{R}}$ be the finite set of field embeddings of K into \mathbb{R} . A *modulus for K* is a function $\mathfrak{m}: V_{K, \mathbb{R}} \sqcup \mathcal{P}_K \rightarrow \mathbb{N}$ such that the restriction $\mathfrak{m}_{\infty} := \mathfrak{m}|_{V_{K, \mathbb{R}}}$ of \mathfrak{m} to $V_{K, \mathbb{R}}$ is $\{0, 1\}$ -valued and the restriction $\mathfrak{m}|_{\mathcal{P}_K}$ of \mathfrak{m} to \mathcal{P}_K is finitely supported. Then $\mathfrak{m}_0 := \prod_{\mathfrak{p} \in \mathcal{P}_K} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{m})}$ is a non-zero ideal of R , and it is customary to write \mathfrak{m} as $\mathfrak{m} = \mathfrak{m}_{\infty} \mathfrak{m}_0$ and to say w *divides* \mathfrak{m}_{∞} , denoted by $w \mid \mathfrak{m}_{\infty}$, when $\mathfrak{m}_{\infty}(w) = 1$. If \mathfrak{m} is a modulus for K , we let $\mathcal{I}_{\mathfrak{m}} \subseteq \mathcal{I}$ denote the group of fractional ideals of K that are coprime to \mathfrak{m}_0 . Given a modulus \mathfrak{m} for K , there exists a finite abelian extension $K(\mathfrak{m})$ of K , called the *ray class field mod \mathfrak{m}* , and a surjective group homomorphism $r_{K(\mathfrak{m})/K}: \mathcal{I}_{\mathfrak{m}} \rightarrow \text{Gal}(K(\mathfrak{m})/K)$ whose kernel is precisely the subgroup of $\mathcal{I}_{\mathfrak{m}}$ consisting of principal ideals (x) where $x \in K^*$ is coprime to \mathfrak{m}_0 and satisfies $v_{\mathfrak{p}}(x-1) \geq v_{\mathfrak{p}}(\mathfrak{m}_0)$ for all $\mathfrak{p} \mid \mathfrak{m}_0$ and $w(x) > 0$ for all $w \mid \mathfrak{m}_{\infty}$ (see [30, Chapter V, Section 3]). The map $r_{K(\mathfrak{m})/K}$ is called the *Artin map*, and the subgroup of elements in K^* whose principal ideals lie in $\ker(r_{K(\mathfrak{m})/K})$ is denoted by $K_{\mathfrak{m}, 1}$. The quotient $\text{Cl}_K(\mathfrak{m}) := \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, 1}) \cong$

$\text{Gal}(K(\mathfrak{m})/K)$ is called the *ray class group mod \mathfrak{m}* . Moreover, given any finite abelian extension L of K , there exists a modulus \mathfrak{m} for K such that $L \subseteq K(\mathfrak{m})$. Thus, ray class fields of K are analogous to the cyclotomic extensions of \mathbb{Q} . However, it is not known how to explicitly describe these ray class fields in general.

Two special cases deserve particular attention: When $\mathfrak{m} = \mathfrak{m}_0 = (1)$ is the trivial modulus for K , then $K(1)$ is the *Hilbert class field of K* , and the Artin map defines an isomorphism from $\text{Cl}_K = \mathcal{I}/i(K^*)$ onto $\text{Gal}(K(1)/K)$; when $\mathfrak{m} = \mathfrak{m}_\infty = (\infty)$ is the modulus for K that is supported on all of the real embeddings of K , then $K(\infty)$ is the *narrow Hilbert class field of K* , and the Artin map defines an isomorphism from $\mathcal{I}/i(K_+^*)$ onto $\text{Gal}(K(\infty)/K)$ where K_+^* is the subgroup of K^* consisting of those elements that are positive in every real embedding of K ; such elements are called *totally positive*, and the group $\text{Cl}_K^+ := \mathcal{I}/i(K_+^*)$ is called the *narrow class group of K* .

The group K_+^* appeared in the study of Bost–Connes type systems because it plays a special role the adelic approach to class field theory when one wishes to work only with finite places (see [18, Section 1]). One may consider the monoid $R_+^\times := R^\times \cap K_+^*$ of non-zero totally positive algebraic integers in K . It acts naturally on the additive group of R , so one may form $R \rtimes R_+^\times$. The potential importance of the monoid $R \rtimes R_+^\times$ was initially suggested by Laca, based on his work on Bost–Connes type systems [18], and it was this insight that eventually led me to the more general construction, which is explained below.

Overview of the results

Article A. Let K be a number field with ring of integers R . In my first paper [A] (see Chapter 5 below), I initiate the study of a new class of semigroup C^* -algebras by considering the left regular C^* -algebras of submonoids of $R \rtimes R^\times$ obtained by restricting the multiplicative part to lie in a submonoid of R^\times that is defined by generalized congruence conditions which may involve positivity requirements under the various real embeddings of K . For instance, the submonoid R_+^\times of R^\times is such a monoid. Let \mathfrak{m} be a modulus for K and let $R_{\mathfrak{m},1} := R^\times \cap K_{\mathfrak{m},1}$ be the monoid of non-zero algebraic integers that generate ideals lying in the kernel of the Artin map for the ray class field $K(\mathfrak{m})$. Let $\mathcal{I}_{\mathfrak{m}}^+ \subseteq \mathcal{I}_{\mathfrak{m}}^1$ be the submonoid of non-zero integral ideals that are coprime to \mathfrak{m}_0 ; the ray class group $\text{Cl}_K(\mathfrak{m})$ can be equivalently defined as the quotient of $\mathcal{I}_{\mathfrak{m}}^+$ by the equivalence relation $\mathfrak{a} \sim \mathfrak{b}$ if there exists $a, b \in R_{\mathfrak{m},1}$ such that $a\mathfrak{a} = b\mathfrak{b}$. My initial idea was to consider $R \rtimes R_{\mathfrak{m},1}$ since I anticipated that the action of $R_{\mathfrak{m},1}$ on $\mathcal{I}_{\mathfrak{m}}^+$ would somehow be encoded in the left regular C^* -algebra of $R \rtimes R_{\mathfrak{m},1}$. I soon found out that the monoids $R_{\mathfrak{m},1}$, indeed a more general class of monoids, called congruence monoids, had already appeared in the literature on non-unique factorizations in rings [11], and this thesis is based on this class of monoids.

Let \mathfrak{m} be a modulus for K . The *group of residues modulo \mathfrak{m}* is the multiplicative group $(R/\mathfrak{m})^* := \left(\prod_{w|\mathfrak{m}_\infty} \{\pm 1\} \right) \times (R/\mathfrak{m}_0)^*$ where $(R/\mathfrak{m}_0)^*$ is the multiplicative group of residues modulo \mathfrak{m}_0 . For $a \in R^\times$ coprime to \mathfrak{m}_0 , let $[a]_{\mathfrak{m}} := ((\text{sign}(w(a)))_{w|\mathfrak{m}_\infty}, a + \mathfrak{m}_0) \in (R/\mathfrak{m})^*$ be the *residue of a modulo \mathfrak{m}* . Given a group Γ of residues modulo \mathfrak{m} , the associated *congruence monoid* is the multiplicative monoid

$$R_{\mathfrak{m},\Gamma} := \{a \in R^\times : a \text{ coprime to } \mathfrak{m}_0, [a]_{\mathfrak{m}} \in \Gamma\}$$

which consists of all those non-zero algebraic integers in R coprime to \mathfrak{m}_0 whose residue modulo \mathfrak{m} lies in Γ . If \mathfrak{m} is trivial, so that Γ is also trivial, then $R_{\mathfrak{m},\Gamma} = R^\times$, and if \mathfrak{m}_0 and Γ are trivial and \mathfrak{m}_∞ takes the value 1 at every real embedding of K ,

¹The superscript “+” here means “positive” with respect to the canonical lattice ordering on $\mathcal{I}_{\mathfrak{m}}$; it has nothing to do with totally positive elements.

then $R_{\mathfrak{m},\Gamma} = R_+^\times$.

Now let $M \subseteq R^\times$ be a congruence monoid as defined above. Then M acts on the additive group of R by multiplication, so we may form $R \rtimes M$ and consider the C^* -algebra $C_\lambda^*(R \rtimes M)$. I first show that $R \rtimes M$ is left Ore with group of left quotients $Q \rtimes \langle M \rangle$ where $Q = M^{-1}R$ is the localization of R at M and $\langle M \rangle = M^{-1}M$ is the subgroup of left quotients of M in the multiplicative group K^* of K . Then I compute the semilattice $\mathcal{J}_{R \rtimes M}$ of constructible right ideals of $R \rtimes M$ and prove that $R \rtimes M$ satisfies the independence condition. As before, let $\mathcal{I}_{\mathfrak{m}}$ denote the group of fractional ideals of K that are coprime to \mathfrak{m}_0 , and let $\mathcal{I}_{\mathfrak{m}}^+ \subseteq \mathcal{I}_{\mathfrak{m}}$ be the monoid of non-zero integral ideals that are coprime to \mathfrak{m}_0 . The semilattice $\mathcal{J}_{R \rtimes M}$ is isomorphic to the semilattice of sets $\{(x + \mathfrak{a}) \times \mathfrak{a}^\times : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+ \} \cup \{\emptyset\}$ under intersection, where $\mathfrak{a}^\times := \mathfrak{a} \setminus \{0\}$. I then show that the canonical homomorphism from $C^*(R \rtimes M)$ to $C_\lambda^*(R \rtimes M)$ is an isomorphism, and give a description of $C_\lambda^*(R \rtimes M)$ as the C^* -algebra $C^*((Q \rtimes \langle M \rangle) \rtimes \Omega)$ of the partial transformation groupoid $(Q \rtimes \langle M \rangle) \rtimes \Omega$, where Ω is an ‘adelic’ space that is homeomorphic to the spectrum of the canonical Cartan subalgebra $D_\lambda(R \rtimes M)$. This generalizes the analysis from [5, Section 5] and [26, Section 2].

Motivated by Laca and Raeburn’s work on C^* -algebras of quasi-lattice ordered monoids [20], I establish faithfulness criteria for representations of $C_\lambda^*(R \rtimes M)$ in terms of certain ‘defect’ projections in $D_\lambda(R \rtimes M)$. Let $i: \langle M \rangle \rightarrow \mathcal{I}_{\mathfrak{m}}$ by $i(a) = aR$, and put $C := \mathcal{I}_{\mathfrak{m}}/i(\langle M \rangle)$; then C is a quotient of the ray class group $\text{Cl}_K(\mathfrak{m})$. In particular, C is a finite abelian group. For each class $\mathfrak{k} \in C$, choose an integral ideal $\mathfrak{a}_{\mathfrak{k}} \in \mathfrak{k}$. The faithfulness theorem asserts that a representation ψ of $C_\lambda^*(R \rtimes M)$ in a C^* -algebra B is faithful if for each class \mathfrak{k} , ψ is non-zero on all ‘defect projections’ obtained from $E_{\mathfrak{a}_{\mathfrak{k}} \times (\mathfrak{a}_{\mathfrak{k}})^\times}$ by subtracting finitely many sub-projections which are of the form $E_{(y+\mathfrak{a}) \times \mathfrak{a}^\times}$. This result is new even in the case of the full $ax + b$ -semigroup.

Following [10], I compute the primitive ideal space of $C_\lambda^*(R \rtimes M)$: Let $\mathcal{S} = \{\mathfrak{p} \in \mathcal{P}_K : \mathfrak{p} \mid \mathfrak{m}_0\}$ be the support of \mathfrak{m}_0 , and put $\mathcal{P}_K^{\mathfrak{m}} := \mathcal{P}_K \setminus \mathcal{S}$. For each prime \mathfrak{p} in $\mathcal{P}_K^{\mathfrak{m}}$, let $f_{\mathfrak{p}}$ denote the order of the class of \mathfrak{p} in C , so that $\mathfrak{p}^{f_{\mathfrak{p}}} = t_{\mathfrak{p}}R$ for some $t_{\mathfrak{p}} \in M$, and for each non-empty subset A of $\mathcal{P}_K^{\mathfrak{m}}$, let I_A be the ideal of $C_\lambda^*(R \rtimes M)$ generated by the projections $1 - \sum_{x \in R/t_{\mathfrak{p}}R} \lambda_{(x,t_{\mathfrak{p}})} \lambda_{(x,t_{\mathfrak{p}}}^*$; also let $I_\emptyset = \{0\}$. Then the

map $A \mapsto I_A$ is a homeomorphism from $2^{\mathcal{P}_K^m}$ onto $\text{Prim}(C_\lambda^*(R \rtimes M))$, where $2^{\mathcal{P}_K^m}$ is given the power-cofinite topology. This shows that the quotients by primitive ideals are obtained by imposing certain ‘Cuntz-like’ relations.

My approach to proving the above parameterization of primitive ideals is a bit different from the approach in the case of the full $ax + b$ -semigroup in [10] in that I use general results on groupoid C*-algebras developed by Sims and Williams [32], which could be applied directly to the groupoid model $C^*((Q \rtimes \langle M \rangle) \rtimes \Omega)$ of $C_\lambda^*(R \rtimes M)$ rather than passing to a dilated system and then using general results on crossed product C*-algebras. The explicit description of primitive ideals in terms of range projections of the generating isometries is new even in the case of the full $ax + b$ -semigroup. This description of the primitive ideal space shows in particular that the non-zero minimal primitive ideals $I_{\{\mathfrak{p}\}}$ are in one-to-one correspondence with the (non-zero) primes \mathfrak{p} of R not dividing \mathfrak{m}_0 , and that $C_\lambda^*(R \rtimes M)$ has a unique maximal ideal, $I_{\mathcal{P}_K^m}$. I give a description of the quotient by this maximal ideal as a semigroup crossed product which is similar to the crossed product description of the ring C*-algebra of R given in [9] (see also [23]). Indeed, if $\hat{R}_S := \prod_{\mathfrak{p} \in \mathcal{P}_K \setminus S} R_{\mathfrak{p}}$ where $R_{\mathfrak{p}}$ is the ring of \mathfrak{p} -adic integers in the \mathfrak{p} -adic completion $K_{\mathfrak{p}}$ of K , then there is a canonical isomorphism $C_\lambda^*(R \rtimes M)/I_{\mathcal{P}_K^m} \cong C(\hat{R}_S) \rtimes (R \rtimes M)$ where $R \rtimes M$ acts on \hat{R}_S through the diagonal embedding $R \hookrightarrow \hat{R}_S$.

The initial data needed for the main construction in this thesis consists of a triple $(R, \mathfrak{m}, \Gamma)$. Using the faithfulness criteria I show that the construction is functorial in the sense that if we are given $(R, \mathfrak{m}, \Gamma)$ and $(R', \mathfrak{m}', \Gamma')$ and an injective ring homomorphism $\iota: R \hookrightarrow R'$ such that $\iota(R_{\mathfrak{m}, \Gamma}) \subseteq R'_{\mathfrak{m}', \Gamma'}$, then there exists an injective *-homomorphism $C_\lambda^*(R \rtimes R_{\mathfrak{m}, \Gamma}) \hookrightarrow C_\lambda^*(R' \rtimes R'_{\mathfrak{m}', \Gamma'})$ such that $\lambda_{(b,a)} \mapsto \lambda_{(\iota(b), \iota(a))}$ for every $(b, a) \in R \rtimes R_{\mathfrak{m}, \Gamma}$. This extends the functoriality result from [5].

Article B. The C*-algebra $C_\lambda^*(R \rtimes M)$ carries a canonical time evolution σ such that $\sigma_t(\lambda_{(b,a)}) = N(a)^{it} \lambda_{(b,a)}$ for all $(b, a) \in R \rtimes R_{\mathfrak{m}, \Gamma}$ and $t \in \mathbb{R}$. In my second paper [B] (see Chapter 6 below) I study KMS states for this time evolution. I first generalize the phase transition theorem from [5], using the general results from [31]. For this, I use that the isomorphism $C_\lambda^*(R \rtimes M) \cong C^*((Q \rtimes \langle M \rangle) \rtimes \Omega)$ from [A]

is \mathbb{R} -equivariant with respect to the time evolution, also denoted by σ , on $C^*((Q \rtimes \langle M \rangle) \rtimes \Omega)$ arising from the cocycle $((g, k), w) \mapsto \log N(k)$, so that it suffices to consider the C^* -dynamical system $(C^*((Q \rtimes \langle M \rangle) \rtimes \Omega), \sigma)$. To ease notation, let $\mathcal{G} := (Q \rtimes \langle M \rangle) \rtimes \Omega$.

Neshveyev's theorem [31, Theorem 1.3] for KMS states on groupoid C^* -algebras says that to compute the KMS states for the system $(C^*(\mathcal{G}), \sigma)$ one must first compute all probability measures μ on Ω that satisfy the scaling condition $\mu((n, k)E) = N(k)^{-\beta} \mu(E)$ for $(n, k) \in Q \rtimes \langle M \rangle$ and Borel sets $E \subseteq \Omega$ in the domain of (n, k) . Then every KMS state ϕ will always sit over such a measure and will be given by $\phi(f) = \int_{\Omega} \phi_x(f|_{\mathcal{G}_x^x}) d\mu(x)$ where $\{\phi_x\}$ is a measurable field of states on the C^* -algebras of the isotropy groups \mathcal{G}_x^x (see [31, Section 1] for the precise formulation).

It is easy to see that there are no probability measures on Ω satisfying the scaling condition for $\beta < 1$. For each $\beta > 2$, measure-theoretic arguments using the Borel-Cantelli lemma show that extremal probability measures on Ω satisfying the scaling condition are in one-to-one correspondence with ideal classes in C , namely, the measure corresponding to the class $\mathfrak{k} \in C$ is given by $\mu_{\beta, \mathfrak{k}} = \frac{1}{\zeta_{\mathfrak{k}}(\beta-1)} \sum_{\mathfrak{a} \in \mathfrak{k}, \mathfrak{a} \subseteq R, \mathfrak{a} \subseteq R/\mathfrak{a}} N(\mathfrak{a})^{-\beta} \delta_{[x, \mathfrak{a}]}$ where $\zeta_{\mathfrak{k}}(s) = \sum_{\mathfrak{a} \in \mathfrak{k}, \mathfrak{a} \subseteq R} N(\mathfrak{a})^{-s}$ is the partial Dedekind zeta function associated with the class \mathfrak{k} and $\delta_{[x, \mathfrak{a}]}$ is the point-mass measure at $[x, \mathfrak{a}] \in \Omega$. Since this measure is concentrated on the orbit of the point $[0, \mathfrak{a}_{\mathfrak{k}}]$ for any fixed integral ideal $\mathfrak{a}_{\mathfrak{k}} \in \mathfrak{k}$, and the isotropy group of the point $[0, \mathfrak{a}_{\mathfrak{k}}]$ is $\mathfrak{a}_{\mathfrak{k}} \rtimes M^*$, it follows from Neshveyev's theorem that the KMS states sitting over $\mu_{\beta, \mathfrak{k}}$ are parameterized by tracial states on the group C^* -algebra $C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes M^*)$. From this, one can deduce that the simplex of KMS_{β} states is isomorphic to the simplex of tracial states on $\bigoplus_{\mathfrak{k} \in C} C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes M^*)$. This argument is similar to that given in [31, Section 3] except that I do not rely on results for Bost–Connes type systems in order to compute the probability measures that satisfy the scaling condition.

For each $\beta \in [1, 2]$, I prove that there is a unique σ - KMS_{β} state ϕ_{β} on $C_{\lambda}^*(R \rtimes M)$. The state ϕ_{β} factors through the conditional expectation $C_{\lambda}^*(R \rtimes M) \rightarrow D_{\lambda}(R \rtimes M)$ and is determined by the values $\phi_{\beta}(E_{(x+\mathfrak{a}) \times \mathfrak{a}^{\times}}) = N(\mathfrak{a})^{-\beta}$ for $x \in R$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. The proof of uniqueness for the full $ax+b$ -semigroup in [5] uses asymptotic results on partial Dedekind zeta functions associated with ideal classes in the ideal class group

of K , which are not available for the partial zeta functions associated with classes in C . However, I am able to use ideas from the uniqueness of high temperature KMS states for Bost–Connes type systems [15] to prove uniqueness. Moreover, inspired by the type computations in [17], I use ideas from the ergodicity result of Lagarias and Neshveyev [22] to prove that ϕ_β is of type III₁ for every $\beta \in [1, 2]$.

Recent work by Laca, Larsen, and Neshveyev [16] gives a general method for computing ground states on groupoid C*-algebras. Using their result, I prove that there is an affine isomorphism of the σ -ground state space of $C_\lambda^*(R \rtimes M)$ onto the state space of the C*-algebra $\bigoplus_{\mathfrak{k} \in C} M_{k_{\mathfrak{k}} \cdot N(\mathfrak{a}_{\mathfrak{k},1})}(C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes M^*))$ where $k_{\mathfrak{k}}$ is the number of norm-minimizing ideals in the class \mathfrak{k} .

From the above analysis, I deduce that the boundary quotient of $C_\lambda^*(R \rtimes M)$ has a unique KMS₁ state that is of type III₁, and no KMS _{β} states for $\beta \neq 1$. Then I prove phase transition theorems for the left regular C*-algebras $C_\lambda^*(M)$ and $C_\lambda^*(M/M^*)$ of the multiplicative monoids M and M/M^* , respectively.

Further work

K-theory. Whenever a new class of C^* -algebras is introduced, it is a natural problem to compute their K-theory. In the preprint [3], Li and I studied K-theoretic invariants of the C^* -algebra $C_\lambda^*(R \rtimes M)$. Our first result is an application of the general results for K-theory of semigroup C^* -algebras from [6, 7] which gives a formula for the K-theory of $C_\lambda^*(R \rtimes M)$ in terms of the K-theory of group C^* -algebras of the isotropy groups which already appeared above. Specifically, if for each class $\mathfrak{k} \in C$ we choose $\mathfrak{a}_\mathfrak{k} \in \mathfrak{k}$, then here is an isomorphism $K_*(C_\lambda^*(R \rtimes M)) \cong \bigoplus_{\mathfrak{k} \in C} K_*(C^*(\mathfrak{a}_\mathfrak{k} \rtimes M^*))$. It is interesting to observe the similarity between this K-theory formula and the parameterization space for low temperature KMS states given in [B, Theorem 3.2]. The groups $K_*(C^*(\mathfrak{a}_\mathfrak{k} \rtimes M^*))$ are difficult to compute in general, but we did establish that for every $\mathfrak{a} \in \mathcal{I}_m$, the groups $K_*(C^*(\mathfrak{a} \rtimes M^*))$ and $K_*(C^*(R \rtimes M^*))$ are isomorphic after inverting $|\text{tor}(M^*)|$.

Let $K(\mathfrak{m})^{\bar{\Gamma}}$ be the subfield of $K(\mathfrak{m})$ that is fixed by every automorphism in the image of $i(\langle M \rangle)$ in $\text{Gal}(K(\mathfrak{m})/K)$ under $r_{K(\mathfrak{m})/K}$. Then via Galois theory, the group C may be identified with $\text{Gal}(K(\mathfrak{m})^{\bar{\Gamma}}/K)$. Following the approach taken in [26] for the case of the full $ax + b$ -semigroup, we computed, for almost all prime ideals \mathfrak{p} of R , the torsion order of the K_0 -class of the identity in $C_\lambda^*(R \rtimes M)/I_{\{\mathfrak{p}\}}$, and then used these numbers to recover information about both the norm of \mathfrak{p} and the inertia degree of \mathfrak{p} in the class field $K(\mathfrak{m})^{\bar{\Gamma}}$. As part of our analysis, we resolved a problem that had been left open in [26]; in fact, we proved that one can characterize the number of roots of unity in M from K-theoretic invariants associated with $C_\lambda^*(R \rtimes M)$. Our main result shows that the C^* -algebra $C_\lambda^*(R \rtimes M)$ remembers a great deal of the initial input data. Using this information, we proved the following theorem which partially characterizes the initial data in terms of $C_\lambda^*(R \rtimes M)$. Suppose that K and L are number fields with rings of algebraic integers R and S . Let \mathfrak{m} and \mathfrak{n} be moduli for K and L , and let Γ and Λ be subgroups of $(R/\mathfrak{m})^*$ and $(S/\mathfrak{n})^*$, respectively. If there is an isomorphism $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma}) \cong C_\lambda^*(S \rtimes S_{\mathfrak{n},\Lambda})$, then K and L are arithmetically equivalent (that is, have the same Dedekind zeta functions), and $K(\mathfrak{m})^{\bar{\Gamma}}$ and $L(\mathfrak{n})^{\bar{\Lambda}}$ are Kronecker equivalent. In this situation, if K or L is Galois over \mathbb{Q} , then K and L must be equal; moreover, if in addition we know that both the class fields

$K(\mathfrak{m})^{\bar{\Gamma}}$ and $L(\mathfrak{n})^{\bar{\Lambda}}$ are Galois over \mathbb{Q} , then we can even prove that $K(\mathfrak{m})^{\bar{\Gamma}} = L(\mathfrak{n})^{\bar{\Lambda}}$, $R^* \cdot (R_{\mathfrak{n}} \cap R_{\mathfrak{m},\Gamma}) = S^* \cdot (S_{\mathfrak{m}} \cap S_{\mathfrak{n},\Lambda})$, and $\mathcal{I}_{\mathfrak{m}}/i(\langle R_{\mathfrak{m},\Gamma} \rangle) \cong \mathcal{I}_{\mathfrak{n}}/i(\langle S_{\mathfrak{n},\Lambda} \rangle)$.

Building on techniques developed in [28, 29], we also proved that the C*-algebra $C_{\lambda}^*(R \rtimes M)$ is \mathcal{O}_{∞} -stable; this property is important in the classification program for C*-algebras (see, for instance, [12]).

Lastly, we showed that the techniques from [27] can be generalized and improved slightly to obtain the following result on Cartan pairs: Let K and L be number fields with rings of algebraic integers R and S , respectively, and suppose we are given data \mathfrak{m} , Γ and \mathfrak{n} , Λ for K and L , respectively. If $(C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma}), D_{\lambda}(R \rtimes R_{\mathfrak{m},\Gamma})) \cong (C_{\lambda}^*(S \rtimes S_{\mathfrak{n},\Lambda}), D_{\lambda}(S \rtimes S_{\mathfrak{n},\Lambda}))$, then K and L are arithmetically equivalent, $K(\mathfrak{m})^{\bar{\Gamma}}$ and $L(\mathfrak{n})^{\bar{\Lambda}}$ are arithmetically equivalent, and $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma}) \cong \mathcal{I}_{\mathfrak{n}}/i(L_{\mathfrak{n},\Lambda})$.

Partition functions. In the preprint [1], Laca, Takeishi, and I studied the fine structure of the simplices of low temperature KMS states for the system $(C_{\lambda}^*(R \rtimes M), \sigma)$. The key notion underlying this work is that of a *partition function*; these functions are invariants of extremal type I KMS states, and they often carry a great deal of information. For low temperature (that is, large β), the system $(C_{\lambda}^*(R \rtimes M), \sigma)$ has a wealth of KMS states by [B, Theorem 3.2], and we proved that there is a natural dichotomy of the extremal low temperature KMS states according to type; they are either type I_{∞} or type II_{∞} . For each $\beta > 2$, we proved that the extremal type I KMS_{β} states are parameterized by the disjoint union $\bigsqcup_{\mathfrak{k} \in C} \mathcal{E}_I(\text{T}(C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes M^*)))$ where $\mathcal{E}_I(\text{T}(C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes M^*)))$ is the set of extremal tracial states on $C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes M^*)$ that are of type I. Moreover, the set $\mathcal{E}_I(\text{T}(C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes M^*)))$ is parameterized by pairs (O, χ) where $O \subseteq \widehat{\mathfrak{a}}_{\mathfrak{k}}$ is a finite orbit for the action $M^* \curvearrowright \widehat{\mathfrak{a}}_{\mathfrak{k}}$ and χ is a character of the isotropy group of any point in O . We showed that the partition function of the KMS state $\varphi_{\mathfrak{k}, O, \chi}$ corresponding to the tripe (\mathfrak{k}, O, χ) is given by $Z_{\varphi_{\mathfrak{k}, O, \chi}}(s) = |O| \underline{N}(\mathfrak{k})^s \zeta_{\mathfrak{k}}(s-1)$ for $\Re(s) > 2$ where $\underline{N}(\mathfrak{k}) := \min\{N(\mathfrak{a}) : \mathfrak{a} \in \mathfrak{k}, \mathfrak{a} \subseteq R\}$. The main result of [1] is the following.

Theorem ([1, Theorem 7.6]). *Let K be a number field with ring of integers R , \mathfrak{m} a modulus for K , Γ a subgroup of $(R/\mathfrak{m})^*$, and M the associated congruence monoid. Let $\pi_0(\Sigma_{\beta, I}^{\min})$ denote the set of connected components of $\Sigma_{\beta, I}^{\min}$, and let $\bar{\varphi}$ denote the*

connected component of $\varphi \in \Sigma_{\beta, I}^{\min}$. Then

$$(i) \quad |\pi_0(\Sigma_{\beta, I}^{\min})| = \begin{cases} |\mathcal{I}_{\mathfrak{m}}/i(\langle M \rangle)| & \text{if } M^* = \{1\}, \\ |\text{tor}(M^*)| \cdot |\mathcal{I}_{\mathfrak{m}}/i(\langle M \rangle)| \cdot |\mathcal{F}_R| & \text{if } M^* \neq \{1\}, \end{cases}$$

where \mathcal{F}_R is the set of fixed points for the action of M^* on \widehat{R} ;

(ii) the partition function Z_φ depends only on the connected component of φ , and

$$\sum_{\bar{\varphi} \in \pi_0(\Sigma_{\beta, I}^{\min})} \tilde{Z}_\varphi(s) = \begin{cases} \zeta_{K, \mathfrak{m}}(s-1) & \text{if } M^* = \{1\}, \\ |\text{tor}(M^*)| \cdot |\mathcal{F}_R| \cdot \zeta_{K, \mathfrak{m}}(s-1) & \text{if } M^* \neq \{1\}, \end{cases}$$

where the sum is taken over any set of representatives and $\Re s > 2$ and \tilde{Z}_φ is a ‘normalized’ version of Z_φ (see [1, Definition 7.5]);

$$(iii) \quad \lim_{s \rightarrow \infty} \sum_{\bar{\varphi} \in \pi_0(\Sigma_{\beta, I}^{\min})} \tilde{Z}_\varphi(s) = \begin{cases} 1 & \text{if } M^* = \{1\}, \\ |\text{tor}(M^*)| \cdot |\mathcal{F}_R| & \text{if } M^* \neq \{1\}. \end{cases}$$

Using this theorem, we were able to recover many of the number-theoretic invariants that were recovered in [3], but with entirely different techniques. Of course, we also kept track of the canonical time evolution, whereas Li and I considered only the C^* -algebra.

Hecke algebras. Laca and I are currently studying Hecke C^* -algebras associated with the same initial number-theoretic data used to define a congruence monoid; this provides another concrete link between class field theory and operator algebras.

C*-ALGEBRAS FROM ACTIONS OF CONGRUENCE MONOIDS ON RINGS OF ALGEBRAIC INTEGERS*

CHRIS BRUCE

ABSTRACT. Let K be a number field with ring of integers R . Given a modulus \mathfrak{m} for K and a group Γ of residues modulo \mathfrak{m} , we consider the semi-direct product $R \rtimes R_{\mathfrak{m}, \Gamma}$ obtained by restricting the multiplicative part of the full $ax+b$ -semigroup over R to those algebraic integers whose residue modulo \mathfrak{m} lies in Γ , and we study the left regular C*-algebra of this semigroup. We give two presentations of this C*-algebra and realize it as a full corner in a crossed product C*-algebra. We also establish a faithfulness criterion for representations in terms of projections associated with ideal classes in a quotient of the ray class group modulo \mathfrak{m} , and we explicitly describe the primitive ideals using relations only involving the range projections of the generating isometries; this leads to an explicit description of the boundary quotient. Our results generalize and strengthen those of Cuntz, Deninger, and Laca and of Echterhoff and Laca for the C*-algebra of the full $ax+b$ -semigroup. We conclude by showing that our construction is functorial in the appropriate sense; in particular, we prove that the left regular C*-algebra of $R \rtimes R_{\mathfrak{m}, \Gamma}$ embeds canonically into the left regular C*-algebra of the full $ax+b$ -semigroup. Our methods rely heavily on Li's theory of semigroup C*-algebras.

1. INTRODUCTION

1.1. Historical context. Cuntz pioneered the study of C*-algebras associated with $ax+b$ -semigroups over the ring \mathbb{Z} in [Cun]; his work was motivated by the construction of Bost and Connes in [Bo-Co]. Cuntz introduced a C*-algebra $\mathcal{Q}_{\mathbb{N}}$ defined

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using generators and relations involving the additive group of \mathbb{Z} and the multiplicative semigroup $\mathbb{N}^\times := \mathbb{N} \setminus \{0\}$. The C^* -algebra $\mathcal{Q}_{\mathbb{N}}$ can be canonically (and faithfully) represented on $\ell^2(\mathbb{Z})$, $\mathcal{Q}_{\mathbb{N}}$ is simple and purely infinite, and admits a unique KMS state for a canonical time evolution, see [Cun]. Cuntz showed that $\mathcal{Q}_{\mathbb{N}}$ can be realized as a full corner in the crossed product C^* -algebra for the action of the $ax + b$ -group $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ on the ring $\mathbb{A}_{\mathbb{Q},f}$ of finite adeles over \mathbb{Q} and then discussed its K -theory. Another C^* -algebra $\mathcal{Q}_{\mathbb{Z}}$ was defined in [Cun] using an analogous presentation but with the larger multiplicative semigroup $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$ of all non-zero integers in place of \mathbb{N}^\times .

Laca and Raeburn initiated the study of Toeplitz algebras in this context, see [La-Rae3]. They showed that the semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$ is quasi-lattice ordered, and they studied phase transitions for a canonical time evolution on its left regular C^* -algebra $C_\lambda^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ (which they called the “Toeplitz algebra” of $\mathbb{N} \rtimes \mathbb{N}^\times$). They also exhibited Cuntz’s $\mathcal{Q}_{\mathbb{N}}$ as the boundary quotient of $C_\lambda^*(\mathbb{N} \rtimes \mathbb{N}^\times)$. In a subsequent paper, Laca and Neshveyev parameterized the Nica spectrum of $\mathbb{N} \rtimes \mathbb{N}^\times$ in terms of an adelic space and computed the type of each equilibrium state at high temperature, see [La-Nesh].

Building on [Cun], Cuntz and Li introduced the so-called ring C^* -algebras in [Cun-Li1] (see also [Li1]). In particular, given a ring of integers R in a number field K , Cuntz and Li defined a C^* -algebra $\mathfrak{A}[R]$ using generators and relations generalizing those used in [Cun] to define $\mathcal{Q}_{\mathbb{Z}}$, so that for the ring \mathbb{Z} , their construction gave the C^* -algebra $\mathcal{Q}_{\mathbb{Z}}$. They showed that $\mathfrak{A}[R]$ also has a canonical (and faithful) representation on $\ell^2(R)$, and proved that $\mathfrak{A}[R]$ is simple and purely infinite. They gave a description of $\mathfrak{A}[R]$ as a canonical full corner in the crossed product for the action of the $ax + b$ -group $K \rtimes K^\times$ on the ring $\mathbb{A}_{K,f}$ of finite adeles over K , and used this description to make a connection with Bost-Connes type systems for arbitrary number fields as defined in [L-L-N]. The problem of computing the K -theory of $\mathfrak{A}[R]$ was particularly difficult; it was solved in the case that K has only two roots of unity by Cuntz and Li in [Cun-Li2] using a duality theorem for global fields, and then in full generality by Li and Lück in [Li-Lü].

Cuntz, Deninger, and Laca defined Toeplitz algebras associated with rings of integers of arbitrary number fields in [C-D-L]. Given a number field K with ring of integers R , they defined a C^* -algebra $\mathfrak{T}[R]$ using generators and relations similar to those

used to define $\mathfrak{A}[R]$, but without certain “tightness” relations. They proved that $\mathfrak{T}[R]$ is canonically isomorphic to the left regular C^* -algebra $C_\lambda^*(R \rtimes R^\times)$ of the $ax + b$ -semigroup $R \rtimes R^\times$ where the multiplicative semigroup $R^\times := R \setminus \{0\}$ acts on (the additive group of) R by multiplication. In [C-D-L], the left regular C^* -algebra of $R \rtimes R^\times$ is denoted by \mathfrak{T} and is called the “Toeplitz algebra” of $R \rtimes R^\times$. Cuntz, Deninger, and Laca studied phase transitions for a canonical time evolution on $C_\lambda^*(R \rtimes R^\times)$, and they proved that the associated C^* -dynamical system exhibits several interesting properties. They gave a description of $C_\lambda^*(R \rtimes R^\times)$ as a full corner in a crossed product for an action of the $ax + b$ -group $K \rtimes K^\times$ on a certain adelic space, and proved that their construction was functorial for inclusions of rings of integers. They also showed that the ring C^* -algebra $\mathfrak{A}[R]$ of R appeared naturally as a quotient of $C_\lambda^*(R \rtimes R^\times)$.

Since [C-D-L] appeared, the C^* -algebras of $ax + b$ -semigroups over rings of algebraic integers have been studied intensively. They inspired Neshveyev to prove a powerful general result on KMS states for groupoid C^* -algebras, see [Nesh], where Neshveyev also gives an alternative approach to proving the phase transition theorem from [C-D-L]. These C^* -algebras also provided a motivating class of examples for Li’s theory of semigroup C^* -algebras developed in [Li1, Li2] (see also [C-E-L-Y, Chapter 5]). In [Ech-La], Echterhoff and Laca developed general results on primitive ideal spaces of crossed products, then used these results to compute the primitive ideal space of $C_\lambda^*(R \rtimes R^\times)$. Cuntz, Echterhoff, and Li proved a general formula for the K -theory of a large class of semigroup C^* -algebras in [C-E-L1, C-E-L2] which, as a particular case, gives a formula for the K -theory of $C_\lambda^*(R \rtimes R^\times)$. They also showed in [C-E-L1] that $C_\lambda^*(R \rtimes R^\times)$ is purely infinite, has the ideal property, but does not have real rank zero. Building on these works, Li gave an explicit description of the primitive ideals in $C_\lambda^*(R \rtimes R^\times)$ in [Li4] and used K -theoretic invariants to show that one can recover the Dedekind zeta function of K from $C_\lambda^*(R \rtimes R^\times)$, provided that one knows the number of roots of unity in K . Continuing his investigation, Li showed in [Li5] that one can recover both the Dedekind zeta function of K and the ideal class group $\text{Cl}(K)$ of K from $C_\lambda^*(R \rtimes R^\times)$ together with its canonical diagonal sub- C^* -algebra. Li also studied the semigroup C^* -algebras of $ax + b$ -semigroups for more general classes of rings in [Li6], where he showed that some of the results on ideal structure, pure infiniteness, and K -theory can be generalized; in [Li7], he gives an alternative approach to pure infiniteness of these $ax + b$ -semigroup C^* -algebras

using partial transformation groupoids. Recently, Laca and Warren in [La-War] have used Neshveyev's characterization of traces on crossed products from [Nesh, Section 2] to describe the low temperature KMS equilibrium states from the phase transition theorem in [C-D-L] in terms of ergodic invariant measures for groups of linear toral automorphisms. As a result, this revealed a connection with the generalized Furstenberg conjecture in ergodic theory.

1.2. Overview of the construction. In this paper, we generalize the construction from [C-D-L] by considering the C*-algebras of a larger class of semigroups. The construction of these semigroups depends not only on a number field K , but also on additional number-theoretic data that arise naturally in the study of the ray class fields of K , that is, in class field theory. Namely, given a number field K with ring of integers R , a modulus \mathfrak{m} for K , and a group Γ of residues modulo \mathfrak{m} , the associated congruence monoid $R_{\mathfrak{m},\Gamma}$ is the multiplicative monoid of algebraic integers in R that reduce to an element of Γ modulo \mathfrak{m} . We form the semi-direct product $R \rtimes R_{\mathfrak{m},\Gamma}$ where $R_{\mathfrak{m},\Gamma}$ acts on R by multiplication, and investigate the left regular C*-algebra of this semigroup. We formulate and prove the appropriate generalizations of several of the results mentioned above for the full $ax + b$ -semigroup. In addition, we give a new faithfulness criterion for representations, see Section 6.

We now briefly explain our construction in the special case of the number field $K = \mathbb{Q}$, see Section 3 for a detailed discussion of the general case. Let $\mathcal{P}_{\mathbb{Q}}$ denote the set of rational prime numbers, and let w be the unique embedding $w : \mathbb{Q} \hookrightarrow \mathbb{R}$. A *modulus* for \mathbb{Q} is a function $\mathfrak{m} : \{w\} \sqcup \mathcal{P}_{\mathbb{Q}} \rightarrow \mathbb{N}$ such that $\mathfrak{m}(w) \in \{0, 1\}$ and $\mathfrak{m}(p) = 0$ for all but finitely many primes $p \in \mathcal{P}_{\mathbb{Q}}$. Denote by m the positive integer $\prod_{p \in \mathcal{P}_{\mathbb{Q}}} p^{\mathfrak{m}(p)}$. The *multiplicative group of residues modulo \mathfrak{m}* is $(\mathbb{Z}/\mathfrak{m})^* := \{\pm 1\} \times (\mathbb{Z}/m\mathbb{Z})^*$ where $(\mathbb{Z}/m\mathbb{Z})^*$ is the multiplicative group of invertible elements in the ring $\mathbb{Z}/m\mathbb{Z}$. For $a \in \mathbb{Z}$ such that $\gcd(a, m) = 1$, the *residue of a modulo \mathfrak{m}* is

$$[a]_{\mathfrak{m}} := (\text{sign}(a), a + m\mathbb{Z}) \in (\mathbb{Z}/\mathfrak{m})^*$$

where $\text{sign}(a) := a/|a|$. Dealing with moduli allows us to speak of congruence relations that can involve positivity conditions. Let $\Gamma \subseteq (\mathbb{Z}/\mathfrak{m})^*$ be a subgroup, and let

$$\mathbb{Z}_{\mathfrak{m},\Gamma} := \{a \in \mathbb{Z}^{\times} : \gcd(a, m) = 1, [a]_{\mathfrak{m}} \in \Gamma\}$$

where $\mathbb{Z}^\times := \mathbb{Z} \setminus \{0\}$. Since Γ is a group, $\mathbb{Z}_{m,\Gamma}$ is a unital semigroup under multiplication. Such semigroups are called *congruence monoids*, see [HK, Definition 5] and [G-HK]. Notice that $\mathbb{Z}_{m,\Gamma}$ is a disjoint union of arithmetic progressions; for example, if Γ is the trivial group, then $\mathbb{Z}_{m,\Gamma} = 1 + m\mathbb{N}$. We form the semi-direct product semigroup $\mathbb{Z} \rtimes \mathbb{Z}_{m,\Gamma}$ with respect to the action of $\mathbb{Z}_{m,\Gamma}$ on (the additive group of) \mathbb{Z} given by multiplication. The left regular C^* -algebra of $\mathbb{Z} \rtimes \mathbb{Z}_{m,\Gamma}$ is the sub- C^* -algebra of $\mathcal{B}(\ell^2(\mathbb{Z} \rtimes \mathbb{Z}_{m,\Gamma}))$ generated by the isometries $\lambda_{(b,a)}$ for $(b,a) \in \mathbb{Z} \rtimes \mathbb{Z}_{m,\Gamma}$ defined via the left translation action of $\mathbb{Z} \rtimes \mathbb{Z}_{m,\Gamma}$ on itself. In this article, we study C^* -algebras of semigroups of this kind and their analogues for general number fields.

It is very natural to consider C^* -algebras associated with semigroups of the form $R \rtimes M$ where M is a subsemigroup of R^\times . For $K = \mathbb{Q}$ and $R = \mathbb{Z}$, such C^* -algebras have already been considered in two special cases: Larsen and Li in [Lar-Li] considered the *2-adic ring C^* -algebra* associated with the semigroup $\mathbb{Z} \rtimes [2)$ where $[2) := \{1, 2, 2^2, 2^3, \dots\}$, and Barlak, Omland, and Stammeier in [B-O-S] considered C^* -algebras associated with semigroups of the form $\mathbb{Z} \rtimes M$ where M is a subsemigroup of \mathbb{N}^\times generated by a non-empty family of relative prime numbers. If we consider the special case where $\Gamma = \{1\} \times (\mathbb{Z}/m\mathbb{Z})^*$, then $\mathbb{Z}_{m,\Gamma}$ is the subsemigroup of \mathbb{N}^\times generated by the prime numbers that do not divide m , so that our $\mathbb{Z} \rtimes \mathbb{Z}_{m,\Gamma}$ is a semigroup of the type considered in [B-O-S].

Some of the analysis in Sections 3, 4, and 5 can likely be generalized to other semigroups of the form $R \rtimes M$. However, results in later sections of this paper rely heavily on M being a congruence monoid, which shows that actions of congruence monoids give rise to particularly nice semigroups, and we thus focus on this case from the beginning to avoid unnecessary technical difficulties. The author plans to consider more general semigroups of the form $R \rtimes M$ in a future work.

1.3. Outlook. We now briefly mention two works that build directly on the results of this paper. The semigroup C^* -algebras that we consider here carry canonical time evolutions coming from the norm map on K , and a computation of the KMS and ground states of the associated C^* -dynamical systems is worked out in [Bru]. There, the finite group $\mathcal{I}_m/i(K_{m,\Gamma})$, which appears first in Section 3 below, plays an important role. For instance, for each $\beta > 2$, the simplex of KMS_β states for the canonical time evolution on $C_\lambda^*(R \rtimes R_{m,\Gamma})$ decomposes over $\mathcal{I}_m/i(K_{m,\Gamma})$, whereas

uniqueness for β in the critical interval [1, 2] relies on classical properties of the L -functions associated with characters of $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$, see [Bru, Theorem 3.2].

Another natural problem is to determine whether the analyses from [Li-Lü, Li4, Li5] on K -theoretic invariants can be carried out for C^* -algebras arising from actions of congruence monoids on rings of algebraic integers. This is investigated in [Bru-Li], where we show that the left regular semigroup C^* -algebra $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ contains subtle number-theoretic information about K and about a certain class field (i.e., finite abelian extension) of K that is naturally associated with the data (\mathfrak{m}, Γ) , see [Bru-Li, Theorem 5.5]. Even in the case of the full $ax + b$ -semigroup over R , said theorem is novel since no connection with class field theory had been made previously. It is further shown in [Bru-Li, Section 3] that $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ is purely infinite in a very strong sense.

1.4. Organization of this paper. We begin in Section 2 with a brief discussion of notation and preliminaries for semigroup C^* -algebras in Section 2.1 and for moduli of algebraic number fields in Section 2.2. In Section 3, we define $R \rtimes R_{\mathfrak{m},\Gamma}$ and take a first step towards understanding $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$; namely, we compute the semilattice of constructible right ideals of $R \rtimes R_{\mathfrak{m},\Gamma}$ and prove that this semilattice satisfies the independence condition from [Li2], see Proposition 3.4. This puts us in a setting where we can use general results from Li's theory of semigroup C^* -algebras from [Li2, Li3] (see also [Li6] and [C-E-L-Y, Chapter 5]).

We begin our study of the left regular C^* -algebra $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ in Section 4 where we give two presentations for $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ in terms of explicit generators and relations, see Propositions 4.1 and 4.3. In Section 5, we realize $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ as a full corner in a crossed product and hence also as the C^* -algebra of a groupoid, see Equation (3) and Proposition 5.4. Then, in Section 6, we follow the approach of [La-Rae1, Theorem 3.7] to establish a faithfulness criterion for representations of $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ in terms of spanning projections of the canonical diagonal sub- C^* -algebra, see Theorem 6.1.

Section 7 contains an explicit description of the primitive ideal space of $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$, which generalizes [Ech-La, Theorem 3.6], see Theorem 7.1. However, in the proof of Theorem 7.1, we use a general result by Sims and Williams for groupoid C^* -algebras, see [Sims-Wil, Lemma 4.6], rather than working with crossed product C^* -algebras

as in [Ech-La]. We also give an explicit presentation of the primitive ideals using relations that only involve the range projections of the generating isometries. This presentation is motivated by the description of the primitive ideals of $C_\lambda^*(R \rtimes R^\times)$ given in [Li4, Section 3] and [Li5]. We then prove in Section 8 that the boundary quotient of $C_\lambda^*(R \rtimes R_{m,\Gamma})$ can be realized as a semigroup crossed product; this generalizes the semigroup crossed product description for the ring C^* -algebra of R .

In Section 9, we show that the number-theoretic input for our construction carries a canonical partial order, and that our construction respects this order, that is, it is functorial in the appropriate sense, see Propositions 9.2 and 9.5.

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2. PRELIMINARIES

2.1. The left regular C^* -algebra of a semigroup. Let P be a unital subsemigroup of a countable group G , and let $\{\delta_x : x \in P\}$ be the canonical orthonormal basis for $\ell^2(P)$. Each $p \in P$ gives rise to an isometry λ_p in $\mathcal{B}(\ell^2(P))$ such that $\lambda_p(\delta_x) = \delta_{px}$ for all $x \in P$. The *left regular C^* -algebra of P* is $C_\lambda^*(P) := C^*(\{\lambda_p : p \in P\})$. The canonical “diagonal” sub- C^* -algebra of $C_\lambda^*(P)$ is $D_\lambda(P) := C_\lambda^*(P) \cap \ell^\infty(P)$, where we view $\ell^\infty(P)$ as sub- C^* -algebra of $\mathcal{B}(\ell^2(P))$ in the canonical way. Since P embeds into a group, $D_\lambda(P)$ coincides with the smallest unital sub- C^* -algebra of $\ell^\infty(P)$ that is invariant under conjugation by the isometries λ_p for $p \in P$ and the co-isometries λ_p^* for $p \in P$; however, to see this we must introduce some ideas from [Li2].

For each subset $X \subseteq P$ and $p \in P$, let

$$pX := \{px : x \in X\} \quad \text{and} \quad p^{-1}(X) := (p^{-1}X) \cap P = \{p^{-1}x \in G : x \in X\} \cap P.$$

Consider the smallest collection \mathcal{J}_P of subsets of P such that

- \emptyset and P are in \mathcal{J}_P ;

- if X is in \mathcal{J}_P and p is in P , then pX and $p^{-1}(X)$ are in \mathcal{J}_P ;
- if $X, Y \in \mathcal{J}_P$, then $X \cap Y \in \mathcal{J}_P$.

It is shown in [Li2, Section 3] that the first two conditions imply the third. Members of \mathcal{J}_P are called *constructible right ideals of P* , see [Li2, Section 2] and [Li3, Definition 2.1]. We refer the reader to [Li2] or [Li1, Section A.2] for a discussion of the motivation for considering constructible ideals and some of the history leading up to their conception.

Since P embeds in a group, the results of [Li2, Section 3] show that

$$D_\lambda(P) = \overline{\text{span}}(\{E_X : X \in \mathcal{J}_P\})$$

where $E_X \in \mathcal{B}(\ell^2(P))$ is the orthogonal projection onto the subspace $\ell^2(X) \subseteq \ell^2(P)$. At this point, it is not difficult to see that $D_\lambda(P)$ is indeed the smallest unital sub-C*-algebra D of $\ell^\infty(P)$ such that $p \in P$ and $d \in D$ implies $\lambda_p d \lambda_p^* \in D$ and $\lambda_p^* d \lambda_p \in D$.

Following [Li2, Definition 2.26], we say that \mathcal{J}_P is *independent* or P satisfies the *independence condition* if $\bigcup_{i=1}^m X_i = X$ for $X, X_1, \dots, X_m \in \mathcal{J}_P$ implies $X = X_i$ for some $1 \leq i \leq m$. Semigroups satisfying the independence condition are particularly tractable; indeed, if P satisfies the independence condition, then the diagonal C*-algebra $D_\lambda(P)$ enjoys a certain universal property, which we will discuss in Section 4. Much of Section 3 is devoted to establishing that the class of semigroups under consideration in this paper satisfy the independence condition.

2.2. Moduli and ray classes. Let K be a number field with ring of integers R , and let $R^\times := R \setminus \{0\}$ denote the multiplicative semigroup of non-zero elements in R . Let \mathcal{P}_K denote the set of all non-zero prime ideals of R , and let \mathcal{I} denote the group of fractional ideals of K . For $\mathfrak{a} \in \mathcal{I}$, there is a unique factorization $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathcal{P}_K} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$ where $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$, and $v_{\mathfrak{p}}(\mathfrak{a}) = 0$ for all but finitely many \mathfrak{p} ; for $x \in K^\times := K \setminus \{0\}$, we let $v_{\mathfrak{p}}(x) := v_{\mathfrak{p}}(xR)$. Let $i : K^\times \rightarrow \mathcal{I}$ be the group homomorphism $i(x) := xR$; the *ideal class group of K* is given by $\text{Cl}(K) := \mathcal{I}/i(K^\times)$.

If $[K : \mathbb{Q}]$ is the degree of K over \mathbb{Q} , then there are exactly $[K : \mathbb{Q}]$ embeddings of K into the complex numbers; these come in two flavours: there are the *real embeddings* $w : K \hookrightarrow \mathbb{R}$ and the *complex embeddings* $w : K \hookrightarrow \mathbb{C}$ such that $w(K) \not\subseteq \mathbb{R}$. We let

$V_{K,\mathbb{R}}$ be the (finite) set of real embeddings of K . A *modulus* \mathfrak{m} for K is a function $\mathfrak{m} : V_{K,\mathbb{R}} \sqcup \mathcal{P}_K \rightarrow \mathbb{N}$ such that

- $\mathfrak{m}_\infty := \mathfrak{m}|_{V_{K,\mathbb{R}}} : V_{K,\mathbb{R}} \rightarrow \mathbb{N}$ takes values in $\{0, 1\}$;
- $\mathfrak{m}|_{\mathcal{P}_K} : \mathcal{P}_K \rightarrow \mathbb{N}$ is finitely supported, that is, $\mathfrak{m}(\mathfrak{p}) = 0$ for all but finitely many \mathfrak{p} .

Let \mathfrak{m}_0 be the ideal $\mathfrak{m}_0 := \prod_{\mathfrak{p}} \mathfrak{p}^{\mathfrak{m}(\mathfrak{p})}$ of R . It is conventional to write \mathfrak{m} as a formal product $\mathfrak{m} = \mathfrak{m}_\infty \mathfrak{m}_0$. The set of moduli for K carries a canonical partial order; by definition, $\mathfrak{m} \leq \mathfrak{n}$ if and only if $\mathfrak{m}_\infty(w) \leq \mathfrak{n}_\infty(w)$ for all $w \in V_{K,\mathbb{R}}$ and $\mathfrak{m}(\mathfrak{p}) \leq \mathfrak{n}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{P}_K$; this is nothing more than the usual partial order on \mathbb{N} -valued functions. Traditionally, one says that \mathfrak{m} *divides* \mathfrak{n} if $\mathfrak{m} \leq \mathfrak{n}$ and writes $\mathfrak{m} \mid \mathfrak{n}$ instead of $\mathfrak{m} \leq \mathfrak{n}$. In particular, a prime \mathfrak{p} divides \mathfrak{m} if and only if $\mathfrak{m}(\mathfrak{p}) > 0$, and a real embedding w divides \mathfrak{m} if and only if $\mathfrak{m}_\infty(w) = 1$. Thus, we will write $w \mid \mathfrak{m}_\infty$ to indicate that \mathfrak{m}_∞ takes the value one at the real embedding w . The *multiplicative group of residues modulo* \mathfrak{m} is

$$(R/\mathfrak{m})^* := \prod_{w \mid \mathfrak{m}_\infty} \{\pm 1\} \times (R/\mathfrak{m}_0)^*.$$

If \mathfrak{m}_∞ is trivial, that is, if $\mathfrak{m}(w) = 0$ for all real embeddings w , then $(R/\mathfrak{m})^* = (R/\mathfrak{m}_0)^*$, and if $\mathfrak{m}|_{\mathcal{P}_K}$ is trivial, so that $\mathfrak{m}_0 = R$, then $(R/\mathfrak{m})^* = \prod_{w \mid \mathfrak{m}_\infty} \{\pm 1\}$. If \mathfrak{m} is trivial, then $(R/\mathfrak{m})^*$ is simply the trivial group.

Note that it does not make sense to talk about additive classes modulo \mathfrak{m} . By the Chinese Remainder Theorem, $(R/\mathfrak{m}_0)^* \cong \prod_{\mathfrak{p} \mid \mathfrak{m}_0} (R/\mathfrak{p}^{\mathfrak{m}(\mathfrak{p})})^*$. Let

$$R_{\mathfrak{m}} := \{a \in R^\times : v_{\mathfrak{p}}(a) = 0 \text{ for all } \mathfrak{p} \text{ such that } \mathfrak{p} \mid \mathfrak{m}_0\}$$

be the multiplicative semigroup of non-zero algebraic integers that are coprime to the ideal \mathfrak{m}_0 . If $a \in R_{\mathfrak{m}}$, then a is invertible modulo \mathfrak{m}_0 , and we define its *residue modulo* \mathfrak{m} to be

$$[a]_{\mathfrak{m}} := ((\text{sign}(w(a)))_{w \mid \mathfrak{m}_\infty}, a + \mathfrak{m}_0) \in (R/\mathfrak{m})^*,$$

where $\text{sign}(t) := t/|t|$ for any non-zero real number t .

Lemma 2.1. *The map $R_{\mathfrak{m}} \rightarrow (R/\mathfrak{m})^*$ given by $a \mapsto [a]_{\mathfrak{m}}$ is a surjective semigroup homomorphism.*

Proof. It is easy to see that $[ab]_{\mathfrak{m}} = [a]_{\mathfrak{m}}[b]_{\mathfrak{m}}$ for all $a, b \in R_{\mathfrak{m}}$. Let $(\epsilon, b + \mathfrak{m}_0) \in (R/\mathfrak{m})^*$. By [Nar, Proposition 2.2(i)], the coset $1 + \mathfrak{m}_0$ contains (infinitely many) elements of any given signature. Thus, we can find $c \in 1 + \mathfrak{m}_0$ such that $(\text{sign}(w(bc)))_{w|\mathfrak{m}_\infty} = \epsilon$. Since $bc \in R_{\mathfrak{m}}$, and $bc + \mathfrak{m}_0 = b + \mathfrak{m}_0$, we have $[bc]_{\mathfrak{m}} = (\epsilon, b + \mathfrak{m}_0)$. \square

Let $K_{\mathfrak{m}} := \{a \in K^\times : v_{\mathfrak{p}}(a) = 0 \text{ for all } \mathfrak{p} \mid \mathfrak{m}_0\}$ be the (multiplicative) subgroup of K^\times consisting of non-zero elements of K whose corresponding principal fractional ideal is coprime to \mathfrak{m}_0 .

Lemma 2.2. *The group of (left) quotients $R_{\mathfrak{m}}^{-1}R_{\mathfrak{m}} := \{a/b : a, b \in R_{\mathfrak{m}}\}$ of $R_{\mathfrak{m}}$ in K^\times coincides with $K_{\mathfrak{m}}$. Therefore, the semigroup homomorphism $R_{\mathfrak{m}} \rightarrow (R/\mathfrak{m})^*$ given by $a \mapsto [a]_{\mathfrak{m}}$ has a unique extension to a (surjective) group homomorphism $K_{\mathfrak{m}} \rightarrow (R/\mathfrak{m})^*$, which we denote by $x \mapsto [x]_{\mathfrak{m}}$.*

Proof. Clearly, $R_{\mathfrak{m}}^{-1}R_{\mathfrak{m}} \subseteq K_{\mathfrak{m}}$. Let $x \in K_{\mathfrak{m}}$. Then $xR = \mathfrak{a}/\mathfrak{b}$ with \mathfrak{a} and \mathfrak{b} integral ideals coprime to \mathfrak{m}_0 , and \mathfrak{a} and \mathfrak{b} represent the same class \mathfrak{k} in $\text{Cl}(K)$. Choose an integral ideal \mathfrak{c} in \mathfrak{k}^{-1} such that \mathfrak{c} is coprime to \mathfrak{m}_0 . Then there are $a, b \in R_{\mathfrak{m}}$ such that $\mathfrak{a}\mathfrak{c} = aR$ and $\mathfrak{b}\mathfrak{c} = bR$. Now, $xR = \mathfrak{a}/\mathfrak{b} = \mathfrak{a}\mathfrak{c}/\mathfrak{b}\mathfrak{c} = aR/bR$, so that $x = au/b$ for some $u \in R^*$, which shows the reverse inclusion.

If $x \in K_{\mathfrak{m}}$, then by Lemma 2.2, we can write $x = a/b$ with $a, b \in R_{\mathfrak{m}}$, and $[x]_{\mathfrak{m}}$ is given by $[x]_{\mathfrak{m}} = [a]_{\mathfrak{m}}[b]_{\mathfrak{m}}^{-1}$. A standard argument shows that this gives a well-defined group homomorphism. \square

Moduli play a central role in the ideal-theoretic formulation of class field theory, see [Mil, Chapter V]. Let $\mathcal{I}_{\mathfrak{m}}$ denote the group of fractional ideals of K that are coprime to \mathfrak{m}_0 , and let $i : K_{\mathfrak{m}} \rightarrow \mathcal{I}_{\mathfrak{m}}$ be the canonical homomorphism given by $a \mapsto aR$. Let $K_{\mathfrak{m},1} := \{x \in K_{\mathfrak{m}} : [x]_{\mathfrak{m}} = 1\}$, so that $K_{\mathfrak{m}}/K_{\mathfrak{m},1} \cong (R/\mathfrak{m})^*$. The group $K_{\mathfrak{m},1}$ is called the *ray modulo \mathfrak{m}* , and the group $\text{Cl}_{\mathfrak{m}}(K) := \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},1})$ is the *ray class group modulo \mathfrak{m}* . Let $R_{\mathfrak{m},1} := R \cap K_{\mathfrak{m},1}$, let R^* denote the group of units in R , and let $R_{\mathfrak{m},1}^* := R_{\mathfrak{m},1} \cap R^*$ be the group of invertible elements in $R_{\mathfrak{m},1}$. A relationship between ray class groups and the usual ideal class group is demonstrated by the following standard result.

Proposition 2.3 ([Mil, Chapter V, Theorem 1.7]). *For every modulus \mathfrak{m} , there is a five-term exact sequence*

$$1 \rightarrow R_{\mathfrak{m},1}^* \rightarrow R^* \rightarrow (R/\mathfrak{m})^* \rightarrow \text{Cl}_{\mathfrak{m}}(K) \rightarrow \text{Cl}(K) \rightarrow 1.$$

Hence, $\text{Cl}_{\mathfrak{m}}(K)$ is a finite group of order

$$h_{\mathfrak{m}} := h \cdot [R^* : R_{\mathfrak{m},1}^*]^{-1} \cdot 2^{r_0} \cdot N(\mathfrak{m}_0) \prod_{\mathfrak{p}|\mathfrak{m}_0} (1 - N(\mathfrak{p})^{-1})$$

where $h := |\text{Cl}(K)|$ is the class number of K , r_0 denotes the number of real embeddings w of K for which $\mathfrak{m}(w) = 1$, and $N(\mathfrak{p}) := |R/\mathfrak{p}|$ is the norm of \mathfrak{p} .

3. SEMIGROUPS DEFINED BY ACTIONS OF CONGRUENCE MONOIDS ON RINGS OF ALGEBRAIC INTEGERS

Let K be a number field with ring of integers R , and fix a modulus \mathfrak{m} for K . For each subgroup Γ of $(R/\mathfrak{m})^*$, let

$$R_{\mathfrak{m},\Gamma} := \{a \in R_{\mathfrak{m}} : [a]_{\mathfrak{m}} \in \Gamma\}.$$

Clearly $R_{\mathfrak{m},\Gamma}$ is a subsemigroup of $R_{\mathfrak{m}}$ containing the semigroup $R_{\mathfrak{m},1} = R_{\mathfrak{m},\{1\}}$. For $\Gamma = (R/\mathfrak{m})^*$, we have $R_{\mathfrak{m},\Gamma} = R_{\mathfrak{m}}$.

Remark 3.1. Semigroups of the form $R_{\mathfrak{m},\Gamma}$ are called *congruence monoids*, see [HK, Definition 5] and [G-HK].

Proposition 3.2. *Let $K_{\mathfrak{m},\Gamma} := \{x \in K_{\mathfrak{m}} : [x]_{\mathfrak{m}} \in \Gamma\}$. Then $K_{\mathfrak{m},\Gamma} = R_{\mathfrak{m},\Gamma}^{-1} R_{\mathfrak{m},\Gamma}$ where $R_{\mathfrak{m},\Gamma}^{-1} R_{\mathfrak{m},\Gamma}$ is the group of (left) quotients of $R_{\mathfrak{m},\Gamma}$ in $K_{\mathfrak{m}}$.*

Proof. Clearly, $R_{\mathfrak{m},\Gamma}^{-1} R_{\mathfrak{m},\Gamma} \subseteq K_{\mathfrak{m},\Gamma}$. Let $x \in K_{\mathfrak{m},\Gamma}$. Using Lemma 2.2, we can write $x = a/b$ with $a, b \in R_{\mathfrak{m}}$. Since $[x]_{\mathfrak{m}} = [a]_{\mathfrak{m}}[b]_{\mathfrak{m}}^{-1} \in \Gamma$, there exists $\gamma \in \Gamma$ such that $[a]_{\mathfrak{m}} = [b]_{\mathfrak{m}}\gamma$. By Proposition 2.1, there exists $c \in R_{\mathfrak{m}}$ such that $[c]_{\mathfrak{m}} = [a]_{\mathfrak{m}}^{-1}$. Now, $[ac]_{\mathfrak{m}} = [a]_{\mathfrak{m}}[c]_{\mathfrak{m}} = [1]_{\mathfrak{m}}$ is in Γ , and $[bc]_{\mathfrak{m}} = ([a]_{\mathfrak{m}}\gamma^{-1})[c]_{\mathfrak{m}} = \gamma^{-1}$ is also in Γ , so we have that $x = a/b = ac/bc$ is in $R_{\mathfrak{m},\Gamma}^{-1} R_{\mathfrak{m},\Gamma}$. \square

The semigroup $R_{\mathfrak{m},\Gamma}$ acts on (the additive group of) R by multiplication, and we form the semi-direct product $R \rtimes R_{\mathfrak{m},\Gamma}$. Explicitly, $R \rtimes R_{\mathfrak{m},\Gamma}$ consists of pairs (b, a) with $b \in R$ and $a \in R_{\mathfrak{m},\Gamma}$, and the product of two such pairs is $(b, a)(d, c) := (b + ad, ac)$. Our first observation about $R \rtimes R_{\mathfrak{m},\Gamma}$ is the following.

Proposition 3.3. *The semigroup $R \rtimes R_{\mathfrak{m}, \Gamma}$ is left Ore with enveloping group $(R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma}$ where $R_{\mathfrak{m}}^{-1}R = \{\frac{a}{b} \in K : a \in R, b \in R_{\mathfrak{m}}\}$ denotes the localization of the ring R at $R_{\mathfrak{m}}$. That is, the set of left quotients $(R \rtimes R_{\mathfrak{m}, \Gamma})^{-1}(R \rtimes R_{\mathfrak{m}, \Gamma})$ taken inside $K \rtimes K^{\times}$ coincides with the group $(R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma}$.*

Proof. For $(b, a), (d, c) \in R \rtimes R_{\mathfrak{m}, \Gamma}$, we have

$$(b, a)^{-1}(d, c) = (-ba^{-1}, a^{-1})(d, c) = \left(\frac{d-b}{a}, \frac{c}{a}\right). \quad (1)$$

Hence, $(R \rtimes R_{\mathfrak{m}, \Gamma})^{-1}(R \rtimes R_{\mathfrak{m}, \Gamma})$ lies in $(R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma}$. A direct calculation shows that $(R \rtimes R_{\mathfrak{m}, \Gamma})^{-1}(R \rtimes R_{\mathfrak{m}, \Gamma})$ is a group. Since $(R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma} = (R_{\mathfrak{m}}^{-1}R \rtimes \{1\})(\{0\} \rtimes K_{\mathfrak{m}, \Gamma})$, we will be done once we show that $(R \rtimes R_{\mathfrak{m}, \Gamma})^{-1}(R \rtimes R_{\mathfrak{m}, \Gamma})$ contains the subgroups $R_{\mathfrak{m}}^{-1}R \rtimes \{1\}$ and $\{0\} \rtimes K_{\mathfrak{m}, \Gamma}$.

By considering all products in (1) with $b = d = 0$ and using Proposition 3.2, we see that $\{0\} \rtimes K_{\mathfrak{m}, \Gamma}$ is contained in $(R \rtimes R_{\mathfrak{m}})^{-1}(R \rtimes R_{\mathfrak{m}, \Gamma})$, and by considering all products in (1) with $a = c$, we see that $(R_{\mathfrak{m}, \Gamma}^{-1}R) \rtimes \{1\}$ is contained in $(R \rtimes R_{\mathfrak{m}, \Gamma})^{-1}(R \rtimes R_{\mathfrak{m}, \Gamma})$. It remains to show that $R_{\mathfrak{m}, \Gamma}^{-1}R$ coincides with $R_{\mathfrak{m}}^{-1}R$. The inclusion $R_{\mathfrak{m}, \Gamma}^{-1}R \subseteq R_{\mathfrak{m}}^{-1}R$ is easy to see. Now suppose that $a \in R$ and $b \in R_{\mathfrak{m}}$. By Lemma 2.1, there is a $c \in R_{\mathfrak{m}}$ such that $[c]_{\mathfrak{m}} = [b]_{\mathfrak{m}}^{-1}$, that is, $w(bc) > 0$ for all $w \mid \mathfrak{m}_{\infty}$ and $bc \in 1 + \mathfrak{m}_0$, so that $bc \in R_{\mathfrak{m}, 1}$. Now $a/b = ac/bc$ lies in $R_{\mathfrak{m}, \Gamma}^{-1}R$, so $R_{\mathfrak{m}}^{-1}R \subseteq R_{\mathfrak{m}, \Gamma}^{-1}R$. \square

We now turn to the problem of computing the semilattice $\mathcal{J}_{R \rtimes R_{\mathfrak{m}, \Gamma}}$ of constructible right ideals in $R \rtimes R_{\mathfrak{m}, \Gamma}$. Recall that $\mathcal{I}_{\mathfrak{m}}$ is, by definition, the group of fractional ideals of K that are coprime to \mathfrak{m}_0 . Let $\mathcal{I}_{\mathfrak{m}}^+$ be the submonoid of $\mathcal{I}_{\mathfrak{m}}$ consisting of (non-zero) integral ideals that are coprime to \mathfrak{m}_0 . For $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}$, we set $\mathfrak{a}^{\times} := \mathfrak{a} \setminus \{0\}$. When $\mathfrak{m} = \mathfrak{m}_0 = R$, we will write \mathcal{I} instead of \mathcal{I}_R .

Our goal now is to prove the following result, which generalizes the computation of $\mathcal{J}_{R \rtimes R^{\times}}$ from [Li2, Section 2.4].

Proposition 3.4. *The set $\left(\bigsqcup_{\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+} R/\mathfrak{a}\right) \sqcup \{\emptyset\}$ is a semilattice with respect to intersections. For each $x \in R$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, the set $(x + \mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m}, \Gamma})$ is a constructible right ideal of $R \rtimes R_{\mathfrak{m}, \Gamma}$, and the map*

$$\left(\bigsqcup_{\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+} R/\mathfrak{a}\right) \sqcup \{\emptyset\} \rightarrow \mathcal{J}_{R \rtimes R_{\mathfrak{m}, \Gamma}}$$

given by $x + \mathfrak{a} \mapsto (x + \mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m}, \Gamma})$ and $\emptyset \mapsto \emptyset$ is an isomorphism of semilattices. Moreover, $\mathcal{J}_{R \times R_{\mathfrak{m}, \Gamma}}$ is independent.

We need several preliminary results before we can prove Proposition 3.4. They are contained in the following propositions and lemmas, several of which will also be useful later.

Recall that an element $x \in K^\times$ is *totally positive* if $w(x) > 0$ for every real embedding $w : K \hookrightarrow \mathbb{R}$. Note that if K has no real embeddings, then every element of K^\times is totally positive.

Lemma 3.5. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be distinct non-zero primes of R not dividing \mathfrak{m}_0 and n_1, \dots, n_k be in \mathbb{N} . There is an element x in $R_{\mathfrak{m}, 1}$ such that x is totally positive and $v_{\mathfrak{p}_j}(x) = n_j$ for $j = 1, \dots, k$.*

Proof. For each $1 \leq j \leq k$, let $\pi_{\mathfrak{p}_j} \in \mathfrak{p}_j \setminus \mathfrak{p}_j^2$. By the Chinese Remainder Theorem, there exists $y \in R$ such that

- (1) $y \equiv \pi_{\mathfrak{p}_j}^{n_j} \pmod{\mathfrak{p}_j^{n_j+1}}$;
- (2) $y \equiv 1 \pmod{\mathfrak{m}_0}$.

The first condition says that $v_{\mathfrak{p}_j}(y) = n_j$ for $1 \leq j \leq k$. Choose an integer T in $\mathfrak{m}_0 \mathfrak{p}_1^{n_1+1} \dots \mathfrak{p}_k^{n_k+1}$ such that $x := y + T$ is totally positive. Since $T \in \mathfrak{m}_0 \mathfrak{p}_j^{n_j+1} = \mathfrak{m}_0 \cap \mathfrak{p}_j^{n_j+1}$ for each $1 \leq j \leq k$, x still satisfies (1) and (2), so we are done. \square

The following two lemmas are refinements of well-known results for the case of trivial \mathfrak{m} (in which case Γ must also be trivial), see [C-D-L, Lemma 4.15(a)] and [Li2, Section 2.4].

Lemma 3.6. *Let $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. For each $a \in \mathfrak{a} \cap R_{\mathfrak{m}, 1}$, there exists $b \in \mathfrak{a} \cap R_{\mathfrak{m}, 1}$ such that $\mathfrak{a} = aR + bR$.*

Proof. Write $aR = \mathfrak{a}\mathfrak{c}_a$ for some ideal \mathfrak{c}_a of R . Since a is relatively prime to \mathfrak{m}_0 , we have $\mathfrak{c}_a \in \mathcal{I}_{\mathfrak{m}}^+$. By Lemma 3.5, we can find $b \in \mathfrak{a} \cap R_{\mathfrak{m}, 1}$ such that $v_{\mathfrak{p}}(b) = v_{\mathfrak{p}}(\mathfrak{a})$ for every prime \mathfrak{p} dividing \mathfrak{c}_a . Now write $bR = \mathfrak{a}\mathfrak{c}_b$ for some ideal \mathfrak{c}_b of R . Since $v_{\mathfrak{p}}(\mathfrak{c}_b) = v_{\mathfrak{p}}(b) - v_{\mathfrak{p}}(\mathfrak{a}) = 0$ for all \mathfrak{p} dividing \mathfrak{c}_a , we see that \mathfrak{c}_a and \mathfrak{c}_b are relatively prime, that is, $R = \mathfrak{c}_a + \mathfrak{c}_b$. Thus, $\mathfrak{a} = \mathfrak{a}R = \mathfrak{a}(\mathfrak{c}_a + \mathfrak{c}_b) = aR + bR$. \square

Lemma 3.7. *Let $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. For each $a \in R_{\mathfrak{m},1}$, there exists $b \in R_{\mathfrak{m},1}$ such that $\mathfrak{a} = \frac{a}{b}R \cap R$.*

Proof. Write $aR = \mathfrak{a}\mathfrak{c}_a$ for some ideal \mathfrak{c}_a of R . Since $a \in \mathfrak{c}_a$, Lemma 3.6 implies that there is a $b \in \mathfrak{c}_a \cap R_{\mathfrak{m},1}$ such that $\mathfrak{c}_a = aR + bR$. Since $abR = (aR + bR)(aR \cap bR)$, we have $\mathfrak{a} = aR(\mathfrak{c}_a)^{-1} = aR(aR + bR)^{-1} = b^{-1}(aR \cap bR) = \frac{a}{b}R \cap R$. \square

For any set $X \subseteq R$, we denote by X_+ the subset of all totally positive elements in X , and by $\langle X \rangle$ the ideal of R generated by X .

Lemma 3.8. *Let $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. Then for each subgroup $\Gamma \subseteq (R/\mathfrak{m})^*$, \mathfrak{a} is generated as an ideal by the set $\mathfrak{a} \cap R_{\mathfrak{m},\Gamma}$. Indeed, \mathfrak{a} is generated by the set $(\mathfrak{a} \cap (1 + \mathfrak{m}_0))_+ = \mathfrak{a} \cap (1 + \mathfrak{m}_0)_+$.*

Proof. Since \mathfrak{a} and \mathfrak{m}_0 are coprime, $\mathfrak{a} \cap \mathfrak{m}_0 = \mathfrak{a}\mathfrak{m}_0$, and there exists $x \in \mathfrak{a}$ and $y \in \mathfrak{m}_0$ such that $1 = x + y$. Choose an integer $T \in \mathfrak{a} \cap \mathfrak{m}_0$ such that $x_0 := x + T$ is totally positive. Then $1 = x_0 + y_0$ with $x_0 \in \mathfrak{a}$ and $y_0 := y - T \in \mathfrak{m}_0$. Now,

$$\mathfrak{a} \cap (1 + \mathfrak{m}_0) = \begin{cases} z + \mathfrak{a} \cap \mathfrak{m}_0 = z + \mathfrak{a}\mathfrak{m}_0 & \text{if there exists } z \in \mathfrak{a} \cap (1 + \mathfrak{m}_0) \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence, $\mathfrak{a} \cap (1 + \mathfrak{m}_0) = x_0 + \mathfrak{a}\mathfrak{m}_0$. Since $x_0 \in (x_0 + \mathfrak{a}\mathfrak{m}_0)_+$, it follows that $(\mathfrak{a}\mathfrak{m}_0)_+$ is contained in $\langle (x_0 + \mathfrak{a}\mathfrak{m}_0)_+ \rangle$.

If \mathfrak{b} is any non-zero ideal of R and x an element of \mathfrak{b} , then for sufficiently large $k \in \mathbb{N}^\times$, $x + kN(\mathfrak{b})$ is totally positive. Since $N(\mathfrak{b}) \in \mathfrak{b}_+$, and $x = (x + kN(\mathfrak{b})) - kN(\mathfrak{b})$, we see that any element of a non-zero ideal of R can be written as the difference of two totally positive elements each lying in the ideal. Using this fact, we see that $(\mathfrak{a}\mathfrak{m}_0)_+ \subseteq \langle (x_0 + \mathfrak{a}\mathfrak{m}_0)_+ \rangle$ implies that $\langle (x_0 + \mathfrak{a}\mathfrak{m}_0)_+ \rangle$ contains $\mathfrak{a}\mathfrak{m}_0$.

Since $\mathfrak{a} \supseteq \langle (x_0 + \mathfrak{a}\mathfrak{m}_0)_+ \rangle$, we will be done if we show that \mathfrak{m}_0 and $\langle (x_0 + \mathfrak{a}\mathfrak{m}_0)_+ \rangle$ are coprime. Since $x_0 \in (x_0 + \mathfrak{a}\mathfrak{m}_0)_+$, it suffices to show that $v_{\mathfrak{p}}(x_0) = 0$ for each $\mathfrak{p} \mid \mathfrak{m}_0$. Let $\mathfrak{p} \mid \mathfrak{m}_0$. Then we have $0 = v_{\mathfrak{p}}(1 - x_0 + x_0) \geq \min\{v_{\mathfrak{p}}(1 - x_0), v_{\mathfrak{p}}(x_0)\}$. Now, $1 - x_0 = y_0 \in \mathfrak{m}_0 \subseteq \mathfrak{p}$, which implies that $v_{\mathfrak{p}}(1 - x_0) > 0$. Hence, we must have $v_{\mathfrak{p}}(x_0) = 0$. \square

Proposition 3.9. *The set $\mathcal{I}_{\mathfrak{m}}^+ \sqcup \{\emptyset\}$ is a semilattice with respect to intersections. For each $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, the set $\mathfrak{a} \cap R_{\mathfrak{m},\Gamma}$ is a constructible right ideal of the multiplicative*

semigroup $R_{\mathfrak{m},\Gamma}$, and the map $\mathcal{I}_{\mathfrak{m}}^+ \sqcup \{\emptyset\} \rightarrow \mathcal{J}_{R_{\mathfrak{m},\Gamma}}$ given by $\mathfrak{a} \mapsto \mathfrak{a} \cap R_{\mathfrak{m},\Gamma}$ and $\emptyset \mapsto \emptyset$ is an isomorphism of semilattices. Moreover, $\mathcal{J}_{R_{\mathfrak{m},\Gamma}}$ is independent.

Proof. It is clear that $\mathcal{I}_{\mathfrak{m}}^+ \sqcup \{\emptyset\}$ is a semilattice with respect to intersections. Now let $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. By Lemma 3.7, there exists $a, b \in R_{\mathfrak{m},1}$ such that $\mathfrak{a} = \frac{a}{b}R \cap R$, and so have $\mathfrak{a} \cap R_{\mathfrak{m},\Gamma} = \frac{a}{b}R \cap R_{\mathfrak{m},\Gamma}$. If $x \in R$ such that $\frac{a}{b}x \in R_{\mathfrak{m},\Gamma}$, then x lies in $R_{\mathfrak{m},\Gamma}$; it follows that $\mathfrak{a} \cap R_{\mathfrak{m},\Gamma} = \frac{a}{b}R_{\mathfrak{m},\Gamma} \cap R_{\mathfrak{m},\Gamma}$, which clearly lies in $\mathcal{J}_{R_{\mathfrak{m},\Gamma}}$. This settles the second claim.

To show surjectivity, it suffices to show that $\mathcal{J} := \{\mathfrak{a} \cap R_{\mathfrak{m},\Gamma} : \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\} \subseteq \mathcal{J}_{R_{\mathfrak{m},\Gamma}} \cup \{\emptyset\}$ satisfies the characterizing properties of $\mathcal{J}_{R_{\mathfrak{m},\Gamma}}$ (see Section 2.1). Clearly, \emptyset and $R_{\mathfrak{m},\Gamma}$ lie in \mathcal{J} . Let $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ and $x \in K_{\mathfrak{m},\Gamma}$. If $xa = y \in R_{\mathfrak{m},\Gamma}$ for some $a \in \mathfrak{a}$, then $[a]_{\mathfrak{m}} = [x]_{\mathfrak{m}}^{-1}[y]_{\mathfrak{m}} \in \Gamma$, so $a \in R_{\mathfrak{m},\Gamma}$. Thus,

$$x(\mathfrak{a} \cap R_{\mathfrak{m},\Gamma}) \cap R_{\mathfrak{m},\Gamma} = x\mathfrak{a} \cap xR_{\mathfrak{m},\Gamma} \cap R_{\mathfrak{m},\Gamma} = x\mathfrak{a} \cap R_{\mathfrak{m},\Gamma} = (x\mathfrak{a} \cap R) \cap R_{\mathfrak{m},\Gamma}$$

lies in \mathcal{J} , which proves that \mathcal{J} satisfies the desired properties. Hence, $\mathcal{J}_{R_{\mathfrak{m},\Gamma}} \subseteq \mathcal{J}$ which shows that the map $\mathfrak{a} \mapsto \mathfrak{a} \cap R_{\mathfrak{m},\Gamma}$ is surjective.

Suppose now that $\mathfrak{a} \cap R_{\mathfrak{m},\Gamma} = \mathfrak{b} \cap R_{\mathfrak{m},\Gamma}$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_{\mathfrak{m}}^+$. Then Lemma 3.8 implies that $\mathfrak{a} = \mathfrak{b}$, so this map is also injective.

It remains to show independence. Suppose that $\mathfrak{a}, \mathfrak{a}_1, \dots, \mathfrak{a}_k \in \mathcal{I}_{\mathfrak{m}}^+$ are distinct ideals such that $\mathfrak{a}_i \cap R_{\mathfrak{m},\Gamma} \subseteq \mathfrak{a} \cap R_{\mathfrak{m},\Gamma}$ for $i = 1, \dots, k$. We need to show that $\bigcup_{i=1}^k \mathfrak{a}_i \cap R_{\mathfrak{m},\Gamma} \subsetneq \mathfrak{a} \cap R_{\mathfrak{m},\Gamma}$. By Lemma 3.8, the inclusion $\mathfrak{a}_i \cap R_{\mathfrak{m},\Gamma} \subseteq \mathfrak{a} \cap R_{\mathfrak{m},\Gamma}$ implies that $\mathfrak{a}_i \subseteq \mathfrak{a}$. Since $\mathfrak{a}_i \neq \mathfrak{a}$, we even have $\mathfrak{a}_i \subsetneq \mathfrak{a}$ for $1 \leq i \leq k$. Thus, there are positive integers $N \leq M$, distinct non-zero primes $\mathfrak{p}_1, \dots, \mathfrak{p}_N, \mathfrak{p}_{N+1}, \dots, \mathfrak{p}_M$, and non-negative integers $n_1, \dots, n_N, n_{i,1}, \dots, n_{i,M}$, for $1 \leq i \leq k$, with $n_j \leq n_{i,j}$ for all $1 \leq j \leq N$, $1 \leq i \leq M$, such that

$$\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_N^{n_N} \quad \text{and} \quad \mathfrak{a}_i = \mathfrak{p}_1^{n_{i,1}} \cdots \mathfrak{p}_N^{n_{i,N}} \cdots \mathfrak{p}_M^{n_{i,M}}.$$

By Lemma 3.5, there exists $x \in R_{\mathfrak{m},1}$ such that $v_{\mathfrak{p}_j}(x) = n_j$ for $j = 1, \dots, N$ and $v_{\mathfrak{p}_i}(x) = 0$ for $i = N+1, \dots, M$. It follows that $x \in \mathfrak{a}$ and $x \notin \mathfrak{a}_i$ for $i = 1, \dots, k$. Thus, $x \in \mathfrak{a} \cap R_{\mathfrak{m},1} \setminus \bigcup_{i=1}^k \mathfrak{a}_i$. Since $R_{\mathfrak{m},1} \subseteq R_{\mathfrak{m},\Gamma}$, it follows that $x \in \mathfrak{a} \cap R_{\mathfrak{m},\Gamma} \setminus \bigcup_{i=1}^k \mathfrak{a}_i$, so we are done. \square

We are now ready to prove Proposition 3.4.

Proof of Proposition 3.4. If $x + \mathbf{a}, y + \mathbf{b}$ lie in $\bigsqcup_{\mathbf{a} \in \mathcal{I}_m^+} R/\mathbf{a}$, then

$$(x + \mathbf{a}) \cap (y + \mathbf{b}) = \begin{cases} z + \mathbf{a} \cap \mathbf{b} & \text{if } z \in (x + \mathbf{a}) \cap (y + \mathbf{b}) \\ \emptyset & \text{if } (x + \mathbf{a}) \cap (y + \mathbf{b}) = \emptyset. \end{cases}$$

Thus, $(\bigsqcup_{\mathbf{a} \in \mathcal{I}_m^+} R/\mathbf{a}) \sqcup \{\emptyset\}$ is a semilattice with respect to intersections

Let $\mathbf{a} \in \mathcal{I}_m^+$. By Lemma 3.7, we can write $\mathbf{a} = \frac{a}{b}R \cap R$ for some $a, b \in R_{m,1}$. As in the proof of Proposition 3.9, we have $\mathbf{a} \cap R_{m,\Gamma} = \frac{a}{b}R_{m,\Gamma} \cap R_{m,\Gamma}$. Thus,

$$[(0, \frac{a}{b})R \times R_{m,\Gamma}] \cap R \times R_{m,\Gamma} = \mathbf{a} \times (\mathbf{a} \cap R_{m,\Gamma}),$$

and for $x \in R$ we have $(x, 0)(\mathbf{a} \times (\mathbf{a} \cap R_{m,\Gamma})) = (x + \mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma})$. Hence, $(x + \mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma})$ is in $\mathcal{J}_{R \times R_{m,\Gamma}}$ for all $x \in R$ and $\mathbf{a} \in \mathcal{I}_m^+$.

To show surjectivity, it suffices to show that $\tilde{\mathcal{J}} := \{(x + \mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma}) : x \in R, \mathbf{a} \in \mathcal{I}_m^+ \} \cup \{\emptyset\}$ satisfies the characterizing properties of $\mathcal{J}_{R \times R_{m,\Gamma}}$ (see Section 2.1). It is easy to see that $\tilde{\mathcal{J}}$ is closed under taking finite intersections. Let $(x + \mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma}) \in \tilde{\mathcal{J}}$ and $(b, a) \in R \times R_{m,\Gamma}$. Then

$$(b, a)[(x + \mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma})] = (b + ax + a\mathbf{a}) \times (a\mathbf{a} \cap R_{m,\Gamma})$$

lies in $\tilde{\mathcal{J}}$. Moreover, for any $c \in R_{m,\Gamma}$,

$$(0, c)^{-1}[(x + \mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma})] \cap (R \times R_{m,\Gamma}) = [c^{-1}(x + \mathbf{a}) \times (c^{-1}\mathbf{a} \cap R_{m,\Gamma})] \cap (R \times R_{m,\Gamma}).$$

Now, $c^{-1}(x + \mathbf{a}) \cap R = c^{-1}((x + \mathbf{a}) \cap cR)$ is either empty or of the form $c^{-1}z + c^{-1}\mathbf{a} \cap R$ for some $z \in (x + \mathbf{a}) \cap cR$, and $c^{-1}(\mathbf{a} \cap R_{m,\Gamma}) \cap R_{m,\Gamma} = c^{-1}\mathbf{a} \cap R_{m,\Gamma}$. It follows that $\tilde{\mathcal{J}}$ satisfies the conditions in Section 2.1, which concludes the proof of surjectivity.

Injectivity follows as in the proof of Proposition 3.9, and independence of $\mathcal{J}_{R \times R_{m,\Gamma}}$ follow from independence of $\mathcal{J}_{R_{m,\Gamma}}$. \square

We conclude this section by giving several corollaries. The first simply says that Proposition 3.4 generalizes the computation of $\mathcal{J}_{R \times R^\times}$ from [Li2, Section 2.4].

Corollary 3.10 ([Li2, Section 2.4]). *We have*

$$\mathcal{J}_{R \times R^\times} = \{(x + \mathbf{a}) \times \mathbf{a}^\times : x \in R, \mathbf{a} \in \mathcal{I}^+ \} \cup \{\emptyset\}$$

where $\mathbf{a}^\times := \mathbf{a} \setminus \{0\}$. Moreover, $\mathcal{J}_{R \times R^\times}$ is independent.

Proof. Apply Proposition 3.4 for the case of trivial \mathfrak{m} and Γ . \square

As before, let $i : K_{\mathfrak{m}} \rightarrow \mathcal{I}_{\mathfrak{m}}$ denote the map $i(x) = xR$. Then the group $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ is a quotient of the finite group $\text{Cl}_{\mathfrak{m}}(K)$, hence is finite; indeed, $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma}) \cong \text{Cl}_{\mathfrak{m}}(K)/\bar{\Gamma}$ where $\bar{\Gamma} = i(K_{\mathfrak{m},\Gamma})/i(K_{\mathfrak{m},1})$. Recall that a semigroup is right LCM if all of its constructible right ideals are principal.

Corollary 3.11. *The semigroup $R \rtimes R_{\mathfrak{m},\Gamma}$ is right LCM if and only if the group $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ is trivial.*

Proof. By Proposition 3.4, $R \rtimes R_{\mathfrak{m},\Gamma}$ is right LCM if and only if every integral ideal $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ is principal and generated by some $a \in R_{\mathfrak{m},\Gamma}$. This is equivalent to $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ being trivial. \square

Let $K = \mathbb{Q}$, so that $R = \mathbb{Z}$. Let $m \in \mathbb{N}^{\times}$ be a positive natural number, and let $\mathfrak{m} = \mathfrak{m}_{\infty}\mathfrak{m}_0$ where \mathfrak{m}_{∞} takes the value one at the only real embedding of \mathbb{Q} and $\mathfrak{m}_0(p) := v_p(m)$. Then a calculation shows that $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},1}) \cong (\mathbb{Z}/m\mathbb{Z})^*$. Thus, Corollary 3.11 shows that, even in the case $K = \mathbb{Q}$, the semigroup $R \rtimes R_{\mathfrak{m},\Gamma}$ is usually not right LCM.

We also have:

Corollary 3.12. *The map $\mathcal{J}_{R \rtimes R_{\mathfrak{m},\Gamma}} \rightarrow \mathcal{J}_{R \rtimes R}$ given by $(x + \mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m},\Gamma}) \mapsto (x + \mathfrak{a}) \times \mathfrak{a}^{\times}$ and $\emptyset \mapsto \emptyset$ is an injective map of semilattices.*

Proof. The map $(x + \mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m},\Gamma}) \mapsto (x + \mathfrak{a}) \times \mathfrak{a}^{\times}$ is well-defined by Proposition 3.9, and it is not difficult to see that it is a map of semilattices. Injectivity follows from Proposition 3.4. \square

4. PRESENTATIONS FOR $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$.

Let K be a number field with ring of integers R . Also let \mathfrak{m} be a modulus for K , let $S := \{\mathfrak{p} \in \mathcal{P}_K : \mathfrak{p} \mid \mathfrak{m}_0\}$ be the *support* of \mathfrak{m}_0 , and let $\Gamma \subseteq (R/\mathfrak{m})^*$ be a subgroup. These will remain fixed throughout this section.

We begin with a short discussion of semigroup crossed products. Let P be a subsemigroup of a countable group G as in Section 2.1, and suppose that α is an action

of P on a unital C^* -algebra D by injective $*$ -endomorphisms. The triple (D, P, α) is called a *semigroup dynamical system*. A *covariant representation of (D, P, α)* in a unital C^* -algebra B is a pair (π, V) where $\pi : D \rightarrow B$ is a unital $*$ -homomorphism, and $V : P \rightarrow \text{Isom}(B)$ is a semigroup homomorphism satisfying the *covariance condition*

$$V_p \pi(d) V_p^* = \pi(\alpha_p(d)) \quad \text{for all } p \in P \text{ and } d \in D. \quad (2)$$

Here, $\text{Isom}(B)$ denotes the semigroup of isometries in B . Given a semigroup dynamical system (D, P, α) , the *semigroup crossed product $D \rtimes_\alpha P$* , as defined in [La-Rae1, Definition 2.2], is the universal unital C^* -algebra for covariant representations of (D, P, α) ; that is, $D \rtimes_\alpha P$ is a unital C^* -algebra, and there is a covariant representation (i_D, v) of (D, P, α) in $D \rtimes_\alpha P$ such that

- $D \rtimes_\alpha P = C^*(\{i_D(d) : d \in D\} \cup \{v_p : p \in P\})$;
- for any covariant representation (π, V) of (D, P, α) in a C^* -algebra B , there exists a representation $\pi \times V : D \rtimes_\alpha P \rightarrow B$ such that $(\pi \times V) \circ i_D = \pi$ and $(\pi \times V) \circ v = V$.

Following [Li2], we now show how to canonically associate a semigroup dynamical system with P . By definition, a *semilattice* is a commutative semigroup in which every element is an idempotent; the collection \mathcal{J}_P is a semilattice with semigroup operation given by intersection of subsets. The *C^* -algebra of \mathcal{J}_P* , as defined in [Li-Nor, Section 2], is the universal C^* -algebra $C_u^*(\mathcal{J}_P)$ generated by projections $\{e_X : X \in \mathcal{J}_P\}$ such that

$$e_\emptyset = 0 \quad \text{and} \quad e_X e_Y = e_{X \cap Y} \quad \text{for all } X, Y \in \mathcal{J}_P.$$

Note that $C_u^*(\mathcal{J}_P)$ is unital with unit e_P . Since the collection $\{e_X : X \in \mathcal{J}_P\}$ of generating projections is closed under multiplication, we have $C_u^*(\mathcal{J}_P) = \overline{\text{span}}(\{e_X : X \in \mathcal{J}_P\})$. The universal property of $C_u^*(\mathcal{J}_P)$ implies existence of a $*$ -homomorphism $C_u^*(\mathcal{J}_P) \rightarrow D_\lambda(P)$ determined on the spanning projections by $e_X \mapsto E_X$ where $E_X \in \mathcal{B}(\ell^2(P))$ is, as in Section 2.1, the orthogonal projection from $\ell^2(P)$ onto $\ell^2(X) \subseteq \ell^2(P)$. By [Li2, Proposition 2.24], this map is an isomorphism if and only if P satisfies the independence condition.

The semigroup P acts on the semilattice \mathcal{J}_P by left multiplication, $p : X \mapsto pX$, which gives rise to an action of P on the (commutative) C^* -algebra $C_u^*(\mathcal{J}_P)$ of the

semilattice \mathcal{J}_P by injective *-endomorphisms α_p that are determined on the generating projections by $\alpha_p(e_X) = e_{pX}$. Thus, we get the semigroup dynamical system $(C_u^*(\mathcal{J}_P), P, \alpha)$. From the definition of $C_u^*(\mathcal{J}_P)$ we see that the crossed product $C_u^*(\mathcal{J}_P) \rtimes_\alpha P$ is the universal C*-algebra generated by isometries $\{v_p : p \in P\}$ and projections $\{e_X : X \in \mathcal{J}_P\}$ such that

- (I) $v_p v_q = v_{pq}$ and $v_p e_X v_p^* = e_{pX}$ for all $p, q \in P$ and $X \in \mathcal{J}_P$;
- (II) $e_\emptyset = 0$, $e_P = 1$, and $e_X e_Y = e_{X \cap Y}$ for all $X, Y \in \mathcal{J}_P$.

This is precisely the presentation for the (full) semigroup C*-algebra $C^*(P)$ of P as given in [Li2, Definition 2.2], so $C^*(P) = C_u^*(\mathcal{J}_P) \rtimes_\alpha P$, see [Li2, Lemma 2.14].

Let $\lambda : p \mapsto \lambda_p \in \text{Isom}(C_\lambda^*(P))$ be the left regular representation of P , and let η be the canonical *-homomorphism $\eta : C_u^*(\mathcal{J}_P) \rightarrow D_\lambda(P)$ such that $\eta(e_X) = E_X$. Then the pair (η, λ) is a covariant representation of $(C_u^*(\mathcal{J}_P), P, \alpha)$ in $C_\lambda^*(P)$. The associated representation $C^*(P) \rightarrow C_\lambda^*(P)$ determined by $v_p \mapsto \lambda_p$ and $e_X \mapsto E_X$ is called the *left regular representation of $C^*(P)$* .

We now turn to the special case of $P_{\mathfrak{m}, \Gamma} := R \rtimes R_{\mathfrak{m}, \Gamma}$. First, note that by Proposition 3.4, the semigroup $P_{\mathfrak{m}, \Gamma}$ satisfies the independence condition, so [Li2, Proposition 2.24] asserts that the canonical *-homomorphism $C_u^*(\mathcal{J}_{P_{\mathfrak{m}, \Gamma}}) \rightarrow D_\lambda(P_{\mathfrak{m}, \Gamma})$ is an isomorphism.

Proposition 4.1. *The left regular representation $C^*(P_{\mathfrak{m}, \Gamma}) \rightarrow C_\lambda^*(P_{\mathfrak{m}, \Gamma})$ is an isomorphism.*

Proof. By Proposition 3.3, $P_{\mathfrak{m}, \Gamma}$ is left Ore with solvable, hence amenable, enveloping group $(R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma}$, and $P_{\mathfrak{m}, \Gamma}$ satisfies the independence condition by Proposition 3.4; hence, [Li2, Section 3.1] combined with [Li3, Theorem 6.1] implies our claim. \square

From now on, we will use Proposition 4.1 to identify $C^*(P_{\mathfrak{m}, \Gamma}) = C_u^*(\mathcal{J}_{P_{\mathfrak{m}, \Gamma}}) \rtimes P_{\mathfrak{m}, \Gamma}$ with $C_\lambda^*(P_{\mathfrak{m}, \Gamma})$. We also have:

Proposition 4.2. *The canonical inclusion of semilattices $\mathcal{J}_{P_{\mathfrak{m}, \Gamma}} \rightarrow \mathcal{J}_{R \rtimes R^\times}$ from Corollary 3.12 gives rise to an injective *-homomorphism $C_u^*(\mathcal{J}_{P_{\mathfrak{m}, \Gamma}}) \rightarrow C_u^*(\mathcal{J}_{R \rtimes R^\times})$*

such that $e_{(x+\mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m}, \Gamma})} \mapsto e_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}$. Moreover, this map is equivariant for the obvious $P_{\mathfrak{m}, \Gamma}$ -actions.

Proof. Existence of such a $*$ -homomorphism follows immediately from the universal property of $C_u^*(\mathcal{J}_{P_{\mathfrak{m}, \Gamma}})$. Equivariance is obvious, and injectivity follows Proposition 3.4 and [C-E-L-Y, Proposition 5.6.21]. \square

To avoid cumbersome notation, we will often identify $C_u^*(\mathcal{J}_{P_{\mathfrak{m}, \Gamma}})$ with its image in $C_u^*(\mathcal{J}_{R \rtimes R^\times})$ under the canonical inclusion from Proposition 4.2. Thus, we will write $e_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}$ rather than $e_{(x+\mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m}, \Gamma})}$ for a canonical spanning projection of $C_u^*(\mathcal{J}_{P_{\mathfrak{m}, \Gamma}})$.

Our next result gives a presentation for $C^*(P_{\mathfrak{m}, \Gamma})$ that is, for the particular case of trivial \mathfrak{m} , entirely analogous to the presentation given in [C-D-L, Definition 2.1], see also [Li2, Section 2.4].

Proposition 4.3. *For $x \in R$, let $u^x := v_{(x, 1)}$, for $a \in R_{\mathfrak{m}, \Gamma}$, let $s_a := v_{(0, a)}$, and for $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, let $e_{\mathfrak{a}} := e_{\mathfrak{a} \times (\mathfrak{a} \cap R_{\mathfrak{m}, \Gamma})}$. Then:*

- (Ta) *The u^x are unitary and satisfy $u^x u^y = u^{x+y}$, the s_a are isometries and satisfy $s_a s_b = s_{ab}$. Moreover, $s_a u^x = u^{ax} s_a$ for all $x, y \in R$ and $a, b \in R_{\mathfrak{m}, \Gamma}$.*
- (Tb) *The $e_{\mathfrak{a}}$ are projections and satisfy $e_{\mathfrak{a}} e_{\mathfrak{b}} = e_{\mathfrak{a} \cap \mathfrak{b}}$, $e_R = 1$.*
- (Tc) *We have $s_a e_{\mathfrak{b}} s_a^* = e_{\mathfrak{a} \cap \mathfrak{b}}$.*
- (Td) *For $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, $\begin{cases} u^x e_{\mathfrak{a}} = e_{\mathfrak{a}} u^x & \text{for } x \in \mathfrak{a}, \text{ and} \\ e_{\mathfrak{a}} u^x e_{\mathfrak{a}} = 0 & \text{for } x \notin \mathfrak{a}. \end{cases}$*

Moreover, $C^*(P_{\mathfrak{m}, \Gamma})$ is universal in the following sense: if B is a C^* -algebra containing elements U^x for $x \in R$, S_a for $a \in R_{\mathfrak{m}, \Gamma}$, and $E_{\mathfrak{a}}$ for $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ satisfying the obvious ‘‘uppercase’’ analogues of (Ta)–(Td), then there is a unique $*$ -homomorphism $C^*(P_{\mathfrak{m}, \Gamma}) \rightarrow B$ such that $u^x \mapsto U^x$, $s_a \mapsto S_a$, and $e_{\mathfrak{a}} \mapsto E_{\mathfrak{a}}$.

Proof. A calculation analogous to that given in [Li2, Section 2.4] shows that the relations (Ta)–(Td) are satisfied.

If $\{U^x : x \in R\}$, $\{S_a : a \in R_{\mathfrak{m}, \Gamma}\}$, and $\{E_{\mathfrak{a}} : \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\}$ are elements in a C^* -algebra B satisfying ‘‘uppercase’’ analogues of (Ta)–(Td), let $V_{(x, a)} := U^x S_a$ and

$E_{x+\mathfrak{a}} := U^x E_{\mathfrak{a}} U^{-x}$ for $x \in R$, $a \in R_{\mathfrak{m}, \Gamma}$, and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. A calculation verifies that these elements satisfying the defining relations (I) and (II) for $C^*(P_{\mathfrak{m}, \Gamma})$, so the universal property of $C^*(P_{\mathfrak{m}, \Gamma})$ gives us a $*$ -homomorphism $C^*(P_{\mathfrak{m}, \Gamma}) \rightarrow B$ such that $v_{(x,a)} \mapsto V_{(x,a)}$ and $e_{(x+\mathfrak{a}) \times \mathfrak{a} \times} \mapsto E_{x+\mathfrak{a}}$. \square

5. DESCRIPTION AS A FULL CORNER IN A CROSSED PRODUCT.

We will now describe $C^*(R \rtimes R_{\mathfrak{m}, \Gamma})$ as a full corner in a crossed product, and thus also as a groupoid C^* -algebra. Since $R \rtimes R_{\mathfrak{m}, \Gamma}$ is left Ore by Proposition 3.3, this could be derived from [La, Theorem 2.1.1]. However, for the present setting, the results from [Li3, Section 4] give us a concrete realization of the ‘‘dilated system’’ which will be more convenient for our purposes.

5.1. The Toeplitz condition. Let P be a subsemigroup of a group G as in Section 2.1. Let λ^G denote the left regular representation of G on $\ell^2(G)$, and for each subset $Y \subseteq G$, let $E_Y \in \mathcal{B}(\ell^2(G))$ be the corresponding multiplication operator, that is, E_Y is the orthogonal projection onto $\ell^2(Y) \subseteq \ell^2(G)$. Let $\mathcal{J}_{P \subseteq G}$ be the smallest collection of subsets of G that contains \mathcal{J}_P , is closed under left translation by elements in G , and is closed under finite intersections. Let $D_{P \subseteq G} := \overline{\text{span}}(\{E_X : X \in \mathcal{J}_{P \subseteq G}\})$. Then $D_{P \subseteq G}$ is a sub- C^* -algebra of $\ell^\infty(G)$, and, as explained in [C-E-L1, Section 2.5], we can identify $D_{P \subseteq G} \rtimes_r G$ with the sub- C^* -algebra of $\mathcal{B}(\ell^2(G))$ given by $\overline{\text{span}}(\{E_Y \lambda_g^G : Y \in \mathcal{J}_{P \subseteq G}, g \in G\})$. By [Li3, Lemma 3.8], the projection E_P is full in $D_{P \subseteq G} \rtimes_r G$. We always have the containment $C_\lambda^*(P) \subseteq E_P(D_{P \subseteq G} \rtimes_r G)E_P$, where we view $C_\lambda^*(P)$ as a sub- C^* -algebra of $\mathcal{B}(\ell^2(G))$. The reverse containment need not hold in general. By [Li3, Definition 4.1], the inclusion $P \subseteq G$ satisfies the *left Toeplitz condition* provided that for each $g \in G$, the compression $E_P \lambda_g^G E_P$ of λ_g^G by E_P is either zero or of the form $E_P \lambda_g^G E_P = \lambda_{p_1}^* \lambda_{q_1} \cdots \lambda_{p_n}^* \lambda_{q_n}$ for some $p_1, q_1, \dots, p_n, q_n \in P$. If $P \subseteq G$ satisfies the left Toeplitz condition, then [Li3, Lemma 3.9] guarantees that $C_\lambda^*(P) = E_P(D_{P \subseteq G} \rtimes_r G)E_P$.

Now assume that $P \subseteq G$ satisfies the left Toeplitz condition, and let $\Omega_{P \subseteq G} := \text{Spec}(D_{P \subseteq G})$. By [Li3, Lemma 4.2(i)], we have $D_P = E_P D_{P \subseteq G} E_P$, so there is a canonical inclusion $\Omega_P \subseteq \Omega_{P \subseteq G}$, and $E_P(D_{P \subseteq G} \rtimes_r G)E_P \cong 1_{\Omega_P}(C_0(\Omega_{P \subseteq G}) \rtimes_r$

G) 1_{Ω_P} . We also have

$$1_{\Omega_P}(C_0(\Omega_{P \subseteq G} \rtimes_r G))1_{\Omega_P} \cong C_r^*(G \times \Omega_P)$$

where $G \times \Omega_P := \{(g, w) \in G \times \Omega_P : gw \in \Omega_P\}$ is the reduction of the transformation groupoid $G \times \Omega_{P \subseteq G}$ by the compact open set Ω_P . Our notation for the reduction groupoid is justified by the fact that the groupoid $G \times \Omega_P$ can be canonically identified with the transformation groupoid for a canonical *partial action* of G on Ω_P , see [Li7, Section 2].

We now return to the case of $R \rtimes R_{\mathfrak{m}, \Gamma} \subseteq (R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma}$. To avoid cumbersome notation, we let $P_{\mathfrak{m}, \Gamma} := R \rtimes R_{\mathfrak{m}, \Gamma}$ and $G_{\mathfrak{m}, \Gamma} := (R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma}$. Since $P_{\mathfrak{m}, \Gamma}$ is left Ore by Proposition 3.3, the inclusion $P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}$ satisfies the left Toeplitz condition by [Li3, Section 8.3]. From the discussion above, we have isomorphisms

$$C_\lambda^*(P_{\mathfrak{m}, \Gamma}) \cong 1_{\Omega_{P_{\mathfrak{m}, \Gamma}}} (C_0(\Omega_{P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}}) \rtimes_r G_{\mathfrak{m}, \Gamma}) 1_{\Omega_{P_{\mathfrak{m}, \Gamma}}} \cong C_r^*(G_{\mathfrak{m}, \Gamma} \times \Omega_{P_{\mathfrak{m}, \Gamma}}).$$

Our aim now is to describe the diagonal sub-C*-algebra $D_{P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}} \cong C_0(\Omega_{P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}})$.

Proposition 5.1. *We have $\mathcal{J}_{P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}} = \{(x + \mathfrak{a}) \times \mathfrak{a}^\times : x \in K, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}\} \cup \{\emptyset\}$.*

Proof. Since $P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}$ is left Toeplitz, [Li3, Lemma 4.2] implies that $\mathcal{J}_{P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}} = \{gX : g \in G, X \in \mathcal{J}_{P_{\mathfrak{m}, \Gamma}}\}$. Hence, $\mathcal{J}_{P_{\mathfrak{m}, \Gamma} \subseteq G_{\mathfrak{m}, \Gamma}} = \{(y + \mathfrak{a}) \times \mathfrak{a}^\times : y \in R_{\mathfrak{m}}^{-1}R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}\} \cup \{\emptyset\}$, so the inclusion “ \subseteq ” holds.

To prove the reverse inclusion, let $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}$ and $y \in K$. We need to find $x \in R_{\mathfrak{m}}^{-1}R$ such that $x + \mathfrak{a} = y + \mathfrak{a}$. By strong approximation ([Nar, Theorem 6.28]), there exists $x \in K$ such that

- $v_{\mathfrak{p}}(x - y) \geq v_{\mathfrak{p}}(\mathfrak{a})$ for all $\mathfrak{p} \mid \mathfrak{a}$;
- $v_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \mid \mathfrak{m}_0$.

That is, $x + \mathfrak{a} = y + \mathfrak{a}$ and x is integral at every prime that divides \mathfrak{m}_0 . Write $xR = \mathfrak{b}/\mathfrak{c}$ where \mathfrak{b} and \mathfrak{c} are coprime integral ideals. Then, because $v_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \mid \mathfrak{m}_0$, \mathfrak{c} is coprime to \mathfrak{m}_0 and thus defines a class $[\mathfrak{c}]$ in $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}})$; let \mathfrak{d} be an integral ideal in the inverse class $[\mathfrak{c}]^{-1}$, so that $\mathfrak{c}\mathfrak{d} = bR$ for some $b \in R_{\mathfrak{m}}$. The class of \mathfrak{d} in $\text{Cl}(K)$ coincides with the inverse of the class of \mathfrak{c} in $\text{Cl}(K)$, and \mathfrak{b} and \mathfrak{c} are in the same ideal class in $\text{Cl}(K)$, so there exists $a \in R$ such that $\mathfrak{b}\mathfrak{d} = aR$. Now we

have

$$xR = \mathfrak{b}/\mathfrak{c} = \mathfrak{b}\mathfrak{d}/\mathfrak{c}\mathfrak{d} = aR/bR = (a/b)R,$$

so $x = au/b$ for some $u \in R^*$ which shows that $x \in R_{\mathfrak{m}}^{-1}R$. Since $x + \mathfrak{a} = y + \mathfrak{a}$, we are done. \square

5.2. An adelic description of the spectrum of the diagonal. We will now describe $C(\Omega_{P_{\mathfrak{m},\Gamma}})$ and $C_0(\Omega_{P_{\mathfrak{m},\Gamma} \subseteq G_{\mathfrak{m},\Gamma}})$ as functions on certain adelic spaces; this is motivated by [La-Nesh, Section 1] and [C-D-L, Section 5], also see [Li4, Section 2].

Each non-zero prime ideal \mathfrak{p} of R defines a normalized *absolute value* $|\cdot|_{\mathfrak{p}}$ on K^\times ; explicitly, $|x|_{\mathfrak{p}} := N(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}$. We let $K_{\mathfrak{p}}$ denote the corresponding completion of K and $R_{\mathfrak{p}} = \{x \in K_{\mathfrak{p}} : |x|_{\mathfrak{p}} \leq 1\}$ the ring of integers in $K_{\mathfrak{p}}$. The *ring of finite adeles over K* is

$$\mathbb{A}_f := \left\{ \mathfrak{a} = (a_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}} K_{\mathfrak{p}} : a_{\mathfrak{p}} \in R_{\mathfrak{p}} \text{ for all but finitely many } \mathfrak{p} \right\}.$$

Equipped with the restricted product topology with respect to the compact open subsets $R_{\mathfrak{p}} \subseteq K_{\mathfrak{p}}$, \mathbb{A}_f is a locally compact ring. Let \hat{R} denote the compact subring $\prod_{\mathfrak{p}} R_{\mathfrak{p}}$ consisting of *integral adeles*. We can modify this definition to work with only the primes not dividing \mathfrak{m} . Let $S := \{\mathfrak{p} \in \mathcal{P}_K : \mathfrak{p} \mid \mathfrak{m}_0\}$ be the support of \mathfrak{m}_0 , and put

$$\mathbb{A}_S := \left\{ \mathfrak{a} = (a_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p} \notin S} K_{\mathfrak{p}} : a_{\mathfrak{p}} \in R_{\mathfrak{p}} \text{ for all but finitely many } \mathfrak{p} \right\}.$$

Also equip \mathbb{A}_S with the restricted product topology. Denote by \hat{R}_S the compact subring $\prod_{\mathfrak{p} \notin S} R_{\mathfrak{p}}$ of \mathbb{A}_S , and let $\hat{R}_S^* := \prod_{\mathfrak{p} \notin S} R_{\mathfrak{p}}^*$ be the group of units in \hat{R}_S . The compact group \hat{R}_S^* acts on \mathbb{A}_S by multiplication, and we will let $\bar{\mathfrak{a}}$ denote the image of $\mathfrak{a} \in \mathbb{A}_S$ under the quotient map $\mathbb{A}_S \rightarrow \mathbb{A}_S/\hat{R}_S^*$. There is a diagonal embedding of additive groups $K \hookrightarrow \mathbb{A}_S$, so K acts on \mathbb{A}_S by translation. Moreover, the image of $K_{\mathfrak{m},\Gamma}$ under this embedding is contained in the multiplicative group \mathbb{A}_S^* of units in \mathbb{A}_S , so $K_{\mathfrak{m},\Gamma}$ acts on \mathbb{A}_S by multiplication. This action descends to an action of $K_{\mathfrak{m},\Gamma}$ on the quotient \mathbb{A}_S/\hat{R}_S^* given by $k\bar{\mathfrak{a}} = \overline{k\mathfrak{a}}$. Hence, the locally compact space $\mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*$ carries a canonical action of $G_{\mathfrak{m},\Gamma}$ given by $(n, k)(\mathfrak{b}, \bar{\mathfrak{a}}) = (n + k\mathfrak{b}, k\bar{\mathfrak{a}})$.

Remark 5.2. The space \hat{R}_S/\hat{R}_S^* can be canonically identified with $\prod_{\mathfrak{p} \notin S} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}}$, which may be thought of as the space of “super ideals coprime to \mathfrak{m}_0 ”, and we can

identify \mathcal{I}_m^+ with its canonical image in \hat{R}_S/\hat{R}_S^* via $\mathbf{a} \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathbf{a})}$. Similarly, \mathbb{A}_S/\hat{R}_S^* may be thought of as the space of “super fractional ideals coprime to \mathfrak{m}_0 ”.

We define an equivalence relation on $\mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*$ by $(\mathbf{b}, \bar{\mathbf{a}}) \sim (\mathbf{d}, \bar{\mathbf{c}})$ if $\bar{\mathbf{a}} = \bar{\mathbf{c}}$ and $\mathbf{b} - \mathbf{d} \in \bar{\mathbf{a}}\hat{R}_S$. The action of $G_{\mathfrak{m},\Gamma}$ descends to a well-defined action on the locally compact quotient space

$$\Omega_K^{\mathfrak{m}} := (\mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*)/\sim.$$

This equivalence relation restricts to an equivalence relation on the compact subset $\hat{R}_S \times \hat{R}_S/\hat{R}_S^* \subseteq \mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*$, and the quotient space

$$\Omega_R^{\mathfrak{m}} := (\hat{R}_S \times \hat{R}_S/\hat{R}_S^*)/\sim$$

is a compact subset of $\Omega_K^{\mathfrak{m}}$.

Proposition 5.3. *There are $G_{\mathfrak{m},\Gamma}$ -equivariant isomorphisms $D_{P_{\mathfrak{m},\Gamma}} \cong C(\Omega_R^{\mathfrak{m}})$ and $D_{P_{\mathfrak{m},\Gamma} \subseteq G_{\mathfrak{m},\Gamma}} \cong C_0(\Omega_K^{\mathfrak{m}})$ such that the following diagram commutes*

$$\begin{array}{ccc} D_{P_{\mathfrak{m},\Gamma}} & \hookrightarrow & D_{P_{\mathfrak{m},\Gamma} \subseteq G_{\mathfrak{m},\Gamma}} \\ \downarrow \cong & & \downarrow \cong \\ C(\Omega_R^{\mathfrak{m}}) & \hookrightarrow & C_0(\Omega_K^{\mathfrak{m}}) \end{array}$$

where the horizontal arrows are the canonical inclusions, and the vertical arrows are determined by

$$e_{(x+\mathbf{a}) \times \mathbf{a}^\times} \mapsto \mathbf{1}_{\{[\mathbf{b}, \bar{\mathbf{a}}]: v_{\mathfrak{p}}(\bar{\mathbf{a}}) \geq v_{\mathfrak{p}}(\mathbf{a}) \text{ and } v_{\mathfrak{p}}(\mathbf{b}-x) \geq v_{\mathfrak{p}}(\mathbf{a}) \text{ for all } \mathfrak{p} \notin S\}}.$$

Proof. From Lemma 5.1, we have $\mathcal{J}_{P_{\mathfrak{m},\Gamma} \subseteq G_{\mathfrak{m},\Gamma}} = \{(x+\mathbf{a}) \times \mathbf{a}^\times : x \in K, \mathbf{a} \in \mathcal{I}_m\} \cup \{\emptyset\}$. When \mathfrak{m} is trivial, the result follows from the analysis in [Li4, Section 2], and the general case goes through almost verbatim. \square

An immediate consequence, we have isomorphisms

$$C_\lambda^*(P_{\mathfrak{m},\Gamma}) \cong 1_{\Omega_R^{\mathfrak{m}}}(C_0(\Omega_K^{\mathfrak{m}}) \rtimes_r G_{\mathfrak{m},\Gamma}) 1_{\Omega_R^{\mathfrak{m}}} \cong C_\lambda^*(G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}}) \quad (3)$$

where $G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}} = \{(g, w) \in G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}} : gw \in \Omega_R^{\mathfrak{m}}\}$ is the reduction groupoid of the transformation groupoid $G_{\mathfrak{m},\Gamma} \rtimes \Omega_K^{\mathfrak{m}}$ with respect to the compact open set $\Omega_R^{\mathfrak{m}}$.

Proposition 5.4. *There is an isomorphism*

$$\vartheta : C^*(P_{\mathfrak{m},\Gamma}) \cong C^*(G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}})$$

that is determined on generators by $\vartheta(v_{(b,a)}) = 1_{\{(b,a)\} \times \Omega_R^m}$ for $(b, a) \in P_{m,\Gamma}$.

Proof. Since $G_{m,\Gamma}$ is amenable, there is a canonical isomorphism $C^*(G_{m,\Gamma} \rtimes \Omega_{P_{m,\Gamma}}) \cong C_r^*(G_{m,\Gamma} \rtimes \Omega_{P_{m,\Gamma}})$. Hence, the result follows from Proposition 4.1 combined with (3). \square

6. FAITHFUL REPRESENTATIONS OF $C^*(R \rtimes R_{m,\Gamma})$.

6.1. A criterion for faithfulness. As before, we will use the notation $P_{m,\Gamma} := R \rtimes R_{m,\Gamma}$ and $G_{m,\Gamma} := (R_m^{-1}R) \rtimes K_{m,\Gamma}$. Also let $S := \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{m}_0\}$ be the support of \mathfrak{m}_0 and put $\mathcal{P}_K^m := \mathcal{P}_K \setminus S$.

Following the approach of [La-Rae1, Theorem 3.7], we next establish a faithfulness criterion for representations of $C^*(P_{m,\Gamma})$ in terms of spanning projections of the diagonal.

Theorem 6.1. *For each class $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, choose an integral ideal $\mathfrak{a}_\mathfrak{k} \in \mathfrak{k}$. Suppose ψ is a representation of $C^*(P_{m,\Gamma})$ in a C^* -algebra B . Then ψ is injective if and only if for each $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, we have*

$$\psi \left(\prod_{i=1}^m (e_{\mathfrak{a}_\mathfrak{k} \times \mathfrak{a}_\mathfrak{k}^\times} - e_{(y_i + \mathfrak{a}_i) \times \mathfrak{a}_i^\times}) \right) \neq 0$$

for all $y_1, \dots, y_m \in R$ and $\mathfrak{a}_1, \dots, \mathfrak{a}_m \in \mathcal{I}_m^+$ such that $y_i + \mathfrak{a}_i \subsetneq \mathfrak{a}_\mathfrak{k}$ for $1 \leq i \leq m$.

We need a preliminary result.

Proposition 6.2. *A representation ψ of $C^*(P_{m,\Gamma})$ is faithful if and only if it is faithful on $C_u^*(\mathcal{J}_{P_{m,\Gamma}})$.*

Proof. Since the isomorphism $C^*(P_{m,\Gamma}) \cong C^*(G_{m,\Gamma} \rtimes \Omega_R^m)$ from Proposition 5.4 carries $C_u^*(\mathcal{J}_{P_{m,\Gamma}})$ isomorphically onto $C(\Omega_R^m)$, it suffices to prove that a representation ψ of the C^* -algebra $C^*(G_{m,\Gamma} \rtimes \Omega_R^m)$ is faithful if and only if it is faithful on $C(\Omega_R^m)$.

Since $G_{m,\Gamma} \rtimes \Omega_R^m$ is amenable, by [Exel, 4.4 Theorem], it suffices to show that $G_{m,\Gamma} \rtimes \Omega_R^m$ is essentially principal; in the terminology from [Exel], this means that we need to show that the interior of the isotropy bundle of $G_{m,\Gamma} \rtimes \Omega_R^m$ coincides with the

unit space of $G \times \Omega_R^m$. For this, it suffices to show that the set of points in Ω_R^m with trivial isotropy is dense in Ω_R^m ; this is a special case of the subsequent result. \square

For each $w \in \Omega_R^m$, let $G_{m,\Gamma}.w := \{gw : (g, w) \in \Omega_R^m\}$ be the orbit of w ; its closure $\overline{G_{m,\Gamma}.w}$ is called the *quasi-orbit* of w . The following proposition is more than we need; its full strength will be used in Section 7 below.

Proposition 6.3 (cf. [Ech-La, Lemmas 3.1, 3.4 and Corollary 3.5]). *For $\bar{\mathbf{a}} \in \hat{R}_S/\hat{R}_S^*$, let $Z(\bar{\mathbf{a}}) := \{\mathfrak{p} \in \mathcal{P}_K^m : \bar{\mathbf{a}}_{\mathfrak{p}} = 0\}$, and for each set $A \subseteq \mathcal{P}_K^m$, let $C_A := \{[\mathbf{b}, \bar{\mathbf{a}}] \in \Omega_R^m : A \subseteq Z(\bar{\mathbf{a}})\}$. Then*

- (1) *the quasi-orbit of a point $[\mathbf{b}, \bar{\mathbf{a}}] \in \Omega_R^m$ is equal to $C_{Z(\bar{\mathbf{a}})}$;*
- (2) *for any closed $G_{m,\Gamma}$ -invariant subset $C \subseteq \Omega_R^m$, the set of points in C with trivial isotropy is dense in C .*

In particular, the set of points in Ω_R^m with trivial isotropy is dense in Ω_R^m .

Proof. The proof of the first part is similar to the proof of [Ech-La, Lemma 3.1], but differs in a few places, so we include it here.

Clearly, we have $[\mathbf{b}, \bar{\mathbf{a}}] \in C_{Z(\bar{\mathbf{a}})}$. Since $C_{Z(\bar{\mathbf{a}})}$ is closed and $G_{m,\Gamma}$ -invariant, it follows that the quasi-orbit of $[\mathbf{b}, \bar{\mathbf{a}}]$ is contained in $C_{Z(\bar{\mathbf{a}})}$. Thus, we only need to show that $C_{Z(\bar{\mathbf{a}})}$ is contained in the quasi-orbit of $[\mathbf{b}, \bar{\mathbf{a}}]$. Let $[\mathbf{d}, \bar{\mathbf{c}}] \in C_{Z(\bar{\mathbf{a}})}$. Any open set containing $[\mathbf{d}, \bar{\mathbf{c}}]$ contains the image under the quotient map $\pi : \hat{R}_S \times \hat{R}_S/\hat{R}_S^* \rightarrow \Omega_R^m$ of an (open) set $W_1 \times W_2$ where $W_1 \subseteq \hat{R}_S$ is an open set of the form

$$W_1 = \{\mathbf{e} \in \hat{R}_S : v_{\mathfrak{p}}(\mathbf{e} - \mathbf{d}) \geq v_{\mathfrak{p}}(\mathbf{a}) \text{ for all } \mathfrak{p} \in \mathcal{P}_K^m\}$$

for some integral ideal $\mathbf{a} \in \mathcal{I}_m^+$, and $W_2 \subseteq \hat{R}_S/\hat{R}_S^*$ is an open set of the form

$$W_2 = \{\bar{\mathbf{e}} \in \hat{R}_S/\hat{R}_S^* : v_{\mathfrak{p}}(\bar{\mathbf{e}}) = v_{\mathfrak{p}}(\bar{\mathbf{c}}) \text{ for } \mathfrak{p} \in F \setminus Z(\bar{\mathbf{c}}) \text{ and } v_{\mathfrak{p}}(\bar{\mathbf{e}}) \geq n_{\mathfrak{p}} \text{ for } \mathfrak{p} \in F \cap Z(\bar{\mathbf{c}})\}$$

for some finite set $F \subseteq \mathcal{P}_K^m$ and non-negative integers $n_{\mathfrak{p}}$ for $\mathfrak{p} \in F \cap Z(\bar{\mathbf{c}})$. By Lemma 3.5, we can find $b \in R_{m,1}$ such that $v_{\mathfrak{p}}(b) = v_{\mathfrak{p}}(\bar{\mathbf{a}})$ for $\mathfrak{p} \in F \setminus Z(\bar{\mathbf{c}})$. Now use Lemma 3.5 again to choose $a \in R_{m,1}$ such that

- $v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(\bar{\mathbf{c}})$ for $\mathfrak{p} \in F \setminus Z(\bar{\mathbf{c}})$;
- $v_{\mathfrak{p}}(a) = n_{\mathfrak{p}} + v_{\mathfrak{p}}(b)$ for $\mathfrak{p} \in F \cap Z(\bar{\mathbf{c}})$;
- $v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(b)$ for $\mathfrak{p} \in F^c$ with $v_{\mathfrak{p}}(b) > 0$.

Let $k := a/b$. Then $k \in K_{m,1}$, $k\bar{\mathbf{a}} \in \hat{R}_S/\hat{R}_S^*$, and $k\bar{\mathbf{a}} \in W_2$. By strong approximation ([Nar, Theorem 6.28]), K is dense in \mathbb{A}_S , so there exists $y \in K$ such that $y+k\mathbf{b} \in W_1$. As in the proof of Lemma 5.1, we can find $x \in R_m^{-1}R$ such that $x - y \in \mathbf{a}$. Then $x + k\mathbf{b} \in W_1$, so we have that $(x, k)[\mathbf{b}, \bar{\mathbf{a}}] \subseteq \pi(W_1 \times W_2)$. Hence, $C_{Z(\bar{\mathbf{a}})}$ is contained in the quasi-orbit of $[\mathbf{b}, \bar{\mathbf{a}}]$.

An argument analogous to that given in the proof of [Ech-La, Lemma 3.4] now shows that for any $A \subseteq \mathcal{P}_K^m$, there exists $[\mathbf{d}, \bar{\mathbf{c}}] \in C_A$ such that the isotropy group of $[\mathbf{d}, \bar{\mathbf{c}}]$ is trivial and $Z(\bar{\mathbf{c}}) = A$. This implies part (2), so we are done. \square

We are now ready for the proof of Theorem 6.1.

Proof of Theorem 6.1. By Proposition 6.2, it suffices to prove that the restriction of ψ to $C_u^*(\mathcal{J}_{P_{m,\Gamma}})$ is injective. For this, by [C-E-L-Y, Proposition 5.6.21], it is enough to show that

$$\psi \left(e_{(y+\mathbf{a}) \times \mathbf{a}^\times} - \bigvee_{i=1}^m e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times} \right) = \psi \left(\prod_{i=1}^m (e_{(y+\mathbf{a}) \times \mathbf{a}^\times} - e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times}) \right) \neq 0 \quad (4)$$

for $y, y_1, \dots, y_m \in R$, $\mathbf{a}, \mathbf{a}_1, \dots, \mathbf{a}_m \in \mathcal{I}_m^+$ such that $y_i + \mathbf{a}_i \subsetneq y + \mathbf{a}$ for $1 \leq i \leq m$. Here, $\bigvee_{i=1}^m e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times}$ is the smallest projection in $C_u^*(\mathcal{J}_{P_{m,\Gamma}})$ that dominates each $e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times}$, see [C-E-L-Y, Lemma 5.6.20].

We will exploit the covariance condition, see (2). For each $(b, a) \in P_{m,\Gamma}$, let $W_{(b,a)} := \psi(v_{(b,a)})$, and observe that

$$\begin{aligned} W_{(y,1)}^* \psi \left(\prod_{i=1}^m (e_{(y+\mathbf{a}) \times \mathbf{a}^\times} - e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times}) \right) W_{(y,1)} &= \psi \left(\prod_{i=1}^m (v_{(y,1)}^* (e_{(y+\mathbf{a}) \times \mathbf{a}^\times} - e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times}) v_{(y,1)}) \right) \\ &= \psi \left(\prod_{i=1}^m (v_{(y,1)}^* e_{(y+\mathbf{a}) \times \mathbf{a}^\times} v_{(y,1)} - v_{(y,1)}^* e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times} v_{(y,1)}) \right) \\ &= \psi \left(\prod_{i=1}^m (e_{\mathbf{a} \times \mathbf{a}^\times} - e_{(y_i-y+\mathbf{a}_i) \times \mathbf{a}^\times}) \right). \end{aligned}$$

Since $W_{(y,1)}$ is a unitary, it follows that $\psi(\prod_{i=1}^m (e_{(y+\mathbf{a}) \times \mathbf{a}^\times} - e_{(y_i+\mathbf{a}_i) \times \mathbf{a}^\times}))$ is non-zero if and only if $\psi(\prod_{i=1}^m (e_{\mathbf{a} \times \mathbf{a}^\times} - e_{(y_i-y+\mathbf{a}_i) \times \mathbf{a}^\times}))$ is non-zero; hence, it is enough to show that (4) holds when $y = 0$.

Let $y_1, \dots, y_m \in R$ and $\mathfrak{a}, \mathfrak{a}_1, \dots, \mathfrak{a}_m \in \mathcal{I}_m^+$ be such that $y_i + \mathfrak{a}_i \subsetneq \mathfrak{a}$ for $1 \leq i \leq m$. If $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$ is the class containing \mathfrak{a} , then there exists $a, b \in R_{m,\Gamma}$ such that $a\mathfrak{a} = b\mathfrak{a}_\mathfrak{k}$. We have

$$\begin{aligned} W_{(0,a)}\psi(e_{\mathfrak{a} \times \mathfrak{a}^\times})W_{(0,a)}^* &= \psi(v_{(0,a)}e_{\mathfrak{a} \times \mathfrak{a}^\times}v_{(0,a)}^*) \\ &= \psi(e_{a\mathfrak{a} \times (a\mathfrak{a})^\times}) \\ &= \psi(e_{b\mathfrak{a}_\mathfrak{k} \times (b\mathfrak{a}_\mathfrak{k})^\times}) \\ &= \psi(v_{(0,b)}e_{\mathfrak{a}_\mathfrak{k} \times \mathfrak{a}_\mathfrak{k}^\times}v_{(0,b)}^*) \\ &= W_{(0,b)}\psi(e_{\mathfrak{a}_\mathfrak{k} \times \mathfrak{a}_\mathfrak{k}^\times})W_{(0,b)}^*. \end{aligned}$$

Now, $y_i + \mathfrak{a}_i \subseteq \mathfrak{a}$ implies that $ay_i + a\mathfrak{a}_i \subseteq a\mathfrak{a} = b\mathfrak{a}_\mathfrak{k}$. Hence, there exists $\tilde{y}_i \in \mathfrak{a}_\mathfrak{k}$ such that $ay_i = b\tilde{y}_i$. From this, we see that $a\mathfrak{a}_i \subseteq b(\mathfrak{a}_\mathfrak{k} - \tilde{y}_i)$ which implies that $\tilde{\mathfrak{a}}_i := \frac{a}{b}\mathfrak{a}_i$ is an integral ideal. Since $a, b \in R_{m,\Gamma}$, we see also that $\tilde{\mathfrak{a}}_i$ is coprime to \mathfrak{m}_0 , so that $\tilde{\mathfrak{a}}_i$ lies in \mathcal{I}_m^+ . Since $a(y_i + \mathfrak{a}_i) = b(\tilde{y}_i + \tilde{\mathfrak{a}}_i)$, we have

$$\begin{aligned} W_{(0,a)}\psi(e_{(y_i + \mathfrak{a}_i) \times \mathfrak{a}_i^\times})W_{(0,a)}^* &= \psi(e_{a(y_i + \mathfrak{a}_i) \times (a\mathfrak{a}_i)^\times}) \\ &= \psi(e_{b(\tilde{y}_i + \tilde{\mathfrak{a}}_i) \times (b\tilde{\mathfrak{a}}_i)^\times}) \\ &= W_{(0,b)}\psi(e_{(\tilde{y}_i + \tilde{\mathfrak{a}}_i) \times \tilde{\mathfrak{a}}_i^\times})W_{(0,b)}^*. \end{aligned}$$

Conjugating by an isometry defines an injective *-homomorphism, so

$$\begin{aligned} \psi\left(\prod_{i=1}^n (e_{\mathfrak{a} \times \mathfrak{a}^\times} - e_{(y_i + \mathfrak{a}_i) \times \mathfrak{a}_i^\times})\right) = 0 &\iff W_{(0,a)}\psi\left(\prod_{i=1}^n (e_{\mathfrak{a} \times \mathfrak{a}^\times} - e_{(y_i + \mathfrak{a}_i) \times \mathfrak{a}_i^\times})\right)W_{(0,a)}^* = 0 \\ &\iff \psi\left(\prod_{i=1}^n (v_{(0,a)}e_{\mathfrak{a} \times \mathfrak{a}^\times}v_{(0,a)}^* - v_{(0,a)}e_{(y_i + \mathfrak{a}_i) \times \mathfrak{a}_i^\times}v_{(0,a)}^*)\right) = 0 \\ &\iff \psi\left(\prod_{i=1}^n (v_{(0,b)}e_{\mathfrak{a}_\mathfrak{k} \times \mathfrak{a}_\mathfrak{k}^\times}v_{(0,b)}^* - v_{(0,b)}e_{(\tilde{y}_i + \tilde{\mathfrak{a}}_i) \times \tilde{\mathfrak{a}}_i^\times}v_{(0,b)}^*)\right) = 0 \\ &\iff W_{(0,b)}\psi\left(\prod_{i=1}^n (e_{\mathfrak{a}_\mathfrak{k} \times \mathfrak{a}_\mathfrak{k}^\times} - e_{(\tilde{y}_i + \tilde{\mathfrak{a}}_i) \times \tilde{\mathfrak{a}}_i^\times})\right)W_{(0,b)}^* = 0 \\ &\iff \psi\left(\prod_{i=1}^n (e_{\mathfrak{a}_\mathfrak{k} \times \mathfrak{a}_\mathfrak{k}^\times} - e_{(\tilde{y}_i + \tilde{\mathfrak{a}}_i) \times \tilde{\mathfrak{a}}_i^\times})\right) = 0. \end{aligned}$$

Since $\psi\left(\prod_{i=1}^n (e_{\mathfrak{a}_\mathfrak{k} \times \mathfrak{a}_\mathfrak{k}^\times} - e_{(\tilde{y}_i + \tilde{\mathfrak{a}}_i) \times \tilde{\mathfrak{a}}_i^\times})\right)$ is non-zero by assumption, $\psi\left(\prod_{i=1}^n (e_{\mathfrak{a} \times \mathfrak{a}^\times} - e_{(y_i + \mathfrak{a}_i) \times \mathfrak{a}_i^\times})\right)$ must also be non-zero. Hence, ψ is injective on $C_u^*(\mathcal{J}_{P_{m,\Gamma}})$ as desired. \square

As an immediate consequence, we obtain the following reformulation.

Corollary 6.4. *Suppose that B is a C^* -algebra containing elements U^x for $x \in R$, S_a for $a \in R_{m,\Gamma}$, and $E_{\mathbf{a}}$ for $\mathbf{a} \in \mathcal{I}_m^+$ satisfying the “uppercase” analogues of (Ta)–(Td) from Proposition 4.3, and let $\psi : C^*(P_{m,\Gamma}) \rightarrow B$ be the unique $*$ -homomorphism such that $\psi(U^x) = U^x$, $\psi(S_a) = S_a$, and $\psi(E_{\mathbf{a}}) = E_{\mathbf{a}}$. Then ψ is an isomorphism onto the sub- C^* -algebra of B generated by $\{U^x x \in R\}$, $\{S_a : a \in R_{m,\Gamma}\}$, and $\{E_{\mathbf{a}} : \mathbf{a} \in \mathcal{I}_m^+\}$ if and only if for each $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, we have*

$$\prod_{i=1}^m (E_{\mathbf{a}_i} - U^{y_i} E_{\mathbf{a}_i} U^{-y_i}) \neq 0$$

for all $y_1, \dots, y_m \in R$ and $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathcal{I}_m^+$ such that $y_i + \mathbf{a}_i \subsetneq \mathbf{a}_i$ for $1 \leq i \leq m$.

Thus, Theorem 6.1 may be viewed as a uniqueness result, analogous to a Cuntz-Krieger uniqueness theorem.

6.2. Representations coming from ideal classes. Using the inclusion from Corollary 3.12, we will view $\mathcal{J}_{P_{m,\Gamma}}$ as a subsemilattice of $\mathcal{J}_{R \rtimes R^\times}$. The canonical action of $P_{m,\Gamma}$ on $\mathcal{J}_{P_{m,\Gamma}}^\times$ given by $(b, a)[(x + \mathbf{a}) \times \mathbf{a}^\times] = (b + ax + a\mathbf{a}) \times (a\mathbf{a})^\times$ gives rise to an isometric representation V of $P_{m,\Gamma}$ on the Hilbert space $\mathcal{H} := \ell^2(\mathcal{J}_{P_{m,\Gamma}}^\times)$; namely, $V : P_{m,\Gamma} \rightarrow \text{Isom}(\mathcal{H})$ is determined on the canonical orthonormal basis by $V_{(b,a)} \delta_{(x+\mathbf{a}) \times \mathbf{a}^\times} = \delta_{(b+ax+a\mathbf{a}) \times (a\mathbf{a})^\times}$.

Proposition 6.5. *For each class $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, the subspace $\mathcal{H}_{\mathfrak{k}} := \overline{\text{span}}(\{\delta_{(z+\mathbf{b}) \times \mathbf{b}^\times} : \mathbf{b} \in \mathfrak{k}\}) \subseteq \mathcal{H}$ is invariant under $V_{(b,a)}$ for all $(b, a) \in P_{m,\Gamma}$. Let $V_{(b,a)}^{\mathfrak{k}}$ be the restriction of $V_{(b,a)}$ to $\mathcal{H}_{\mathfrak{k}}$. For $x \in R$ and $\mathbf{a} \in \mathcal{I}_m^+$, let $P_{x+\mathbf{a}}^{\mathfrak{k}}$ be the orthogonal projection from $\mathcal{H}_{\mathfrak{k}}$ onto the subspace $\overline{\text{span}}(\{\delta_{(z+\mathbf{b}) \times \mathbf{b}^\times} : z + \mathbf{b} \subseteq x + \mathbf{a}\})$. Then there is a representation $\psi_{\mathfrak{k}} : C^*(P_{m,\Gamma}) \rightarrow \mathcal{B}(\mathcal{H}_{\mathfrak{k}})$ such that $\psi_{\mathfrak{k}}(v_{(b,a)}) = V_{(b,a)}^{\mathfrak{k}}$ and $\psi_{\mathfrak{k}}(e_{(x+\mathbf{a}) \times \mathbf{a}^\times}) = P_{x+\mathbf{a}}^{\mathfrak{k}}$. Moreover, $\psi_{\mathfrak{k}}$ is faithful.*

Proof. It is easy to see that $\mathcal{H}_{\mathfrak{k}}$ is invariant. A calculation shows that the collections $\{V_{(b,a)}^{\mathfrak{k}} : (b, a) \in P_{m,\Gamma}\}$ and $\{0\} \cup \{P_{x+\mathbf{a}}^{\mathfrak{k}} : x \in R, \mathbf{a} \in \mathcal{I}_m^+\}$ satisfy the defining relations (I) and (II) for $C^*(P_{m,\Gamma})$, so existence of $\psi_{\mathfrak{k}}$ follows from the defining universal property of $C^*(P_{m,\Gamma})$. For each $\tilde{\mathfrak{k}} \in \mathcal{I}_m/i(K_{m,\Gamma})$, let $\mathbf{a}_{\tilde{\mathfrak{k}}} \in \tilde{\mathfrak{k}}$ be an integral ideal. By Theorem 6.1, injectivity of $\psi_{\tilde{\mathfrak{k}}}$ will follow if we show that for each $\tilde{\mathfrak{k}}$, we have $\prod_{i=1}^m (P_{\mathbf{a}_{\tilde{\mathfrak{k}}}}^{\tilde{\mathfrak{k}}} - P_{y_i+\mathbf{a}_i}^{\tilde{\mathfrak{k}}}) \neq 0$ for any $y_1, \dots, y_m \in R$ and $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathcal{I}_m^+$ such that

$y_i + \mathfrak{a}_i \subsetneq \mathfrak{a}_{\tilde{\mathfrak{k}}}$. For this, it suffices to find $\mathfrak{b} \in \mathfrak{k}$ such that $\mathfrak{b} \subseteq \mathfrak{a}_{\tilde{\mathfrak{k}}}$ and $y_i + \mathfrak{a}_i \not\subseteq \mathfrak{b}$. By [Mil, Theorem 7.2], the class $\mathfrak{k}\tilde{\mathfrak{k}}^{-1}$ contains infinitely many prime ideals, so we can choose a prime $\mathfrak{p} \in \mathfrak{k}\tilde{\mathfrak{k}}^{-1}$ such that $y_1, y_2, \dots, y_m \notin \mathfrak{p}$. Then $\mathfrak{b} := \mathfrak{p}\mathfrak{a}_{\tilde{\mathfrak{k}}} \in \mathfrak{k}$ clearly satisfies $\mathfrak{b} \subsetneq \mathfrak{a}_{\tilde{\mathfrak{k}}}$, and we also have $y_i + \mathfrak{a}_i \not\subseteq \mathfrak{b}$ because $\mathfrak{b} \subseteq \mathfrak{p}$, and $y_i + \mathfrak{a}_i \subseteq \mathfrak{b}$ would imply $y_i \in \mathfrak{b}$. \square

Remark 6.6. In the case of trivial \mathfrak{m} , it is shown in [C-D-L, Section 4] that the direct sum $\bigoplus_{\mathfrak{k} \in \text{Cl}(K)} \psi_{\mathfrak{k}}$ is faithful.

7. THE PRIMITIVE IDEAL SPACE

Given a C^* -algebra B , let $\text{Prim}(B)$ denote the primitive ideal space of B . If $X \subseteq B$ is any subset, we let $\langle X \rangle_B$ denote the (closed, two-sided) ideal of B generated by X ; by convention, $\langle \emptyset \rangle := \{0\}$.

Continuing with the notation from the previous section, we let $\mathcal{P}_K^{\mathfrak{m}} := \mathcal{P}_K \setminus S$ denote the collection of (non-zero) prime ideals of R that do not divide \mathfrak{m}_0 , let $P_{\mathfrak{m}, \Gamma} = R \rtimes R_{\mathfrak{m}, \Gamma}$, and let $G_{\mathfrak{m}, \Gamma} = (R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m}, \Gamma}$.

Equip $2^{\mathcal{P}_K^{\mathfrak{m}}}$ with the power-cofinite topology. Recall that a base for the power-cofinite topology is given by the sets $U_F := \{T \in 2^{\mathcal{P}_K^{\mathfrak{m}}} : T \cap F = \emptyset\}$ for $F \subseteq 2^{\mathcal{P}_K^{\mathfrak{m}}}$ finite. We may view both $2^{\mathcal{P}_K^{\mathfrak{m}}}$ and $\text{Prim}(C^*(P_{\mathfrak{m}, \Gamma}))$ as partially ordered sets with respect to the orders given by inclusion of subsets and inclusion of ideals, respectively. The following theorem is a strengthening and generalization of [Ech-La, Theorem 3.6]. Our explicit description is motivated by the explicit description of the primitive ideals of $C^*(R \rtimes R^\times)$ given in [Li4, Li5].

Theorem 7.1. *For each $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}$, let $f_{\mathfrak{p}}$ denote the order of $[\mathfrak{p}] \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})$, so that $\mathfrak{p}^{f_{\mathfrak{p}}} = t_{\mathfrak{p}}R$ for some $t_{\mathfrak{p}} \in R_{\mathfrak{m}, \Gamma}$. For each subset $A \subseteq \mathcal{P}_K^{\mathfrak{m}}$, let*

$$I_A := \left\langle \left\{ 1 - \sum_{x \in R/t_{\mathfrak{p}}R} v_{(x, t_{\mathfrak{p}})} v_{(x, t_{\mathfrak{p}})}^* : \mathfrak{p} \in A \right\} \right\rangle_{C^*(P_{\mathfrak{m}, \Gamma})}.$$

Then I_A is a primitive ideal, and the map $2^{\mathcal{P}_K^{\mathfrak{m}}} \rightarrow \text{Prim}(C^(P_{\mathfrak{m}, \Gamma}))$ given by $A \mapsto I_A$ is an order-preserving homeomorphism.*

Before we can prove Theorem 7.1, we need a preliminary result.

Each open $G_{\mathfrak{m},\Gamma}$ -invariant subset $U \subseteq \Omega_R^{\mathfrak{m}}$ gives rise to the ideal $C^*(G_{\mathfrak{m},\Gamma} \times U) \subseteq C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}})$. In particular, for each point $w \in \Omega_R^{\mathfrak{m}}$, the set $\Omega_R^{\mathfrak{m}} \setminus \overline{G_{\mathfrak{m},\Gamma}.w}$ is open and $G_{\mathfrak{m},\Gamma}$ -invariant where, as before, $G_{\mathfrak{m},\Gamma}.w := \{gw : (g, w) \in \Omega_R^{\mathfrak{m}}\}$ is the orbit of w , and $\overline{G_{\mathfrak{m},\Gamma}.w}$ is the closure of $G_{\mathfrak{m},\Gamma}.w$, which is called the quasi-orbit of w . The *quasi-orbit space* is given by $\mathcal{Q}(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}) := \Omega_R^{\mathfrak{m}} / \sim$ where $w \sim w'$ if $\overline{G_{\mathfrak{m},\Gamma}.w} = \overline{G_{\mathfrak{m},\Gamma}.w'}$; this space was described in Proposition 6.3 above.

Lemma 7.2. *For each $x \in \Omega_R^{\mathfrak{m}}$, the ideal $C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus \overline{G_{\mathfrak{m},\Gamma}.w}))$ is primitive, and the map $\Omega_R^{\mathfrak{m}} \rightarrow \text{Prim}(C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}))$ given by $w \mapsto C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus \overline{G_{\mathfrak{m},\Gamma}.w}))$ descends to a homeomorphism $\mathcal{Q}(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}) \simeq \text{Prim}(C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}))$.*

Moreover, if $\vartheta : C^*(P_{\mathfrak{m},\Gamma}) \cong C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}})$ is the isomorphism from Proposition 5.4, then $\vartheta(I_A) = C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_A))$ for every $A \subseteq \mathcal{P}_K^{\mathfrak{m}}$.

Proof. Each ideal $C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus \overline{G_{\mathfrak{m},\Gamma}.w}))$ is primitive by [Sims-Wil, Lemma 4.5].

The groupoid $G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}$ is second countable, étale, and amenable. By Proposition 6.3(2), we may apply [Sims-Wil, Lemma 4.6] to conclude that the map $\Omega_R^{\mathfrak{m}} \rightarrow \text{Prim}(C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}))$ given by $w \mapsto C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus \overline{G_{\mathfrak{m},\Gamma}.w}))$ descends to a homeomorphism $\mathcal{Q}(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}) \simeq \text{Prim}(C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}))$.

We now turn to the second claim. Let $A \subseteq \mathcal{P}_K^{\mathfrak{m}}$. For each $\mathfrak{p} \in A$, we have that

$$\vartheta\left(1 - \sum_{x \in R/t_{\mathfrak{p}}R} v_{(x,t_{\mathfrak{p}})} v_{(x,t_{\mathfrak{p}}}^*\right) = 1_{\{[\mathfrak{b},\bar{\mathfrak{a}}]:v_{\mathfrak{p}}(\bar{\mathfrak{a}}) < f_{\mathfrak{p}}\}}$$

lies in $C_0(\Omega_R^{\mathfrak{m}} \setminus C_A)$. Hence, $\vartheta(I_A) \subseteq C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_A))$.

We know that $\vartheta(I_A) = \bigcap_J J$ where J runs over all primitive ideals of $C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}})$ that contain $\vartheta(I_A)$, so to show that $C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_A))$ is contained in $\vartheta(I_A)$, it suffices to show that any primitive ideal that contains $\vartheta(I_A)$ must also contain $C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_A))$. Suppose $J \in \text{Prim}(C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}))$ with $\vartheta(I_A) \subseteq J$. By part (1), $J = C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_B))$ for some $B \subseteq \mathcal{P}_K^{\mathfrak{m}}$. Now, we have $1_{\{[\mathfrak{b},\bar{\mathfrak{a}}]:v_{\mathfrak{p}}(\bar{\mathfrak{a}}) < f_{\mathfrak{p}}\}} \in C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_B))$ for all $\mathfrak{p} \in A$ which implies that $1_{\{[\mathfrak{b},\bar{\mathfrak{a}}]:v_{\mathfrak{p}}(\bar{\mathfrak{a}}) < f_{\mathfrak{p}}\}}$ vanishes on C_B for all $\mathfrak{p} \in A$; hence, $A \subseteq B$. Thus, $C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_A)) \subseteq J$. \square

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. By Proposition 6.3(1) and Lemma 7.2(1), the map $A \mapsto C^*(G_{\mathfrak{m},\Gamma} \times (\Omega_R^{\mathfrak{m}} \setminus C_A))$ is an order-preserving bijection from $2^{\mathcal{P}_K^{\mathfrak{m}}}$ onto $\text{Prim}(C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}}))$. The proof that this map is a homeomorphism is analogous to the proof of [La-Rae2, Proposition 2.4]. Thus, Theorem 7.1 follows from Lemma 7.2(2). \square

Corollary 7.3. *The ideal $I_{\mathcal{P}_K^{\mathfrak{m}}}$ is the unique maximal ideal of $C^*(P_{\mathfrak{m},\Gamma})$, and the map $\mathfrak{p} \mapsto I_{\{\mathfrak{p}\}}$ defines a bijection from $\mathcal{P}_K^{\mathfrak{m}}$ onto the set of minimal primitive ideals of $C^*(P_{\mathfrak{m},\Gamma})$.*

Proof. This follows from Theorem 7.1 since the bijection $A \mapsto I_A$ is inclusion-preserving. \square

8. THE BOUNDARY QUOTIENT

By Corollary 7.3, the ideal $I_{\mathcal{P}_K^{\mathfrak{m}}}$ is the unique maximal ideal of $C^*(P_{\mathfrak{m},\Gamma})$. The C^* -algebra $C^*(P_{\mathfrak{m},\Gamma})/I_{\mathcal{P}_K^{\mathfrak{m}}}$ is the *boundary quotient* of $C^*(P_{\mathfrak{m},\Gamma})$, as defined in [Li3, Section 7] (see also [Li7, Chapter 5.7]). We now give a description of $C^*(P_{\mathfrak{m},\Gamma})/I_{\mathcal{P}_K^{\mathfrak{m}}}$ as a semigroup crossed product. This generalizes the well-known semigroup crossed product description of the ring C^* -algebra of R .

Each $(b, a) \in P_{\mathfrak{m},\Gamma}$ gives rise to an injective continuous map $\hat{R}_S \rightarrow \hat{R}_S$ given by $(b, a)\mathbf{x} := b + a\mathbf{x}$; let $\tau_{(b,a)}$ be the corresponding $*$ -endomorphism of $C(\hat{R}_S)$. Then $(C(\hat{R}_S), P_{\mathfrak{m},\Gamma}, \tau)$ is a semigroup dynamical system, so we may form the crossed product C^* -algebra $C(\hat{R}_S) \rtimes_{\tau} P_{\mathfrak{m},\Gamma}$. For $(b, a) \in P_{\mathfrak{m},\Gamma}$, let $w_{(b,a)}$ be the corresponding isometry in $C(\hat{R}_S) \rtimes_{\tau} P_{\mathfrak{m},\Gamma}$.

Proposition 8.1. *There is a surjective $*$ -homomorphism $\pi : C^*(P_{\mathfrak{m},\Gamma}) \rightarrow C(\hat{R}_S) \rtimes_{\tau} P_{\mathfrak{m},\Gamma}$ such that*

$$\pi(v_{(b,a)}) = w_{(b,a)} \quad \text{and} \quad \pi(e_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}) = 1_{x+\hat{\mathfrak{a}}}$$

for all $(b, a) \in P_{\mathfrak{m},\Gamma}$ and $(x + \mathfrak{a}) \times \mathfrak{a}^\times \in \mathcal{J}_{P_{\mathfrak{m},\Gamma}}^\times$, where $\hat{\mathfrak{a}}$ denotes the closed ideal of \hat{R}_S generated by \mathfrak{a} . Moreover, $\ker \pi = I_{\mathcal{P}_K^{\mathfrak{m}}}$, so we get an isomorphism $C^*(P_{\mathfrak{m},\Gamma})/I_{\mathcal{P}_K^{\mathfrak{m}}} \cong C(\hat{R}_S) \rtimes_{\tau} P_{\mathfrak{m},\Gamma}$.

Proof. Consider the collection of projections $\{1_{x+\hat{\mathfrak{a}}} : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\}$ and the collection of isometries $\{w_{(b,a)} : (b, a) \in P_{\mathfrak{m},\Gamma}\}$. A calculation verifies that these collections

satisfy the defining relations (I) and (II) for $C^*(P_{\mathfrak{m},\Gamma})$, so the universal property of $C^*(P_{\mathfrak{m},\Gamma})$ gives us a *-homomorphism $\pi : C^*(P_{\mathfrak{m},\Gamma}) \rightarrow C(\hat{R}_S) \rtimes_{\tau} P_{\mathfrak{m},\Gamma}$ such that

$$\pi(v_{(b,a)}) = w_{(b,a)} \quad \text{and} \quad \pi(e_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}) = 1_{x+\hat{\mathfrak{a}}}$$

for all $(b, a) \in P_{\mathfrak{m},\Gamma}$ and $(x + \mathfrak{a}) \times \mathfrak{a}^\times \in \mathcal{J}_{P_{\mathfrak{m},\Gamma}}^\times$. Since $\text{span}\{1_{x+\hat{\mathfrak{a}}} : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\}$ is dense in $C(\hat{R}_S)$, we see that

$$\{1_{x+\hat{\mathfrak{a}}} : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\} \cup \{w_{(b,a)} : (b, a) \in P_{\mathfrak{m},\Gamma}\}$$

generates $C(\hat{R}_S) \rtimes_{\tau} P_{\mathfrak{m},\Gamma}$ as a C^* -algebra, so π is surjective.

It remains to show that $\ker \pi = I_{\mathcal{P}_K^{\mathfrak{m}}}$. Since $I_{\mathcal{P}_K^{\mathfrak{m}}}$ is a maximal ideal, it suffices to show that $I_{\mathcal{P}_K^{\mathfrak{m}}} \subseteq \ker \pi$. For every $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, the canonical embedding $R \hookrightarrow \hat{R}_S$ induces an isomorphism $R/\mathfrak{a} \cong \hat{R}_S/\hat{\mathfrak{a}}$, so $\hat{R}_S = \bigsqcup_{x \in R/\mathfrak{a}} (x + \hat{\mathfrak{a}})$. Hence,

$$\pi\left(1 - \sum_{x \in R/t_{\mathfrak{p}}R} v_{(x,t_{\mathfrak{p}})} v_{(x,t_{\mathfrak{p}})}^*\right) = 1 - \sum_{x \in R/t_{\mathfrak{p}}R} 1_{x+t_{\mathfrak{p}}\hat{R}_S} = 0.$$

Since the projections $1 - \sum_{x \in R/t_{\mathfrak{p}}R} v_{(x,t_{\mathfrak{p}})} v_{(x,t_{\mathfrak{p}})}^*$ for $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}$ generate $I_{\mathcal{P}_K^{\mathfrak{m}}}$, we are done. \square

9. FUNCTORIALITY

As before, let K be a number field with ring of integers R . Recall that the number-theoretic data for our construction consists of a pair (\mathfrak{m}, Γ) where \mathfrak{m} a modulus for K and Γ a subgroup of $(R/\mathfrak{m})^*$. The set of such pairs carries a canonical partial order, which we now describe.

Let \mathfrak{m} and \mathfrak{n} be moduli for K , and let Γ and Λ be subgroups of $(R/\mathfrak{m})^*$ and $(R/\mathfrak{n})^*$, respectively. Denote by $\text{pr}_{\mathfrak{m}} : R_{\mathfrak{m}} \rightarrow (R/\mathfrak{m})^*$ and $\text{pr}_{\mathfrak{n}} : R_{\mathfrak{n}} \rightarrow (R/\mathfrak{n})^*$ the canonical projection maps. Recall that $\mathfrak{m} \mid \mathfrak{n}$ if $\mathfrak{m}_0 \mid \mathfrak{n}_0$ and $\mathfrak{m}_{\infty} \leq \mathfrak{n}_{\infty}$. If $\mathfrak{m} \mid \mathfrak{n}$, then we have a canonical inclusion of semigroups $R_{\mathfrak{n}} \subseteq R_{\mathfrak{m}}$, and a canonical surjective group homomorphism $\pi_{\mathfrak{n},\mathfrak{m}} : (R/\mathfrak{n})^* \rightarrow (R/\mathfrak{m})^*$ such that the following diagram commutes:

$$\begin{array}{ccc} R_{\mathfrak{n}} & \xrightarrow{\text{incl}} & R_{\mathfrak{m}} \\ \downarrow \text{pr}_{\mathfrak{n}} & & \downarrow \text{pr}_{\mathfrak{m}} \\ (R/\mathfrak{n})^* & \xrightarrow{\pi_{\mathfrak{n},\mathfrak{m}}} & (R/\mathfrak{m})^* \end{array} \quad (5)$$

We define $(\mathfrak{m}, \Gamma) \leq (\mathfrak{n}, \Lambda)$ if and only if $\mathfrak{m} \mid \mathfrak{n}$ and $\pi_{\mathfrak{n}, \mathfrak{m}}(\Lambda) \subseteq \Gamma$. We will show next that our construction respects this ordering, that is, it is functorial in the appropriate sense. First, we need a lemma.

Lemma 9.1. *Let \mathfrak{m} be a modulus for K , and suppose that w is a real embedding of K . Then $w \mid \mathfrak{m}_\infty$ if and only if $w(x) > 0$ for all $x \in R_{\mathfrak{m}, 1}$.*

Proof. First, suppose that $w(x) > 0$ for all $x \in R_{\mathfrak{m}, 1}$, and assume that $w \nmid \mathfrak{m}_\infty$. By definition,

$$R_{\mathfrak{m}, 1} = \{x \in 1 + \mathfrak{m}_0 : v(x) > 0 \text{ for all } v \mid \mathfrak{m}_\infty\},$$

and [Nar, Proposition 2.2(i)] asserts that the coset $1 + \mathfrak{m}_0$ contains (infinitely many) elements of every signature. Hence, there exists $a \in R_{\mathfrak{m}, 1}$ with $v(a) > 0$ for every $v \mid \mathfrak{m}_\infty$ and $w(a) < 0$. This contradicts that $w(x) > 0$ for all $x \in R_{\mathfrak{m}, 1}$, so we must have $w \mid \mathfrak{m}_\infty$. The other direction is obvious. \square

Proposition 9.2. *Let \mathfrak{m} and \mathfrak{n} be moduli for K , and let Γ and Λ be subgroups of $(R/\mathfrak{m})^*$ and $(R/\mathfrak{n})^*$, respectively. Then*

- (1) $R_{\mathfrak{n}, \Lambda} \subseteq R_{\mathfrak{m}, \Gamma}$ if and only if $(\mathfrak{m}, \Gamma) \leq (\mathfrak{n}, \Lambda)$.
- (2) If the equivalent conditions from (1) are satisfied, so that there is a canonical inclusion of semigroups $\iota : R \rtimes R_{\mathfrak{n}, \Lambda} \hookrightarrow R \rtimes R_{\mathfrak{m}, \Gamma}$, then there is an injective $*$ -homomorphism $C^*(R \rtimes R_{\mathfrak{n}, \Lambda}) \rightarrow C^*(R \rtimes R_{\mathfrak{m}, \Gamma})$ such that $v_{(b, a)} \mapsto v_{(\iota(b), \iota(a))}$.

Proof. (1): First, note that $R_{\mathfrak{n}, \Lambda} \subseteq R_{\mathfrak{m}, \Gamma}$ implies that $R_{\mathfrak{n}, 1} \subseteq R_{\mathfrak{m}, 1}$. We will now show that $\mathfrak{m}_\infty \leq \mathfrak{n}_\infty$. Suppose w is a real embedding of K such that $\mathfrak{m}_\infty(w) = 1$. Since $R_{\mathfrak{n}, 1} \subseteq R_{\mathfrak{m}, 1}$, we must have $w(x) > 0$ for all $x \in R_{\mathfrak{n}, 1}$, so $w \mid \mathfrak{n}_\infty$ by Lemma 9.1.

Next we show that $\mathfrak{m}_0 \mid \mathfrak{n}_0$. The inclusion $R_{\mathfrak{n}, 1} \subseteq R_{\mathfrak{m}, 1}$ implies that $(1 + \mathfrak{n}_0)_+ \subseteq (1 + \mathfrak{m}_0)_+$, which in turn implies that $(\mathfrak{n}_0)_+ \subseteq (\mathfrak{m}_0)_+$. Since ideals are generated by the totally positive elements that they contain (see the proof of Lemma 3.8), we have $\mathfrak{n}_0 \subseteq \mathfrak{m}_0$.

Using commutativity of (5) and that $R_{\mathfrak{n}, \Lambda} \subseteq R_{\mathfrak{m}, \Gamma}$, we have

$$\pi_{\mathfrak{n}, \mathfrak{m}}(\Lambda) = \pi_{\mathfrak{n}, \mathfrak{m}}(\text{pr}_{\mathfrak{n}}(R_{\mathfrak{n}, \Lambda})) = \text{pr}_{\mathfrak{m}}(R_{\mathfrak{n}, \Lambda}) \subseteq \text{pr}_{\mathfrak{m}}(R_{\mathfrak{m}, \Gamma}) = \Gamma,$$

as desired.

For the converse, suppose $(\mathfrak{m}, \Gamma) \leq (\mathfrak{n}, \Lambda)$, so that $\mathfrak{m} \mid \mathfrak{n}$ and $\pi_{\mathfrak{n}, \mathfrak{m}}(\Lambda) \subseteq \Gamma$. Then $\text{pr}_{\mathfrak{m}}^{-1}(\pi_{\mathfrak{n}, \mathfrak{m}}(\Lambda)) \subseteq \text{pr}_{\mathfrak{m}}^{-1}(\Gamma) = R_{\mathfrak{m}, \Gamma}$, and commutativity of (5) implies $\text{pr}_{\mathfrak{m}}(R_{\mathfrak{n}, \Lambda}) = \pi_{\mathfrak{n}, \mathfrak{m}}(\Lambda)$, so we have $R_{\mathfrak{n}, \Lambda} \subseteq \text{pr}_{\mathfrak{m}}^{-1}(\pi_{\mathfrak{n}, \mathfrak{m}}(\Lambda)) \subseteq R_{\mathfrak{m}, \Gamma}$.

(2): Assume $R_{\mathfrak{n}, \Lambda} \subseteq R_{\mathfrak{m}, \Gamma}$. Then $\mathfrak{m} \mid \mathfrak{n}$ by part (1) which implies that $\mathcal{I}_{\mathfrak{n}}^+ \subseteq \mathcal{I}_{\mathfrak{m}}^+$. The collections $\{e_{(x+\mathfrak{a}) \times \mathfrak{a}^\times} : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{n}}^+ \} \cup \{0\}$ and $\{v_{(\iota(b), \iota(a))} : (b, a) \in R \rtimes R_{\mathfrak{n}, \Lambda}\}$ of projections and isometries, respectively, in $C^*(R \rtimes R_{\mathfrak{m}, \Gamma})$ satisfy the defining relations (I) and (II) for $C^*(R \rtimes R_{\mathfrak{n}, \Lambda})$, so the universal property of $C^*(R \rtimes R_{\mathfrak{n}, \Lambda})$ gives us a *-homomorphism $\psi : C^*(R \rtimes R_{\mathfrak{n}, \Lambda}) \rightarrow C^*(R \rtimes R_{\mathfrak{m}, \Gamma})$ such that $\psi(v_{(b, a)}) = v_{(\iota(b), \iota(a))}$ for all $(b, a) \in R \rtimes R_{\mathfrak{n}, \Lambda}$.

The projections $\{e_{(x+\mathfrak{a}) \times \mathfrak{a}^\times} : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{n}}^+\}$ are linearly independent in $C_u^*(\mathcal{J}_{R \rtimes R_{\mathfrak{m}, \Gamma}})$ by Proposition 3.4, so the hypotheses of Theorem 6.1 are satisfied; hence, ψ is injective. \square

In particular, if we take \mathfrak{m} to be trivial, so that Γ must also be trivial, then we obtain the following result.

Corollary 9.3. *For each modulus \mathfrak{n} and each subgroup $\Lambda \subseteq (R/\mathfrak{n})^*$, there is an injective *-homomorphism $C^*(R \rtimes R_{\mathfrak{n}, \Lambda}) \rightarrow C^*(R \rtimes R^\times)$ such that $v_{(b, a)} \mapsto v_{(\iota(b), \iota(a))}$ where $\iota : R \rtimes R_{\mathfrak{n}, \Lambda} \hookrightarrow R \rtimes R^\times$ is the canonical inclusion.*

We can also ask what happens as the number field varies. Let K and K' be number fields with rings of integers R and R' , respectively.

Lemma 9.4. *Suppose that \mathfrak{m} is a modulus for K and that there is an inclusion of number fields $i : K \hookrightarrow K'$. Define a modulus $\tilde{\mathfrak{m}}$ for K' by $\tilde{\mathfrak{m}}_\infty(w') := \mathfrak{m}_\infty(w' \circ i)$ for each real embedding $w' : K' \hookrightarrow \mathbb{R}$ and $\tilde{\mathfrak{m}}_0 := i(\mathfrak{m}_0)R'$ where $i(\mathfrak{m}_0)R'$ is the ideal of R' generated by $i(\mathfrak{m}_0)$. For each modulus \mathfrak{m}' of K' , we have $i(R_{\mathfrak{m}, 1}) \subseteq R'_{\mathfrak{m}', 1}$ if and only if $\mathfrak{m}' \mid \tilde{\mathfrak{m}}$.*

Proof. Suppose that $i(R_{\mathfrak{m}, 1}) \subseteq R'_{\mathfrak{m}', 1}$. Then for each $w' \mid \mathfrak{m}'_\infty$, we see that $w' \circ i(x) > 0$ for every $x \in R_{\mathfrak{m}, 1}$, so $(w' \circ i) \mid \mathfrak{m}_\infty$ by Lemma 9.1. That is, $w' \mid \mathfrak{m}'_\infty$ implies $w' \mid \tilde{\mathfrak{m}}_\infty$, so we have $\mathfrak{m}'_\infty \mid \tilde{\mathfrak{m}}_\infty$. The inclusion $i(R_{\mathfrak{m}, 1}) \subseteq R'_{\mathfrak{m}', 1}$ also implies that $(1 + i(\mathfrak{m}_0))_+ \subseteq (1 + \mathfrak{m}'_0)_+$ where $(1 + i(\mathfrak{m}_0))_+$ and $(1 + \mathfrak{m}'_0)_+$ denote the sets of totally positive elements in $1 + i(\mathfrak{m}_0)$ and $1 + \mathfrak{m}'_0$, respectively. It follows that $\tilde{\mathfrak{m}}_0 = i(\mathfrak{m}_0)R'$

is contained in \mathfrak{m}'_0 , that is, $\mathfrak{m}'_0 \mid \tilde{\mathfrak{m}}_0$. Thus, we have shown $i(R_{\mathfrak{m},1}) \subseteq R'_{\mathfrak{m}',1}$ implies $\mathfrak{m}' \mid \tilde{\mathfrak{m}}$.

For the converse, suppose that $\mathfrak{m}' \mid \tilde{\mathfrak{m}}$. Let $a \in R_{\mathfrak{m},1}$, so that $a \in 1 + \mathfrak{m}_0$ and $w(a) > 0$ for every $w \mid \mathfrak{m}_\infty$. We have $1 + \tilde{\mathfrak{m}}_0 \subseteq 1 + \mathfrak{m}'_0$, and if $w' \mid \mathfrak{m}'_\infty$, then $w' \mid \tilde{\mathfrak{m}}$, so that $(w' \circ i) \mid \mathfrak{m}_\infty$. Hence, $i(a) \in 1 + \mathfrak{m}'_0$, and if $w' \mid \mathfrak{m}'_\infty$, then $(w' \circ i)(a) > 0$. That is, $i(a) \in R'_{\mathfrak{m}',1}$. Hence, $i(R_{\mathfrak{m},1}) \subseteq R'_{\mathfrak{m}',1}$, as desired. \square

In the setup from Lemma 9.4, suppose that $\mathfrak{m}' \mid \tilde{\mathfrak{m}}$. The inclusion $i|_R : R \hookrightarrow R'$ induces homomorphisms $(R/\mathfrak{m}_0)^* \rightarrow (R'/\tilde{\mathfrak{m}}_0)^*$ and $\prod_{w \mid \mathfrak{m}_\infty} \{\pm 1\} \rightarrow \prod_{(w' \circ i) \mid \mathfrak{m}_\infty} \{\pm 1\}$. Combining these, gives us a homomorphism $\varphi : (R/\mathfrak{m})^* \rightarrow (R'/\tilde{\mathfrak{m}})^*$. These maps give rise to the following commutative diagram

$$\begin{array}{ccccc}
R_{\mathfrak{m}} & \xhookrightarrow{i|_{R_{\mathfrak{m}}}} & R'_{\tilde{\mathfrak{m}}} & \xhookrightarrow{\text{incl}} & R'_{\mathfrak{m}'} \\
\downarrow \text{pr}_{\mathfrak{m}} & & \downarrow \text{pr}_{\tilde{\mathfrak{m}}} & & \downarrow \text{pr}_{\mathfrak{m}'} \\
(R/\mathfrak{m})^* & \xrightarrow{\varphi} & (R'/\tilde{\mathfrak{m}})^* & \xrightarrow{\pi_{\tilde{\mathfrak{m}},\mathfrak{m}'}} & (R'/\mathfrak{m}')^*.
\end{array} \tag{6}$$

Proposition 9.5. *Let \mathfrak{m} and \mathfrak{m}' be moduli for K and K' , respectively, and let Γ and Γ' be subgroups of $(R/\mathfrak{m})^*$ and $(R'/\mathfrak{m}')^*$, respectively. Suppose that there is an inclusion of number fields $i : K \hookrightarrow K'$. Then, using the notation from the preceding discussion, we have the following:*

- (1) $i(R_{\mathfrak{m},\Gamma}) \subseteq R'_{\mathfrak{m}',\Gamma'}$ if and only if $\mathfrak{m}' \mid \tilde{\mathfrak{m}}$ and $\pi_{\tilde{\mathfrak{m}},\mathfrak{m}'} \circ \varphi(\Gamma) \subseteq \Gamma'$.
- (2) If the equivalent conditions in (1) are satisfied, so that there is an inclusion $\iota : K \rtimes K^\times \hookrightarrow K' \rtimes (K')^\times$ that restricts to an inclusion $R \rtimes R_{\mathfrak{m},\Gamma} \hookrightarrow R' \rtimes R'_{\mathfrak{m}',\Gamma'}$, then there is an injective $*$ -homomorphism $C^*(R \rtimes R_{\mathfrak{m},\Gamma}) \hookrightarrow C^*(R' \rtimes R'_{\mathfrak{m}',\Gamma'})$ such that $v_{(b,a)} \mapsto v_{(\iota(b),\iota(a))}$ for all $(b,a) \in R \rtimes R_{\mathfrak{m},\Gamma}$.

Proof. (1): Suppose that $i(R_{\mathfrak{m},\Gamma}) \subseteq R'_{\mathfrak{m}',\Gamma'}$. Then $i(R_{\mathfrak{m},1}) \subseteq R'_{\mathfrak{m}',1}$, so Lemma 9.4 implies that $\mathfrak{m}' \mid \tilde{\mathfrak{m}}$. Let $\gamma \in \Gamma$, and write $\gamma = [a]_{\mathfrak{m}}$ for some $a \in R_{\mathfrak{m},\Gamma}$. Using commutativity of (6), we have

$$\pi_{\tilde{\mathfrak{m}},\mathfrak{m}'} \circ \varphi(\gamma) = \pi_{\tilde{\mathfrak{m}},\mathfrak{m}'}([i(a)]_{\tilde{\mathfrak{m}}}) = [i(a)]_{\mathfrak{m}'}$$

Since $i(a)$ lies in $R'_{\mathfrak{m}',\Gamma'}$ by assumption, we have $[i(a)]_{\mathfrak{m}'} \in \Gamma'$. Hence, $\pi_{\tilde{\mathfrak{m}},\mathfrak{m}'} \circ \varphi(\Gamma) \subseteq \Gamma'$.

For the converse, suppose that $\mathfrak{m}' \mid \tilde{\mathfrak{m}}$ and $\pi_{\tilde{\mathfrak{m}}, \mathfrak{m}'} \circ \varphi(\Gamma) \subseteq \Gamma'$. Let $a \in R_{\mathfrak{m}, \Gamma}$. We need to show that $i(a)$ lies in $R'_{\mathfrak{m}', \Gamma'}$, that is, we need to show that $[i(a)]_{\mathfrak{m}'}$ lies in Γ' . By commutativity of (6), we have $[i(a)]_{\mathfrak{m}'} = \pi_{\tilde{\mathfrak{m}}, \mathfrak{m}'} \circ \varphi([a]_{\mathfrak{m}})$. Since $[a]_{\mathfrak{m}} \in \Gamma$ and $\pi_{\tilde{\mathfrak{m}}, \mathfrak{m}'} \circ \varphi(\Gamma) \subseteq \Gamma'$, we have $\pi_{\tilde{\mathfrak{m}}, \mathfrak{m}'} \circ \varphi([a]_{\mathfrak{m}}) \in \Gamma'$, as desired.

(2): By [C-D-L, Proposition 3.2 and Theorem 4.13], there is an injective $*$ -homomorphism $\psi : C^*(R \rtimes R^\times) \rightarrow C^*(R' \rtimes (R')^\times)$ such that $\psi(v_{(b,a)}) = v_{(\iota(b), \iota(a))}$. Let θ and θ' by the canonical injective $*$ -homomorphisms $\theta : C^*(R \rtimes R_{\mathfrak{m}, \Gamma}) \rightarrow C^*(R \rtimes R^\times)$ and $\theta' : C^*(R' \rtimes R'_{\Gamma'}) \rightarrow C^*(R' \rtimes (R')^\times)$ from Proposition 9.2. There is a (unique) $*$ -homomorphism ρ such that the following diagram commutes:

$$\begin{array}{ccc} C^*(R \rtimes R^\times) & \xhookrightarrow{\psi} & \text{im}(\theta') \\ \uparrow \theta & & \cong \downarrow (\theta'|_{\text{im}(\theta')})^{-1} \\ C^*(R \rtimes R_{\mathfrak{m}, \Gamma}) & \xrightarrow{\rho} & C^*(R' \rtimes R'_{\mathfrak{m}', \Gamma'}) \end{array}$$

Moreover, it is not difficult to see that ρ is injective and $\rho(v_{(b,a)}) = v_{(\iota(b), \iota(a))}$. \square

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PHASE TRANSITIONS ON C*-ALGEBRAS FROM ACTIONS OF CONGRUENCE MONOIDS ON RINGS OF ALGEBRAIC INTEGERS*

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ABSTRACT. We compute the KMS (equilibrium) states for the canonical time evolution on C*-algebras from actions of congruence monoids on rings of algebraic integers. We show that for each $\beta \in [1, 2]$, there is a unique KMS_β state, and we prove that it is a factor state of type III₁. There are phase transitions at $\beta = 2$ and $\beta = \infty$ involving a quotient of a ray class group. Our computation of KMS and ground states generalizes the results of Cuntz, Deninger, and Laca for the full $ax + b$ -semigroup over a ring of integers, and our type classification generalizes a result of Laca and Neshveyev in the case of the rational numbers and a result of Neshveyev in the case of arbitrary number fields.

1. INTRODUCTION

Given a number field K with ring of integers R , Cuntz, Deninger, and Laca studied phase transitions for the canonical time evolution on the left regular C*-algebra $C_\lambda^*(R \rtimes R^\times)$ of the (full) $ax + b$ -semigroup $R \rtimes R^\times$ over R , see [13]. Their results built on work of Laca and Raeburn in [29] and Laca and Neshveyev in [26] on the similar semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$ associated to the number field \mathbb{Q} . The classification of KMS and ground states from [13] showed that the C*-dynamical system associated with $C_\lambda^*(R \rtimes R^\times)$ exhibits several interesting properties; for example, the parameterization spaces for both the ground states and the low temperature KMS states decomposes over the ideal class group $\text{Cl}(K)$ of K , and uniqueness for the high temperature

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KMS states is related to the distribution of ideals over the group $\text{Cl}(K)$. In [43], Neshveyev developed general results for computing KMS states on C*-algebras of non-principal groupoids, and gave an alternative computation of the KMS states on $C_\lambda^*(R \rtimes R^\times)$.

The construction from [13] was recently generalized in [5] by restricting the multiplicative part of $R \rtimes R^\times$ to lie in certain subsemigroups of R^\times . Specifically, given a modulus $\mathfrak{m} = \mathfrak{m}_\infty \mathfrak{m}_0$ for K and a group Γ of residues modulo \mathfrak{m} , one considers the left regular C*-algebra $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ of the semi-direct product $R \rtimes R_{\mathfrak{m},\Gamma}$ where $R_{\mathfrak{m},\Gamma} \subseteq R^\times$ is the congruence monoid consisting of algebraic integers in R^\times that reduce to an element of Γ modulo \mathfrak{m} . For each number field K , the construction produces infinitely many non-isomorphic C*-algebras as \mathfrak{m} and Γ vary. For the special case of trivial \mathfrak{m} , in which case Γ must also be trivial, one gets the full $ax + b$ -semigroup C*-algebra studied in [13]. And for the special case where \mathfrak{m}_∞ is supported at all real embeddings of K and \mathfrak{m}_0 is trivial, one gets the semigroup $R \rtimes R_+^\times$ where R_+^\times is the subsemigroup of R^\times consisting of (non-zero) totally positive algebraic integers.

The main result of this paper is the computation of all KMS and ground states on the left regular C*-algebra $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ for the canonical time evolution σ coming from the norm map on K , including a classification of type for the high temperature KMS states, see Theorem 3.2 for the precise statement. As a consequence, we obtain that the boundary quotient of $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ admits a unique KMS state, which is of type III₁. Moreover, the techniques needed to prove Theorem 3.2 also lead to a computation of all the KMS and ground states on the left regular C*-algebras of the monoids $R_{\mathfrak{m},\Gamma}$ and $R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*$ where $R_{\mathfrak{m},\Gamma}^* := R_{\mathfrak{m},\Gamma} \cap R^*$ is the group of units in $R_{\mathfrak{m},\Gamma}$.

In order to explain our main result, we must first discuss a few number-theoretic preliminaries, see Section 2.1 for more details. Let $\mathcal{I}_\mathfrak{m}$ denote the group of fractional ideals in K that are coprime to the modulus \mathfrak{m} , and let $K_{\mathfrak{m},\Gamma} = R_{\mathfrak{m},\Gamma}^{-1} R_{\mathfrak{m},\Gamma} \subseteq K^\times$ be the group of (left) quotients of $R_{\mathfrak{m},\Gamma}$. For each $x \in K_{\mathfrak{m},\Gamma}$, let $i(x) := xR$ be the principal fractional ideal in K generated by x , so that $i(K_{\mathfrak{m},\Gamma})$ is a subgroup of $\mathcal{I}_\mathfrak{m}$. The quotient group $\mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})$ will appear throughout this paper. In the case that \mathfrak{m} and Γ are trivial, $\mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})$ coincides with the ideal class group $\text{Cl}(K)$.

In general, $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ is a quotient of the ray class group modulo \mathfrak{m} and thus is always a finite group.

We show that the C^* -dynamical system $(C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma}), \mathbb{R}, \sigma)$ exhibits phase transitions at $\beta = 2$ and $\beta = \infty$. For each $\beta \in [1, 2]$, we prove that there is a unique σ - KMS_β state on $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$. Our proof of uniqueness for $\beta \in [1, 2]$ uses the well-known fact that the L -functions associated with non-trivial characters of $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ do not have poles at 1. This technique is inspired by the proofs of uniqueness for high temperature KMS states on Bost–Connes type systems, see, for example, [3, Section 7], [41, Proposition], and [22, Theorem 2.1(ii)]. To maneuver ourselves into a position where we can use these methods, we expand on an idea from [43]. In the special case of trivial \mathfrak{m} and Γ , when our uniqueness result coincides with that in [13, Theorem 6.7], our approach is close to that taken in [43, Section 3] and is rather different than that taken in [13]. We then prove that for each $\beta \in [1, 2]$, the (unique) KMS_β state ϕ_β on $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ is a factor state of type III_1 . Indeed, we prove that the von Neumann algebra generated by the GNS representation of ϕ_β is isomorphic to the injective factor of type III_1 with separable predual. This builds on [26] and also generalizes the result asserted in [43, Section 3] on type for the high temperature KMS states on the left regular C^* -algebra $C_\lambda^*(R \rtimes R^\times)$ of the full $ax + b$ -semigroup, see Remark 4.2 below for more on this. Our computation of the type uses ideas from [26] and [32]; it relies on a general version of the prime ideal theorem for classes in $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$.

We now discuss the case where $\beta \in (2, \infty]$. For each class $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$, choose an integral ideal $\mathfrak{a}_\mathfrak{k} \in \mathfrak{k}$ representing \mathfrak{k} . The group $R_{\mathfrak{m},\Gamma}^*$ of units of $R_{\mathfrak{m},\Gamma}$ acts on $\mathfrak{a}_\mathfrak{k}$ by multiplication, so we may form the semi-direct product $\mathfrak{a}_\mathfrak{k} \rtimes R_{\mathfrak{m},\Gamma}^*$. For each $\beta \in (2, \infty]$, we prove that the set of KMS_β states decomposes over the finite set $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$; specifically, the extremal KMS_β states are parameterized by pairs (\mathfrak{k}, τ) where \mathfrak{k} is a class in $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ and τ is an extremal tracial state on the group C^* -algebra $C^*(\mathfrak{a}_\mathfrak{k} \rtimes R_{\mathfrak{m},\Gamma}^*)$. Moreover, the parameter space for the ground states also decomposes over $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$, but the extreme points are given by pairs (\mathfrak{k}, ϕ) where $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ and ϕ is an extremal state on a matrix algebra over $C^*(\mathfrak{a}_\mathfrak{k} \rtimes R_{\mathfrak{m},\Gamma}^*)$, so there are usually ground states that are not KMS_∞ states. For the special case of trivial \mathfrak{m} and Γ , we recover the main parameterization results from [13,

Sections 7&8]. Our computation uses general results for KMS and ground states on groupoid C*-algebras from [43] and [23], respectively.

The boundary quotient of $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ also carries a canonical time evolution. We prove that the associated C*-dynamical system admits a unique KMS state, which is of type III₁ and has inverse temperature $\beta = 1$. In the special case where \mathfrak{m} and Γ are trivial, the boundary quotient coincides with the ring C*-algebra of R and we recover the known uniqueness result in that case, see [12] for the case $K = \mathbb{Q}$ and [13, Theorem 6.7] for the case of a general number field.

The techniques and results used to prove Theorem 3.2 also lead to a phase transition theorem for the canonical time evolution on the left regular C*-algebra $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ of a congruence monoid and also the left regular C*-algebra $C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$ of the semigroup $R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*$ of principal integral ideals of R that are generated by an element of $R_{\mathfrak{m},\Gamma}$. These simpler C*-dynamical systems also exhibit several phase transitions, and the group $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ also appears in this context. Additionally, there is a phase transition at $\beta = 0$; the reason for this is that the spectrum of the diagonal in the case of the multiplicative monoids contains a unique fixed point, whereas in the case of $R \rtimes R_{\mathfrak{m},\Gamma}$, there are no fixed points. This generalizes and expounds the result from [13, Remark 7.5] (see also [14, Remark 6.6.5]).

This paper is organized as follows. Section 2 contains preliminaries: in Section 2.1, we recall some well-known concepts from algebraic number theory, including that of moduli and ray class groups, and we fix some notation that will be used throughout this article; in Section 2.2, we review the necessary background on congruence monoids and C*-algebras from actions of congruence monoids on rings of algebraic integers from [5]. In Section 3, we first introduce a canonical time evolution σ on $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$, and state our main theorem on phase transitions; this result gives a parameterization of all KMS and ground states of the C*-dynamical system $(C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma}), \mathbb{R}, \sigma)$, including the type for all high temperature KMS states, see Theorem 3.2. Sections 3.2 through 3.6 contain the proof of the parameterization results in Theorem 3.2. The claim about type is proven in Section 4, see Theorem 4.1. In Section 5, we use Theorem 3.2 to compute the KMS and ground states on the boundary quotient of $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$, see Theorem 5.1. Section 6 contains our phase transition theorems for the left regular C*-algebras $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ and $C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$.

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2. PRELIMINARIES

2.1. Moduli for number fields and ray class groups. Let K be a number field with ring of integers R . Let \mathcal{P}_K denote the set of non-zero prime ideals of R , so that each fractional ideal \mathfrak{a} of K can be written as $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathcal{P}_K} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$ where $v_{\mathfrak{p}}(\mathfrak{a}) \in \mathbb{Z}$ and $v_{\mathfrak{p}}(\mathfrak{a}) = 0$ for all but finitely many \mathfrak{p} . For $x \in K^\times := K \setminus \{0\}$, the set xR is the principal fractional ideal of K generated by x , and write $v_{\mathfrak{p}}(x)$ instead of $v_{\mathfrak{p}}(xR)$.

Let $V_{K, \mathbb{R}}$ be the (finite) set of real embeddings $K \hookrightarrow \mathbb{R}$. A *modulus* for K is a function $\mathfrak{m} : V_{K, \mathbb{R}} \sqcup \mathcal{P}_K \rightarrow \mathbb{N}$ such that

- $\mathfrak{m}_\infty := \mathfrak{m}|_{V_{K, \mathbb{R}}}$ takes values in $\{0, 1\}$;
- $\mathfrak{m}|_{\mathcal{P}_K}$ is finitely supported, that is, $\mathfrak{m}(\mathfrak{p}) = 0$ for all but finitely many $\mathfrak{p} \in \mathcal{P}_K$.

Let \mathfrak{m}_0 be the ideal of R defined by $\mathfrak{m}_0 := \prod_{\mathfrak{p} \in \mathcal{P}_K} \mathfrak{p}^{\mathfrak{m}(\mathfrak{p})}$. It is conventional to write \mathfrak{m} as a formal product $\mathfrak{m} = \mathfrak{m}_\infty \mathfrak{m}_0$, and to write $w \mid \mathfrak{m}_\infty$ when $w \in V_{K, \mathbb{R}}$ is such that $\mathfrak{m}(w) = 1$. For background on moduli for number fields, we refer the reader to [38, Chapter V, Section 1].

The *multiplicative group of residues modulo \mathfrak{m}* is

$$(R/\mathfrak{m})^* := \left(\prod_{w|\mathfrak{m}_\infty} \{\pm 1\} \right) \times (R/\mathfrak{m}_0)^*$$

where $(R/\mathfrak{m}_0)^*$ denotes the multiplicative group of units of the ring R/\mathfrak{m}_0 . We let $R^\times := R \setminus \{0\}$ be the multiplicative semigroup of non-zero algebraic integers in K , and we let

$$R_{\mathfrak{m}} := \{x \in R^\times : v_{\mathfrak{p}}(x) = 0 \text{ for all } \mathfrak{p} \text{ with } \mathfrak{p} \mid \mathfrak{m}_0\}$$

be the multiplicative semigroup of (non-zero) algebraic integers that are coprime to \mathfrak{m}_0 . For $a \in R_{\mathfrak{m}}$, let $[a]_{\mathfrak{m}}$ denote the *residue of a modulo \mathfrak{m}*

$$[a]_{\mathfrak{m}} := ((\text{sign}(w(a)))_{w|\mathfrak{m}_\infty}, a + \mathfrak{m}_0) \in (R/\mathfrak{m})^*$$

where $\text{sign}(w(a)) := w(a)/|w(a)|$.

The map $a \mapsto [a]_{\mathfrak{m}}$ extends uniquely to a surjective group homomorphism from the group of quotients $R_{\mathfrak{m}}^{-1}R_{\mathfrak{m}}$ of $R_{\mathfrak{m}}$ onto $(R/\mathfrak{m})^*$, see [5, Lemma 2.1]. By [5, Lemma 2.2], $R_{\mathfrak{m}}^{-1}R_{\mathfrak{m}}$ coincides with the group $K_{\mathfrak{m}} := \{x \in K^\times : v_{\mathfrak{p}}(x) = 0 \text{ for all } \mathfrak{p} \text{ with } \mathfrak{p} \mid \mathfrak{m}_0\}$.

The *ray modulo \mathfrak{m}* is the kernel of the map $a \mapsto [a]_{\mathfrak{m}}$, $K_{\mathfrak{m}} \rightarrow (R/\mathfrak{m})^*$; it is denoted by $K_{\mathfrak{m},1}$. Let $\mathcal{I}_{\mathfrak{m}}$ denote the group of fractional ideals of K that are coprime to \mathfrak{m}_0 , and for $x \in K^\times$, we let $i(x) := xR$ denote the fractional ideal of K generated by x ; the *ray class group modulo \mathfrak{m}* is $\text{Cl}_{\mathfrak{m}}(K) := \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},1})$. If \mathfrak{m} is trivial, that is, if $\mathfrak{m}_\infty \equiv 0$ and $\mathfrak{m}_0 = R$, then $\mathcal{I}_{\mathfrak{m}}$ equals the group \mathcal{I} of all fractional ideals of K , and the ray modulo \mathfrak{m} is simply the multiplicative group K^\times of non-zero elements in K . In this case, the ray class group modulo \mathfrak{m} coincides with the *ideal class group* $\text{Cl}(K) = \mathcal{I}/i(K^\times)$ of K . The canonical homomorphism $\text{Cl}_{\mathfrak{m}}(K) \rightarrow \text{Cl}(K)$ is surjective, and the ray class group $\text{Cl}_{\mathfrak{m}}(K)$ is always finite, see [38, Chapter V, Theorem 1.7] for more on the relationship between $\text{Cl}_{\mathfrak{m}}(K)$ and $\text{Cl}(K)$. We mention in passing that ray class groups play an important role in the ideal-theoretic formulation of class field theory.

2.2. \mathbf{C}^* -algebras from actions of congruence monoids on rings of integers.

We now recall the construction from [5]. Let K be a number field with ring of integers R , and let \mathfrak{m} be a modulus for K . For a subgroup $\Gamma \subseteq (R/\mathfrak{m})^*$, let

$$R_{\mathfrak{m},\Gamma} := \{a \in R_{\mathfrak{m}} : [a]_{\mathfrak{m}} \in \Gamma\}.$$

Then $R_{\mathfrak{m},\Gamma}$ is a multiplicative subsemigroup of R^\times ; such semigroups are called *congruence monoids* in the literature on semigroups, see, for example, [16] or [17]. By [5, Proposition 3.1], the group of quotients $R_{\mathfrak{m},\Gamma}^{-1}R_{\mathfrak{m},\Gamma}$ coincides with

$$K_{\mathfrak{m},\Gamma} := \{x \in K_{\mathfrak{m}} : [x]_{\mathfrak{m}} \in \Gamma\}.$$

The group $i(K_{\mathfrak{m},\Gamma})$ of principal fractional ideals generated by elements of $K_{\mathfrak{m},\Gamma}$ has finite index in $\mathcal{I}_{\mathfrak{m}}$; indeed, the quotient $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ can be canonically identified with the quotient $\text{Cl}_{\mathfrak{m}}(K)/\bar{\Gamma}$ where $\bar{\Gamma} := i(K_{\mathfrak{m},\Gamma})/i(K_{\mathfrak{m},1})$.

The semigroup $R_{\mathfrak{m},\Gamma}$ acts on (the additive group of) R by multiplication, so we may form the semi-direct product semigroup $R \rtimes R_{\mathfrak{m},\Gamma}$. For each $(b, a) \in R \rtimes R_{\mathfrak{m},\Gamma}$, let $\lambda_{(b,a)}$ be the isometry in $\mathcal{B}(\ell^2(R \rtimes R_{\mathfrak{m},\Gamma}))$ determined by $\lambda_{(b,a)}(\varepsilon_{(y,x)}) = \varepsilon_{(b+ay, ax)}$ where $\{\varepsilon_{(y,x)} : (y, x) \in R \rtimes R_{\mathfrak{m},\Gamma}\}$ is the canonical orthonormal basis for $\ell^2(R \rtimes R_{\mathfrak{m},\Gamma})$. The *left regular C*-algebra* $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ of $R \rtimes R_{\mathfrak{m},\Gamma}$ is the sub-C*-algebra of $\mathcal{B}(\ell^2(R \rtimes R_{\mathfrak{m},\Gamma}))$ generated by the left regular representation of $R \rtimes R_{\mathfrak{m},\Gamma}$. That is,

$$C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma}) := C^*(\{\lambda_{(b,a)} : (b, a) \in R \rtimes R_{\mathfrak{m},\Gamma}\}).$$

We refer the reader to [33, 34] or [14, Chapter 5] for the general theory of semigroup C*-algebras. Let $\mathcal{I}_{\mathfrak{m}}^+$ denote the non-zero integral ideals of R that are coprime to \mathfrak{m}_0 , and for each $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ and $x \in R$, let $E_{(x+\mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m},\Gamma})}$ be the orthogonal projection from $\ell^2(R \rtimes R_{\mathfrak{m},\Gamma})$ onto the subspace $\ell^2((x + \mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m},\Gamma}))$. The C*-algebra $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ has a canonical “diagonal” sub-C*-algebra given by

$$D_\lambda(R \rtimes R_{\mathfrak{m},\Gamma}) = \overline{\text{span}}(\{E_{(x+\mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m},\Gamma})} : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\}),$$

see [5, Proposition 3.3] and [33, Section 3].

3. EQUILIBRIUM STATES

3.1. The phase transition theorem. Let B be a C*-algebra. A *time evolution* on B is a group homomorphism $\gamma : \mathbb{R} \rightarrow \text{Aut}(B)$ such that for each fixed $x \in B$, the map $t \mapsto \gamma_t(x)$ is continuous. The triple (B, \mathbb{R}, γ) is called a *C*-dynamical system*. There is a standard notion of equilibrium in this context, namely that of KMS and ground states. We now recall the relevant definitions.

Let (B, \mathbb{R}, γ) be a C*-dynamical system. An element $x \in B$ is *γ -analytic* if the map $\mathbb{R} \rightarrow B$ given by $t \mapsto \gamma_t(x)$ extends to an entire function $z \mapsto \gamma_z(x)$ from \mathbb{C} to B .

Let $\beta \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$. A state φ on B is a γ -KMS $_{\beta}$ state, or a KMS state at inverse temperature β for γ , if it satisfies the KMS $_{\beta}$ condition

$$\varphi(xy) = \varphi(y\gamma_{i\beta}(x)) \quad (1)$$

for all γ -analytic elements x, y in a γ -invariant subset with dense linear span. The parameter β is often called the *inverse temperature*, see [4, Chapter 5] for motivation from quantum statistical mechanics. The γ -KMS $_0$ states are defined to be the γ -invariant traces on B ; these are the “infinite temperature” or “chaotic” states.

If B is unital, then [4, Theorem 5.3.30(1)&(2)] asserts that for each $\beta \in \mathbb{R}$, the set Σ_{β} of KMS $_{\beta}$ states of the system (B, \mathbb{R}, γ) is a (possibly empty) convex weak*-compact subset of the state space $S(B)$ of B that is also a Choquet simplex. Moreover, by [4, Theorem 5.3.30(3)], a KMS $_{\beta}$ state ϕ is an extreme point of Σ_{β} if and only if ϕ is a factor state, that is, if the von Neumann algebra $\pi_{\phi}(B)''$ generated by the GNS representation π_{ϕ} of ϕ is a factor.

For $\beta = \infty$, that is, for “zero temperature”, there are two different notions of equilibrium states. A state φ on B is a γ -KMS $_{\infty}$ -state if it is the weak*-limit of a net $(\varphi_i)_i$ where φ_i is a γ -KMS $_{\beta_i}$ -state for each i and $\beta_i \rightarrow \infty$, and a state φ on B is a γ -ground state if the map

$$z \mapsto \varphi(x\gamma_z(y))$$

is bounded on the upper half-plane for all γ -analytic elements x, y in a γ -invariant subset with dense linear span. Every KMS $_{\infty}$ state is a ground state by [4, Proposition 5.3.23], but there may be ground states that are not KMS $_{\infty}$ states, as we shall see in Theorem 3.2(iii)&(iv) below. Also see [23, Corollary 1.8] for a more general explanation of why the set of KMS $_{\infty}$ states may be properly contained in the set of ground states, and [29, Theorem 7.1(3)&(4)] and [13, Section 8] for examples of this phenomenon. Note that the distinction between KMS $_{\infty}$ states and ground states was first made in [9, Definition 3.7], and is not observed in [4].

If B is unital, then the set Σ_{∞} of KMS $_{\infty}$ states of the system (B, \mathbb{R}, γ) is a convex weak*-compact subset of $S(B)$ by [9, Proposition 3.8], whereas the set of ground states need not be a simplex, see [29, Remark 7.2(v)].

We now return to the C*-algebra $C_{\lambda}^*(R \rtimes R_{\mathfrak{m}, \Gamma})$. For a non-zero ideal \mathfrak{a} of R , we let $N(\mathfrak{a}) := |R/\mathfrak{a}|$ denote the norm of \mathfrak{a} , which is always finite. The map $a \mapsto N(aR)$

defines a semigroup homomorphism from R^\times to the multiplicative semigroup $\mathbb{N}^\times := \mathbb{N} \setminus \{0\}$ of positive integers, and the map $R \rtimes R_{\mathfrak{m},\Gamma} \rightarrow \mathbb{R}_+^*$ given by $(b, a) \mapsto N(a)$ is a semigroup homomorphism. For each $t \in \mathbb{R}$, let U_t denote the diagonal unitary on $\ell^2(R \rtimes R_{\mathfrak{m},\Gamma})$ that is determined on the canonical basis by

$$U_t(\varepsilon_{(b,a)}) = N(a)^{it} \varepsilon_{(b,a)}.$$

Then $t \mapsto U_t$ defines a unitary representation $\mathbb{R} \rightarrow \mathcal{U}(\ell^2(R \rtimes R_{\mathfrak{m},\Gamma}))$, and a routine argument shows that the group $\{U_t : t \in \mathbb{R}\}$ of unitaries implements a time evolution on $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$; specifically, we have the following result.

Proposition 3.1. *There is a time evolution $\sigma : \mathbb{R} \rightarrow \text{Aut}(C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma}))$ such that*

$$\sigma_t(\lambda_{(b,a)}) = N(a)^{it} \lambda_{(b,a)} \quad \text{for all } (b, a) \in R \rtimes R_{\mathfrak{m},\Gamma} \text{ and } t \in \mathbb{R}. \quad (2)$$

Let $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ be an ideal class. An integral ideal $\mathfrak{a} \in \mathfrak{k}$ is said to be *norm-minimizing in the class \mathfrak{k}* if $N(\mathfrak{a}) \leq N(\mathfrak{b})$ for every other integral ideal \mathfrak{b} in \mathfrak{k} ; note that each ideal class \mathfrak{k} contains only finitely many norm-minimizing ideals. Norm-minimizing ideals appeared in [30] during the investigation of phase transitions for C^* -dynamical systems associated with Hecke C^* -algebras, and then later in [13, Section 8] and [23].

Let $R_{\mathfrak{m},\Gamma}^* := R_{\mathfrak{m},\Gamma} \cap R^*$ be the group of invertible elements in $R_{\mathfrak{m},\Gamma}$. For each fractional ideal $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}$, the group $R_{\mathfrak{m},\Gamma}^*$ acts on (the additive group of) \mathfrak{a} by multiplication, so we may form the semi-direct product group $\mathfrak{a} \rtimes R_{\mathfrak{m},\Gamma}^*$.

View $\ell^\infty(R \rtimes R_{\mathfrak{m},\Gamma})$ as a sub- C^* -algebra of $\mathcal{B}(\ell^2(R \rtimes R_{\mathfrak{m},\Gamma}))$ in the canonical way, and let \mathcal{E} be the restriction to $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ of the canonical faithful conditional expectation $\mathcal{B}(\ell^2(R \rtimes R_{\mathfrak{m},\Gamma})) \rightarrow \ell^\infty(R \rtimes R_{\mathfrak{m},\Gamma})$. It follows from [33, Lemma 3.11] that the range of \mathcal{E} is equal to $D_\lambda(R \rtimes R_{\mathfrak{m},\Gamma})$. The main result of this paper is the following phase transition theorem.

Theorem 3.2. *Let K be a number field, \mathfrak{m} a modulus for K , and Γ a subgroup of $(R/\mathfrak{m})^*$. For each $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$, choose a norm-minimizing ideal $\mathfrak{a}_{\mathfrak{k},1}$ in \mathfrak{k} .*

- (i) *There are no σ -KMS $_\beta$ states on $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ for $\beta < 1$.*
- (ii) *For each $\beta \in [1, 2]$, there is a unique σ -KMS $_\beta$ state ϕ_β on $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$. The state ϕ_β factors through the expectation $\mathcal{E} : C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma}) \rightarrow D_\lambda(R \rtimes R_{\mathfrak{m},\Gamma})$*

and is determined by the values

$$\phi_\beta(E_{(x+\mathfrak{a}) \times (\mathfrak{a} \cap R_{\mathfrak{m}, \Gamma})}) = N(\mathfrak{a})^{-\beta} \quad \text{for } x \in R \text{ and } \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+.$$

Moreover, ϕ_β is of type III₁; indeed, the von Neumann algebra $\pi_{\phi_\beta}(C_\lambda^*(R \rtimes R_{\mathfrak{m}, \Gamma}))''$ generated by the GNS representation π_{ϕ_β} of ϕ_β is isomorphic to the injective factor of type III₁ with separable predual.

- (iii) For each $\beta \in (2, \infty)$, there is an affine isomorphism of the simplex of tracial states on the C^* -algebra

$$\bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})} C^*(\mathfrak{a}_{\mathfrak{k}, 1} \rtimes R_{\mathfrak{m}, \Gamma}^*)$$

onto the simplex of σ -KMS _{β} states on $C_\lambda^*(R \rtimes R_{\mathfrak{m}, \Gamma})$.

- (iv) There is an affine isomorphism of the σ -ground state space of $C_\lambda^*(R \rtimes R_{\mathfrak{m}, \Gamma})$ onto the state space of the C^* -algebra

$$\bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})} M_{k_{\mathfrak{k}} \cdot N(\mathfrak{a}_{\mathfrak{k}, 1})}(C^*(\mathfrak{a}_{\mathfrak{k}, 1} \rtimes R_{\mathfrak{m}, \Gamma}^*))$$

where $k_{\mathfrak{k}}$ is the number of norm-minimizing ideals in the class \mathfrak{k} .

Before continuing to the proof, we make several remarks.

- Remark 3.3.* (a) For the particular case of trivial \mathfrak{m} and Γ , Theorem 3.2 recovers the parameterization results obtained in [13, Sections 6 and 7].
- (b) If \mathfrak{a} and \mathfrak{b} lie in the same class $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})$, so that there is a $k \in K_{\mathfrak{m}, \Gamma}$ with $\mathfrak{a} = k\mathfrak{b}$, then the map $x \mapsto kx$ defines an $R_{\mathfrak{m}, \Gamma}^*$ -equivariant isomorphism $\mathfrak{a} \cong \mathfrak{b}$, so that $\mathfrak{a} \rtimes R_{\mathfrak{m}, \Gamma}^* \cong \mathfrak{b} \rtimes R_{\mathfrak{m}, \Gamma}^*$. Thus, in parts (iii) and (iv), we could replace each C^* -algebra $C^*(\mathfrak{a}_{\mathfrak{k}, 1} \rtimes R_{\mathfrak{m}, \Gamma}^*)$ with $C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes R_{\mathfrak{m}, \Gamma}^*)$ for any other ideal $\mathfrak{a}_{\mathfrak{k}}$ in the class \mathfrak{k} .
- (c) In light of Theorem 3.2(iii), it is natural to ask if one can explicitly describe the simplex of traces on the group C^* -algebra $C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes R_{\mathfrak{m}, \Gamma}^*)$ for a fixed class $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})$ and integral ideal $\mathfrak{a}_{\mathfrak{k}} \in \mathfrak{k}$. It turns out that, even for trivial \mathfrak{m} and Γ , this is a difficult problem that is related to the generalized Furstenberg conjecture, see [31] and [6, § 5].

- (d) In joint work with Xin Li [7, Theorem 4.1], we prove that the K -theory of the C^* -algebra $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ decomposes as

$$K_*(C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})) \cong \bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} K_*(C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes R_{\mathfrak{m},\Gamma}^*))$$

where $\mathfrak{a}_{\mathfrak{k}}$ is an integral ideal in the class \mathfrak{k} . It is interesting that the C^* -algebra $\bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} C^*(\mathfrak{a}_{\mathfrak{k}} \rtimes R_{\mathfrak{m},\Gamma}^*)$ appears in both the K -theory formula and the parameterization of the low temperature KMS states. For the $ax+b$ -semigroup $R \rtimes R^\times$, this has already been discussed by Cuntz in [14, Chapter 6, Section 6].

- (e) An alternative method for computing the low temperature KMS states on $C_\lambda^\times(R \rtimes R^\times)$ is discussed in [14, Chapter 6, Section 6]. Presumably, it could also be used here.
- (f) It follows from Theorem 3.2(iii)&(iv) that there are usually ground states which are not KMS_∞ states, that is, there is a phase transition at $\beta = \infty$. For example, let $K = \mathbb{Q}$, so that $R = \mathbb{Z}$. Let $m \in \mathbb{N}^\times$ be a positive natural number, and let $\mathfrak{m} = \mathfrak{m}_\infty \mathfrak{m}_0$ where \mathfrak{m}_∞ takes the value one at the only real embedding of \mathbb{Q} and $\mathfrak{m}_0(p) := v_p(m)$. Then a calculation shows that the map $(\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},1})$ given by $[a]_m \mapsto [a\mathbb{Z}]$ is a well-defined isomorphism $(\mathbb{Z}/m\mathbb{Z})^* \cong \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},1})$ and that each class $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},1})$ contains a unique norm-minimizing ideal of norm $n_{\mathfrak{k}}$ where $n_{\mathfrak{k}}$ is the smallest positive integer in the residue class modulo m corresponding to \mathfrak{k} under the above isomorphism. Moreover, in this situation, the isotropy groups appearing in Theorem 3.2(iii)&(iv) are all isomorphic to \mathbb{Z} , so Theorem 3.2 implies that the KMS_∞ states are parameterized by traces on the commutative C^* -algebra

$$\bigoplus_{\mathfrak{k} \in (\mathbb{Z}/m\mathbb{Z})^*} C^*(\mathbb{Z}) \cong \bigoplus_{\mathfrak{k} \in (\mathbb{Z}/m\mathbb{Z})^*} C(\mathbb{T}),$$

whereas the ground states are parameterized by states on the C^* -algebra

$$\bigoplus_{\mathfrak{k} \in (\mathbb{Z}/m\mathbb{Z})^*} M_{n_{\mathfrak{k}}}(C^*(\mathbb{Z})) \cong \bigoplus_{\mathfrak{k} \in (\mathbb{Z}/m\mathbb{Z})^*} M_{n_{\mathfrak{k}}}(C(\mathbb{T})).$$

- (g) For $\beta > 2$, the extremal σ - KMS_β states on $C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$ are either type I or type II. However, the techniques needed to deal with the case $\beta >$

2 are rather different since these states usually do not factor through the expectation \mathcal{E} , see [6]

This section and the next are devoted to the proof of Theorem 3.2, which we break up into several parts. The next five subsections contain some preliminaries, and the proofs of parts (i) through (iv), excluding the type computation. The proof that the von Neumann algebra $\pi_{\phi_\beta}(C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma}))''$ is isomorphic to the injective factor of type III₁ with separable predual is given in Section 4.

3.2. Preliminaries for the proof. The semigroup $R \rtimes R_{\mathfrak{m},\Gamma}$ canonically embeds into the group $(R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m},\Gamma}$ where $(R_{\mathfrak{m}}^{-1}R) = \{\frac{a}{b} : a \in R, b \in R_{\mathfrak{m}}\}$ is the localization of R at $R_{\mathfrak{m}}$. By [5, Proposition 3.2], the semigroup $R \rtimes R_{\mathfrak{m},\Gamma}$ is left Ore and its group of left quotients coincides with $(R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m},\Gamma}$. That is, $(R \rtimes R_{\mathfrak{m},\Gamma})^{-1}(R \rtimes R_{\mathfrak{m},\Gamma}) = (R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m},\Gamma}$. To simplify notation, let

$$P_{\mathfrak{m},\Gamma} := R \rtimes R_{\mathfrak{m},\Gamma} \quad \text{and} \quad G_{\mathfrak{m},\Gamma} := (R_{\mathfrak{m}}^{-1}R) \rtimes K_{\mathfrak{m},\Gamma}.$$

Also let $S := \{\mathfrak{p} \in \mathcal{P}_K : \mathfrak{p} \mid \mathfrak{m}_0\}$ be the *support of \mathfrak{m}_0* , which is a finite set of primes, and put $\mathcal{P}_K^{\mathfrak{m}} := \mathcal{P}_K \setminus S$.

3.2.1. A groupoid model. The material below on adeles and a groupoid model for $C_\lambda^*(P_{\mathfrak{m},\Gamma})$ is from [5, Section 5]. It was motivated by similar results from [13, Section 5] for the special case where \mathfrak{m} and Γ are trivial.

For each non-zero prime ideal \mathfrak{p} of R , let $K_{\mathfrak{p}}$ be the corresponding \mathfrak{p} -adic completion of K and $R_{\mathfrak{p}}$ the ring of integers in $K_{\mathfrak{p}}$. Let

$$\mathbb{A}_S := \left\{ \mathbf{a} = (a_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}} K_{\mathfrak{p}} : a_{\mathfrak{p}} \in R_{\mathfrak{p}} \text{ for all but finitely many } \mathfrak{p} \right\}$$

equipped with the restricted product topology with respect to the compact open subsets $R_{\mathfrak{p}} \subseteq K_{\mathfrak{p}}$. Denote by \hat{R}_S the compact subring $\prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}} R_{\mathfrak{p}}$, and let $\hat{R}_S^* := \prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}} R_{\mathfrak{p}}^*$ be the group of units of \hat{R}_S . The compact group \hat{R}_S^* acts on \mathbb{A}_S by multiplication, and we let $\bar{\mathbf{a}}$ denote the image of $\mathbf{a} \in \mathbb{A}_S$ under the quotient mapping $\mathbb{A}_S \rightarrow \mathbb{A}_S / \hat{R}_S^*$. Define an equivalence relation on $\mathbb{A}_S \times \mathbb{A}_S / \hat{R}_S^*$ by

$$(\mathbf{b}, \bar{\mathbf{a}}) \sim (\mathbf{d}, \bar{\mathbf{c}}) \quad \text{if } \bar{\mathbf{a}} = \bar{\mathbf{c}} \text{ and } \mathbf{b} - \mathbf{d} \in \bar{\mathbf{a}} \hat{R}_S^*.$$

Via the diagonal embedding, the groups $R_m^{-1}R_m$ and $K_{m,\Gamma}$ act on \mathbb{A}_S by translation and multiplication, respectively. The canonical action of $G_{m,\Gamma}$ on $\mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*$ given by $(n, k)(\mathbf{b}, \bar{\mathbf{a}}) = (n + k\mathbf{b}, k\bar{\mathbf{a}})$ descends to a well-defined action on the locally compact Hausdorff quotient space

$$\Omega_K^m := (\mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*)/\sim.$$

By restricting the above equivalence relation to the subset $\hat{R}_S \times \hat{R}_S/\hat{R}_S^* \subseteq \mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*$, we obtain the compact open subset

$$\Omega_R^m := (\hat{R}_S \times \hat{R}_S/\hat{R}_S^*)/\sim$$

of Ω_K^m . Let $G_{m,\Gamma} \times \Omega_R^m$ be the reduction of the transformation groupoid $G_{m,\Gamma} \times \Omega_K^m$ by the set Ω_R^m , that is,

$$G_{m,\Gamma} \times \Omega_R^m = \{(g, w) \in G_{m,\Gamma} \times \Omega_K^m : gw \in \Omega_R^m\}.$$

Our choice of notation for the reduction groupoid comes from the fact that $G_{m,\Gamma} \times \Omega_R^m$ can be canonically identified with a *partial transformation groupoid*, see [36, Section 3.3].

Proposition 3.4 ([5, Propositions 4.1 and 5.3]). *There is an isomorphism*

$$\vartheta : C_\lambda^*(P_{m,\Gamma}) \cong C^*(G_{m,\Gamma} \times \Omega_R^m)$$

that is determined on generators by $\vartheta(\lambda_{(b,a)}) = 1_{\{(b,a)\} \times \Omega_R^m}$ for $(b, a) \in P_{m,\Gamma}$.

Proof. The key result needed here is [5, Proposition 5.3]. The proof briefly sketched in [5] follows arguments from [35, Section 2] closely. Here, we shall sketch another more direct proof, which is closer to the proof of the analogous result given in [13, Section 5] for the case of the full $ax + b$ -semigroup.

Using Li's theory of semigroup C*-algebras, it is shown in [5, Section 5] that $C_\lambda^*(P_{m,\Gamma})$ can be canonically identified with the groupoid C*-algebra $C_r^*(G_{m,\Gamma} \times \Omega_{P_{m,\Gamma}})$ of the partial transformation groupoid $G_{m,\Gamma} \times \Omega_{P_{m,\Gamma}}$ where $\Omega_{P_{m,\Gamma}}$ is the spectrum of the commutative C*-algebra

$$D_\lambda(P_{m,\Gamma}) = \overline{\text{span}}(\{E_{(x+\mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma})} : x \in R, \mathbf{a} \in \mathcal{I}_m^+\}),$$

$E_{(x+\mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma})} \in \mathcal{B}(\ell^2(R \times R_{m,\Gamma}))$ is the orthogonal projection onto the subspace $\ell^2((x + \mathbf{a}) \times (\mathbf{a} \cap R_{m,\Gamma}))$, and the partial action of $G_{m,\Gamma}$ on $D_\lambda(P_{m,\Gamma})$ is determined by

$(n, k)E_{(x+a) \times (a \cap R_{m,\Gamma})} = E_{(n+kx+ka) \times (ka) \cap R_{m,\Gamma}}$ for $(n, k) \in G_{m,\Gamma}$ and $x \in R$, $\mathbf{a} \in \mathcal{I}_m^+$ such that $k\mathbf{a} \in \mathcal{I}_m^+$ and $n+kx \in R$.

For each coset $x + \mathbf{a}$ where $\mathbf{a} \in \mathcal{I}_m^+$ and $x \in R$, let

$$V_{(x+a) \times a^\times} := \{[\mathbf{b}, \bar{\mathbf{a}}] \in \Omega_R^m : v_{\mathbf{p}}(\bar{\mathbf{a}}) \geq v_{\mathbf{p}}(\mathbf{a}), v_{\mathbf{p}}(\mathbf{b} - x) \geq v_{\mathbf{p}}(\mathbf{a}) \text{ for all } \mathbf{p} \in \mathcal{P}_K^m\}$$

where $\mathbf{a}^\times := \mathbf{a} \setminus \{0\}$. Then a calculation shows that $V_{(x+a) \times a^\times} \cap V_{(y+b) \times b^\times} = V_{[(x+a) \cap (y+b)] \times (a \cap b)^\times}$ where $V_\emptyset := \emptyset$. By [5, Proposition 3.4] and [34, Corollary 2.7], there exists a *-homomorphism $D_\lambda(P_{m,\Gamma}) \rightarrow C(\Omega_R^m)$ such that $E_{(x+a) \times (a \cap R_{m,\Gamma})} \mapsto 1_{V_{(x+a) \times a^\times}}$. This map is injective by [14, Proposition 5.6.21], and one can directly check, as was done in [13, Proposition 5.2] for the full $ax + b$ -semigroup, that this map is also surjective. Since it is also $G_{m,\Gamma}$ -equivariant, it follows that $C_r^*(G_{m,\Gamma} \rtimes \Omega_{P_{m,\Gamma}}) \cong C^*(G_{m,\Gamma} \rtimes \Omega_R^m)$, and it is not difficult to see that $\lambda_{(b,a)}$ is mapped to $1_{\{(b,a)\} \times \Omega_R^m}$ for all $(b, a) \in P_{m,\Gamma}$. \square

3.2.2. Quasi-invariant measures on Ω_R^m . The multiplicative map $R^\times \rightarrow \mathbb{N}^\times$ given by $a \mapsto N(aR) = |R/aR|$ has a unique extension to a group homomorphism $K^* \rightarrow \mathbb{Q}_+^*$ that we also denote by N . Let c_N be the real-valued one-cocycle $G_{m,\Gamma} \rtimes \Omega_R^m \rightarrow \mathbb{R}$ given by $c_N((n, k), w) = \log N(k)$, so that [44, Proposition 5.1] gives us a time evolution σ^{c_N} on $C^*(G_{m,\Gamma} \rtimes \Omega_R^m)$ such that

$$\sigma_t^{c_N}(f)((n, k), w) = N(k)^{it} f((n, k), w) \quad \text{for all } f \in C_c(G_{m,\Gamma} \rtimes \Omega_R^m) \text{ and } t \in \mathbb{R}. \quad (3)$$

Lemma 3.5. *Under the isomorphism $\vartheta : C_\lambda^*(P_{m,\Gamma}) \cong C^*(G_{m,\Gamma} \rtimes \Omega_R^m)$ from Proposition 3.4, the time evolution σ from (2) is conjugated to σ^{c_N} , that is, $\sigma_t^{c_N} = \vartheta \circ \sigma_t \circ \vartheta^{-1}$ for every $t \in \mathbb{R}$.*

Proof. We have $\vartheta(\lambda_{(b,a)}) = 1_{\{(b,a)\} \times \Omega_R^m}$ for $(b, a) \in P_{m,\Gamma}$, and a short calculation shows that $\sigma_t^{c_N}(1_{\{(b,a)\} \times \Omega_R^m}) = N(a)^{it} 1_{\{(b,a)\} \times \Omega_R^m}$ for all $t \in \mathbb{R}$. Thus, we have $\vartheta \circ \sigma_t \circ \vartheta^{-1}(1_{\{(b,a)\} \times \Omega_R^m}) = \sigma_t^{c_N}(1_{\{(b,a)\} \times \Omega_R^m})$ for all $t \in \mathbb{R}$. Since the collection $\{\lambda_{(b,a)} : (b, a) \in P_{m,\Gamma}\}$ generates $C_\lambda^*(P_{m,\Gamma})$ as a C*-algebra, this is enough. \square

Lemma 3.5 implies that there is an isomorphism of C*-dynamical systems

$$(C_\lambda^*(P_{m,\Gamma}), \mathbb{R}, \sigma) \cong (C^*(G_{m,\Gamma} \rtimes \Omega_R^m), \mathbb{R}, \sigma^{c_N}),$$

so we may work with the latter system for our computations of KMS and ground states. From now on, we will write σ rather than σ^{c_N} for the time evolution on $C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}})$.

Any state ϕ on $C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}})$ defines a probability measure μ on $\Omega_R^{\mathfrak{m}}$ by restricting ϕ to $C(\Omega_R^{\mathfrak{m}})$ and then applying the Riesz representation theorem to the state $\phi|_{C(\Omega_R^{\mathfrak{m}})}$. It is well-known, going back to [44, Proposition 5.4], that if ϕ is a σ -KMS $_{\beta}$ state, then the KMS $_{\beta}$ condition (1) forces the measure μ to be *quasi-invariant with Radon-Nikodym cocycle given by $e^{-\beta c_N} = N^{-\beta}$* , that is, μ must satisfy the scaling condition

$$\mu((n, k)Z) = N(k)^{-\beta} \mu(Z) \quad (4)$$

for all $(n, k) \in G_{\mathfrak{m},\Gamma}$ and Borel sets $Z \subseteq \Omega_R^{\mathfrak{m}}$ such that $(n, k)Z \subseteq \Omega_R^{\mathfrak{m}}$. Moreover, the set of probability measures that satisfy (4) forms a (possibly empty) Choquet simplex, see, for example, [45, Exercise 3.3.1].

There may be many σ -KMS $_{\beta}$ states on $C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}})$ that define the same quasi-invariant measure on $\Omega_R^{\mathfrak{m}}$, and [43, Theorem 1.3] gives a parameterization of all such σ -KMS $_{\beta}$ states in terms of traces on the C*-algebras of certain isotropy groups. Thus, to compute the σ -KMS $_{\beta}$ states on $C^*(G_{\mathfrak{m},\Gamma} \times \Omega_R^{\mathfrak{m}})$ for $\beta < \infty$, we must first compute, for each fixed $\beta \in \mathbb{R}$, the simplex of all probability measures μ on $\Omega_R^{\mathfrak{m}}$ that satisfy (4). It is easy to see that there are no such measures for $\beta < 1$, as explained in Section 3.3 below, and for this reason we restrict to the case $\beta \geq 1$ now.

Lemma 3.6. *Let $\mathcal{J} := \{(x + \mathfrak{a}) \times \mathfrak{a}^{\times} : x \in R, \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\}$, and for $(x + \mathfrak{a}) \times \mathfrak{a}^{\times} \in \mathcal{J}$, let*

$$V_{(x+\mathfrak{a}) \times \mathfrak{a}^{\times}} := \{[\mathfrak{b}, \bar{\mathfrak{a}}] \in \Omega_R^{\mathfrak{m}} : v_{\mathfrak{p}}(\bar{\mathfrak{a}}) \geq v_{\mathfrak{p}}(\mathfrak{a}), v_{\mathfrak{p}}(\mathfrak{b} - x) \geq v_{\mathfrak{p}}(\mathfrak{a}) \text{ for all } \mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}\}.$$

If μ is a probability measure on $\Omega_R^{\mathfrak{m}}$ and $\beta \geq 1$, then μ satisfies (4) if and only if

$$\mu((n, k)V_X) = N(k)^{-\beta} \mu(V_X) \quad (5)$$

for all $(n, k) \in G_{\mathfrak{m},\Gamma}$ and $X \in \mathcal{J}$ such that $(n, k)X = \{(n + kx, ky) : (x, y) \in X\}$ lies in \mathcal{J} .

Moreover, for $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, let

$$U_{\mathfrak{a}} := \hat{\mathfrak{a}}\hat{R}_S/\hat{R}_S^* = \{\bar{\mathfrak{a}} \in \hat{R}_S/\hat{R}_S^* : v_{\mathfrak{p}}(\bar{\mathfrak{a}}) \geq v_{\mathfrak{p}}(\mathfrak{a}) \text{ for all } \mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}\}.$$

Then for $k \in K_{\mathfrak{m}, \Gamma}$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ such that $k\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, we have $kU_{\mathfrak{a}} = U_{k\mathfrak{a}}$, and a probability measure ν on \hat{R}_S/\hat{R}_S^* satisfies

$$\nu(kZ) = N(k)^{-(\beta-1)}\nu(Z) \quad (6)$$

for every $k \in K_{\mathfrak{m}, \Gamma}$ and every Borel set $Z \subseteq \hat{R}_S/\hat{R}_S^*$ such that $kZ \subseteq \hat{R}_S/\hat{R}_S^*$ if and only if

$$\nu(kU_{\mathfrak{a}}) = N(k)^{-(\beta-1)}\nu(U_{\mathfrak{a}})$$

for all $k \in K_{\mathfrak{m}, \Gamma}$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ such that $k\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$.

Proof. A calculation shows that V_X is the support of the projection $\vartheta(E_X) \in C(\Omega_R^{\mathfrak{m}})$. The result follows from the fact that the projections $\{\vartheta(E_X) : X \in \mathcal{J}\} \cup \{0\}$ span a dense sub- $*$ -algebra of $C(\Omega_R^{\mathfrak{m}})$.

A short calculation shows that for $k \in K_{\mathfrak{m}, \Gamma}$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ such that $k\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, we have $kU_{\mathfrak{a}} = U_{k\mathfrak{a}}$. The last claim follows from the fact that the projections $\{1_{U_{\mathfrak{a}}} : \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+\}$ span a dense sub- $*$ -algebra of $C(\hat{R}_S/\hat{R}_S^*)$. \square

Our next result is inspired by the proof of [26, Proposition 2.1] and [43, Section 3].

Proposition 3.7. *Let π denote the quotient map $\hat{R}_S \times \hat{R}_S/\hat{R}_S^* \rightarrow \Omega_R^{\mathfrak{m}}$, and let m denote the normalized Haar measure on \hat{R}_S . Given a probability measure ν on \hat{R}_S/\hat{R}_S^* , form the product measure $m \times \nu$, and let $\pi_*(m \times \nu)$ denote the probability measure on $\Omega_R^{\mathfrak{m}}$ obtained by pushing forward $m \times \nu$ under π . For each fixed $\beta \geq 1$, the map $\nu \mapsto \pi_*(m \times \nu)$ defines an affine bijection from the set of probability measures ν on \hat{R}_S/\hat{R}_S^* satisfying Equation (6) onto the set of probability measures on $\Omega_R^{\mathfrak{m}}$ satisfying (4).*

Proof. Suppose that ν is a probability measure on \hat{R}_S/\hat{R}_S^* satisfying (6), and let $\mu := \pi_*(m \times \nu)$. We need to show that μ satisfies (4). By Lemma 3.6, it suffices to show that μ satisfies (5).

If $(n, k) \in G_{\mathfrak{m}, \Gamma}$ and $X = (x + \mathfrak{a}) \times \mathfrak{a}^{\times} \in \mathcal{J}$ are such that $(n, k)X \in \mathcal{J}$, then $(n + kx + k\mathfrak{a}\hat{R}_S) \times U_{k\mathfrak{a}} = \pi^{-1}(V_{(n+kx+k\mathfrak{a}) \times (k\mathfrak{a})^{\times}})$, so we have

$$\mu((n, k)V_X) = m(n+kx+k\mathfrak{a}\hat{R}_S)\nu(U_{k\mathfrak{a}}) = N(k\mathfrak{a})^{-1}N(k)^{-(\beta-1)}\nu(U_{\mathfrak{a}}) = N(k)^{-\beta}N(\mathfrak{a})^{-1}\nu(U_{\mathfrak{a}}).$$

For every $x \in R$,

$$\mu(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}) = m(x + \mathfrak{a}\hat{R}_S)\nu(U_{\mathfrak{a}}) = N(\mathfrak{a})^{-1}\nu(U_{\mathfrak{a}}),$$

so we see that μ satisfies (5). Since $\nu(U_{\mathfrak{a}}) = N(\mathfrak{a})\mu(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times})$, and ν is determined by its values on the sets $U_{\mathfrak{a}}$, we also conclude that the map $\nu \mapsto \pi_*(m \times \nu)$ is injective.

It remains to check surjectivity. Suppose μ is a probability measure on $\Omega_R^{\mathfrak{m}}$ satisfying (4), and let $q : \Omega_R^{\mathfrak{m}} \rightarrow \hat{R}_S/\hat{R}_S^*$ denote the surjective map given by $q([\mathfrak{b}, \bar{\mathfrak{a}}]) = \bar{\mathfrak{a}}$, so that $q \circ \pi = \pi_2$ is the projection from $\hat{R}_S \times \hat{R}_S^*$ onto the second coordinate. To show surjectivity, it is enough to show that

- (1) $q_*\mu$ satisfies (6);
- (2) $\mu = \pi_*(m \times q_*\mu)$.

For $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, let $V_{R \times \mathfrak{a}^\times} := \bigsqcup_{y \in R/\mathfrak{a}} V_{(y+\mathfrak{a}) \times \mathfrak{a}^\times}$. Since μ satisfies (4),

$$\mu(V_{R \times \mathfrak{a}^\times}) = \sum_{y \in R/\mathfrak{a}} \mu(V_{(y+\mathfrak{a}) \times \mathfrak{a}^\times}) = N(\mathfrak{a})\mu(V_{\mathfrak{a} \times \mathfrak{a}^\times}). \quad (7)$$

Let $k \in K_{\mathfrak{m}, \Gamma}$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ be such that $k\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. Then $q^{-1}(U_{k\mathfrak{a}}) = V_{R \times (k\mathfrak{a})^\times}$, so

$$\begin{aligned} q_*\mu(U_{k\mathfrak{a}}) &= \mu(V_{R \times (k\mathfrak{a})^\times}) = N(k\mathfrak{a})\mu(V_{k\mathfrak{a} \times (k\mathfrak{a})^\times}) \\ &= N(k\mathfrak{a})N(k)^{-\beta}\mu(V_{\mathfrak{a} \times \mathfrak{a}^\times}) \quad (\text{using that } \mu \text{ satisfies (4)}) \\ &= N(k)^{-(\beta-1)}\mu(V_{R \times \mathfrak{a}^\times}) \\ &= N(k)^{-(\beta-1)}q_*\mu(U_{\mathfrak{a}}). \end{aligned}$$

Thus 1. holds. To show 2., it suffices to show that $\mu(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}) = \pi_*(m \times q_*\mu)(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times})$ for all $(x + \mathfrak{a}) \times \mathfrak{a}^\times \in \mathcal{J}$. We have

$$\pi_*(m \times q_*\mu)(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}) = (m \times q_*\mu)((x + \mathfrak{a}\hat{R}_S) \times U_{\mathfrak{a}}) = N(\mathfrak{a})^{-1}q_*\mu(U_{\mathfrak{a}}) = N(\mathfrak{a})^{-1}\mu(V_{R \times \mathfrak{a}^\times}).$$

Using (7) and (4), we have $\mu(V_{R \times \mathfrak{a}^\times}) = N(\mathfrak{a})\mu(V_{\mathfrak{a} \times \mathfrak{a}^\times}) = N(\mathfrak{a})\mu(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times})$. Hence, $\mu(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times}) = \pi_*(m \times q_*\mu)(V_{(x+\mathfrak{a}) \times \mathfrak{a}^\times})$ for all $(x + \mathfrak{a}) \times \mathfrak{a}^\times \in \mathcal{J}$, as desired.

It is not difficult to check that the map $\nu \mapsto m \times \nu$ is affine, and since the push-forward map $m \times \nu \mapsto \pi_*(m \times \nu)$ is also affine, we see that $\nu \mapsto \pi_*(m \times \nu)$ is affine. \square

3.3. The easy case: part (i).

Proof of Theorem 3.2(i). Suppose that ϕ is a σ -KMS $_\beta$ state on $C_\lambda^*(P_{\mathfrak{m},\Gamma})$. For each $x \in R$ and each $a \in R_{\mathfrak{m},\Gamma}$, the KMS $_\beta$ condition (1) yields

$$\phi(\lambda_{(x,a)}\lambda_{(x,a)}^*) = N(a)^{-\beta}\phi(\lambda_{(x,a)}^*\lambda_{(x,a)}) = N(a)^{-\beta}.$$

Hence,

$$0 \leq \phi\left(1 - \sum_{x \in R/aR} \lambda_{(x,a)}\lambda_{(x,a)}^*\right) = 1 - \sum_{x \in R/aR} \phi(\lambda_{(x,a)}\lambda_{(x,a)}^*) = 1 - N(a)^{1-\beta},$$

so we must have $\beta \geq 1$. □

3.4. Uniqueness in the critical interval and the proof of part (ii). Let $\nu_0 := \delta_{\bar{\mathbf{0}}}$ be the unit mass concentrated at the point $\bar{\mathbf{0}} \in \hat{R}_S/\hat{R}_S^*$. For each $\beta \in (0, \infty)$, let $\nu_{\beta, \mathfrak{p}}$ be the probability measure on $\hat{R}_{\mathfrak{p}}/\hat{R}_{\mathfrak{p}}^* \cong \mathfrak{p}^{\mathbb{N} \cup \{\infty\}}$ given by $\nu_{\beta, \mathfrak{p}} = (1 - N(\mathfrak{p})^{-\beta}) \sum_{n=0}^{\infty} N(\mathfrak{p})^{-n\beta} \delta_{\mathfrak{p}^n}$, and let $\nu_\beta := \prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}} \nu_{\beta, \mathfrak{p}}$.

Lemma 3.8. *For each $\beta \in [0, \infty)$, the measure ν_β satisfies*

$$\nu(kZ) = N(k)^{-\beta}\nu(Z) \tag{8}$$

for every $k \in K_{\mathfrak{m},\Gamma}$ and every Borel set $Z \subseteq \hat{R}_S/\hat{R}_S^*$ such that $kZ \subseteq \hat{R}_S/\hat{R}_S^*$. Moreover, $\nu_\beta(U_{\mathfrak{a}}) = N(\mathfrak{a})^{-\beta}$ for all $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ where $U_{\mathfrak{a}} = \mathfrak{a}\hat{R}_S/\hat{R}_S^*$.

Proof. As pointed out in the proof of Proposition 3.7, to show ν_β satisfies (8), it suffices to show that ν_β satisfies

$$\nu_\beta(kU_{\mathfrak{a}}) = N(k)^{-\beta}\nu_\beta(U_{\mathfrak{a}})$$

for all $k \in K_{\mathfrak{m},\Gamma}$ and $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$ such that $k\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. Since $\bar{\mathbf{0}} \in U_{\mathfrak{a}}$ for all $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$, it is easy to see that ν_0 satisfies this condition. Now let $\beta \in (0, \infty)$. For any $\mathfrak{b} \in \mathcal{I}_{\mathfrak{m}}^+$, a calculation shows that $\nu_\beta(U_{\mathfrak{b}}) = N(\mathfrak{b})^{-\beta}$ which settles the second claim. Using this, we have

$$\nu_\beta(kU_{\mathfrak{a}}) = \nu_\beta(U_{k\mathfrak{a}}) = N(k\mathfrak{a})^{-\beta} = N(k)^{-\beta}N(\mathfrak{a})^{-\beta} = N(k)^{-\beta}\nu_\beta(U_{\mathfrak{a}})$$

as desired. □

The crux in computing the KMS_β states for $\beta \in [1, 2]$ is the following purely measure-theoretic result.

Theorem 3.9. *For each $\beta \in [0, 1]$, ν_β is the unique probability measure on \hat{R}_S/\hat{R}_S^* satisfying (8).*

To prove Theorem 3.9, we will expand on an idea of Neshveyev's from the end of [43, Section 3], which will put us in a setting where we can employ techniques analogous to those used for Bost–Connes type systems.

We need two preliminary results. The first puts us in a situation where we can work with the lattice group \mathcal{I}_m of all fractional ideals coprime to \mathfrak{m}_0 , rather than the more complicated group $K_{m,\Gamma}$. The following result is motivated by the general techniques from [25] on extending KMS weights.

Lemma 3.10. *View \hat{R}_S/\hat{R}_S^* as a subset of $\mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ via the identification $\hat{R}_S/\hat{R}_S^* \simeq \{[R]\} \times \hat{R}_S/\hat{R}_S^*$. Then each probability measure ν on \hat{R}_S/\hat{R}_S^* satisfying (8) has a unique extension to a finite measure $\tilde{\nu}$ on $\mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ satisfying*

$$\tilde{\nu}(\mathfrak{a}Z) = N(\mathfrak{a})^{-\beta} \tilde{\nu}(Z) \quad (9)$$

for all $\mathfrak{a} \in \mathcal{I}_m$ and Borel sets $Z \subseteq \mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ such that $\mathfrak{a}Z \subseteq \mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ where $\mathfrak{a}Z = \{(\mathfrak{a}\mathfrak{k}, \mathfrak{a}\bar{\mathfrak{a}}) : (\mathfrak{k}, \bar{\mathfrak{a}}) \in Z\}$.

Proof. Our proof is similar to that of [21, Lemma 2.2]. For $\mathfrak{a} \in \mathcal{I}_m$, let $[\mathfrak{a}]$ denote the class of \mathfrak{a} in $\mathcal{I}_m/i(K_{m,\Gamma})$, and for each integral ideal \mathfrak{a} , let $Y_{\mathfrak{a}} := \{[R]\} \times U_{\mathfrak{a}}$, so that $\hat{R}_S/\hat{R}_S^* \simeq Y_R \subseteq \mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$.

Suppose that ν is a probability measure on \hat{R}_S/\hat{R}_S^* satisfying (8). We first show that there can be at most one measure μ on $\mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ that both satisfies (9) and extends ν . Indeed, suppose that μ is such a measure, and for each class $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, choose an integral ideal $\mathfrak{a}_{\mathfrak{k}} \in \mathfrak{k}$; for $\mathfrak{k} = [R]$, take $\mathfrak{a}_{\mathfrak{k}} = R$. Then

$$\mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^* = \bigsqcup_{\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})} \mathfrak{a}_{\mathfrak{k}}^{-1} Y_{\mathfrak{a}_{\mathfrak{k}}},$$

so, for any Borel set Z ,

$$\mu(Z) = \sum_{\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})} \mu(Z \cap \mathfrak{a}_{\mathfrak{k}}^{-1} Y_{\mathfrak{a}_{\mathfrak{k}}}).$$

Since μ satisfies (9), $\mu(Z \cap \mathfrak{a}_\mathfrak{k}^{-1} Y_{\mathfrak{a}_\mathfrak{k}}) = \mu(\mathfrak{a}_\mathfrak{k}^{-1}(\mathfrak{a}_\mathfrak{k} Z \cap Y_{\mathfrak{a}_\mathfrak{k}})) = N(\mathfrak{a}_\mathfrak{k})^\beta \mu(\mathfrak{a}_\mathfrak{k} Z \cap Y_{\mathfrak{a}_\mathfrak{k}})$. Since $Y_{\mathfrak{a}_\mathfrak{k}} \subseteq Y_R$, we see that μ is determined by its restriction to Y_R . Thus, there can be at most one measure on $\mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ that both satisfies (9) and extends ν . We now proceed to construct this extension. Define $\tilde{\nu}$ on $\mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ by

$$\tilde{\nu}(Z) = \sum_{\mathfrak{k} \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})} N(\mathfrak{a}_\mathfrak{k})^\beta \nu(\mathfrak{a}_\mathfrak{k} Z \cap Y_{\mathfrak{a}_\mathfrak{k}})$$

for Borel sets $Z \subseteq \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma}) \times \hat{R}_S/\hat{R}_S^*$. A short calculation shows that $\tilde{\nu}$ is a finite measure extending ν . We need to show that $\tilde{\nu}$ satisfies (9). For each \mathfrak{k} , let $b_\mathfrak{k} \in K_{\mathfrak{m},\Gamma}$ be such that $\mathfrak{a}\mathfrak{a}_\mathfrak{k} = b_\mathfrak{k}\mathfrak{a}_{\mathfrak{a}\cdot\mathfrak{k}}$. We have

$$\begin{aligned} \tilde{\nu}(\mathfrak{a}Z) &= \sum_{\mathfrak{k} \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})} N(\mathfrak{a}_\mathfrak{k})^\beta \nu(\mathfrak{a}_\mathfrak{k}\mathfrak{a}Z \cap Y_{\mathfrak{a}_\mathfrak{k}}) \\ &= N(\mathfrak{a})^{-\beta} \sum_{\mathfrak{k} \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})} N(\mathfrak{a}_\mathfrak{k}\mathfrak{a})^\beta \nu(\mathfrak{a}_\mathfrak{k}\mathfrak{a}Z \cap Y_{\mathfrak{a}_\mathfrak{k}}) \\ &= N(\mathfrak{a})^{-\beta} \sum_{\mathfrak{k} \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})} N(\mathfrak{a}_\mathfrak{k}\mathfrak{a})^\beta \nu(\mathfrak{a}_\mathfrak{k}\mathfrak{a}Z \cap Y_{\mathfrak{a}\mathfrak{a}_\mathfrak{k}}) \quad (\text{since } \mathfrak{a}_\mathfrak{k}(\mathfrak{a}Z \cap Y_R) = \mathfrak{a}_\mathfrak{k}(\mathfrak{a}Z \cap Y_{\mathfrak{a}})) \\ &= N(\mathfrak{a})^{-\beta} \sum_{\mathfrak{k} \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})} N(b_\mathfrak{k}\mathfrak{a}_{\mathfrak{a}\cdot\mathfrak{k}})^\beta \nu(b_\mathfrak{k}\mathfrak{a}_{\mathfrak{a}\cdot\mathfrak{k}}Z \cap Y_{b_\mathfrak{k}\mathfrak{a}_{\mathfrak{a}\cdot\mathfrak{k}}}) \\ &= N(\mathfrak{a})^{-\beta} \sum_{\mathfrak{k} \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})} N(b_\mathfrak{k}\mathfrak{a}_{\mathfrak{a}\cdot\mathfrak{k}})^\beta N(b_\mathfrak{k})^{-\beta} \nu(\mathfrak{a}_{\mathfrak{a}\cdot\mathfrak{k}}Z \cap Y_{\mathfrak{a}_{\mathfrak{a}\cdot\mathfrak{k}}}) \quad (\text{using (8)}) \\ &= N(\mathfrak{a})^{-\beta} \sum_{\tilde{\mathfrak{k}} \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma})} N(\mathfrak{a}_{\tilde{\mathfrak{k}}})^\beta \nu(\mathfrak{a}_{\tilde{\mathfrak{k}}}Z \cap Y_{\mathfrak{a}_{\tilde{\mathfrak{k}}}}) \\ &= N(\mathfrak{a})^{-\beta} \tilde{\nu}(Z). \end{aligned}$$

This concludes the proof. □

The following ergodicity results is the key step towards Theorem 3.9.

Proposition 3.11. *Let $\beta \in (0, 1]$ and suppose that ν is a probability measure on $\mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma}) \times \hat{R}_S/\hat{R}_S^*$ satisfying (9). Then the closed subspace*

$$H = \{f \in L^2(\mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma}) \times \hat{R}_S/\hat{R}_S^*, \nu) : f(\mathfrak{a}z) = f(z) \text{ for } \mathfrak{a} \in \mathcal{I}_\mathfrak{m}^+, z \in \mathcal{I}_\mathfrak{m}/i(K_{\mathfrak{m},\Gamma}) \times \hat{R}_S/\hat{R}_S^*\}$$

of $L^2(\mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*, \nu)$ consisting of \mathcal{I}_m^+ -invariant functions coincides with the constant functions. That is, the partial action of \mathcal{I}_m on $(\mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*, \nu)$ is ergodic.

Proof. The proof is similar to that of [22, Theorem 2.1(ii)]. Let P be the orthogonal projection from $L^2(\mathcal{I}_m/i(K_{m,\Gamma}) \times \hat{R}_S/\hat{R}_S^*, \nu)$ onto H ; we need to show that Pf is a constant function for every f . For this, it suffices to compute P at pull-backs of functions on

$$\mathcal{I}_m/i(K_{m,\Gamma}) \times \prod_{p \in F} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}}$$

for every non-empty finite subset $F \subseteq \mathcal{P}_K^m$. Now fix such an F , and let $\mathcal{I}_{F^c}^+$ be the free submonoid of \mathcal{I}_m^+ generated by the primes in F . Up to a set of measure zero, $\mathcal{I}_m/i(K_{m,\Gamma}) \times \prod_{p \in F} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}}$ coincides with

$$\bigsqcup_{\mathfrak{a} \in \mathcal{I}_{F^c}^+} \mathfrak{a}(\mathcal{I}_m/i(K_{m,\Gamma}) \times \underbrace{\{(1, \dots, 1)\}}_{|F|\text{-times}}).$$

Since $\mathcal{I}_m/i(K_{m,\Gamma}) \times \{(1, \dots, 1)\} \simeq \mathcal{I}_m/i(K_{m,\Gamma})$ is a finite group, it suffices to compute, for each fixed $\mathfrak{a} \in \mathcal{I}_{F^c}^+$ and character $\tilde{\chi}$ of $\mathcal{I}_m/i(K_{m,\Gamma})$, Pf where f is the pull-back of

$$\mathcal{I}_m/i(K_{m,\Gamma}) \times \prod_{p \in F} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}} \ni (\mathfrak{k}, \bar{\mathfrak{a}}) \mapsto \begin{cases} \tilde{\chi}([\mathfrak{a}]^{-1}\mathfrak{k}) & \text{if } (\mathfrak{k}, \bar{\mathfrak{a}}) \in \mathfrak{a}(\mathcal{I}_m/i(K_{m,\Gamma}) \times \underbrace{\{(1, \dots, 1)\}}_{|F|\text{-times}}) \\ 0 & \text{otherwise.} \end{cases}$$

The character $\chi : \mathcal{I}_m \rightarrow \mathbb{T}$ defined by $\chi(\mathfrak{a}) := \tilde{\chi}([\mathfrak{a}])$ satisfies $\chi(i(K_{m,1})) = \{1\}$, and is thus a (generalized) Dirichlet character modulo \mathfrak{m} .

For each finite subset $\tilde{F} \subseteq \mathcal{P}_K^m$, let $P_{\tilde{F}}$ be the orthogonal projection onto the subspace $H_{\tilde{F}}$ consisting of $\mathcal{I}_{\tilde{F}^c}^+$ -invariant functions, so that the projection P is the decreasing strong operator limit of the net $(P_{\tilde{F}})_{\tilde{F}}$. Also let

$$W_{\tilde{F}} := \mathcal{I}_m/i(K_{m,\Gamma}) \times \underbrace{\{(1, \dots, 1)\}}_{|\tilde{F}|\text{-times}} \times \prod_{p \in \tilde{F}^c} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}},$$

so that the sets $\mathfrak{b}W_{\tilde{F}}$ are disjoint for $\mathfrak{b} \in \mathcal{I}_{\tilde{F}^c}^+$, and their union has full ν -measure. Applying [22, Proposition 1.2(2)] with, in the notation from the statement of [22, Proposition 1.2], $Y_0 = W_{\tilde{F}}$, $Y = \mathcal{I}_m/i(K_{m,\Gamma}) \times \prod_{p \in \mathcal{P}_K^m} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}}$, $S = \mathcal{I}_{\tilde{F}^c}^+$ and $G = \mathcal{I}_{\tilde{F}^c}$,

we get that the projection $P_{\tilde{F}}$ is given explicitly by

$$P_{\tilde{F}}f|_{\mathcal{I}_{\tilde{F}^c}^+ w} \equiv \frac{1}{\zeta_{\tilde{F}}(\beta)} \sum_{\mathfrak{b} \in \mathcal{I}_{\tilde{F}^c}^+} N(\mathfrak{b})^{-\beta} f(\mathfrak{b}w)$$

for $w = (\mathfrak{k}, \bar{\mathfrak{a}}) \in W_{\tilde{F}}$ where $\zeta_{\tilde{F}}(\beta) := \prod_{\mathfrak{p} \in \tilde{F}} (1 - N(\mathfrak{p})^{-\beta})^{-1} = \sum_{\mathfrak{b} \in \mathcal{I}_{\tilde{F}^c}^+} N(\mathfrak{b})^{-\beta}$. Now suppose that $\tilde{F} \supseteq F$. Then for $f(\mathfrak{b}w)$ to be non-zero, it is necessary that $\mathfrak{b} \in \mathfrak{a}\mathcal{I}_{(\tilde{F} \setminus F)^c}^+$, and, in this case, $f(\mathfrak{b}w) = \tilde{\chi}([\mathfrak{a}^{-1}\mathfrak{b}]\mathfrak{k}) = \chi(\mathfrak{a}^{-1}\mathfrak{b})\tilde{\chi}(\mathfrak{k})$. Hence, for $w = (\mathfrak{k}, \bar{\mathfrak{a}}) \in W_{\tilde{F}}$, we have

$$P_{\tilde{F}}f|_{\mathcal{I}_{\tilde{F}^c}^+ w} = \frac{1}{\zeta_{\tilde{F}}(\beta)} \sum_{\mathfrak{b} \in \mathfrak{a}\mathcal{I}_{(\tilde{F} \setminus F)^c}^+} N(\mathfrak{b})^{-\beta} \chi(\mathfrak{a}^{-1}\mathfrak{b})\tilde{\chi}(\mathfrak{k}) = \frac{N(\mathfrak{a})^{-\beta}\tilde{\chi}(\mathfrak{k})}{\zeta_{\tilde{F}}(\beta)} \sum_{\mathfrak{c} \in \mathcal{I}_{(\tilde{F} \setminus F)^c}^+} N(\mathfrak{c})^{-\beta} \chi(\mathfrak{c}). \quad (10)$$

If χ is the trivial character, then the right-hand side of (10) equals

$$N(\mathfrak{a})^{-\beta} \frac{\prod_{\mathfrak{p} \in \tilde{F} \setminus F} (1 - N(\mathfrak{p})^{-\beta})^{-1}}{\prod_{\mathfrak{p} \in \tilde{F}} (1 - N(\mathfrak{p})^{-\beta})^{-1}} = N(\mathfrak{a})^{-\beta} \prod_{\mathfrak{p} \in F} (1 - N(\mathfrak{p})^{-\beta}),$$

so $Pf = \lim_{\tilde{F}} P_{\tilde{F}}f$ is constant. Now suppose that χ is non-trivial. Then,

$$\begin{aligned} \|P_{\tilde{F}}f\|_{L^2(\nu)}^2 &= \sum_{\mathfrak{b} \in \mathcal{I}_{\tilde{F}^c}^+} \int_{\mathfrak{b}W_{\tilde{F}}} |P_{\tilde{F}}f|^2 d\nu \\ &= \zeta_{\tilde{F}}(\beta) \int_{W_{\tilde{F}}} |P_{\tilde{F}}f|^2 d\nu \quad (\text{using that } \nu \text{ satisfies (9)}) \\ &= \zeta_{\tilde{F}}(\beta) \left| \frac{N(\mathfrak{a})^{-\beta}}{\zeta_{\tilde{F}}(\beta)} \sum_{\mathfrak{c} \in \mathcal{I}_{(\tilde{F} \setminus F)^c}^+} N(\mathfrak{c})^{-\beta} \chi(\mathfrak{c}) \right|^2 \nu(W_{\tilde{F}}) \quad (\text{using (10)}) \\ &= \left(N(\mathfrak{a})^{-\beta} \frac{\prod_{\mathfrak{p} \in \tilde{F}} |1 - N(\mathfrak{p})^{-\beta}|}{\prod_{\mathfrak{p} \in \tilde{F} \setminus F} |1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-\beta}|} \right)^2 \quad (\text{since } \nu(W_{\tilde{F}}) = \zeta_{\tilde{F}}(\beta)^{-1}). \end{aligned}$$

Hence,

$$\|Pf\|_{L^2(\nu)} = \lim_{\tilde{F}} \|P_{\tilde{F}}f\|_{L^2(\nu)} = N(\mathfrak{a})^{-\beta} \lim_{\tilde{F}} \frac{\prod_{\mathfrak{p} \in \tilde{F}} |1 - N(\mathfrak{p})^{-\beta}|}{\prod_{\mathfrak{p} \in \tilde{F} \setminus F} |1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-\beta}|}. \quad (11)$$

For each \tilde{F} , the function

$$\beta \mapsto \frac{\prod_{\mathfrak{p} \in \tilde{F}} |1 - N(\mathfrak{p})^{-\beta}|}{\prod_{\mathfrak{p} \in \tilde{F} \setminus F} |1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-\beta}|}$$

is increasing on $(0, \infty)$ since for $\mathfrak{p} \in \tilde{F} \setminus F$, the function $\beta \mapsto \frac{|1 - N(\mathfrak{p})^{-\beta}|}{|1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-\beta}|}$ is increasing on $(0, \infty)$. For $\beta > 1$, the limit $\lim_{\tilde{F}} \frac{\prod_{\mathfrak{p} \in \tilde{F}} |1 - N(\mathfrak{p})^{-\beta}|}{\prod_{\mathfrak{p} \in \tilde{F} \setminus F} |1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-\beta}|}$ exists and is equal to

$$\frac{|L(\chi, \beta)| \prod_{\mathfrak{p} \in F \cup S} |(1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-\beta})|}{\zeta_K(\beta) \prod_{\mathfrak{p} \in S} (1 - N(\mathfrak{p})^{-\beta})}$$

where $L(\chi, \beta)$ is the (generalized) Dirichlet L -function associated with χ and $\zeta_K(\beta)$ is the Dedekind zeta function of K . Now as $\beta \rightarrow 1^+$, $L(\chi, \beta)$ tends to a finite value, see, for example, [38, Chapter VI, Corollary 2.11], whereas $\zeta_K(\beta)$ has a pole at $\beta = 1$ by [38, Chapter VI, Corollary 2.12]. Therefore, the right hand side of (11) converges to zero for all $\beta \in (0, 1]$, so $\|Pf\|_{L^2(\nu)} = 0$. In particular, Pf is constant. \square

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. We first deal with the case $\beta = 0$. Suppose ν is a $K_{\mathfrak{m}, \Gamma}$ -invariant probability measure on \hat{R}_S / \hat{R}_S^* . Then, in particular, we have $\nu(a\hat{R}_S / \hat{R}_S^*) = 1$ for every $a \in R_{\mathfrak{m}, \Gamma}$, which implies that $\nu(\bigcap_a a\hat{R}_S / \hat{R}_S^*) = 1$. Since $\bigcap_a a\hat{R}_S / \hat{R}_S^* = \{\mathbf{0}\}$, we have $\nu = \delta_{\mathbf{0}}$, as desired.

Now let $\beta \in (0, 1]$. By Lemma 3.10, it suffices to show that the probability measure $\bar{\nu}_\beta$ on $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma}) \times \hat{R}_S / \hat{R}_S^*$ obtained by normalizing $\tilde{\nu}_\beta$ is the unique probability measure satisfying (9). The set of probability measures that satisfy (9) forms a simplex Σ , and Proposition 3.11 says that all measures in Σ are ergodic. A non-trivial convex combination of measures is never ergodic, so we have $\Sigma = \{\bar{\nu}_\beta\}$. \square

We are now ready for the proof of uniqueness for $\beta \in [1, 2]$.

Proof of the existence and uniqueness statement in Theorem 3.2(ii). Let π denote the quotient map $\hat{R}_S \times \hat{R}_S / \hat{R}_S^* \rightarrow \Omega_R^{\mathfrak{m}}$ and m the normalized Haar measure on \hat{R}_S . For $\beta \in [1, 2]$, let $\mu_\beta := \pi_*(m \times \nu_{\beta-1})$ be push-forward of the product measure $m \times \nu_{\beta-1}$ under π . It follows from Theorem 3.9 combined with Proposition 3.7 that μ_β is the unique probability measure on $\Omega_R^{\mathfrak{m}}$ satisfying (4).

We now show that the set of points in Ω_R^m with non-trivial isotropy has μ_β -measure zero. Our proof is almost the same as that of [43, Lemma 3.3], but we include it for completeness. For $\beta = 1$, we can identify the measure space (Ω_R^m, μ_1) with (\hat{R}_S, m) , and the partial action of $G_{m,\Gamma}$ on (\hat{R}_S, m) is given by the usual “ $ax + b$ ” action, that is, by $(n, k)\mathbf{a} = n + k\mathbf{a}$ for $(n, k) \in G_{m,\Gamma}$ and $\mathbf{a} \in \hat{R}_S$ such that $n + k\mathbf{a} \in \hat{R}_S$. In this case, each non-identity element of $G_{m,\Gamma}$ has at most one fixed point in \hat{R}_S , and every point in \hat{R}_S has m -measure zero. It follows that the set of points in Ω_R^m with non-trivial isotropy has μ_1 -measure zero.

Now let $\beta \in (1, 2]$. To show that the set of points in Ω_R^m with non-trivial isotropy has μ_β -measure zero, it suffices to show that for each non-trivial element $\gamma = (n, k) \in G_{m,\Gamma}$, the set of points in $[\mathbf{b}, \bar{\mathbf{a}}] \in \Omega_R^m$ fixed by γ has μ_β -measure zero. As was done in a similar situation in the proof of [26, Proposition 2.1], we can disintegrate μ_β with respect to the canonical projection map $\Omega_R^m \rightarrow \hat{R}_S/\hat{R}_S^*$ to get that

$$\int_{\Omega_R^m} f([\mathbf{b}, \bar{\mathbf{a}}]) d\mu_\beta([\mathbf{b}, \bar{\mathbf{a}}]) = \int_{\hat{R}_S/\hat{R}_S^*} \left(\int_{\hat{R}_S/\bar{\mathbf{a}}\hat{R}_S} f([\mathbf{b}, \bar{\mathbf{a}}]) d\lambda_{\bar{\mathbf{a}}}(\dot{\mathbf{b}}) \right) d\nu_\beta(\bar{\mathbf{a}})$$

for each $f \in C(\Omega_R^m)$ where for ν_β -a.e. $\bar{\mathbf{a}} \in \hat{R}_S/\hat{R}_S^*$, the probability measure $\lambda_{\bar{\mathbf{a}}}$ is equal to the normalized Haar measure $m_{\bar{\mathbf{a}}}$ on the quotient $\hat{R}_S/\bar{\mathbf{a}}\hat{R}_S$ and where $\dot{\mathbf{b}}$ denotes the image of \mathbf{b} under the quotient map $\hat{R}_S \rightarrow \hat{R}_S/\bar{\mathbf{a}}\hat{R}_S$. Hence, to show that set of points in $[\mathbf{b}, \bar{\mathbf{a}}] \in \Omega_R^m$ fixed by γ has μ_β -measure zero, it is enough to show that for ν_β -a.e. $\bar{\mathbf{a}} \in \hat{R}_S/\hat{R}_S^*$, the set $A_{\gamma, \bar{\mathbf{a}}} := \{\dot{\mathbf{b}} \in \hat{R}_S/\bar{\mathbf{a}}\hat{R}_S : \gamma[\mathbf{b}, \mathbf{a}] = [\mathbf{b}, \mathbf{a}]\}$ has $m_{\bar{\mathbf{a}}}$ -measure zero. Let

$$\mathbb{A}_S^* := \left\{ \mathbf{a} = (\mathbf{a}_p)_p \in \prod_{p \in \mathcal{P}_K \setminus S} K_p^* : \mathbf{a}_p \in R_p^* \text{ for all but finitely many } p \right\},$$

so that $(\mathbb{A}_S^* \cap \hat{R}_S)/\hat{R}_S^*$ can be identified with the countable set \mathcal{I}_m^+ . Since $\beta \in (1, 2]$, the set

$$\{\bar{\mathbf{a}} \in \hat{R}_S/\hat{R}_S^* : \bar{\mathbf{a}} \notin (\mathbb{A}_S^* \cap \hat{R}_S)/\hat{R}_S^* \text{ and } \bar{\mathbf{a}}_p \neq 0 \text{ for all } p\}$$

has full ν_β -measure, so we only need to show that $m_{\bar{\mathbf{a}}}(A_{\gamma, \bar{\mathbf{a}}}) = 0$ for all $\bar{\mathbf{a}} \notin (\mathbb{A}_S^* \cap \hat{R}_S)/\hat{R}_S^*$ such that $\bar{\mathbf{a}}_p \neq 0$ for all p . The set $A_{\gamma, \bar{\mathbf{a}}}$ is empty unless $k\bar{\mathbf{a}} = \bar{\mathbf{a}}$, in which case the condition $\bar{\mathbf{a}}_p \neq 0$ for all p forces $k \in R_{m,\Gamma}^*$. Now we see that $A_{\gamma, \bar{\mathbf{a}}} = \{\dot{\mathbf{b}} \in \hat{R}_S/\bar{\mathbf{a}}\hat{R}_S : (k-1)\dot{\mathbf{b}} = \dot{\mathbf{b}}\}$. If $k = 1$, then $A_{\gamma, \bar{\mathbf{a}}}$ is empty unless $n \in \bar{\mathbf{a}}\hat{R}_S$; in this case, we must have $n = 0$ because of the assumption that $\bar{\mathbf{a}} \notin (\mathbb{A}_S^* \cap \hat{R}_S)/\hat{R}_S^*$, which then implies $\gamma = (0, 1)$. Since we assumed γ to be non-trivial, we see that $A_{\gamma, \bar{\mathbf{a}}}$ can be

non-empty only when $k \neq 1$. In this case, $k - 1$ lies in $R_{\mathfrak{p}}^*$ for all \mathfrak{p} outside some finite set $F \subseteq \mathcal{P}_K^{\mathfrak{m}}$, and thus any $\dot{\mathbf{c}} \in A_{\gamma, \bar{\mathbf{a}}}$ is uniquely determined for all $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}} \setminus F$. We can now obtain the inequality

$$m_{\bar{\mathbf{a}}}(A_{\gamma, \bar{\mathbf{a}}}) \leq \prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}} \setminus F} |R_{\mathfrak{p}}/\bar{\mathbf{a}}_{\mathfrak{p}}R_{\mathfrak{p}}|^{-1},$$

and the product on the right hand side diverges to zero because of the assumption that there are infinitely many \mathfrak{p} for which $\bar{\mathbf{a}}_{\mathfrak{p}}$ does not lie in $R_{\mathfrak{p}}^*$. This finishes our proof that the set of points in $\Omega_R^{\mathfrak{m}}$ with non-trivial isotropy has μ_{β} -measure zero.

Now [43, Theorem 1.3] implies that $\mu_{\beta} \circ \mathcal{E}$ is the unique σ -KMS $_{\beta}$ state on $C^*(G_{\mathfrak{m}, \Gamma} \times \Omega_R^{\mathfrak{m}})$ for $\beta \in [1, 2]$. Moreover, since $\mu_{\beta}(V_{(x+\mathfrak{a}) \times \mathfrak{a}^{\times}}) = N(\mathfrak{a})^{-\beta}$, we are done. \square

Remark 3.12. If \mathfrak{m} and Γ are trivial, or if $\mathfrak{m} = \mathfrak{m}_{\infty}$ consists of all the real embeddings of K and Γ is trivial, then Theorem 3.9 can be deduced from [43, Theorem 3.1]. However, even in this case, our proof here is different: in [43, Section 3], the special case of Theorem 3.9 is obtained by using known results from [27] for the Hecke C^* -dynamical system associated with the Hecke pair $(K \rtimes K_+^*, R \rtimes R_+^*)$, whereas we give a more direct proof.

3.5. Low temperature KMS states: the proof of part (iii). The map $\mathfrak{a} \mapsto \prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})}$ canonically identifies $\mathcal{I}_{\mathfrak{m}}^+$ with a subset of $\prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}}$. Composing with the canonical homeomorphism $\prod_{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}} \simeq \hat{R}_S/\hat{R}_S^*$, we may view $\mathcal{I}_{\mathfrak{m}}^+$ as a subset of \hat{R}_S/\hat{R}_S^* . The image of $\mathcal{I}_{\mathfrak{m}}^+$ in \hat{R}_S/\hat{R}_S^* consists of those ‘‘super ideals’’ that are coprime to \mathfrak{m}_0 and have only finitely many divisors, that is, with the set

$$\{\bar{\mathbf{a}} \in \hat{R}_S/\hat{R}_S^* : \bar{\mathbf{a}} \in U_{\mathfrak{a}} \text{ for only finitely many } \mathfrak{a}\}$$

where $U_{\mathfrak{a}} = \mathfrak{a}\hat{R}_S/\hat{R}_S^*$. We will show that for each $\beta > 1$, every probability measure on \hat{R}_S/\hat{R}_S^* that satisfies (8) must be concentrated on this countable set, and thus is a convex combination of measures that are concentrated on orbits for the partial action of $K_{\mathfrak{m}, \Gamma}$ on \hat{R}_S/\hat{R}_S^* . These orbits are precisely the sets $\mathfrak{k} \cap \mathcal{I}_{\mathfrak{m}}^+$ for $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})$.

The *partial zeta function* associated with a class $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})$ is the Dirichlet series

$$\zeta_{\mathfrak{k}}(s) := \sum_{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_{\mathfrak{m}}^+} N(\mathfrak{a})^{-s},$$

which converges for all complex numbers s with real part greater than 1.

Lemma 3.13. For each $\beta \in (1, \infty)$ and each class $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, let $\nu_{\beta,\mathfrak{k}}$ be the probability measure on \hat{R}_S/\hat{R}_S^* given by

$$\nu_{\beta,\mathfrak{k}} := \frac{1}{\zeta_{\mathfrak{k}}(\beta)} \sum_{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_m^+} N(\mathfrak{a})^{-\beta} \delta_{\mathfrak{a}}$$

where $\delta_{\mathfrak{a}}$ denotes the unit mass concentrated at the point $\mathfrak{a} \in \hat{R}_S/\hat{R}_S^*$. Then each measure $\nu_{\beta,\mathfrak{k}}$ satisfies (8). Moreover, any probability measure ν that satisfies (8) for $\beta \in (1, \infty)$ is a convex combination of measures from $\{\nu_{\beta,\mathfrak{k}} : \mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})\}$.

Proof. For $\beta \in (1, \infty)$ and $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, a calculation shows that the measure $\nu_{\beta,\mathfrak{k}}$ satisfies (8).

Now fix $\beta \in (1, \infty)$, and let ν be a probability measure on \hat{R}_S/\hat{R}_S^* that satisfies (8). Recall that the inverse of a fractional ideal \mathfrak{a} in \mathcal{I}_m is given by $\mathfrak{a}^{-1} := \{x \in K : x\mathfrak{a} \subseteq R\}$. For $\mathfrak{a} \in \mathcal{I}_m^+$ and $x \in \mathfrak{a}^{-1} \cap K_{m,\Gamma}$, we have $xU_{\mathfrak{a}} = U_{x\mathfrak{a}}$. Now,

$$\begin{aligned} \sum_{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_m^+} \nu(U_{\mathfrak{a}}) &= \sum_{x \in (\mathfrak{a}_{\mathfrak{k}}^{-1} \cap K_{m,\Gamma})/R_{m,\Gamma}^*} \nu(U_{x\mathfrak{a}_{\mathfrak{k}}}) \\ &= \sum_{x \in (\mathfrak{a}_{\mathfrak{k}}^{-1} \cap K_{m,\Gamma})/R_{m,\Gamma}^*} N(x)^{-\beta} \nu(U_{\mathfrak{a}_{\mathfrak{k}}}) \quad (\text{using (8)}) \\ &= \sum_{x \in (\mathfrak{a}_{\mathfrak{k}}^{-1} \cap K_{m,\Gamma})/R_{m,\Gamma}^*} N(x\mathfrak{a}_{\mathfrak{k}})^{-\beta} N(\mathfrak{a}_{\mathfrak{k}})^{\beta} \nu(U_{\mathfrak{a}_{\mathfrak{k}}}) \\ &= \zeta_{\mathfrak{k}}(\beta) N(\mathfrak{a}_{\mathfrak{k}})^{\beta} \nu(U_{\mathfrak{a}_{\mathfrak{k}}}). \end{aligned}$$

Thus, since $\beta > 1$,

$$\sum_{\mathfrak{a} \in \mathcal{I}_m^+} \nu(U_{\mathfrak{a}}) = \sum_{\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})} \zeta_{\mathfrak{k}}(\beta) N(\mathfrak{a}_{\mathfrak{k}})^{\beta} \nu(U_{\mathfrak{a}_{\mathfrak{k}}}) < \infty,$$

so the Borel-Cantelli lemma implies that ν is concentrated on the set

$$\{\bar{\mathfrak{a}} \in \hat{R}_S/\hat{R}_S^* : \bar{\mathfrak{a}} \in U_{\mathfrak{a}} \text{ for only finitely many } \mathfrak{a}\}.$$

This set coincides with the canonical copy of \mathcal{I}_m^+ in \hat{R}_S/\hat{R}_S^* . Since ν satisfies (8) and \mathcal{I}_m^+ is countable, the set of points that have positive ν -measure must be a (disjoint) union of orbits for the partial action of $K_{m,\Gamma}$ on \mathcal{I}_m^+ , and ν is a convex combination of its normalized restrictions to these orbits; moreover, these orbits are precisely the sets $\mathfrak{k} \cap \mathcal{I}_m^+$ for $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, and a calculation shows that $\nu_{\beta,\mathfrak{k}}$ is the only

probability measure that both satisfies (8) and is concentrated on $\mathfrak{k} \cap \mathcal{I}_m^+$, so we are done. \square

We are now ready for the proof of Theorem 3.2(iii).

Proof of Theorem 3.2(iii). As before, let π denote the quotient map $\hat{R}_S \times \hat{R}_S / \hat{R}_S^* \rightarrow \Omega_R^m$, and let m be the normalized Haar measure on \hat{R}_S . For each $\beta > 2$ and each class $\mathfrak{k} \in \mathcal{I}_m / i(K_{m,\Gamma})$, let $\mu_{\beta,\mathfrak{k}} := \pi_*(m \times \nu_{\beta-1,\mathfrak{k}})$ be the push-forward of the product measure $m \times \nu_{\beta-1,\mathfrak{k}}$ under π . By Proposition 3.7, the map $\nu_{\beta-1} \mapsto \pi_*(m \times \nu_{\beta-1,\mathfrak{k}})$ establishes an affine bijection from the simplex of probability measures on \hat{R}_S / \hat{R}_S^* that satisfy (6) onto the simplex of probability measures on Ω_R^m that satisfy (4); hence, by Lemma 3.13, every probability measure on Ω_R^m that satisfies (4) is a convex combination of measures from the set $\{\pi_*(m \times \nu_{\beta-1,\mathfrak{k}}) : \mathfrak{k} \in \mathcal{I}_m / i(K_{m,\Gamma})\}$.

For each $\mathfrak{a} \in \mathcal{I}_m^+$, there are exactly $N(\mathfrak{a})$ points $[\mathfrak{b}, \mathfrak{a}]$ in Ω_R^m with second component equal to \mathfrak{a} . Indeed, we can always write $[\mathfrak{b}, \mathfrak{a}] = [x, \mathfrak{a}]$ for some $x \in \hat{R}_S / \mathfrak{a}\hat{R}_S \cong R / \mathfrak{a}$ with $\mathfrak{b} - x \in \mathfrak{a}\hat{R}_S$. Hence, for each $\mathfrak{k} \in \mathcal{I}_m / i(K_{m,\Gamma})$, the set $\{[b, \mathfrak{a}] \in \Omega_R^m : \mathfrak{a} \in \mathfrak{k}\}$ is countable. Moreover, the partial action of $G_{m,\Gamma}$ on $\{[b, \mathfrak{a}] \in \Omega_R^m : \mathfrak{a} \in \mathfrak{k}\}$ is transitive, and the measure $\mu_{\beta,\mathfrak{k}}$ is concentrated on $\{[b, \mathfrak{a}] \in \Omega_R^m : \mathfrak{a} \in \mathfrak{k}\}$. This set contains the point $[0, \mathfrak{a}_{\mathfrak{k},1}]$, which has isotropy group $\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{m,\Gamma}^*$, and the σ -KMS $_\beta$ states ϕ satisfying $\phi|_{C(\Omega_R^m)} = \mu_{\beta,\mathfrak{k}}$ are in one-to-one correspondence with the tracial states of the group C*-algebra $C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{m,\Gamma}^*)$ by [43, Corollary 1.4]. Explicitly, the σ -KMS $_\beta$ state $\phi_{\beta,\mathfrak{k},\tau}$ corresponding to a tracial state τ of $C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{m,\Gamma}^*)$ is given as follows. (This explicit description comes from the proofs of [43, Theorem 1.3 & Corollary 1.4].) For each point $[x, \mathfrak{a}]$ in the orbit $O_{\mathfrak{k}}$ of $[0, \mathfrak{a}_{\mathfrak{k},1}]$, there exists $\gamma \in G_{m,\Gamma}$ such that $\gamma[x, \mathfrak{a}] = [0, \mathfrak{a}_{\mathfrak{k},1}]$. Conjugating by γ then defines an isomorphism of the isotropy group $(x, 1)\mathfrak{a} \rtimes R_{m,\Gamma}^*(-x, 1)$ of $[x, \mathfrak{a}]$ onto $\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{m,\Gamma}^*$, which in turn gives rise to an isomorphism of group C*-algebras $C^*((x, 1)\mathfrak{a} \rtimes R_{m,\Gamma}^*(-x, 1)) \cong C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{m,\Gamma}^*)$. Let $\tau_{x,\mathfrak{a}}$ be the tracial state of $C^*((x, 1)\mathfrak{a} \rtimes R_{m,\Gamma}^*(-x, 1))$ given by the composition of τ with the isomorphism $C^*((x, 1)\mathfrak{a} \rtimes R_{m,\Gamma}^*(-x, 1)) \cong C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{m,\Gamma}^*)$. Then

$$\phi_{\beta,\mathfrak{k},\tau}(f) = \int_{O_{\mathfrak{k}}} \sum_{g \in R_{m,\Gamma}^*} f(g, [x, \mathfrak{a}]) \tau_{x,\mathfrak{a}}(u_g) d\mu_{\beta,\mathfrak{k}}([x, \mathfrak{a}]) \quad \text{for } f \in C_c(G_{m,\Gamma} \times \Omega_R^m). \quad (12)$$

(The formula given in (12) can be further simplified, but shall leave it in the more compact form above for convenience.) A calculation shows that the map $\tau \mapsto \phi_{\beta,\mathfrak{k},\tau}$

from the simplex of tracial states on $C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{\mathfrak{m},\Gamma}^*)$ to the simplex of σ -KMS $_{\beta}$ states on $C^*(G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}})$ is affine. If τ is a convex combination $\tau = \sum_{\mathfrak{k}} \lambda_{\mathfrak{k}} \tau_{\mathfrak{k}}$ where $\tau_{\mathfrak{k}}$ is a tracial state on $C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{\mathfrak{m},\Gamma}^*)$ for each \mathfrak{k} , then we let $\phi_{\beta,\mathfrak{k},\tau} := \sum_{\mathfrak{k}} \lambda_{\mathfrak{k}} \phi_{\beta,\mathfrak{k},\tau_{\mathfrak{k}}}$. Then the map from the simplex of tracial states of $\bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{\mathfrak{m},\Gamma}^*)$ onto the simplex of σ -KMS $_{\beta}$ states on $C^*(G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}})$ given $\tau \mapsto \phi_{\beta,\mathfrak{k},\tau}$ is affine; it follows from [43, Theorem 1.3] that $\tau \mapsto \phi_{\beta,\mathfrak{k},\tau}$ is also a bijection. Weak* continuity follows from the explicit formula (12); since both simplices are weak* compact, this is enough to guarantee that $\tau \mapsto \phi_{\beta,\mathfrak{k},\tau}$ is a homeomorphism, which concludes the proof of Theorem 3.2(iii). \square

3.6. Ground states: the proof of part (iv). We will first use [23, Theorem 1.9] to identify the ground states of $(C^*(G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}}), \mathbb{R}, \sigma)$ with the states on the C^* -algebra of the *boundary groupoid* of the cocycle c^N , see [23, Section 1] for the general definition. In our special situation, this boundary groupoid has a particularly explicit description, which is given in the following result.

Proposition 3.14. *Let $G_{\mathfrak{m},\Gamma,1} := \{(n, k) \in G_{\mathfrak{m},\Gamma} : N(k) = 1\}$ be the kernel of the homomorphism $G_{\mathfrak{m},\Gamma} \rightarrow \mathbb{R}_+^*$ given by $(n, k) \mapsto N(k)$, and let*

$$(\Omega_R^{\mathfrak{m}})_0 := \Omega_R^{\mathfrak{m}} \setminus \left(\bigcup_{(n,k): N(k) > 1} (n, k) \Omega_R^{\mathfrak{m}} \right).$$

Then the map $\psi \mapsto \phi_{\psi}$ defined by

$$\phi_{\psi}(f) = \psi(f|_{G_{\mathfrak{m},\Gamma,1} \rtimes (\Omega_R^{\mathfrak{m}})_0}) \quad \text{for } f \in C_c(G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}})$$

is an affine isomorphism of the state space of $C^(G_{\mathfrak{m},\Gamma,1} \rtimes (\Omega_R^{\mathfrak{m}})_0)$ onto the σ -ground state space of $C^*(G_{\mathfrak{m},\Gamma} \rtimes \Omega_R^{\mathfrak{m}})$ where $G_{\mathfrak{m},\Gamma,1} \rtimes (\Omega_R^{\mathfrak{m}})_0$ is the reduction groupoid of $G_{\mathfrak{m},\Gamma,1} \rtimes \Omega_K^{\mathfrak{m}}$ with respect to the compact subset $(\Omega_R^{\mathfrak{m}})_0 \subseteq \Omega_K^{\mathfrak{m}}$.*

Proof. This is a direct application of [23, Theorem 1.9]. \square

We are now ready for the proof of Theorem 3.2(iv).

Proof of Theorem 3.2(iv). For each class $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$, let $\mathfrak{a}_{\mathfrak{k},1}, \dots, \mathfrak{a}_{\mathfrak{k},k_{\mathfrak{k}}}$ denote the norm-minimizing ideals in \mathfrak{k} . In light of Proposition 3.14, it suffices to prove that

there is an isomorphism

$$C^*(G_{\mathfrak{m},\Gamma,1} \times (\Omega_R^{\mathfrak{m}})_0) \cong \bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} M_{k_{\mathfrak{k}} \cdot N(\mathfrak{a}_{\mathfrak{k},1})}(C^*(\mathfrak{a}_{\mathfrak{k},1} \rtimes R_{\mathfrak{m},\Gamma}^*)).$$

We first claim that

$$(\Omega_R^{\mathfrak{m}})_0 = \{[\mathfrak{b}, \bar{\mathfrak{a}}] \in \Omega_R^{\mathfrak{m}} : \bar{\mathfrak{a}} = \mathfrak{a}_{\mathfrak{k},j} \text{ for some } \mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma}), 1 \leq j \leq k_{\mathfrak{k}}\}.$$

“ \subseteq ”: For each prime $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}$, let $f_{\mathfrak{p}}$ denote the order of the class $[\mathfrak{p}]$ of \mathfrak{p} in $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$, so that there exists $t_{\mathfrak{p}} \in R_{\mathfrak{m},\Gamma}$ such that $\mathfrak{p}^{f_{\mathfrak{p}}} = t_{\mathfrak{p}}R$. Let $[\mathfrak{b}, \bar{\mathfrak{a}}] \in \Omega_R^{\mathfrak{m}}$, and suppose that $v_{\mathfrak{p}}(\bar{\mathfrak{a}}) = \infty$ for some $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}$, so that $t_{\mathfrak{p}}\bar{\mathfrak{a}} = \bar{\mathfrak{a}}$. By the strong approximation theorem, there exists $x \in R_{\mathfrak{m}}^{-1}R_{\mathfrak{m}}$ such that $v_{\mathfrak{p}}(x + \mathfrak{b}) \geq f_{\mathfrak{p}}$. Then $t_{\mathfrak{p}}^{-1}(x + \mathfrak{b}) \in \hat{R}_S$, and

$$[\mathfrak{b}, \bar{\mathfrak{a}}] = (-x, t_{\mathfrak{p}})[t_{\mathfrak{p}}^{-1}(x + \mathfrak{b}), \bar{\mathfrak{a}}] \in (-x, t_{\mathfrak{p}})\Omega_R^{\mathfrak{m}}.$$

Since $N(t_{\mathfrak{p}}) = f_{\mathfrak{p}} > 1$, we see that $[\mathfrak{b}, \bar{\mathfrak{a}}] \notin (\Omega_R^{\mathfrak{m}})_0$. Therefore, if $[\mathfrak{b}, \bar{\mathfrak{a}}] \in (\Omega_R^{\mathfrak{m}})_0$, then $v_{\mathfrak{p}}(\bar{\mathfrak{a}}) < \infty$ for every $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}$.

Next, we will show that if $[\mathfrak{b}, \bar{\mathfrak{a}}] \in (\Omega_R^{\mathfrak{m}})_0$, then $\bar{\mathfrak{a}}$ is divisible by only finitely many primes. Suppose $[\mathfrak{b}, \bar{\mathfrak{a}}] \in \Omega_R^{\mathfrak{m}}$ is such that $\{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}} : v_{\mathfrak{p}}(\bar{\mathfrak{a}}) > 0\}$ is infinite. Then there are finitely many distinct primes $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_N$ in $\{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}} : v_{\mathfrak{p}}(\bar{\mathfrak{a}}) > 0\}$ such that the ideal $\prod_{j=1}^N \mathfrak{p}_j$ is principal. Let $a \in R_{\mathfrak{m}}$ be such that $aR = \prod_{j=1}^N \mathfrak{p}_j$. Then $a^{-1}\bar{\mathfrak{a}}$ lies in \hat{R}_S , and by the strong approximation theorem, there exists $x \in R_{\mathfrak{m}}^{-1}R_{\mathfrak{m}}$ such that $v_{\mathfrak{p}}(a^{-1}(x + \mathfrak{b})) \geq 0$ for every $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}$, so that $a^{-1}(x + \mathfrak{b}) \in \hat{R}_S$. Now

$$[\mathfrak{b}, \bar{\mathfrak{a}}] = (-x, a)[a^{-1}(x + \mathfrak{b}), a^{-1}\bar{\mathfrak{a}}] \in (-x, a)\Omega_R^{\mathfrak{m}}.$$

Since $N(a) > 1$, we see that $[\mathfrak{b}, \bar{\mathfrak{a}}] \notin (\Omega_R^{\mathfrak{m}})_0$. Thus, if $[\mathfrak{b}, \bar{\mathfrak{a}}] \in (\Omega_R^{\mathfrak{m}})_0$, then $v_{\mathfrak{p}}(\bar{\mathfrak{a}}) = 0$ for all but finitely many $\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}}$.

The above two facts imply that if $[\mathfrak{b}, \bar{\mathfrak{a}}] \in (\Omega_R^{\mathfrak{m}})_0$, then $\bar{\mathfrak{a}} = \mathfrak{a}$ for some $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}^+$. In this case, there exist $k \in K_{\mathfrak{m},\Gamma}$ and $1 \leq j \leq k_{[\mathfrak{a}]}$ such that $\mathfrak{a} = k\mathfrak{a}_{\mathfrak{k},j}$, and there exists $x \in R$ such that $[\mathfrak{b}, \mathfrak{a}] = [x, \mathfrak{a}]$. It remains to show that k has norm 1. Since $(\Omega_R^{\mathfrak{m}})_0$ is $G_{\mathfrak{m},\Gamma,1}$ -invariant, we see that $(0, k)[\mathbf{0}, \mathfrak{a}_{\mathfrak{k},j}] = [\mathbf{0}, k\mathfrak{a}_{\mathfrak{k},j}] = (-x, 1)[x, k\mathfrak{a}_{\mathfrak{k},j}]$ lies in $(\Omega_R^{\mathfrak{m}})_0$, so we must have $N(k) = 1$, that is, \mathfrak{a} must be norm-minimizing in \mathfrak{k} . This finishes our proof of the first inclusion.

“ \supseteq ”: Let $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$, $1 \leq j \leq k_{\mathfrak{k}}$, and suppose that $[x, \mathfrak{a}_{\mathfrak{k},j}] \in \Omega_R^m \cap (n, k)\Omega_R^m$ for some $(n, k) \in G_{m,\Gamma}$. There exists an integral ideal $\mathfrak{b} \in \mathfrak{k}$ such that $\mathfrak{a}_{\mathfrak{k},j} = k\mathfrak{b}$, so minimality of $N(\mathfrak{a}_{\mathfrak{k},j})$ forces $N(k) = 1$. This shows the reverse inclusion and concludes the proof of our claim.

In particular, the above claim shows that $(\Omega_R^m)_0$ is a finite set. Moreover, there is a $G_{m,\Gamma}$ -equivariant decomposition

$$(\Omega_R^m)_0 = \bigsqcup_{\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})} X_{\mathfrak{k}}$$

where $X_{\mathfrak{k}} = \{[x, \mathfrak{a}_{\mathfrak{k},j}] : x \in R/\mathfrak{a}_{\mathfrak{k},j}, j = 1, \dots, k_{\mathfrak{k}}\}$ is the orbit of any $[b, \mathfrak{a}_{\mathfrak{k},j}]$ under the partial action of $G_{m,\Gamma,1}$; it follows that we have the direct sum decomposition

$$C^*(G_{m,\Gamma,1} \times (\Omega_R^m)_0) \cong \bigoplus_{\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})} C^*(G_{m,\Gamma,1} \times X_{\mathfrak{k}}).$$

For each class \mathfrak{k} , $G_{m,\Gamma,1} \times X_{\mathfrak{k}}$ is a transitive groupoid, and the isotropy group of the point $[0, \mathfrak{a}_{\mathfrak{k},1}]$ is $\mathfrak{a}_{\mathfrak{k},1} \times R_{m,\Gamma}^*$; therefore, it follows that $C^*(G_{m,\Gamma,1} \times X_{\mathfrak{k}}) \cong M_{|X_{\mathfrak{k}}|}(C^*(\mathfrak{a}_{\mathfrak{k},1} \times R_{m,\Gamma}^*))$, see, for example, [39, Theorem 3.1]. Since $|X_{\mathfrak{k}}| = k_{\mathfrak{k}} \cdot N(\mathfrak{a}_{\mathfrak{k},1})$, we are done. \square

4. TYPE III₁ FACTORS AND THE DISTRIBUTION OF PRIME IDEALS IN $\mathcal{I}_m/i(K_{m,\Gamma})$

Each extremal σ -KMS _{β} state ϕ on $C_\lambda^*(R \rtimes R_{m,\Gamma})$ is a factor state, that is, the von Neumann algebra $\pi_\phi(C_\lambda^*(R \rtimes R_{m,\Gamma}))''$ generated by the GNS representation π_ϕ of ϕ is a factor, see [4, Theorem 5.3.30(3)]. It is therefore a natural problem to determine the type of the factors arising from extremal σ -KMS _{β} states on $C_\lambda^*(R \rtimes R_{m,\Gamma})$. The main result of this section is the following theorem, which, in light of the uniqueness of the injective factor of type III₁ with separable predual, see [8] and [18], completes the proof of Theorem 3.2(ii).

Theorem 4.1. *For each $\beta \in [1, 2]$, let π_{ϕ_β} be the GNS representation of the σ -KMS _{β} state ϕ_β on $C_\lambda^*(R \rtimes R_{m,\Gamma})$ from Theorem 3.2. Then the von Neumann algebra $\pi_{\phi_\beta}(C_\lambda^*(R \rtimes R_{m,\Gamma}))''$ is an injective factor of type III₁ with separable predual.*

Remark 4.2. It follows from [26, Theorem 3.2] that, for each $\beta \in [1, 2]$, the σ -KMS _{β} state on $C_\lambda^*(\mathbb{Z} \rtimes \mathbb{N}^\times)$ is of type III₁. Moreover, it is asserted in [43, Section 3] that arguments analogous to those used to prove [26, Theorem 3.2] combined with [42,

Corollary 3.2] can be used to show that, for each $\beta \in [1, 2]$, the σ -KMS $_{\beta}$ state on $C_{\lambda}^*(R \rtimes R^{\times})$ is of type III $_1$.

In our more general situation, there are additional difficulties which we will overcome by using techniques from [32, Sections 2&3].

The remainder of this section is devoted to the proof of Theorem 4.1.

We now briefly recall some well-known results about the flow of weights on von Neumann algebra crossed products from [11] (see also [46, Chapter XIII § 2]). The general setup here is similar to that in [26, Section 3] and [42, Section 2], and we will follow the notation therein.

Let X be a second countable, locally compact Hausdorff space and μ a σ -finite measure on X . Suppose that a countably infinite discrete group G acts by nonsingular transformations on the measure space (X, μ) , that is, G acts on X by Borel automorphisms, and for each $g \in G$, the measures μ and $g\mu$ are equivalent where $g\mu$ is the push-forward of μ by g defined by $g\mu(Z) := \mu(g^{-1}Z)$ for every Borel set $Z \subseteq X$.

Assume that the action of G on (X, μ) is essentially free and ergodic, so that the von Neumann algebra crossed product $L^{\infty}(X, \mu) \rtimes G$ is a factor. In this situation, the flow of weights has a particularly explicit description. Indeed, let λ_{∞} denote the Lebesgue measure on \mathbb{R}_+^* ; then there are commuting actions of G and \mathbb{R} on $(\mathbb{R}_+^* \times X, \lambda_{\infty} \times \mu)$ given by

$$g(t, x) = \left(\frac{dg\mu}{d\mu}(gx)t, gx \right) \text{ and } s(t, x) = (e^{-s}t, x) \text{ for } g \in G, s \in \mathbb{R}, (t, x) \in \mathbb{R}_+^* \times X,$$

and the flow of weights on $L^{\infty}(X, \mu) \rtimes G$ is the induced action of \mathbb{R} on the fixed point algebra $L^{\infty}(\mathbb{R}_+^* \times X, \lambda_{\infty} \times \mu)^G$ for the action of G on $L^{\infty}(\mathbb{R}_+^* \times X, \lambda_{\infty} \times \mu)$ arising from the above action of G on $(\mathbb{R}_+^* \times X, \lambda_{\infty} \times \mu)$ (see [46, Chapter XIII § 2 Theorem 2.23]). The factor $L^{\infty}(X, \mu) \rtimes G$ is of type III $_1$ if and only if the action of G on $(\mathbb{R}_+^* \times X, \lambda_{\infty} \times \mu)$ is ergodic.

We now turn to the particular case of interest to us. As before, it will be easiest to work with the C*-algebra $C^*(G_{\mathfrak{m}, \Gamma} \rtimes \Omega_R^{\mathfrak{m}})$. Since ϕ_{β} factors through the expectation \mathcal{E} onto $C(\Omega_R^{\mathfrak{m}})$ and is determined by the probability measure μ_{β} , we have the following standard lemma.

Lemma 4.3. *For each $\beta \in [1, 2]$, let $\tilde{\mu}_\beta$ be the unique quasi-invariant measure on Ω_K^m that extends μ_β and satisfies the obvious analogue of (4) for the action of $G_{m,\Gamma}$ on Ω_K^m . Then*

$$\pi_{\phi_\beta}(C^*(G_{m,\Gamma} \rtimes \Omega_R^m))'' \cong 1_{\Omega_R^m}(L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma})1_{\Omega_R^m}.$$

Therefore, if $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is a factor of type III₁, then $\pi_{\phi_\beta}(C^(G_{m,\Gamma} \rtimes \Omega_R^m))''$ is also a factor of type III₁.*

Hence, to prove Theorem 4.1, it suffices to show that $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is an injective factor of type III₁ with separable predual. Since $G_{m,\Gamma}$ is amenable, $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is injective, and the separability claim is easy to see. This means that we need to prove that $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is a factor of type III₁.

Proposition 4.4. *For each $\beta \in [1, 2]$, the action $G_{m,\Gamma} \curvearrowright (\Omega_K^m, \tilde{\mu}_\beta)$ is essentially free and ergodic. Hence, $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is a factor.*

Proof. Arguments similar to those used in the proof of [43, Lemma 3.3] show that the action $G_{m,\Gamma} \curvearrowright (\Omega_K^m, \tilde{\mu}_\beta)$ is essentially free; note we have already made this observation in the proof of the uniqueness statement in Theorem 3.2(ii).

One can argue directly using Proposition 3.11 to show that the action $G_{m,\Gamma} \curvearrowright (\Omega_K^m, \tilde{\mu}_\beta)$ is ergodic. Alternatively, since Theorem 3.2(ii) says that the state ϕ_β is the unique σ -KMS _{β} state on $C^*(G_{m,\Gamma} \rtimes \Omega_R^m)$, [4, Theorem 5.3.30(3)] implies that $\pi_{\phi_\beta}(C^*(G_{m,\Gamma} \rtimes \Omega_R^m))''$ is a factor. Since $1_{\Omega_R^m}$ is a full projection in $C_0(\Omega_K^m) \rtimes G_{m,\Gamma}$, it follows that $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is also a factor. Thus, the action $G_{m,\Gamma} \curvearrowright (\Omega_K^m, \tilde{\mu}_\beta)$ is ergodic. \square

The following lemma on primes in ideal classes from $\mathcal{I}_m/i(K_{m,\Gamma})$ is the key number-theoretic result needed to compute the flow of weights on $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$. It is a generalization of [42, Lemma 3.3].

Lemma 4.5. *Fix $\beta \in (0, 1]$ and fix a class $\mathfrak{k} \in \mathcal{I}_m/i(K_{m,\Gamma})$. For each $\lambda > 1$ and each $\epsilon > 0$, there exist sequences $(\mathfrak{p}_n)_{n \geq 1}$ and $(\mathfrak{q}_n)_{n \geq 1}$, each consisting of distinct prime ideals in \mathcal{P}_K^m , such that*

$$\left| \frac{N(\mathfrak{q}_n)^\beta}{N(\mathfrak{p}_n)^\beta} - \lambda \right| < \epsilon, \quad \mathfrak{q}_n \mathfrak{p}_n^{-1} \in \mathfrak{k} \text{ for } n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} N(\mathfrak{p}_n)^{-\beta} = \infty. \quad (13)$$

Proof. The proof is similar to that of [42, Lemma 3.3]; it follows ideas from [2] and [1] (also see the proof of [32, Theorem 1.2] for number fields).

The case where $\beta \in (0, 1)$ follows from the case $\beta = 1$, so it suffices to consider only the case $\beta = 1$. Choose $\delta > 0$ such that $1 + \delta < \lambda$ and $\delta\lambda < \epsilon$. Define sets B_n by

$$\begin{aligned} B_{2k} &:= \{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}} : \lambda^{2k} < N(\mathfrak{p}) \leq (1 + \delta)\lambda^{2k}, \mathfrak{p} \in [R]\}, \\ B_{2k+1} &:= \{\mathfrak{p} \in \mathcal{P}_K^{\mathfrak{m}} : \lambda^{2k+1} < N(\mathfrak{p}) \leq (1 + \delta)\lambda^{2k+1}, \mathfrak{p} \in \mathfrak{k}\}. \end{aligned}$$

By our choice of δ , these sets are pairwise disjoint. For a class $\tilde{\mathfrak{k}} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ and $x > 0$, let

$$\pi_{\tilde{\mathfrak{k}}}(x) := |\{\mathfrak{p} \in \tilde{\mathfrak{k}} : \mathfrak{p} \text{ prime and } N(\mathfrak{p}) \leq x\}|$$

be the number of prime ideals in the class $\tilde{\mathfrak{k}}$ whose norms do not exceed x . Given functions f and g , we shall write $f(x) \sim g(x)$ as $x \rightarrow \infty$ if $g(x)$ is non-zero for all sufficiently large x and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$. Note that this is equivalent to $f(x) - g(x) = o(g(x))$. Now [38, Chapter VIII, Theorem 7.2] combined with [40, Chapter 7, Proposition 7.17] imply that

$$\pi_{\tilde{\mathfrak{k}}}(x) \sim \frac{1}{h} \frac{x}{\log x} \text{ as } x \rightarrow \infty$$

where $h := |\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})|$. Since we have

$$\frac{(1 + \delta)x}{\log((1 + \delta)x)} - \frac{x}{\log x} \sim \frac{\delta x}{\log x} \text{ as } x \rightarrow \infty,$$

it follows that

$$\pi_{\tilde{\mathfrak{k}}}((1 + \delta)x) - \pi_{\tilde{\mathfrak{k}}}(x) \sim \frac{\delta}{h} \frac{x}{\log x} \text{ as } x \rightarrow \infty.$$

As this holds for every class $\tilde{\mathfrak{k}}$, we have

$$|B_n| \sim \frac{\delta}{h} \frac{\lambda^n}{n \log \lambda} \text{ as } n \rightarrow \infty. \quad (14)$$

Thus, there exists k_0 such that $|B_{2k+1}| \geq |B_{2k}|$ for all $k \geq k_0$. Now, for each $k \geq k_0$, we can choose a subset $C_{2k+1} \subseteq B_{2k+1}$ such that $|C_{2k+1}| = |B_{2k}|$. Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots$ and $\mathfrak{q}_1, \mathfrak{q}_2, \dots$ be enumerations of the sets $\bigcup_{k \geq k_0} B_{2k}$ and $\bigcup_{k \geq k_0} C_{2k+1}$, respectively, such that $N(\mathfrak{p}_1) \leq N(\mathfrak{p}_2) \leq \dots$, and $N(\mathfrak{q}_1) \leq N(\mathfrak{q}_2) \leq \dots$. Then if $\mathfrak{p}_n \in B_{2k}$ for some $k \geq k_0$, we must have $\mathfrak{q}_n \in B_{2k+1}$, in which case by our choice of δ , we have

$$N(\mathfrak{q}_n \mathfrak{p}_n^{-1}) \in (\lambda - \epsilon, \lambda + \epsilon) \text{ and } \mathfrak{q}_n \mathfrak{p}_n^{-1} \in \mathfrak{k}[R] = \mathfrak{k}.$$

Moreover, using (14), we see that

$$\sum_{n=1}^{\infty} N(\mathfrak{p}_n)^{-1} \geq \sum_{k=k_0}^{\infty} \frac{|B_{2k}|}{(1+\delta)\lambda^{2k}} = \infty.$$

Therefore, the sequences of primes $(\mathfrak{p}_n)_{n \geq 1}$ and $(\mathfrak{q}_n)_{n \geq 1}$ satisfy the desired properties. \square

Our next step is an ergodicity result that will also be used in Section 6. We shall need a general lemma, which we state here in the level of generality from our discussion of von Neumann algebra crossed products. Its proof is routine, so we omit it.

Lemma 4.6. *Let X be a second countable, locally compact Hausdorff space and μ a σ -finite Borel measure on X . Suppose that a countable discrete group G acts on (X, μ) by nonsingular transformations. Let H be a finite index subgroup of G , and assume that $\tilde{\mu}$ is a measure on $G/H \times X$ such that the diagonal action $G \curvearrowright (G/H \times X, \tilde{\mu})$ is nonsingular ergodic and the restriction of $\tilde{\mu}$ to $\{H\} \times X$ coincides with μ . Then the action $H \curvearrowright (X, \mu)$, obtained from the action of G on (X, μ) , is ergodic.*

Proposition 4.7. *For each $\beta \in (0, 1]$, let $\tilde{\nu}_\beta$ be the unique quasi-invariant measure on \mathbb{A}_S/\hat{R}_S^* that extends ν_β and satisfies the obvious analogue of (8) for the action of $K_{\mathfrak{m}, \Gamma}$ on \mathbb{A}_S/\hat{R}_S^* . Then the action of $K_{\mathfrak{m}, \Gamma}$ on $(\mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*, \lambda_\infty \times \tilde{\nu}_\beta)$ given by*

$$k(t, \bar{\mathbf{a}}) = (N(k)^\beta t, k\bar{\mathbf{a}}) \quad \text{for } k \in K_{\mathfrak{m}, \Gamma}, (t, \bar{\mathbf{a}}) \in \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^* \quad (15)$$

is ergodic.

Remark 4.8. If \mathfrak{m}_∞ is supported on all of the real embeddings of K and \mathfrak{m}_0 is trivial, so that $K_{\mathfrak{m}, \Gamma} = K_+^*$ is the multiplicative subgroup of K^* consisting of all (non-zero) totally positive elements, then Proposition 4.7 is precisely [42, Corollary 3.2], which follows from Neshveyev's type computation for the high temperature KMS states on the Bost–Connes system associated with K , see [42, Theorem 3.1].

Proof of Proposition 4.7. Since the subgroup $R_{\mathfrak{m}, \Gamma}^*$ acts trivially, the action of $K_{\mathfrak{m}, \Gamma}$ defines an action of the quotient group $K_{\mathfrak{m}, \Gamma}/R_{\mathfrak{m}, \Gamma}^*$, and it suffices to show that the action of this quotient group is ergodic. Let $\lambda_{\mathfrak{m}, \Gamma}$ denote the normalized Haar measure on $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m}, \Gamma})$. We can view $K_{\mathfrak{m}, \Gamma}/R_{\mathfrak{m}, \Gamma}^*$ as a subgroup of $\mathcal{I}_{\mathfrak{m}}$; by Lemma 4.6,

it is enough to prove that the action of \mathcal{I}_m on $(\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*, \lambda_{m,\Gamma} \times \lambda_\infty \times \tilde{\nu}_\beta)$ given by

$$\mathbf{a}([\mathbf{b}], t, \bar{\mathbf{a}}) = ([\mathbf{a}\mathbf{b}], N(\mathbf{a})^\beta t, \mathbf{a}\bar{\mathbf{a}}) \quad \text{for } \mathbf{a} \in \mathcal{I}_m, ([\mathbf{b}], t, \bar{\mathbf{a}}) \in \mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*$$

is ergodic.

Since the isomorphism $\mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ given $t \mapsto t^\beta$ preserves the measure class of λ_∞ , it suffices to show that the action of \mathcal{I}_m on $(\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*, \lambda_{m,\Gamma} \times \lambda_\infty \times \tilde{\nu}_\beta)$ given by

$$\mathbf{a}([\mathbf{b}], t, \bar{\mathbf{a}}) = ([\mathbf{a}\mathbf{b}], N(\mathbf{a})t, \mathbf{a}\bar{\mathbf{a}}) \quad \text{for } \mathbf{a} \in \mathcal{I}_m, ([\mathbf{b}], t, \bar{\mathbf{a}}) \in \mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^* \tag{16}$$

is ergodic.

Let \mathcal{R} denote the orbit equivalence relation for the canonical action $\mathcal{I}_m \curvearrowright (\mathbb{A}_S/\hat{R}_S^*, \tilde{\nu}_\beta)$ given by $\mathbf{a} : \bar{\mathbf{a}} \mapsto \mathbf{a}\bar{\mathbf{a}}$. This action is essentially free; indeed, the set

$$\{\bar{\mathbf{a}} \in \mathbb{A}_S/\hat{R}_S^* : \text{there exists } \mathfrak{p} \text{ with } v_{\mathfrak{p}}(\bar{\mathbf{a}}) = \infty\}$$

has $\tilde{\nu}_\beta$ -measure zero by the scaling condition, and every point lying in the complement of this set has trivial isotropy. Thus, outside a set of measure zero we can define an $(\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^*)$ -valued 1-cocycle c on \mathcal{R} by

$$c(\bar{\mathbf{a}}, \bar{\mathbf{b}}) = ([\mathbf{a}], N(\mathbf{a})) \text{ if } \mathbf{a}\bar{\mathbf{a}} = \bar{\mathbf{b}}.$$

Then the equivalence relation $\mathcal{R}(c)$ on $\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*$ associated with c as in [15, Section 8] (see also [32, Section 2]) coincides with the orbit equivalence relation for the action of \mathcal{I}_m on $\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*$ given by (16). Therefore, it suffices to show that $\mathcal{R}(c)$ is ergodic. It follows from Proposition 3.11 and Lemma 4.6 that \mathcal{R} is ergodic. Hence, the results of [15, Section 8] imply that $\mathcal{R}(c)$ is ergodic if and only if the asymptotic range $r^*(c)$ of c ([15, Definition 8.2]) coincides with $\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^*$, see [32, Proposition 2.1(iii)].

The proof that $r^*(c) = \mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^*$ relies on [32, Proposition 2.2] and follows the same lines as the computation of the analogous asymptotic range in the proof of [32, Theorem 1.2] for number fields, but with Lemma 4.5 used in place of [32, Corollary 3.3]. We shall give a quick sketch of the argument here. The subset $\hat{R}_S/\hat{R}_S^* \subseteq \mathbb{A}_S/\hat{R}_S^*$ is of $\tilde{\nu}_\beta$ -measure one, and after removing a set of ν_β -measure zero,

we can identify $\hat{R}_S/\hat{R}_S^* \cong \prod_{\mathfrak{p} \in \mathcal{P}_K^m} \mathfrak{p}^{\mathbb{N} \cup \{\infty\}}$ with $\prod_{\mathfrak{p} \in \mathcal{P}_K^m} \mathfrak{p}^{\mathbb{N}}$. The equivalence relation on $\prod_{\mathfrak{p} \in \mathcal{P}_K^m} \mathfrak{p}^{\mathbb{N}}$ obtained by restricting \mathcal{R} to \hat{R}_S/\hat{R}_S^* is given by

$$\bar{\mathbf{a}} \sim \bar{\mathbf{b}} \quad \text{if and only if } v_{\mathfrak{p}}(\bar{\mathbf{a}}) = v_{\mathfrak{p}}(\bar{\mathbf{b}}) \text{ for all but finitely many } \mathfrak{p}.$$

Let $c_{\hat{R}_S/\hat{R}_S^*}$ be the restriction of c to this equivalence relation; it suffices to show that $r^*(c_{\hat{R}_S/\hat{R}_S^*})$ coincides with $\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^*$. The cocycle $c_{\hat{R}_S/\hat{R}_S^*}$ is of product type, as defined in [32, Section 2]. Using [32, Proposition 2.2(iii)], it is enough to show that the asymptotic ratio set (see, for instance, [32, Section 2]) of $c_{\hat{R}_S/\hat{R}_S^*}$ is equal to $\mathcal{I}_m/i(K_{m,\Gamma}) \times \mathbb{R}_+^*$, which is done using Lemma 4.5. \square

We are now ready for the proof of Theorem 4.1.

Proof of Theorem 4.1. Fix $\beta \in [1, 2]$. In light of Lemma 4.3 and Proposition 4.4, we only need to show that the factor $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is of type III₁. That is, we must show that the flow of weights on $L^\infty(\Omega_K^m, \tilde{\mu}_\beta) \rtimes G_{m,\Gamma}$ is trivial. Since

$$\frac{d(n, k)\tilde{\mu}_\beta}{d\tilde{\mu}_\beta}((n, k)w) = N(k)^\beta, \quad \text{for } (n, k) \in G_{m,\Gamma}, w \in \Omega_K^m,$$

this is equivalent to showing that the action of $G_{m,\Gamma}$ on $(\mathbb{R}_+^* \times \Omega_K^m, \lambda_\infty \times \tilde{\mu}_\beta)$ given by

$$(n, k)(t, w) = (N(k)^\beta t, (n, k)w), \quad \text{for } (n, k) \in G_{m,\Gamma}, w \in \Omega_K^m \quad (17)$$

is ergodic. We will now show that it suffices to prove that the action of $K_{m,\Gamma}$ on $(\mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*, \lambda_\infty \times \tilde{\nu}_{\beta-1})$ given by

$$k(t, \bar{\mathbf{a}}) = (N(k)^{\beta-1}t, k\bar{\mathbf{a}}) \quad \text{for } k \in K_{m,\Gamma}, (t, \bar{\mathbf{a}}) \in \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^* \quad (18)$$

is ergodic. Our proof of this fact is a direct generalization of the special case considered in [26, Theorem 3.2], but we include it for the convenience of the reader. Since $(\Omega_K^m, \tilde{\mu}_\beta)$ is a quotient of $(\mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*, \tilde{m} \times \tilde{\nu}_{\beta-1})$ where \tilde{m} is the Haar measure on \mathbb{A}_S normalized such that $\tilde{m}(\hat{R}_S) = 1$, it suffices to show that the action of $G_{m,\Gamma}$ on $(\mathbb{R}_+^* \times \mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*, \lambda_\infty \times \tilde{m} \times \tilde{\nu}_{\beta-1})$ given by

$$(n, k)(t, \mathbf{b}, \bar{\mathbf{a}}) = (N(k)^\beta t, n + k\mathbf{b}, k\bar{\mathbf{a}}), \quad \text{for } (n, k) \in G_{m,\Gamma}, t \in \mathbb{R}, \mathbf{b} \in \mathbb{A}_S, \bar{\mathbf{a}} \in \mathbb{A}_S/\hat{R}_S^*$$

is ergodic. Since $R_m^{-1}R$ is dense in \mathbb{A}_S by the strong approximation theorem, the action of $R_m^{-1}R$ on $(\mathbb{A}_S, \tilde{m})$ by translation is ergodic. Thus, any $(R_m^{-1}R \times \{1\})$ -invariant measurable function on the product space $\mathbb{R}_+^* \times \mathbb{A}_S \times \mathbb{A}_S/\hat{R}_S^*$ does not

depend on the second coordinate. Hence, to prove that the above action is ergodic, it suffices to prove that the action given in Equation (18) is ergodic.

For $\beta = 1$, this follows since $\tilde{\nu}_0 = \delta_{\bar{\mathbf{0}}}$ and $\{N(k) : k \in K_{\mathbf{m},\Gamma}\} = \mathbb{Q}_+^*$, which is dense in \mathbb{R}_+^* , whereas for $\beta \in (1, 2]$, this follows from Proposition 4.7. \square

Remark 4.9. Since the seminal work of Bost and Connes [3], there have been several operator algebraic constructions from number theory that lead to C^* -dynamical systems exhibiting interesting phase transitions where the high temperature KMS states are factor states of type III_1 . See, for example, [3], [10], [19], [2], [41, 42], [26], and [27]. We remark that in all cases, uniqueness of the high temperature KMS states boils down to that fact that certain L -functions do not have poles at 1, and the crucial number-theoretic result needed to compute the type is a version of the prime number theorem.

5. THE BOUNDARY QUOTIENT

By [5, Theorem 7.1], the C^* -algebra $C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma})$ has a unique maximal ideal $I_{\mathcal{P}_K^{\mathbf{m}}}$. The boundary quotient of $C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma})$, as defined in [34, Section 7] (see also [14, Chapter 5.7]), is the quotient $C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma})/I_{\mathcal{P}_K^{\mathbf{m}}}$.

Moreover, [5, Theorem 7.1] gives an explicit description of the ideal $I_{\mathcal{P}_K^{\mathbf{m}}}$. We shall only need to know that $I_{\mathcal{P}_K^{\mathbf{m}}}$ corresponds to the subset of $\Omega_R^{\mathbf{m}}$ given by

$$\Omega_R^{\mathbf{m}} \setminus (\hat{R}_S \times \{\bar{\mathbf{0}}\}) = \{[\mathbf{b}, \bar{\mathbf{a}}] \in \Omega_R^{\mathbf{m}} : \text{there exists } \mathbf{p} \in \mathcal{P}_K^{\mathbf{m}} \text{ with } v_{\mathbf{p}}(\bar{\mathbf{a}}) < \infty\}.$$

Let $\rho : C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma}) \rightarrow C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma})/I_{\mathcal{P}_K^{\mathbf{m}}}$ be the quotient map. For each $t \in \mathbb{R}$, the automorphism σ_t leaves $I_{\mathcal{P}_K^{\mathbf{m}}}$ invariant, so σ defines a time evolution $\bar{\sigma}$ on $C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma})/I_{\mathcal{P}_K^{\mathbf{m}}}$ such that $\bar{\sigma}_t(\rho(\lambda_{(b,a)})) = N(a)^{it}\rho(\lambda_{(b,a)})$ for all $(b, a) \in R \rtimes R_{\mathbf{m},\Gamma}$.

Theorem 5.1. *The C^* -dynamical system $(C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma})/I_{\mathcal{P}_K^{\mathbf{m}}}, \mathbb{R}, \bar{\sigma})$ has a unique $\bar{\sigma}$ -KMS₁ state $\bar{\phi}$, and there are no $\bar{\sigma}$ -KMS _{β} states for $\beta \neq 1$. Moreover, if $\pi_{\bar{\phi}}$ is the GNS representation of $\bar{\phi}$, then $\pi_{\bar{\phi}}(C_\lambda^*(R \rtimes R_{\mathbf{m},\Gamma})/I_{\mathcal{P}_K^{\mathbf{m}}})''$ is isomorphic to the injective factor of type III_1 with separable predual, and $\bar{\phi}$ is determined by the values*

$$\bar{\phi}(\rho(E_{(x+\mathbf{a}) \times (\mathbf{a} \cap R_{\mathbf{m},\Gamma})})) = N(\mathbf{a})^{-1} \quad \text{for all } x \in R \text{ and } \mathbf{a} \in \mathcal{I}_{\mathbf{m}}^+. \quad (19)$$

Proof. If ϕ is a $\bar{\sigma}$ -KMS $_{\beta}$ on $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})/I_{\mathcal{P}_K^{\mathfrak{m}}}$, then the composition $\phi \circ \rho$ is a σ -KMS $_{\beta}$ state on $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ that vanishes on $\ker \rho = I_{\mathcal{P}_K^{\mathfrak{m}}}$. Moreover, the map $\rho^* : \phi \mapsto \phi \circ \rho$ is injective. Suppose that a σ -KMS $_{\beta}$ state ϕ on $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ belongs to the range of ρ^* , and let μ be the quasi-invariant probability measure on $\Omega_R^{\mathfrak{m}}$ determined by $\phi|_{C(\Omega_R^{\mathfrak{m}})}$. From our analysis of quasi-invariant probability measures in Section 3, we know that either $\mu = \mu_{\beta}$ (in the case $\beta \in [1, 2]$) or μ is a convex combination of the measures $\mu_{\beta, \mathfrak{k}}$, $\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ (in the case $\beta \in (2, \infty)$), where μ_{β} is defined in the proof of Theorem 3.2(ii) and $\mu_{\beta, \mathfrak{k}}$ is defined in the proof of Theorem 3.2(iii). Since $\phi|_{C(\Omega_R^{\mathfrak{m}})}$ vanishes on $I_{\mathcal{P}_K^{\mathfrak{m}}} \cap C(\Omega_R^{\mathfrak{m}}) = C_0(\Omega_R^{\mathfrak{m}} \setminus (\hat{R}_S \times \{\bar{\mathbf{0}}\}))$, it follows that μ is concentrated on $\hat{R}_S \times \{\bar{\mathbf{0}}\}$. This happens only for $\beta = 1$ and $\mu = \mu_1$, in which case $\phi = \phi_1$ is the unique σ -KMS $_1$ state on $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})$ from Theorem 3.2(ii). This implies that there are no $\bar{\sigma}$ -KMS $_{\beta}$ states for $\beta \neq 1$.

A direct calculation using [37, Corollary 8.14.4] shows that ϕ_1 vanishes on the ideal $I_{\mathcal{P}_K^{\mathfrak{m}}}$ and thus factors through ρ to define a $\bar{\sigma}$ -KMS $_1$ state on $C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})/I_{\mathcal{P}_K^{\mathfrak{m}}}$; the uniqueness of $\bar{\sigma}$ -KMS $_1$ states now follows from the injectivity of ρ^* .

Since $\pi_{\bar{\phi}} \circ \rho = \pi_{\phi_1}$ where π_{ϕ_1} is the GNS representation of ϕ_1 , it follows from Theorem 3.2(ii) that $\pi_{\bar{\phi}}(C_{\lambda}^*(R \rtimes R_{\mathfrak{m},\Gamma})/I_{\mathcal{P}_K^{\mathfrak{m}}})''$ is isomorphic to the injective factor of type III $_1$ with separable predual. \square

Remark 5.2. For the case of trivial \mathfrak{m} and Γ , the uniqueness claim in Theorem 5.1 follows from [13, Theorem 6.7].

6. PHASE TRANSITIONS ON C*-ALGEBRAS OF MULTIPLICATIVE MONOIDS

For each $a \in R_{\mathfrak{m},\Gamma}$, let λ_a denote the isometry on $\ell^2(R_{\mathfrak{m},\Gamma})$ determined by $\lambda_a(\epsilon_x) = \epsilon_{ax}$ where $\{\epsilon_x : x \in R_{\mathfrak{m},\Gamma}\}$ is the canonical orthonormal basis for $\ell^2(R_{\mathfrak{m},\Gamma})$. Then the left regular C*-algebra of the (commutative) semigroup $R_{\mathfrak{m},\Gamma}$ is the sub-C*-algebra of $\mathcal{B}(\ell^2(R_{\mathfrak{m},\Gamma}))$ generated by these isometries, that is,

$$C_{\lambda}^*(R_{\mathfrak{m},\Gamma}) := C^*(\{\lambda_a : a \in R_{\mathfrak{m},\Gamma}\}).$$

The C*-algebra $C_{\lambda}^*(R_{\mathfrak{m},\Gamma})$ also carries a canonical time evolution σ^{\times} that is determined on the generating isometries by $\sigma_t^{\times}(\lambda_a) = N(a)^{it} \lambda_a$ for $a \in R_{\mathfrak{m},\Gamma}$.

Remark 6.1. Using [5, Proposition 3.9] and Li's theory of semigroup C*-algebras from [33, 34], one can show that there is an injective *-homomorphism

$$C_\lambda^*(R_{\mathfrak{m},\Gamma}) \hookrightarrow C_\lambda^*(R \rtimes R_{\mathfrak{m},\Gamma})$$

such that $\lambda_a \mapsto \lambda_{(0,a)}$ for all $a \in R_{\mathfrak{m},\Gamma}$. Hence, under this embedding, the time evolution σ^\times coincides with the restriction of the time evolution σ to (the image of) $C_\lambda^*(R_{\mathfrak{m},\Gamma})$.

The (commutative) semigroup $R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*$ can be identified with the semigroup of principal ideals that are generated by an element from $R_{\mathfrak{m},\Gamma}$. For each $a \in R_{\mathfrak{m},\Gamma}$, let $\lambda_{aR_{\mathfrak{m},\Gamma}^*}$ denote the corresponding isometry in the left regular C*-algebra $C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$; this C*-algebra also carries a canonical time evolution, which we also denote by σ^\times . It is determined by $\sigma^\times(\lambda_{aR_{\mathfrak{m},\Gamma}^*}) = N(a)^{it}\lambda_{aR_{\mathfrak{m},\Gamma}^*}$ for $aR_{\mathfrak{m},\Gamma}^* \in R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*$.

In this section, we briefly explain how the techniques used to prove Theorem 3.2 also lead to phase transition theorems for the C*-dynamical systems

$$(C_\lambda^*(R_{\mathfrak{m},\Gamma}), \mathbb{R}, \sigma^\times) \quad \text{and} \quad (C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*), \mathbb{R}, \sigma^\times).$$

Namely, we have the following two theorems, the first one for the left regular C*-algebra of a congruence monoid itself, and the second one for left regular C*-algebra of a semigroup of principal ideals that are generated by elements from a congruence monoid.

Theorem 6.2. *Let K be a number field, \mathfrak{m} a modulus for K , and Γ a subgroup of $(R/\mathfrak{m})^*$.*

- (i) *There are no σ^\times -KMS $_\beta$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ for $\beta < 0$.*
- (ii) *The simplex of σ^\times -KMS $_0$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ is isomorphic to the simplex of σ^\times -invariant states on the commutative group C*-algebra $C^*(K_{\mathfrak{m},\Gamma})$.*
- (iii) *For each $\beta \in (0, 1]$, the simplex of σ^\times -KMS $_\beta$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ is isomorphic to the simplex of states on the commutative group C*-algebra $C^*(R_{\mathfrak{m},\Gamma}^*)$. Moreover, if $\psi_{\beta,\chi}$ is the extremal σ^\times -KMS $_\beta$ state corresponding to the character $\chi \in \widehat{R_{\mathfrak{m},\Gamma}^*}$ and $\pi_{\psi_{\beta,\chi}}$ is the GNS representation of $\psi_{\beta,\chi}$, then $\pi_{\psi_{\beta,\chi}}(C_\lambda^*(R_{\mathfrak{m},\Gamma}))''$ is isomorphic to the injective factor of type III $_1$ with separable predual.*

- (iv) For each $\beta > 1$, the simplex of σ^\times -KMS $_\beta$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ is isomorphic to the simplex of states on the commutative C^* -algebra

$$\bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} C^*(R_{\mathfrak{m},\Gamma}^*).$$

- (v) The set of σ^\times -ground states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ is isomorphic to the state space of the C^* -algebra

$$\bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} M_{k_{\mathfrak{k}}}(C^*(R_{\mathfrak{m},\Gamma}^*))$$

where $k_{\mathfrak{k}}$ is the number of norm-minimizing ideals in the class \mathfrak{k} .

Theorem 6.3. Let K be a number field, \mathfrak{m} a modulus for K , and Γ a subgroup of $(R/\mathfrak{m})^*$.

- (i) There are no σ^\times -KMS $_\beta$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$ for $\beta < 0$.
- (ii) The simplex of σ^\times -KMS $_0$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ is isomorphic to the simplex of σ^\times -invariant states on the commutative group C^* -algebra $C^*(K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$.
- (iii) For each $\beta \in (0, 1]$, there is a unique σ^\times -KMS $_\beta$ state ω_β on $C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$. Moreover, if π_{ω_β} is the GNS representation of ω_β , then $\pi_{\omega_\beta}(C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*))''$ is isomorphic to the injective factor of type III $_1$ with separable predual.
- (iv) For each $\beta > 1$, the simplex of σ^\times -KMS $_\beta$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$ is isomorphic to the simplex of states on the finite-dimensional commutative C^* -algebra $\mathbb{C}^{h_{\mathfrak{m},\Gamma}}$ where $h_{\mathfrak{m},\Gamma} := |\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})|$.
- (v) The set of σ^\times -ground states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ is isomorphic to the state space of the C^* -algebra

$$\bigoplus_{\mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} M_{k_{\mathfrak{k}}}(\mathbb{C})$$

where $k_{\mathfrak{k}}$ is the number of norm-minimizing ideals in the class \mathfrak{k} .

Remark 6.4. (a) For the special case of trivial \mathfrak{m} and Γ , the parameterization results in Theorem 6.2(i)-(iv) were already asserted in [13, Remark 7.5].

- (b) An alternative approach to computing the σ^\times -KMS $_\beta$ states on $C_\lambda^*(R^\times)$ and $C_\lambda^*(R^\times/R^*)$ for $\beta > 1$ is given in [14, Remark 6.6.5]. Presumably, the approach taken there could also be used to compute the low temperature KMS states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ and $C_\lambda^*(R_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$.

- (c) Using the canonical isomorphisms $C^*(K_{\mathfrak{m},\Gamma}) \cong C(\widehat{K_{\mathfrak{m},\Gamma}})$ and $C^*(R_{\mathfrak{m},\Gamma}^*) \cong C(\widehat{R_{\mathfrak{m},\Gamma}^*})$ given by the Fourier transform, the parameterizations in Theorem 6.2(iii)&(iv) can be phrased in terms of characters of the discrete abelian groups $K_{\mathfrak{m},\Gamma}$ and $R_{\mathfrak{m},\Gamma}^*$. Specifically,
- for each $\beta \in (0, 1]$, the extremal σ^\times -KMS $_\beta$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ are parameterized by the characters of the discrete abelian group $R_{\mathfrak{m},\Gamma}^*$;
 - for each $\beta > 1$, the extremal σ^\times -KMS $_\beta$ states on $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ are parameterized by pairs (\mathfrak{k}, χ) where \mathfrak{k} is a class in $\mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})$ and χ is character of $R_{\mathfrak{m},\Gamma}^*$.
- (d) An analogous statement involving characters of the discrete abelian group $K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*$ holds for the parameterization given by Theorem 6.3(ii).

The arguments needed to prove these theorems are almost identical, so we will only give a proof of Theorem 6.2.

Proof of Theorem 6.2. The strategy is similar to that used to prove Theorem 3.2, so we will only give a sketch of the arguments. There is a canonical action of the group $K_{\mathfrak{m},\Gamma}$ on \mathbb{A}_S/\hat{R}_S^* , and the C*-algebra of the reduction groupoid

$$K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^* = \{(k, \bar{\mathfrak{a}}) \in K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^* : k\bar{\mathfrak{a}} \in \hat{R}_S/\hat{R}_S^*\} \subseteq K_{\mathfrak{m},\Gamma} \times \mathbb{A}_S/\hat{R}_S^*$$

carries a canonical time evolution, which we also denote by σ^\times , determined by the real-valued 1-cocycle $c^\times : K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^* \rightarrow \mathbb{R}_+^*$ given by $(k, \bar{\mathfrak{a}}) \mapsto N(k)$. Arguments analogous to those given in [5, Section 5] show that the C*-algebra $C_\lambda^*(R_{\mathfrak{m},\Gamma})$ can be canonically and \mathbb{R} -equivariantly identified with the groupoid C*-algebra $C^*(K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^*)$. Hence, it suffices to compute all KMS and ground states of the C*-dynamical system $(C^*(K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^*), \mathbb{R}, \sigma^\times)$.

A short calculation similar to that from the proof of Theorem 3.2(i) shows that assertion (i) holds.

For $\beta \in [0, 1]$, Theorem 3.9 asserts that the measure ν_β defined in Section 3.4 is the unique probability measure on \hat{R}_S/\hat{R}_S^* that satisfies

$$\nu(kZ) = N(k)^{-\beta} \nu(Z)$$

for all $k \in K_{\mathfrak{m},\Gamma}$ and Borel sets $Z \subseteq \hat{R}_S/\hat{R}_S^*$ such that $kZ \subseteq \hat{R}_S/\hat{R}_S^*$. For $\beta = 0$, we have $\nu_\beta = \delta_{\bar{\mathbf{0}}}$, and the isotropy group of the point $\bar{\mathbf{0}}$ is all of $K_{\mathfrak{m},\Gamma}$. Since a state τ

of $C^*(K_{\mathfrak{m},\Gamma})$ is σ^\times -invariant if and only if $\tau(u_k) = 0$ for all $k \in K_{\mathfrak{m},\Gamma}$ with $N(k) \neq 1$, assertion (ii) follows from [43, Theorem 1.3] and [43, Corollary 1.4].

Now suppose $\beta \in (0, 1]$. Then the measure ν_β is concentrated in the set

$$A := \{\bar{\mathbf{a}} \in \hat{R}_S/\hat{R}_S^* : \bar{\mathbf{a}}_{\mathfrak{p}} \neq 0 \text{ for all } \mathfrak{p}\}.$$

Since the isotropy group of any point in this set is equal to $R_{\mathfrak{m},\Gamma}^*$, the parameterization result asserted in (iii) follows from [43, Theorem 1.3] (an argument similar to that used in the proof of Theorem 3.2(iii) is needed to verify that the given parameterization is an isomorphism of simplexes).

The state $\psi_{\beta,\chi}$ corresponding to the character $\chi \in \widehat{R_{\mathfrak{m},\Gamma}^*}$ is given explicitly by

$$\psi_{\beta,\chi}(f) = \int_A \sum_{g \in R_{\mathfrak{m},\Gamma}^*} \chi(g) f(g, \bar{\mathbf{a}}) d\nu_\beta(\bar{\mathbf{a}}) \quad \text{for } f \in C_c(K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^*). \quad (20)$$

(Note that this explicit formula is not given in the statement of [43, Theorem 1.3], but is given in its proof, which uses [43, Theorem 1.1].) Inspired by [24, Proposition 5.2], which came from an idea of Neshveyev [43, Remark 2.5], we shall now describe the von Neumann algebra $\pi_{\psi_{\beta,\chi}}(C^*(K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^*))''$ generated by the GNS representation $\pi_{\psi_{\beta,\chi}}$ of $\psi_{\beta,\chi}$. Let $\chi \in \widehat{R_{\mathfrak{m},\Gamma}^*}$, and choose an extension $\tilde{\chi}$ of χ to $K_{\mathfrak{m},\Gamma}$. There is a *-homomorphism

$$\tilde{\Psi}_{\tilde{\chi}} : C_0(\mathbb{A}_S/\hat{R}_S^*) \rtimes K_{\mathfrak{m},\Gamma} \rightarrow L^\infty(\mathbb{A}_S/\hat{R}_S^*, \tilde{\nu}_\beta) \rtimes (K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$$

such that $\tilde{\Psi}_{\tilde{\chi}}(f u_k) = \tilde{\chi}(k) f u_{\bar{k}}$ for all $f \in C_0(\mathbb{A}_S/\hat{R}_S^*)$ and $k \in K_{\mathfrak{m},\Gamma}$, where \bar{k} denotes the image of k under the quotient map $K_{\mathfrak{m},\Gamma} \rightarrow K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*$ and $\tilde{\nu}_\beta$ is the measure on \mathbb{A}_S/\hat{R}_S^* from the statement of Proposition 4.7.

Let $\Psi_{\tilde{\chi}}$ denote the composition

$$C^*(K_{\mathfrak{m},\Gamma} \times \hat{R}_S/\hat{R}_S^*) \cong 1_{\hat{R}_S/\hat{R}_S^*}(C_0(\mathbb{A}_S/\hat{R}_S^*) \rtimes K_{\mathfrak{m},\Gamma}) 1_{\hat{R}_S/\hat{R}_S^*} \rightarrow L^\infty(\mathbb{A}_S/\hat{R}_S^*, \tilde{\nu}_\beta) \rtimes (K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$$

where the second arrow is the restriction of $\tilde{\Psi}_{\tilde{\chi}}$ to the (full) corner $1_{\hat{R}_S/\hat{R}_S^*}(C_0(\mathbb{A}_S/\hat{R}_S^*) \rtimes K_{\mathfrak{m},\Gamma}) 1_{\hat{R}_S/\hat{R}_S^*}$. A calculation using the explicit formula for $\psi_{\beta,\chi}$ given in Equation (20) shows that $\psi_{\beta,\chi} = \varphi \circ \Psi_{\tilde{\chi}}$ where φ is the canonical normal state on $1_{\hat{R}_S/\hat{R}_S^*}(L^\infty(\mathbb{A}_S/\hat{R}_S^*, \tilde{\nu}_\beta) \rtimes (K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)) 1_{\hat{R}_S/\hat{R}_S^*}$ determined by ν_β . Since the image of $\Psi_{\tilde{\chi}}$ is strong operator dense in the corner $1_{\hat{R}_S/\hat{R}_S^*}(L^\infty(\mathbb{A}_S/\hat{R}_S^*, \tilde{\nu}_\beta) \rtimes (K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)) 1_{\hat{R}_S/\hat{R}_S^*}$,

we get a (non-canonical) isomorphism

$$\pi_{\psi_{\beta,\chi}}(C^*(K_{\mathfrak{m},\Gamma} \rtimes \hat{R}_S/\hat{R}_S^*))'' \cong 1_{\hat{R}_S/\hat{R}_S^*}(L^\infty(\mathbb{A}_S/\hat{R}_S^*, \tilde{\nu}_\beta) \rtimes K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)1_{\hat{R}_S/\hat{R}_S^*}$$

The assertion about injectivity and separability is easy to see, and factoriality follows from extremality of $\psi_{\beta,\chi}$ by [4, Theorem 5.3.30(3)]. To prove our assertion about type, it suffices to show that the flow of weights on $L^\infty(\mathbb{A}_S/\hat{R}_S^*, \tilde{\nu}_\beta) \rtimes (K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*)$ is trivial, and for this it is enough to show that the action of $K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*$ on

$$(\mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*, \lambda_\infty \times \tilde{\nu}_\beta)$$

given by

$$\bar{k}(t, \bar{\mathfrak{a}}) = (N(k)^\beta t, k\bar{\mathfrak{a}}) \quad \text{for } \bar{k} \in K_{\mathfrak{m},\Gamma}/R_{\mathfrak{m},\Gamma}^*, (t, \bar{\mathfrak{a}}) \in \mathbb{R}_+^* \times \mathbb{A}_S/\hat{R}_S^*$$

is ergodic. This follows from Proposition 4.7.

For $\beta \in (1, \infty)$, Lemma 3.13 says that the extremal probability measures that satisfy (8) are precisely the measures $\{\nu_{\beta,\mathfrak{k}} : \mathfrak{k} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})\}$. These measures are concentrated in the set

$$\{\bar{\mathfrak{a}} : \bar{\mathfrak{a}}_{\mathfrak{p}} \neq 0 \text{ for all } \mathfrak{p}\}.$$

Since the isotropy group of any point in this set is $R_{\mathfrak{m},\Gamma}^*$, the parameterization stated in Theorem 6.2(iv) also follows from [43, Theorem 1.3], and arguing as in Theorem 3.2(iii), one shows that this parameterization is an isomorphism of simplexes.

Following the proof of Theorem 3.2(iv), we see that the boundary set of the cocycle c^\times (cf. [23, Section 1]) is equal to

$$(\hat{R}_S/\hat{R}_S^*)_0 := \{\bar{\mathfrak{a}} \in \hat{R}_S/\hat{R}_S^* : \bar{\mathfrak{a}} = \mathfrak{a}_{\mathfrak{k},j} \text{ for some } 1 \leq j \leq k_{\mathfrak{k}}\}$$

where $\mathfrak{a}_{\mathfrak{k},1}, \dots, \mathfrak{a}_{\mathfrak{k},k_{\mathfrak{k}}}$ are the norm-minimizing ideals in the class \mathfrak{k} (see the discussion preceding Theorem 3.2). Let $K_{\mathfrak{m},\Gamma,1} := \{x \in K_{\mathfrak{m},\Gamma} : N(x) = 1\}$. Then [22, Theorem 1.9] asserts that the map $\psi \mapsto \phi_\psi$ defined by

$$\phi_\psi(f) = \psi(f|_{K_{\mathfrak{m},\Gamma,1} \times (\hat{R}_S/\hat{R}_S^*)_0}) \quad \text{for } f \in C_c(K_{\mathfrak{m},\Gamma} \rtimes \hat{R}_S/\hat{R}_S^*)$$

is an affine isomorphism of the state space of $C^*(K_{\mathfrak{m},\Gamma,1} \times (\hat{R}_S/\hat{R}_S^*)_0)$ onto the σ -ground state space of $C^*(K_{\mathfrak{m},\Gamma} \rtimes \hat{R}_S/\hat{R}_S^*)$ where $K_{\mathfrak{m},\Gamma,1} \times (\hat{R}_S/\hat{R}_S^*)_0$ is the reduction groupoid of $K_{\mathfrak{m},\Gamma,1} \times \mathbb{A}_S/\hat{R}_S^*$ with respect to the subset $(\hat{R}_S/\hat{R}_S^*)_0 \subseteq \mathbb{A}_S/\hat{R}_S^*$. Now

arguments similar to those used to prove Theorem 3.2(iv) show that

$$C^*(K_{\mathfrak{m},\Gamma,1} \times (\hat{R}_S/\hat{R}_S^*)_0) \cong \bigoplus_{\mathfrak{t} \in \mathcal{I}_{\mathfrak{m}}/i(K_{\mathfrak{m},\Gamma})} M_{k_{\mathfrak{t}}}(C^*(R_{\mathfrak{m},\Gamma}^*)),$$

which finishes the proof of Theorem 6.2(v). \square

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