

**ON THE STRUCTURE OF  $C^*$ -ALGEBRAS  
ASSOCIATED WITH HYPERBOLIC  
DYNAMICAL SYSTEMS**

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# On the Structure of $C^*$ -algebras Associated with Hyperbolic Dynamical Systems

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*Dedicated to Marc Rieffel on the occasion of  
his sixtieth birthday.*

**Abstract.** We consider the stable, unstable  $C^*$ -algebras and the Ruelle algebras associated to a mixing Smale space. In the case of a shift of finite type, these are the AF-algebras studied by W. Krieger and the (stabilized) Cuntz-Krieger algebras. In the general case, we show that the stable and unstable algebras are simple and amenable. We also show the Ruelle algebras are simple, amenable and purely infinite.

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## 1. Introduction and Statement of the Main Results

Our interest in this paper is in Smale spaces and their associated  $C^*$ -algebras. For more detailed information, we refer the reader to [7,8,11]. We will also give explicit descriptions of the basic ideas in the next section, but for the moment, a Smale space is a homeomorphism,  $\phi$ , of a compact metric space,  $(X, d)$ , having specific properties. Roughly, these mean that  $X$  has local canonical coordinates of contracting and expanding directions for  $\phi$ . We will also assume throughout that  $\phi$  is mixing [7,8].

These systems include Anosov diffeomorphisms (the smooth case), shifts of finite type (the zero dimensional case) and other interesting examples from the theory of self-similar tilings [3].

We consider the notions of stable and unstable equivalence;  $x$  and  $y$  are stably (unstably) equivalent if  $d(\phi^n(x), \phi^n(y))$  tends to zero as  $n$  tends to plus (minus, respectively) infinity. We let  $G_s$  and  $G_u$  denote these equivalence relations, *i.e.* principal groupoids. They may be topologized and given Haar systems so we may consider their  $C^*$ -algebras:

$$S = C^*(G_s), \quad U = C^*(G_u).$$

In the case of a shift of finite type, these are the  $AF$ -algebras considered by Krieger [4].

In general,  $\phi$  induces automorphisms of  $G_s$  and  $G_u$  and we may form groupoids  $G_s \rtimes \mathbb{Z}$  and  $G_u \rtimes \mathbb{Z}$ , whose  $C^*$ -algebras are  $*$ -isomorphic to the crossed products

$$C^*(G_s) \rtimes \mathbb{Z}, \quad C^*(G_u) \rtimes \mathbb{Z}.$$

These are denoted  $R_s$  and  $R_u$ , respectively and we refer to them as the Ruelle algebras. Again for a shift of finite type, these are the (stabilized) Cuntz-Krieger algebras.

Here, we are interested in analyzing the structure of the  $C^*$ -algebras:  $S, U, R_s, R_u$ . We state the main results as follows.

**Theorem 1.1.** *The groupoids  $G_s$  and  $G_u$  are amenable in the sense of Renault. Hence, we have*

$$S = C^*(G_s) \cong C_{\text{red}}^*(G_s),$$

$$U = C^*(G_u) \cong C_{\text{red}}^*(G_u),$$

and these are amenable  $C^*$ -algebras.

**Theorem 1.2.** *The groupoids  $G_s \rtimes \mathbb{Z}$  and  $G_u \rtimes \mathbb{Z}$  are amenable in the sense of Renault. Hence, we have*

$$R_s = C^*(G_s \rtimes \mathbb{Z}) \cong C_{\text{red}}^*(G_s \rtimes \mathbb{Z})$$

$$R_u = C^*(G_u \rtimes \mathbb{Z}) \cong C_{\text{red}}^*(G_u \rtimes \mathbb{Z})$$

and these are amenable  $C^*$ -algebras.

**Theorem 1.3.** *The  $C^*$ -algebras  $S$  and  $U$  are simple.*

**Theorem 1.4.** *The  $C^*$ -algebras  $R_s$  and  $R_u$  are simple.*

**Theorem 1.5.** *The  $C^*$ -algebras  $R_s$  and  $R_u$  are purely infinite.*

The main technique is to use ideas and results of Muhly, Renault and Williams (building on earlier work of Rieffel) regarding equivalence of groupoids and strong Morita equivalence of their  $C^*$ -algebras. A common set-up is to have a groupoid  $G$  with an abstract transversal  $T \subseteq G^\circ$  (the unit space of  $G$ ). In our situation, in considering  $G_s$ , we show how we can use any single unstable equivalence class as a transversal. The subtlety here lies in the fact that such a set is dense in  $X$  and its relative topology is rather unwholesome. It does, however, possess a nice topology in a very natural way. We show how the Muhly-Renault-Williams machine may be adapted to such a situation. Reducing  $G_s$  on such a transversal yields an  $r$ -discrete groupoid, because of the transverse nature of the local stable and unstable co-ordinates. It is then much simpler to analyze these groupoids and translate the results back to the original algebras using the strong Morita equivalence.

## 2. Smale Spaces

Here, we give the basic definitions of a Smale space along with the constructions of the groupoids associated with them. This is taken more or less directly from [10], but we present it for completeness.

Let  $(X, d)$  be a compact metric space and let  $\phi$  be a homeomorphism of  $X$ . We will assume throughout  $(X, \phi)$  is topologically mixing [7]. (This is not part of the usual definition of Smale space.) We assume that we have constants

$$\epsilon_0 > 0, \quad 0 < \lambda_0 < 1$$

and a continuous map

$$(x, y) \in \{(x, y) \in X \times X \mid d(x, y) \leq 2\epsilon_0\} \rightarrow [x, y] \in X$$

satisfying axioms as in [7,8,10,11].

For  $0 < \epsilon \leq \epsilon_0$ , we define

$$V^s(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon, \quad [x, y] = y\}$$

$$V^u(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon, \quad [y, x] = y\}$$

so we have (as an axiom)

$$d(\phi(y), \phi(z)) \leq \lambda_0 d(y, z), \quad y, z \in V^s(x, \epsilon)$$

$$d(\phi^{-1}(y), \phi^{-1}(z)) \leq \lambda_0 d(y, z), \quad y, z \in V^u(x, \epsilon).$$

That is,  $\phi$  contracts on  $V^s(x, \epsilon)$  while  $\phi^{-1}$  contracts on  $V^u(x, \epsilon)$ . The axioms imply that the map sending  $(y, z)$  in  $V^u(x, \epsilon) \times V^s(x, \epsilon)$  to  $[y, z]$  is a homeomorphism onto a neighbourhood of  $x$ . Such a neighbourhood is called a rectangle.

Next, we define, for any  $x$  in  $X$ ,

$$V^s(x) = \bigcup_{n \geq 0} \phi^{-n}(V^s(\phi^n(x), \epsilon))$$

$$V^u(x) = \bigcup_{n \geq 0} \phi^n(V^u(\phi^{-n}(x), \epsilon)),$$

both being independent of  $\epsilon > 0$ . Each set  $\phi^{-n}(V^s(\phi^n(x), \epsilon))$  is given the relative topology of  $X$ , while  $V^s(x)$  is given the inductive limit topology. In this topology it is a locally compact, non-compact Hausdorff space. On the other hand, if we assume that  $(X, \phi)$  is mixing, then  $V^s(x)$  is dense in  $X$  [10]. We treat  $V^u(x)$  in an analogous way.

We recall from [7,8],

$$G_s^o = \{(x, y) \in X \times X \mid y \in V^s(x, \epsilon_0)\}$$

$$G_u^o = \{(x, y) \in X \times X \mid y \in V^u(x, \epsilon_0)\}$$

$$G_s^n = (\phi \times \phi)^{-n} (G_s^o), \quad n \geq 0$$

$$G_u^n = (\phi \times \phi)^n (G_u^o), \quad n \geq 0$$

$$G_s = \bigcup_{n \geq 0} G_s^n$$

$$G_u = \bigcup_{n \geq 0} G_u^n.$$

Then  $G_s$  and  $G_u$  are equivalence relations on  $X$ , called stable and unstable equivalence. Each  $G_s^n, G_u^n$  are given the relative topologies of  $X \times X$  and  $G_s, G_u$  are given the inductive limit topologies. Notice that the  $G_s$ -equivalence class of  $x$  in  $X$  is simply  $V^s(x)$ . Finally, we let

$$G_a^n = G_s^n \cap G_u^n, \quad n \geq 0$$

$$G_a = \bigcup_{n \geq 0} G_a^n.$$

Again, each  $G_a^n$  is given the relative topology of  $X$  while  $G_a$  is given the inductive limit topology.  $G_a$  is also an equivalence relation on  $X$ . For each  $x$  in  $X$ , we denote its  $G_a$ -equivalence class by  $V^a(x)$ ; it is countable and dense in  $X$  if  $(X, \phi)$  is mixing [10].

We regard  $G_s, G_u, G_a$  as principal groupoids. With their topologies they are locally compact and Hausdorff. Moreover,  $G_a$  is  $r$ -discrete and counting measure is a Haar system. Haar systems

$$\{\mu_s^x \mid x \in X\}, \quad \{\mu_u^x \mid x \in X\}$$

for  $G_s$  and  $G_u$ , respectively, are described in [7,8]. We let  $S(X, \phi)$ ,  $U(X, \phi)$  and  $A(X, \phi)$  denote the  $C^*$ -algebras of  $G_s, G_u$  and  $G_a$ , respectively.

The map  $\phi \times \phi$  acts as automorphisms of  $G_s, G_u$  and  $G_a$  (scaling the Haar systems

in the first two). We form the semi-direct products as follows:

$$G_s \rtimes \mathbb{Z} = \{(x, n, y) \mid n \in \mathbb{Z}, (\phi^n(x), y) \in G_s\}$$

$$G_u \rtimes \mathbb{Z} = \{(x, n, y) \mid n \in \mathbb{Z}, (\phi^n(x), y) \in G_u\}$$

$$G_a \rtimes \mathbb{Z} = \{(x, n, y) \mid n \in \mathbb{Z}, (\phi^n(x), y) \in G_a\}$$

with groupoid operations

$$(x, n, y) \cdot (x', n', y') = (x, n + n', y') \text{ if } y = x'$$

$$(x, n, y)^{-1} = (y, -n, x).$$

Observe that  $G_s \subseteq G_s \rtimes \mathbb{Z}$ ,  $G_u \subseteq G_u \rtimes \mathbb{Z}$ ,  $G_a \subseteq G_a \rtimes \mathbb{Z}$  by identifying  $(x, y)$  in  $G_s$  with  $(x, 0, y)$  in  $G_s \rtimes \mathbb{Z}$ , for example.

Notice that  $G_s^o = (G_s \rtimes \mathbb{Z})^o$ ,  $G_u^o = (G_u \rtimes \mathbb{Z})^o$ ,  $G_a^o = (G_a \rtimes \mathbb{Z})^o$ , with the identifications above.

Finally, the map,  $\eta$ , sending  $(x, y, n)$  in  $G_s \times \mathbb{Z}$  to  $(x, n, \phi^n(y))$  in  $G_s \rtimes \mathbb{Z}$  is bijective, and we transfer the product topology from  $G_s \times \mathbb{Z}$  over via this map.

For any  $x$  in  $X$ ,  $(x, 0, x)$  is in the unit space  $(G_s \rtimes \mathbb{Z})^o$  and

$$\begin{aligned} r^{-1} \{(x, 0, x)\} &= \{(x, n, y) \in G_s \rtimes \mathbb{Z}\} \\ &= \bigcup_{n \in \mathbb{Z}} \eta(\{(x, y) \in G_s\} \times \{n\}). \end{aligned}$$

Using this decomposition, we define a Haar system  $\lambda_s^x$  on  $G_s \rtimes \mathbb{Z}$  by setting

$$\lambda_s^x \mid \{(x, y) \mid y \in V^s(x)\} \times \{n\} = \mu_s^x \circ \eta^{-1}.$$

We treat  $G_u \rtimes \mathbb{Z}$  and  $G_a \rtimes \mathbb{Z}$  similarly.

The  $C^*$ -algebras  $C^*(G_s \rtimes \mathbb{Z})$ ,  $C^*(G_u \rtimes \mathbb{Z})$  and  $C^*(G_a \rtimes \mathbb{Z})$  are denoted  $R_s$ ,  $R_u$  and  $R_a$  and are called the Ruelle algebras.

Another description of these algebras is to consider the automorphisms  $\alpha_s$ ,  $\alpha_u$  and  $\alpha_a$  of  $S$ ,  $U$  and  $A$ , respectively, which are induced by the automorphisms  $\phi \times \phi$  of  $G_s$ ,  $G_u$

and  $G_a$  and take the  $C^*$ -crossed products by  $\mathbb{Z}$ . That is,

$$R_s \cong S \rtimes \mathbb{Z}$$

$$R_u \cong U \rtimes \mathbb{Z}$$

$$R_a \cong A \rtimes \mathbb{Z}.$$

### 3. Generalized Transversals

In this section, we present a general result on groupoids. The idea is to show how the techniques of Muhly, Renault and Williams [6] on equivalence of groupoids may be applied to certain situations involving “generalized transversals”. Let us begin by giving a simple example to motivate our result.

Let  $\theta$  be a fixed irrational number between 0 and 1. Let  $G$  be the groupoid of the Kronecker flow on the two-torus,  $\mathbb{T}^2$ , determined by  $\theta$ . That is,

$$G = \mathbb{T}^2 \times \mathbb{R}$$

$$(w_1, w_2, s) \cdot (z_1, z_2, t) = (w_1, w_2, s + t)$$

$$\text{if } z_1 = e^{2\pi i s} w_1 \quad \text{and} \quad z_2 = e^{2\pi i s \theta} w_2.$$

An example of an “abstract transversal” in this situation is

$$T = \mathbb{T} \times \{1\} \times \{0\} \subseteq G^0.$$

The reduction of  $G$  on  $T$  is:

$$G_T^T = \{g \in G \mid r(g), s(g) \in T\}$$

and can, in this case, be identified with

$$\mathbb{T} \times \mathbb{Z}$$

$$(w, k) \cdot (z, \ell) = (w, k + \ell) \quad \text{if } z = e^{2\pi i k \theta} w,$$

in a straightforward way.

Our point is that there is another, less obvious, choice. Pick any irrational  $\alpha$ , between 0 and 1 and unequal to  $\theta$ . Let

$$T = \{(e^{2\pi it}, e^{2\pi i\alpha t}, 0) \in G^0 \mid t \in \mathbb{R}\}$$

which is a line, winding densely in  $\mathbb{T}^2 \cong G^0$  and transverse to each  $G$ -orbit. This  $T$  can also be used as a transversal to  $G$ ; of course, its relative topology in  $G$  is rather horrid. Instead we want to use its natural topology as a line. In this case

$$G_T^T \cong \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}$$

$$(x, k, \ell) \cdot (y, m, n) = (x, k + m, \ell + n)$$

$$\text{if } y = x + k + \ell\theta.$$

The difficulty lies in showing that this “re-topologizing” of the transversal can be incorporated into the equivalence of Muhly, *et al.*

From now on, we assume that  $G$  is a second countable locally compact, Hausdorff groupoid with Haar system. Let  $T$  be a locally compact, second countable, Hausdorff space and let  $f : T \rightarrow G^0$  be a continuous, injective map.

We say that open sets  $U \subseteq G$  and  $V^r \subseteq T$  satisfy (Ar) if, for all  $x$  in  $U$ , there is a unique  $y$  in  $U$  with  $s(x) = s(y)$  and  $r(y) \in f(V^r)$ .

We note that if  $(U, V^r)$  satisfy (Ar) then so does the pair  $(U, f^{-1}r(U) \cap V^r)$ . We will say that open sets  $U \subseteq G$ ,  $V^s \subseteq T$  satisfy (As) if  $(U^{-1}, V^s)$  satisfies (Ar).

Finally, we say that open sets  $U \subseteq G$ ,  $V^r, V^s \subseteq T$  satisfy (A) if

(i) for all  $t$  in  $V^r$ , there is a unique  $y$  in  $U$  with  $r(y) = f(t)$  and  $s(y)$  in  $f(V^s)$

and

(ii) for all  $t$  in  $V^s$ , there is a unique  $y$  in  $U$  with  $s(y) = f(t)$  and  $r(y)$  in  $f(V^r)$ .

On the groupoid  $G$ , space  $T$  and continuous injective map  $f : T \rightarrow G^0$ , we consider the following conditions.

**T1.** For any  $x$  in  $G$  with  $r(x)$  in  $f(T)$ , and open sets

$$x \in U_0 \subseteq G, \quad f^{-1}r(x) \in V_0 \subseteq T,$$

we may find open sets  $U \subseteq G$ ,  $V \subseteq T$  such that  $x \in U \subseteq U_0$ ,  $f^{-1}r(x) \in V \subseteq V_0$  and  $(U, V)$  satisfy (Ar).

**T1'**. For any  $x$  in  $G$  with  $s(x)$  in  $f(T)$ , and open sets

$$x \in U_0 \subseteq G, \quad f^{-1}s(x) \in V_0 \subseteq T,$$

we may find open sets  $U \subseteq G$ ,  $V \subseteq T$  such that  $x \in U \subseteq U_0$ ,  $f^{-1}s(x) \in V \subseteq V_0$  and  $(U, V)$  satisfy (As).

**T2**. For any  $x$  in  $G$  with  $r(x)$  and  $s(x)$  in  $f(T)$  and open sets

$$x \in U_0 \subseteq G$$

$$f^{-1}r(x) \in V_0^r \subseteq T$$

$$f^{-1}s(x) \in V_0^s \subseteq T,$$

there are open sets

$$x \in U \subseteq U_0$$

$$f^{-1}r(x) \in V^r \subseteq V_0^r$$

$$f^{-1}s(x) \in V^s \subseteq V_0^s$$

such that  $(U, V^r, V^s)$  satisfy (A).

**T3**. For any  $x$  in  $G$ , there is  $y$  in  $G$  with  $r(y) = r(x)$ ,  $s(y) \in f(T)$ .

We let

$$G_T = \{x \in G \mid s(x) \in f(T)\}$$

$$G_T^T = \{x \in G \mid r(x), s(x) \in f(T)\}.$$

(Note that these are  $G_{f(T)}$  and  $G_{f(T)}^{f(T)}$ , in the notation of [6].)

**Lemma 3.1.** *Suppose  $G, T, f$  satisfy T1. Then*

(i)  $G, T, f$  satisfy T1'.

(ii)  $G, T, f$  satisfy T2.

(iii) *the collection of sets*

$$U \cap s^{-1}f(V^s) \cap r^{-1}f(V^r),$$

where  $U \subseteq G$ ,  $V^s, V^r \subseteq T$  are open, forms a base for a topology on  $G_T^T$ .

(iv) the collection of sets

$$U \cap s^{-1} f(V^s),$$

where  $U \subseteq G$ ,  $V^s \subseteq T$  are open, forms a base for a topology on  $G_T$ .

*Proof.* The proof of (i) is straightforward and we omit it. As for (ii), consider  $x, U_0, V_0^r, V_0^s$  as in condition T2. We apply T1 and T1' to obtain open sets  $U_1, U_2 \subseteq U_0$  with  $x \in U_1, x \in U_2$ , and open sets  $V_1^r, V_1^s$  in  $T$  with  $f^{-1}r(x) \in V_1^r \subseteq V_0^r, f^{-1}s(x) \in V_1^s \subseteq V_0^s$ , such that  $(U_1, V_1^r)$  satisfy (Ar) while  $(U_2, V_1^s)$  satisfy (As). Writing  $x = r(x)x$ , we may find open sets  $U_3$  and  $U_4$  in  $G$ ,  $r(x) \in U_3, x \in U_4$  and  $U_3U_4 \subseteq U_1 \cap U_2$ . We once again apply T1 and T1' to  $r(x) \in U_3, f^{-1}r(x) \in V_1^r$  and  $x \in U_4, f^{-1}s(x) \in V_1^s$  to obtain open sets  $U_5, U_6$  in  $G$  with  $r(x) \in U_5 \subseteq U_3, x \in U_6 \subseteq U_4$ , and open sets  $V_2^r, V_2^s$  in  $T$  with  $f^{-1}r(x) \in V_2^r \subseteq V_1^r, f^{-1}s(x) \in V_2^s \subseteq V_1^s$ , such that  $(U_5, V_2^r)$  satisfy (Ar) and  $(U_6, V_2^s)$  satisfy (As). We let  $U = U_5U_6, V^r = f^{-1}r(U) \cap V_2^r$  and  $V^s = f^{-1}s(U) \cap V_2^s$ . Let us prove (i) holds in (A). Suppose  $t$  is in  $V^r$ . Then  $f(t) = r(yz)$ , for some  $y$  in  $U_5, z$  in  $U_6$ . By (As), there is  $z'$  in  $U_6$  with  $r(z') = r(z)$  and  $s(z')$  in  $f(V_2^s)$ . Then  $yz'$  is in  $U, r(yz') = f(t)$  and  $s(yz') = s(z') \in f(V^s)$ . As for the uniqueness, suppose  $z_1$  and  $z_2$  are both in  $U$  with  $r(z_1) = r(z_2) = f(t)$  and  $s(z_1), s(z_2)$  are both in  $f(V^s)$ . Then  $z_1, z_2 \in U = U_5U_6 \subseteq U_3U_4 \subseteq U_1 \cap U_2 \subseteq U_2, r(z_1) = r(z_2)$ , and  $s(z_1), s(z_2)$  are in  $f(V^s) \subseteq f(V_1^s)$ . By the uniqueness part of condition (As), we have  $z_1 = z_2$ . The proof of (ii) of (A) is similar. We omit the details.

Parts (iii) and (iv) are routine and we omit the details. ■

**Definition 3.2.** We let  $H$  and  $\Omega$  denote  $G_T^T$  and  $G_T$ , respectively, with the topologies given in the Lemma.

It is immediate that  $H$  and  $\Omega$  are second countable. Also observe that, a sequence  $\{x_n\}$  converges to  $x$  in  $H$  if and only if

$$\lim x_n = x \text{ in } G$$

$$\lim f^{-1}r(x_n) = f^{-1}r(x)$$

$$\text{and } \lim f^{-1}s(x_n) = f^{-1}s(x), \text{ in } T.$$

Similarly,  $\{x_n\}$  converges to  $x$  in  $\Omega$  if and only if

$$\lim x_n = x \text{ in } G$$

$$\lim f^{-1}s(x_n) = f^{-1}s(x) \text{ in } T.$$

The following is an immediate consequence of the definitions.

**Lemma 3.3.** *Suppose  $G, T, f$  satisfy T1. The collection of sets*

$$U \cap r^{-1}f(V^r) \cap s^{-1}f(V^s)$$

where  $U, V^r, V^s$  are open and satisfy (A) forms a base for the topology of  $H$ .

The collection of sets

$$U \cap s^{-1}f(V^s)$$

where  $U \subseteq G, V^s \subseteq T$  are open and satisfy (As), forms a base for the topology of  $\Omega$ .

Our aim is to show that conditions T1 and T2 imply that  $H$  is a locally compact, Hausdorff  $r$ -discrete groupoid with counting measure as a Haar system and that  $\Omega$  is a  $G$ - $H$  equivalence bimodule in the sense of [6].

The following is an immediate consequence of the definitions and we omit the proof.

**Lemma 3.4.** *Suppose  $(U, V^r, V^s)$  are open and satisfy (A) Let*

$$N = U \cap r^{-1}f(V^r) \cap s^{-1}f(V^s).$$

Then

$$r : N \longrightarrow f(V^r)$$

$$s : N \longrightarrow f(V^s)$$

are bijective.

**Lemma 3.5.** *Let  $t$  be in  $T$  and  $x = f(t)$ . Suppose  $(U, V^r, V^s)$  are open, satisfy (A) and  $x \in U, f^{-1}r(x) = f^{-1}s(x) = t \in V^r \cap V^s$ . Then there is  $V \subseteq V^r \cap V^s, t \in V$  and  $V$  open such that*

$$U \cap r^{-1}f(V) \cap s^{-1}f(V) \subseteq G^0.$$

*Proof.* By definition,  $U \cap G^0$  is open in  $G^0$ . As  $f$  is continuous, we may find  $V, t \in V \subseteq V^r \cap V^s$  with  $f(V) \subseteq U \cap G^0$ . Now suppose  $y$  is in  $U \cap r^{-1}f(V) \cap s^{-1}f(V)$ . So  $s(y)$  is in  $f(V)$  and  $f(V) \subseteq U$ , so  $s(y)$  is also in  $U$ . We have:

$$\begin{aligned}
y &\in U, r(y) \in f(V) \subseteq f(V^r) \\
s(y) &\in f(V) \subseteq f(V^s) \\
s(y) &\in U, r(s(y)) = s(y) \in f(V) \subseteq f(V^r) \\
s(s(y)) &= s(y) \in f(V) \subseteq f(V^s) \\
&\text{and } s(y) = s(s(y)).
\end{aligned}$$

Hence by (A) (ii),  $y = s(y)$  by uniqueness. This implies  $y$  is in  $G^0$ . ■

**Theorem.**  *$H$  is a second countable, locally compact, Hausdorff,  $r$ -discrete groupoid, with counting measure as Haar system.*

*Proof.* That  $H = G_T^T$  is a groupoid, in the purely algebraic sense, is immediate. It is also immediate from the lemma and the facts that both  $T$  and  $G$  are second countable, that  $H$  is also. It is straightforward to check that the groupoid operations on  $H$  are continuous and we omit the details.

Lemma 3.5 shows that  $H^0$  is open in  $H$ . The map  $f : T \rightarrow H^0$  is clearly bijective and it is easy to check that it is a homeomorphism.

We will now prove that  $r : H \rightarrow H^0$  is a local homeomorphism; the remaining conclusions follow from this.

First, by 3.3, we have a base for the topology of  $H$  consisting of sets

$$N = U \cap r^{-1}f(V^r) \cap s^{-1}f(V^s)$$

where  $U, V^r, V^s$  are open and satisfy (A). For such a set  $r(N) = f(V^r)$ , which is open in  $H^0$ , as  $f$  is a homeomorphism from  $T$  to  $H^0$ . Therefore,  $r : H \rightarrow H^0$  is open. By 3.4,  $r$  is bijective, and is continuous because the groupoid operations are. Therefore,  $r$  is a homeomorphism from  $N$  to  $f(V^r)$ . This completes the proof. ■

Let us now bring in  $\Omega = G_T$ , with the topology as given earlier.

**Theorem.** *Suppose  $G, T, f$  satisfy T1 and T2. Then  $\Omega$  is a  $G$ - $H$  equivalence bimodule in the sense of [6].*

*Proof.* We must show:

- (i)  $\Omega$  is a left principal  $G$ -space: *i.e.* the left action of  $G$  is free and the map sending  $(x, y)$  in  $G * \Omega$  to  $(xy, y)$  in  $\Omega \times \Omega$  is proper.
- (ii)  $\Omega$  is a right principal  $H$ -space.
- (iii) the  $G$  and  $H$ -actions commute.
- (iv)  $r : \Omega/H \rightarrow G^0$  is a homeomorphism.
- (v)  $s : G \backslash \Omega \rightarrow H^0$  is a homeomorphism.

Notice that condition (iii) and the freeness conditions of (i) and (ii) do not involve any topology. Their proofs are exactly as in [6].

We will make use of the following characterization of proper maps, which is a relative exercise in topology. Let  $X$  and  $Y$  be second countable Hausdorff spaces and let  $\pi : X \rightarrow Y$  be a continuous map. Then  $\pi$  is proper if and only if, for every sequence  $\{x_n\}_1^\infty$  in  $X$  such that  $\{\pi(x_n)\}_{n=1}^\infty$  is convergent in  $Y$ ,  $\{x_n\}_1^\infty$  has a convergent subsequence in  $X$ .

Suppose then that  $\{(x_n, y_n)\}_1^\infty$  is a sequence in  $G * \Omega$  (*i.e.*  $s(x_n) = r(y_n)$ , for all  $n$ ) such that  $\{(x_n y_n, y_n)\}_1^\infty$  has limit  $(z, y)$  in  $\Omega \times \Omega$ . Thus,

$$\lim x_n y_n = z, \quad \text{in } G,$$

$$\lim y_n = y, \quad \text{in } G$$

and

$$\lim f^{-1}s(x_n y_n) = f^{-1}s(z)$$

$$\lim f^{-1}s(y_n) = f^{-1}s(y)$$

in  $T$ . Also  $s(x_n y_n) = s(y_n)$  and hence  $s(z) = s(y)$ . Immediately,  $\{y_n\}$  converges to  $y$  in  $\Omega$  and

$$\begin{aligned} \lim x_n &= \lim x_n y_n y_n^{-1} \\ &= z \cdot y^{-1}, \quad \text{in } G. \end{aligned}$$

We conclude that

$$\lim(x_n, y_n) = (zy^{-1}, y)$$

in  $G * \Omega$ .

We move on to the map

$$(x, y) \in \Omega * H \longrightarrow (x, xy) \in \Omega \times \Omega.$$

Suppose  $\{(x_n, y_n)\}^\infty$  is in  $\Omega * H$  (i.e.  $s(x_n) = r(y_n) \in f(T)$ ,  $s(y_n) \in f(T)$ ) and  $(x_n, x_n y_n)$  converges to  $(x, z)$  in  $\Omega \times \Omega$ . This means that

$$\lim x_n = x \text{ in } G,$$

$$\lim x_n y_n = z \text{ in } G,$$

$$\lim f^{-1}s(x_n) = f^{-1}s(x) \text{ in } T,$$

$$\lim f^{-1}s(x_n y_n) = f^{-1}s(z) \text{ in } T.$$

Then we have

$$\begin{aligned} \lim y_n &= \lim x_n^{-1} x_n y_n \\ &= x^{-1} z \text{ in } G \end{aligned}$$

and

$$\begin{aligned} \lim f^{-1}s(y_n) &= \lim f^{-1}s(x_n y_n) \\ &= f^{-1}s(z) \\ &= f^{-1}s(x^{-1}z) \text{ in } T, \\ \lim f^{-1}r(y_n) &= \lim f^{-1}s(x_n) \\ &= f^{-1}s(x) \\ &= f^{-1}r(x^{-1}z) \text{ in } T. \end{aligned}$$

Hence  $y_n$  converges to  $x^{-1}z$  in  $H$  and  $(x_n, y_n)$  converges to  $(x, x^{-1}z)$  in  $\Omega * H$ . Thus, the map is proper.

To verify (iv) and (v), it suffices to show that  $r : \Omega \rightarrow G^0$  and  $s : \Omega \rightarrow H^0$  are continuous and open. In fact, since  $f : T \rightarrow H^0$  is a homeomorphism, we will discuss

$f^{-1} \circ s : \Omega \rightarrow T$ , rather than  $s$ . Suppose  $\{x_n\}_1^\infty$  is a sequence converging to  $x$  in  $\Omega$ . Then, we have

$$\lim x_n = x \quad \text{in } G$$

$$\lim f^{-1}s(x_n) = f^{-1}s(x) \quad \text{in } T.$$

It follows at once that  $f^{-1}s$  is continuous and  $r$  is continuous on  $G$  and so from  $\Omega$  to  $G^0$ . As for openness, it suffices to consider a set  $U \cap s^{-1}f(V^s)$ ,  $U \subseteq G$  open,  $V^s \subseteq T$  open and  $(U, V^s)$  satisfy (As). It follows from (As) that  $r(U \cap s^{-1}f(V^s)) = r(U)$ , which is open since  $r : G \rightarrow G^0$  is open [9]. Also, we have

$$s(U \cap s^{-1}f(V^s)) = s(U) \cap f(V^s)$$

and

$$f^{-1}s(U \cap s^{-1}f(V^s)) = f^{-1}s(U) \cap V^s$$

which is open in  $T$  since  $s : G \rightarrow G^0$  is open and  $f : T \rightarrow G^0$  is continuous. This completes the proof. ■

#### 4. Reduction of Stable and Unstable Equivalence

The aim of this section is to show that the results of Section 3 may be applied to the groupoids of Section 2. Specifically, we consider  $G = G_s$  and  $G = G_s \rtimes \mathbb{Z}$  as in Section 2 and, for any  $x_0$  in  $X$ , the transversal  $T = V^u(x_0)$ . The map  $f$  is just the inclusion of  $T$  in  $X$ , regarded as the unit space of  $G$ . More accurately, in the case  $G = G_s$

$$f(x) = (x, x), \quad x \in V^u(x_0)$$

and in the case  $G = G_s \rtimes \mathbb{Z}$ ,

$$f(x) = (x, 0, x), \quad x \in V^u(x_0).$$

Let us also note here that the results immediately apply to  $G = G_u$ ,  $G = G_u \rtimes \mathbb{Z}$  and  $T = V^s(x_0)$ , by simply considering the Smale space  $(X, d, \phi^{-1})$  and noting, for example,  $G_s(X, \phi^{-1}) = G_u(X, \phi)$ .

It is worth stressing that the topology on  $V^u(x_0)$  is that given in Section 2 and not the relative topology of  $X$ .

**Lemma 4.1.**

- (a) *Let  $x_0$  be in  $X$ . Define  $f : V^u(x_0) \rightarrow G^0$  by  $f(x) = (x, x)$ ,  $x \in V^u(x_0)$ . Then  $f$  is continuous and injective.*
- (b) *Let  $x$  be in  $X$ ,  $V \subseteq V^u(x, \epsilon_0)$ ,  $W_1, W_2 \subseteq V^s(x, \epsilon_0)$  open in the relative topologies of  $V^u(x)$  and  $V^s(x)$ , respectively, and  $x$  in  $W_1$ . Let*

$$U = \{(x', y') \mid [x', x] = [y', x] \in V, [x, x'] \in W_1, [x, y'] \in W_2\}.$$

*Then  $U$  is an open subset of  $G_s$  and  $(U, V)$  satisfy (Ar).*

*Proof.* The proof of (a) is clear. For (b), it is easy to check that  $U$  is in  $G_s$  and is open. We must check (Ar). Suppose  $(x', y')$  is in  $U$ . Then it is easy to verify that  $([x', x], y')$  is in  $U$ ,  $r([x', x], y') = f([x', x])$  is in  $f(V)$  and  $s([x', x], y') = s(x', y') = (y', y')$ . As for uniqueness, suppose  $(x'', y'')$  is in  $U$ ,  $r(x'', y'')$  is in  $f(V)$  and  $s(x'', y'') = s(x', y')$ . Then we see at once that  $y'' = y'$ . As  $(x'', y'')$  is in  $U$ , and  $x'' \in V$ ,

$$x'' = [x'', x] = [y'', x] = [y', x] = [x', x] = x'.$$

This completes the proof. ■

**Theorem 4.2.** *Let  $(X, d, \phi)$  be a mixing Smale space and let  $x_0$  be in  $X$ . Then  $G = G_s$ ,  $T = V^u(x_0)$  and  $f$  as above satisfy T1 and T3.*

*Proof.* Let us first suppose that  $(x, y)$  is in  $G_s^0$ , with  $x$  in  $V^u(x_0)$ . Suppose also that we have open sets  $(x, y) \in U_0 \subseteq G_s$ ,  $x \in V_0 \subseteq V^u(x_0)$ . First, we may find an open set  $x \in V_1 \subseteq V^u(x, \epsilon_0)$  with  $V_1 \subseteq V_0$ . Next, since the rectangles in  $X$  form a base for its topology, we may find open sets  $x \in V_2 \subseteq V^u(x, \epsilon_0)$ ,  $x \in W_2 \subseteq V^s(x, \epsilon_0)$ ,  $x \in V_3 \subseteq V^u(x, \epsilon_0)$ ,  $y \in W_3 \subseteq V^s(x, \epsilon_0)$  such that  $[V_2, W_2] \times [V_3, W_3]$  contains  $(x, y)$  and is contained in  $U_0$ . Let  $V = V_1 \cap V_2 \cap V_3$  and

$$U = \{(x', y') \mid [x', x] = [y', x] \in V, [x, x'] \in W_2, [x, y'] \in W_3\}.$$

Then  $(x, y) \in U \subseteq U_0$ ,  $x \in V \subseteq V_0$  and  $(U, V)$  satisfies (Ar) by 4.1.

For a general  $(x, y)$  in  $G_s$ , we have  $(x, y)$  is in  $G_s^n$ , for some  $n$ . We may apply the above arguments to  $(\phi^n(x), \phi^n(y))$ ,  $(\phi \times \phi)^n(U_0)$  and  $\phi^n(V_0)$  to obtain the result. We omit the details.

It remains to verify T3. Let  $x$  be any point in  $X$ . As  $V^a(x)$  is dense in  $X$ , we may find  $y$  in  $V^a(x)$  with  $d(x_0, y) < \epsilon_0$ . Then  $(x, [y, x_0])$  is in  $G_s$ ,  $r(x, [y, x_0]) = (x, x)$  and  $s(x, [y, x_0] = [y, x_0])$  is in  $V^u(x_0)$ . Condition T3 follows. ■

**Theorem 4.3.** *Let  $(X, d, \phi)$  be a mixing Smale space and let  $x_0$  be any point of  $X$ . Then  $G = G_s \rtimes \mathbb{Z}$ ,  $T = V^u(x_0)$ ,  $f$  as before satisfy the conditions T1 and T3.*

*Proof.* Property T3 follows easily from the fact that it holds for  $G_s$  and  $G_s \subseteq G_s \rtimes \mathbb{Z}$  with  $G_s^0 = G_s \rtimes \mathbb{Z}^0$ .

As for T1, suppose  $(x, n, y)$  is in  $G_s \rtimes \mathbb{Z}$  and  $U_0, V_0$  are as in T1. Without loss of generality we may assume that

$$U_0 \subseteq \{(x', n, y') \mid (\phi^n(x'), y') \in G_s\}.$$

We may apply T1 for  $G_s$  from 4.2 to  $(\phi^n(x), y) \in G_s$ ,

$$\tilde{U}_0 = \{(\phi^n(x'), y') \mid (x', n, y') \in U_0\}$$

$$\tilde{V}_0 = \phi^n(V_0)$$

to obtain  $\tilde{U}, \tilde{V}$  satisfying (Ar). Now let

$$U = \{(x', n, y') \mid (\phi^n(x'), y') \in \tilde{U}\}$$

$$V = \phi^{-n}(\tilde{V}).$$

It is easy to check  $(U, V)$  satisfies (Ar). ■

**Definition 4.4.** For  $x_0$  in  $X$ , we let  $G_s(x_0)$  denote the groupoid  $H$  of 4.2 in the case  $G = G_s$ ,  $T = V^u(x_0)$ . We let  $G_s(x_0) \rtimes \mathbb{Z}$  denote the groupoid  $H$  in the case

$$G = G_s \rtimes \mathbb{Z}, \quad T = V^u(x_0).$$

Similarly, we define  $G_u(x_0)$  is  $H$  in the case  $G = G_u$  and  $T = V^s(x_0)$  and  $G_u(x_0) \rtimes \mathbb{Z}$  is  $H$  in the case  $G = G_u \rtimes \mathbb{Z}$  and  $T = V^s(x_0)$ .

It is worth noting that all of these groupoids are  $r$ -discrete;  $G_s(x_0)$ ,  $G_u(x_0)$  are also principal. We may identify unit spaces:  $G_s(x_0)^0$ ,  $G_s(x_0) \rtimes \mathbb{Z}^0$  with  $V^u(x_0)$ . Note that the  $G_s(x_0)$ -equivalence class of  $x$  in  $V^u(x_0)$  is  $V^a(x)$ .

The notation  $G_s(x_0) \rtimes \mathbb{Z}$  may be somewhat misleading: this is the semi-direct product groupoid *only* in the case  $V^u(x_0)$  contains a fixed-point of  $\phi$  and hence is  $\phi$ -invariant.

## 5. Proofs of the Main Results

We begin with Theorems 1.1 and 1.2.

*Proof of Theorem 1.1.* We show that  $G_s$  is amenable in the sense of Renault. The case for  $G_u$  is analogous.

We construct a sequence  $\{f_n\}$  in  $C_c(G_s)$  such that  $f_n f_n^*$  converges to 1 uniformly on compact subsets of  $G_s$ . (Note that the other condition of II.3.6 of [9] follows since the unit space of  $G_s$  is compact.) Let  $\Delta$  denote the unit space of  $G_s$ . Let  $g$  in  $C_c(G_s)$  be chosen so that  $g$  is non-negative and strictly positive on  $\Delta$ . Then

$$\begin{aligned} gg^*(x, x) &= \int_{y \in V^s(x)} |g(x, y)|^2 d\mu_s^x(y) \\ &> 0, \quad \text{for all } x \text{ in } X. \end{aligned}$$

Let

$$f(x, y) = gg^*(x, x)^{-\frac{1}{2}} g(x, y), \quad (x, y) \in G_s.$$

Then  $f$  is in  $C_c(G_s)$  and

$$ff^*(x, x) = 1, \quad \text{for all } x \text{ in } X.$$

Define

$$\begin{aligned} f_n &= \lambda^{-\frac{n}{2}} f \circ (\phi \times \phi)^n \\ &= \lambda^{\frac{n}{2}} \alpha_s^{-n}(f) \end{aligned}$$

(with  $\log(\lambda)$  equal to the entropy of  $\phi$ ). Then

$$\begin{aligned} f_n f_n^* &= \lambda^{\frac{n}{2}} \alpha_s^{-n}(f) \lambda^{\frac{n}{2}} \alpha_s^{-n}(f^*) \\ &= \lambda^n \alpha_s^{-n}(ff^*) \\ &= ff^* \circ (\phi \times \phi)^n. \end{aligned}$$

Now for any compact set  $K \subseteq G_s$  and  $\epsilon > 0$ , there is a  $\delta > 0$  so that  $\|ff^* - 1\| < \epsilon$  on the set

$$\Delta_\delta = \{(x, y) \in G_s \mid d(x, y) < \delta\},$$

because  $ff^*$  is continuous and  $ff^* \mid \Delta = 1$ . Choose  $N$  sufficiently large so that

$$(\phi \times \phi)^n(K) \subseteq \Delta_\delta, \quad \text{for all } n \geq N.$$

Then  $n \geq N$  implies  $\|f_n f_n^* - 1\| < \epsilon$  on  $K$ . We have now shown the existence of  $\{f_n\}$  as desired.

The remainder of the proof follows from [2,9]. ■

*Proof of 1.2.* The amenability of  $G_s \rtimes \mathbb{Z}$  results from 1.1 and the amenability of  $\mathbb{Z}$  as follows. Define

$$c : G_s \rtimes \mathbb{Z} \longrightarrow \mathbb{Z}$$

by  $c(x, n, y) = n$ ,  $(x, n, y) \in G_s \rtimes \mathbb{Z}$ . The kernel of  $c$  is  $G_s$  and this situation satisfies the hypotheses of Theorem 5.2.13 [2]. It follows that  $G_s \rtimes \mathbb{Z}$  is amenable. Again, the rest follows from [2,9]. ■

Before beginning the proof of 1.3, we need a dynamical result and a result regarding equivalence of amenable groupoids.

**Lemma 5.1.** *Let  $x_0$  be in  $X$  and  $x$  be in  $V^u(x_0)$ . Then  $V^a(x)$  is dense in  $V^u(x_0)$ , in the new topology introduced in Section 2.*

*Proof.* It suffices to show that  $V^a(x) \cap V^u(x_0, \epsilon_0)$  is dense in  $V^u(x_0, \epsilon_0)$ . Let  $U$  be an open set in  $V^u(x_0, \epsilon_0)$ . Then  $[U, V^s(x_0, \epsilon_0)]$  is open in  $X$ . By the last lemma, we may find  $y$  in  $V^a(x)$  in this set. Let  $z = [y, x_0]$ , which is in  $U$ . Also  $y$  and  $z$  are stably equivalent,

so  $z$  is stably equivalent to  $x$ . Also,  $z$  and  $x$  are both in  $V^u(x_0)$ , hence they are unstably equivalent. Thus  $z$  is in  $V^a(x)$ , as desired. ■

*Proof of 1.3.* First of all  $G_s$  is amenable. Let  $x_0$  be in  $X$ . By [6],  $G_s(x_0)$  is equivalent to  $G_s$ . Hence  $G_s(x_0)$  is also amenable by Theorem 2.2.13 of [2].

Therefore the  $C^*$ -algebras

$$C^*(G_s) \cong C_{\text{red}}^*(G_s)$$

$$C^*(G_s(x_0)) \cong C_{\text{red}}^*(G_s(x_0))$$

have the same ideal structure. Now  $G_s(x_0)$  is an  $r$ -discrete groupoid and so its ideals are described completely by II.4.6 of [9]. In particular, in view of the last lemma,  $C_{\text{red}}^*(G_s(x_0))$  is simple. The conclusion follows. ■

*Proof of 1.4.* The argument begins in the same way as 1.3. We use the fact that  $G_s \rtimes \mathbb{Z}$  is equivalent to  $G_s(x_0) \rtimes \mathbb{Z}$ , which is an  $r$ -discrete groupoid. To apply II.4.6 of [9] to show  $C_{\text{red}}^*(G_s(x_0) \rtimes \mathbb{Z})$  is simple. We must again see two things:  $G_s(x_0) \rtimes \mathbb{Z}$  is minimal and essentially principal. Minimality is the same as for  $G_s(x_0)$ . In both cases, the unit space is  $V^u(x_0)$  and, for a given  $x$  in  $V^u(x_0)$ , its  $G_s(x_0) \rtimes \mathbb{Z}$ -equivalence class contains its  $G_s(x_0)$ -equivalence class which is  $V^a(x)$ , which is already dense in  $V^u(x_0)$ . It remains to show that  $G_s(x_0) \rtimes \mathbb{Z}$  is essentially principal; that is, the isotropy

$$r^{-1} \{(x, 0, x)\} \cap s^{-1} \{(x, 0, x)\} = \{(x, n, x) \in G_s(x_0) \rtimes \mathbb{Z} \mid n \in \mathbb{Z}\}$$

is trivial (equals  $\{(x, 0, x)\}$ ) for a dense set of  $x$  in  $V^u(x_0)$ . We will, in fact, show that the set of  $x$  for which this is non-trivial is countable. As  $V^u(x_0)$  is locally compact and has no isolated points, the conclusion follows.

This will be divided into three Lemmas.

**Lemma 5.2.** *In a mixing Smale space  $(X, \phi)$ , the set of periodic points of period  $n$ ,*

$$\text{Per}_n = \{x \mid \phi^n(x) = x\}$$

*is finite, for any positive integer  $n$ .*

*Proof.* As noted in [8,10,11],  $\phi$  is expansive. That is, there is an  $\epsilon_1 > 0$  so that for any  $x, y$  in  $X$ , if  $d(\phi^k(x), \phi^k(y)) < \epsilon_1$ , for all  $k$  in  $\mathbb{Z}$ , then  $x = y$ . We may then choose  $\epsilon_n$  sufficiently small so that  $d(x, y) < \epsilon_n$  implies  $d(\phi^i(x), \phi^i(y)) < \epsilon_1$ , for  $0 \leq i < n$ . Then it is easy to check that  $\phi^n$  is also expansive, with constant  $\epsilon_n$ . From this, it follows that the distance between any two fixed-points of  $\phi^n$  is at least  $\epsilon_n$ . The result follows since  $X$  is compact. ■

**Lemma 5.3.** *Suppose  $\phi^n(x)$  is in  $V^s(x)$ , for some  $x$  in  $X$ ,  $n \geq 1$ . Then*

$$\lim_{k \rightarrow +\infty} \phi^{nk}(x)$$

*exists and is in  $\text{Per}_n$ .*

*Proof.* Suppose  $z$  is a limit point of  $\{\phi^{nk}(x) \mid k \geq 1\}$ . Then

$$\begin{aligned} \phi^n(z) &= \phi^n\left(\lim_i \phi^{nk_i}(x)\right) \\ &= \lim_i \phi^{nk_i}(\phi^n(x)) \\ &= \lim_i \phi^{nk_i}(x) \\ &= z \end{aligned}$$

since  $x$  and  $\phi^n(x)$  are stably equivalent.

Thus, the limit points of  $\{\phi^{nk}(x) \mid k \geq 1\}$  — which exist as  $X$  is compact — are contained in  $\text{Per}_n$ . We must show there is at most one such point.

Let  $\text{Per}_n = \{x_1, \dots, x_m\}$  (by 5.2) and choose open neighbourhoods  $U_i$  of  $x_i$  such that  $\phi^n(U_i) \cap U_j = \emptyset$  for  $i \neq j$ . If there are infinitely  $k \geq 1$  such that  $\phi^{nk}(x)$  is not in the union of the  $U_i$ , then this sequence has a limit point in  $X - U_1 - U_2 - \dots - U_m$ , by compactness. This limit point is in  $\text{Per}_n$ , but  $\text{Per}_n$  is contained in  $U_1 \cup U_2 \cup \dots \cup U_m$ , a contradiction.

Thus, for some  $k_0 \geq 0$ ,  $\phi^{nk}(x)$  is in  $U_1 \cup \dots \cup U_m$ , for all  $k \geq k_0$ . But as  $\phi^n(U_i) \cap U_j = \emptyset$ , for all  $i \neq j$ ,  $\phi^{nk}(x)$  must all be in the same  $U_i$ , for  $k \geq k_0$ . It follows then that

$$\lim_h \phi^{nh}(x) = x_i$$

as  $x_i$  is the only point of  $\text{Per}_n$  in  $\overline{U_i}$ . This completes the proof. ■

**Lemma 5.4.** *The set of  $x$  in  $V^u(x_0)$  such that  $\phi^n(x)$  is in  $V^s(x)$ , for some  $n \neq 0$ , is countable.*

*Proof.* It is clearly sufficient to prove this for a fixed  $n \neq 0$ .

Suppose  $x$  is such that  $\phi^n(x)$  is in  $V^s(x)$ ,  $x$  in  $V^u(x_0)$ . Then by Lemma 5.3,

$$\lim_{k \rightarrow +\infty} \phi^{nk}(x) = y,$$

for some  $y$  in  $\text{Per}_n$ . It is then easy to see that  $x$  is in  $V^s(y)$ . So the set of  $x$  under consideration is contained in

$$\bigcup_{y \in \text{Per}_n} V^s(y) \cap V^u(x_0).$$

Now,  $\text{Per}_n$  is finite and we noted earlier that for any  $x_0, y$

$$V^s(y) \cap V^u(x_0)$$

is countable. This completes the proof. ■

We can now complete the proof of 1.4 outlined earlier. The groupoid  $G_s(x_0) \rtimes \mathbb{Z}$  is minimal as described above. For a fixed unit  $(x, 0, x)$  with non-trivial isotropy,  $(x, n, x)$  is in  $G_s(x_0) \rtimes \mathbb{Z}$  for some  $n \neq 0$ . This means  $\phi^n(x)$  is in  $V^s(x)$ . The set of such  $x$  is countable. Hence the points of non-trivial isotropy are countable and their complement is dense. That is,  $G_s(x_0) \rtimes \mathbb{Z}$  is essentially principal. The conclusion follows. ■

Toward the proof of 1.5, we begin with the following.

**Proposition 5.5.** *Let  $A$  and  $B$  be simple separable  $C^*$ -algebras which are strongly Morita equivalent. If  $A$  is purely infinite then so is  $B$ .*

*Proof.* Let  $K$  denote the  $C^*$ -algebra of compact operators on the Hilbert space  $\ell^2(\mathbb{N})$ . For each  $i, j$  in  $\mathbb{N}$ ,  $e_{ij}$  denotes the operator

$$(e_{ij} \xi)(k) = \begin{cases} \xi(j) & \text{if } i = k \\ 0 & \text{otherwise,} \end{cases}$$

for  $\xi$  in  $\ell^2(\mathbb{N})$ ,  $k$  in  $\mathbb{N}$ .

As  $A$  and  $B$  are separable and strongly Morita equivalent, we have

$$A \otimes K \cong B \otimes K.$$

First, we show  $A \otimes K$  is purely infinite. Let  $C$  be any hereditary subalgebra of  $A \otimes K$ . Choose  $1 \geq x \geq 0$  in  $C$ . For some  $i$  in  $\mathbb{N}$ ,

$$(1 \otimes e_{ii}) x (1 \otimes e_{ii}) = x_{ii} \otimes e_{ii}$$

is non-zero, where  $0 \leq x_{ii} \leq 1$  is in  $A$ . As  $A$  is purely infinite, there is an infinite projection  $p$  in  $x_{ii} A x_{ii}$ . Then  $p \otimes e_{ii}$  is an infinite projection in  $C$ .

Next, as  $A \otimes K \cong B \otimes K$ , the latter is purely infinite. Finally  $B$  is isomorphic to  $B \otimes e_{11}$  which is a hereditary subalgebra of  $B \otimes K$ , and hence purely infinite. ■

**Definition 5.6** ([1]). A topological groupoid  $G$  is called *locally contracting* if, for every non-empty open set  $U \subseteq G^0$ , there is an open  $G$ -set  $\Delta$  such that

$$r(\bar{\Delta}) \subsetneq s(\Delta) \subseteq U.$$

Also, compare this definition with that of a “local boundary” contained in [5].

**Proposition 5.7.** *For any  $x_0$  in  $X$ ,  $G_s(x_0) \rtimes \mathbb{Z}$  is locally contracting.*

*Proof.* Suppose  $U \subseteq V^u(x_0)$  is non-empty and open. Then for some  $n \geq 1$ ,

$$\phi^{-n}(U) \cap V^u(\phi^{-n}(x_0), \epsilon_0)$$

is non-empty and open in  $V^u(\phi^{-n}(x_0), \epsilon_0)$ . Consider

$$[\phi^{-n}(U) \cap V^u(\phi^{-n}(x_0), \epsilon_0), V^s(\phi^{-n}(x_0), \epsilon_0)]$$

which is open in  $X$ . Thus, it contains a periodic point for  $\phi$ , say  $x_1$ , with  $\phi^N(x_1) = x_1$ , for some  $N \geq 1$ .

Find  $\epsilon_1 > 0$  such that

$$[V^u(x_1, \epsilon_1), \phi^{-n}(x_0)] \subseteq \phi^{-n}(U) \cap V^u(\phi^{-n}(x_0), \epsilon_0).$$

As  $V^u(x_1, \epsilon_1)$  is not discrete, we may find  $m \geq 1$  such that

$$\begin{aligned} \phi^{-mN}(V^u(x_1, \epsilon_1)) &\subseteq V^u(x_1, \epsilon_1 \lambda^{-mN}) \\ &\stackrel{\subseteq}{\neq} V^u(x_1, \epsilon_1). \end{aligned}$$

For each  $y$  in  $V^u(x_1, \epsilon_1)$ ,  $\phi^{-mN}(y)$  is also in the same set and

$$\begin{aligned} [y, \phi^{-n}(x_0)] &\in V^s(y) \\ [y, \phi^{-n}(x_0)] &\in V^n(\phi^{-n}(x_0), \epsilon_0) \\ [\phi^{-mN}(y), \phi^{-n}(x_0)] &\in V^s(\phi^{-mN}(y)) \\ [\phi^{-mN}(y), \phi^{-n}(x_0)] &\in V^u(\phi^{-mN}(y), \epsilon_0). \end{aligned}$$

Let

$$\Delta = \left\{ (\phi^n [\phi^{-mN}(y), \phi^{-n}(x_0)], mN, \phi^n [y, \phi^{-n}(x_0)]) \mid y \in V^u(x_1, \epsilon_1) \right\}.$$

It is easy to check that  $\Delta \subseteq G_s(x_0) \rtimes \mathbb{Z}$  and is a  $G_s(x_0) \rtimes \mathbb{Z}$ -set. Moreover,

$$\begin{aligned} r(\bar{\Delta}) &= \phi^n [\phi^{-mN}(V^u(x_1, \epsilon_1)), \phi^{-n}(x_0)] \\ s(\Delta) &= \phi^n [V^u(x_1, \epsilon_1), \phi^{-n}(x_0)] \end{aligned}$$

and the desired conclusion follows. ■

The proof of 1.5 follows immediately from 5.5, 5.7, Proposition 2.4 of [1] and the fact that  $G_s(x_0) \rtimes \mathbb{Z}$  is essentially principal, as shown in the proof of 1.4.

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