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Certain New Applications of Symmetric q-Calculus for New Subclasses of Multivalent Functions Associated with the Cardioid Domain

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Abstract: In this work, we study some new applications of symmetric quantum calculus in the field of Geometric Function Theory. We use the cardioid domain and the symmetric quantum difference operator to generate new classes of multivalent q-starlike and q-convex functions. We examine a wide range of interesting properties for functions that can be classified into these newly defined classes, such as estimates for the bounds for the first two coefficients, Fekete–Szego-type functional and coefficient inequalities. All the results found in this research are sharp. A number of well-known corollaries are additionally taken into consideration to show how the findings of this research relate to those of earlier studies.

Keywords: analytic functions; symmetric quantum calculus; multivalent functions; symmetric q-difference operator; cardioid domain; multivalent q-starlike and q-convex functions

MSC: 05A30; 30C45; 11B65; 47B38

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1. Introduction and Definitions

Let $\mathcal{A}$ stand for the family of analytic functions μ in the unit disc

$$\mathcal{U} = \{\tau : \tau \in \mathbb{C} \text{ and } |\tau| < 1\},$$

with power series representation

$$\mu(\tau) = \tau + \sum_{t=1}^{\infty} a_{t+1}\tau^{t+1}. \tag{1}$$

For $\mu_1, \mu_2 \in \mathcal{A}$, and μ_1 subordinate to μ_2 in $\mathcal{U}$ is indicated by

$$\mu_1(\tau) \prec \mu_2(\tau), \quad \tau \in \mathcal{U},$$

if a Schwarz function s exists, and belongs to class $\mathcal{B}$ where

$$\mathcal{B} = \{s : s \in \mathcal{A},\ |s(\tau)| < 1 \text{ and } s(0) = 0,\ \tau \in \mathcal{U}\}$$

such that

$$\mu_1(\tau) = \mu_2(s(\tau)),\ \tau \in \mathcal{U}.$$

Likewise, if μ_2 is univalent in $\mathcal{U}$ then

$$\mu_1(\tau) \prec \mu_2(\tau) \Leftrightarrow \mu_1(0) = \mu_2(0) \text{ and } \mu_1(\mathcal{U}) \subset \mu_2(\mathcal{U}).$$

Let class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{h \in \mathcal{A}\text{: } h(0) = 1 \text{ and } Re(\tau) > 0,\ \tau \in \mathcal{U}\}.$$

Let $\mathcal{S}$ stand for the collection of all functions belonging to the class of normalized analytic functions $\mathcal{A}$ that are univalent in $\mathcal{U}$. Based on a geometrical interpretation of image domains, the concept of subordination has been used to build many classes of analytic functions. In this context, some mathematicians have looked into shell-like curves [1–3], the oval- and petal-type domains [4], the conic domain [5,6], the extended conic domains [7], and so on. The function $h(\tau)$

$$h(\tau) = \frac{1 + \vartheta^2\tau^2}{1 - \vartheta\tau - \vartheta^2\tau^2},\ \ \vartheta = \frac{1 - \sqrt{5}}{2} \tag{2}$$

produces the shell-like shape. The conchoid of Maclaurin $h(e^{i\varphi})$ is given below and is the image of the unit circle under the function h:

$$h\left(e^{i\varphi}\right) = h_1(z) + ih_2(z)$$

where

$$h_1(z) = \frac{\sqrt{5}}{2(3 - 2\cos\varphi)},\ \ 0 \le \varphi < 2\pi$$

and

$$h_2(z) = \frac{(4\cos\varphi - 1)\sin\varphi}{2(1 + \cos\varphi)(3 - 2\cos\varphi)},\ \ 0 \le \varphi < 2\pi.$$

The series of $h(z)$ given in (2) is as follows:

$$h(\tau) = 1 + \sum_{t=1}^{\infty}(u_{t-1} + u_{t+1})\vartheta^t\tau^t,$$

where

$$u_t = \frac{(1 - \vartheta)^t - \vartheta^t}{\sqrt{5}}$$

and u_t represents a sequence of constants that are closer to the Fibonacci numbers and are called the Fibonacci coefficients.

The cardioid domain was described by Malik et al. [8], who were inspired by the concept of a circular disc and shell-like curves. Following is their formalization of a new class of analytic functions:

Definition 1 ([8])**.** *Let $\mathcal{CP}(R, T)$ be the class of functions p that satisfy*

$$p(\tau) \prec \overline{p}(R, T, \tau),$$

where $\overline{p}(R,T,\tau)$ *is defined by*

$$\overline{p}(R,T,\tau) = \frac{2R\vartheta^2\tau^2 + (R-1)\vartheta\tau + 2}{2T\vartheta^2\tau^2 + (T-1)\vartheta\tau + 2} \tag{3}$$

with $-1 < T < R \leq 1$ *and* $\vartheta = \frac{1-\sqrt{5}}{2}$, $\tau \in \mathcal{U}$.

Let us pretend $\mathcal{A}_l$ is the set of analytic functions μ with the following power series:

$$\mu(\tau) = \tau^l + \sum_{t=1}^{\infty} a_{t+l}\tau^{t+l}, l \in \mathbb{N}, \quad \tau \in \mathcal{U}. \tag{4}$$

For $l = 1$, then $\mathcal{A}_1 = \mathcal{A}$.

Suppose $\mathcal{S}^*(l)$ represents the family of l-valent starlike functions $\mu \in \mathcal{A}_l$ that satisfy the following condition:

$$Re\left(\frac{\tau\mu'(\tau)}{l\mu(\tau)}\right) > 0,\ \tau \in \mathcal{U},$$

and $\mathcal{K}(l)$ represents the family of l-valent conex functions $\mu \in \mathcal{A}_l$ that satisfies the following condition:

$$Re\left(\frac{\left(\tau\mu'(\tau)\right)'}{l\mu'(\tau)}\right) > 0,\ \tau \in \mathcal{U}.$$

Note that Re indicates the real part and μ' and μ'' represent first and second derivatives with respect to τ. The above two classes $\mathcal{S}^*(l)$ and $\mathcal{K}(l)$ can be written in terms of subordination as follows:

$$\mathcal{S}^*(l) = \left\{\mu \in \mathcal{A}_l : \frac{\tau\mu'(\tau)}{l\mu(\tau)} \prec \frac{1+\tau}{1-\tau}\right\}$$

and

$$\mathcal{K}(l) = \left\{\mu \in \mathcal{A}_l : \frac{1}{l}\left(1 + \frac{\tau\mu''(\tau)}{\mu'(\tau)}\right) \prec \frac{1+\tau}{1-\tau}\right\}.$$

Let $\mathcal{S}_l^*(\alpha, b)$ be the family of l-valent starlike functions of order α, which satisfy the following condition:

$$\Re\left\{1 + \frac{1}{b}\left(\frac{\tau\mu'(\tau)}{l\mu(\tau)} - 1\right) > \alpha,\ \ \tau \in \mathcal{U}\right\},\ b \in \mathbb{C}\backslash\{0\},\ 0 \leq \alpha < 1.$$

Let $\mathcal{K}_l(\alpha, b)$ be the family of l-valent starlike functions of order α, which satisfy the following condition:

$$\Re\left\{1 - \frac{1}{b} + \frac{1}{bl}\left(1 + \frac{\tau\mu''(\tau)}{\mu'(\tau)}\right) > \alpha,\ \ \tau \in \mathcal{U}\right\},\ \mu \in \mathcal{A}_l,\ b \in \mathbb{C}\backslash\{0\},\ 0 \leq \alpha < 1.$$

Note that $\mathcal{S}_l^*(\alpha, b)$ and $\mathcal{K}_l(\alpha, b)$ satisfy the following relation:

$$\mu \in \mathcal{K}_l(\alpha, b) \Leftrightarrow \frac{1}{l}\tau\mu' \in \mathcal{S}_l^*(\alpha, b).$$

Recently, Bulut [9] developed the following classes of l-valent functions associated with the vertical strip domain by using the notion of subordination:

$$\mathcal{S}_{l,b}^{*,\alpha,\beta}=\left\{\mu\in\mathcal{A}_l:1+\frac{1}{b}\left(\frac{1}{l}\frac{\tau\mu'(\tau)}{\mu(\tau)}-1\right)\prec\mu(\alpha,\beta;\tau)\right\}$$

and

$$\mathcal{K}_{l,b}^{\alpha,\beta}=\left\{\mu\in\mathcal{A}_l:1-\frac{1}{b}+\frac{1}{bl}\left(1+\frac{\tau\mu''(\tau)}{\mu'(\tau)}\right)\prec\mu(\alpha,\beta;\tau)\right\},$$

where

$$\mu(\alpha,\beta;\tau)=1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\tau}{1-\tau}\right)$$

and

$$0\leq\alpha<1<\beta,\ b\in\mathbb{C}^*,\ \tau\in\mathcal{U}.$$

Coefficient constraints were determined by Bulut [9] for these new types of functions. Scholars working in the field of Geometric Function Theory (GFT) have used the q-calculus and fractional q-calculus to design and investigate several new classes of analytic and univalent functions. The operator ∂_q of q-calculus was introduced and defined by Jackson [10,11] in 1909. One example is the definition of a preliminary class of q-starlike functions in $\mathcal{U}$ investigated by Ismail et al. [12] who used ∂_q. In a book chapter, Srivastava first employed the fundamental (or q-) hypergeometric functions in GFT (see [13] and for more information [14], and check out [15,16] for additional information on q-calculus operator theory in GFT). Recently, numerous applications of the symmetric q-calculus have been established in the field of fractional calculus and quantum physics [17,18]. Sun et al. introduced and evaluated fractional q-symmetric derivatives for the first time in 2016. Kanas et al. [19] explored a symmetric q-derivative operator and applied this operator to generate a new class of analytic functions and to explore some possible applications of this class of functions in the region of conic. Khan et al. [20] recently extended the new type of conic domain by implementing symmetric calculus notations and the symmetric q-difference operator and established some new results and generated a new class of q-starlike functions with ideas borrowed from symmetric q-calculus. A symmetric q-difference operator for m-fold symmetric functions was recently presented by Khan et al. [21]. Through analysis of this operator, some interesting results were explored for m-fold symmetric bi-univalent functions. Khan et al. introduced the concepts of a multivalent symmetric q-derivative operator in [22], where they also offered various new uses for multivalent q-starlike functions. Here, we give an overview of the symmetric q-difference calculus, including some fundamental definitions, which are used throughout this work.

Definition 2 ([19])**.** *For $t\in\mathbb{N}$ and $0<q<1$, the q-symmetric number is defined as follows:*

$$\widetilde{[t]}_q=\frac{q^{-t}-q^t}{q^{-1}-q},\quad \widetilde{[0]}_q=0.$$

Note that the symmetric q-number cannot be reduced to the q-number.

Definition 3. *For any $t\in\mathbb{Z}^+\cup\{0\}$ and $0<q<1$, the q-symmetric number shifted factorial is defined as follows:*

$$\widetilde{[t]}_q!=\begin{cases}\widetilde{[t]}_q\widetilde{[t-1]}_q\widetilde{[t-2]}_q...\widetilde{[2]}_q\widetilde{[1]}_q, & t\geq1\\ 1 & t=0.\end{cases}$$

Note that

$$\lim_{q\to1-}\widetilde{[t]}_q!=t(t-1)(t-2)...2.1.$$

Definition 4 ([23])**.** *Defining the q-symmetric difference operator as follows and letting $\mu \in \mathcal{A}$, we have*

$$\widetilde{\partial}_q \mu(\tau) = \frac{1}{\tau}\left[\frac{\mu(q\tau) - \mu(q^{-1}\tau)}{q - q^{-1}}\right], \quad \tau \in \mathcal{U}.$$

Note that

$$\widetilde{\partial}_q \tau^t = \widetilde{[t]}_q \tau^{t-1}, \quad \widetilde{\partial}_q\left\{\sum_{t=1}^{\infty} a_t \tau^t\right\} = \sum_{t=1}^{\infty} \widetilde{[t]}_q a_t \tau^{t-1}$$

and

$$\lim_{q \to 1-} \widetilde{\partial}_q \mu(\tau) = \mu'(\tau).$$

Consider the symmetric q-difference operator for $\mu \in \mathcal{A}_l$ as follows:

Definition 5. *Defining the q-symmetric difference operator as follows and letting $\mu \in \mathcal{A}_l$, we have*

$$\widetilde{\partial}_q \mu(\tau) = \frac{\mu(q\tau) - \mu(q^{-1}\tau)}{\tau(q - q^{-1})}, \quad \tau \in \mathcal{U}.$$

Note that

$$\widetilde{\partial}_q(\tau^{t+l}) = \widetilde{[t+l]}_q \tau^{t+l-1}, \quad \widetilde{\partial}_q\left(\sum_{t=1}^{\infty} a_{t+l} \tau^{t+l}\right) = \sum_{t=1}^{\infty} \widetilde{[t+l]}_q a_{t+l} \tau^{t+l-1}, \quad t \in \mathbb{N}, \text{ and } \tau \in \mathcal{U}.$$

We establish two new types of multivalent functions using ideas derived from recent work by [8,9,24].

Definition 6. *The function μ defined in (4) belongs to the class $\mathcal{S}_l^{*,q,b}(R,T)$, if*

$$1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q} \frac{\tau \widetilde{\partial}_q \mu(\tau)}{\mu(\tau)} - 1\right) \prec \overline{p}(R,T;\tau),$$

where $b \in \mathbb{C}\backslash\{0\}$ and $\overline{p}(R,T;\tau)$ is given by (3).

Definition 7. *The function μ defined in (4) belongs to the class $\mathcal{K}_l^{q,b}(R,T)$, if it satisfies*

$$1 - \frac{1}{b} + \frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau \widetilde{\partial}_q \mu(\tau)\right)}{\widetilde{[l]}_q \widetilde{\partial}_q \mu(\tau)}\right) \prec \overline{p}(R,T;\tau),$$

where $b \in \mathbb{C}\backslash\{0\}$ and $\overline{p}(R,T;\tau)$ is given by (3).

2. Set of Lemmas

Our key findings are established using the following lemmas:

Lemma 1 ([8])**.** *Let the function $\overline{p}(R,T;\tau)$ be defined by (3). Then,*
(i) For the disc $|\tau| < \vartheta^2$, the function $\overline{p}(R,T;\tau)$ is univalent.
(ii) If $h(\tau) \prec \overline{p}(R,T;\tau)$, then $\mathrm{Re}(\tau) > \alpha$, where

$$\alpha = \frac{2(R+T-2)\vartheta + 2(2RT - R - T)\vartheta^3 + 16(R+T)\vartheta^2\eta}{4(T-1)(\vartheta + T\vartheta^3) + 32T\vartheta^2\eta},$$

with

$$\eta = \frac{4 + \vartheta^2 - T^2\vartheta^2 - 4T^2\vartheta^4 - (1 - T\vartheta^2)\chi(T)}{4\vartheta\left(1 + T^2\vartheta^2\right)},$$

where

$$\chi(T) = \sqrt{5(2T\vartheta^2 - (T-1)\vartheta + 2)(2T\vartheta^2 + (T-1)\vartheta + 2)}$$

and

$$-1 < T < R \leq 1, \text{ and } \vartheta = \frac{1-\sqrt{5}}{2}.$$

(iii) If $\overline{p}(R,T;\tau) = 1 + \sum_{t=1}^{\infty} \overline{Q}_t \tau^t$, *then*

$$\overline{Q}_t = \begin{cases} (R-T)\frac{\vartheta}{2} & \text{for } t = 1, \\ (R-T)(5-T)\frac{\vartheta^2}{2^2} & \text{for } t = 2, \\ \frac{1-T}{2}\vartheta p_{t-1} - T\vartheta^2 p_{t-2} & \text{for } t = 3,4,5,\ldots \end{cases} \tag{5}$$

where

$$-1 < T < R \leq 1.$$

(iv) Let $h(\tau) \prec \overline{p}(R,T;\tau)$ *and* $h(\tau) = 1 + \sum_{t=1}^{\infty} \delta_t \tau^t$. *Then,*

$$\left|\delta_2 - v\delta_1^2\right| \leq \frac{(R-T)|\vartheta|}{4} \max\{2, |\vartheta(v(R-T) + T - 5)|\}, \quad v \in \mathbb{C}.$$

Lemma 2 ([25]). *Let* $h \in \mathcal{P}$ *and* $h(\tau) = 1 + \sum_{t=1}^{\infty} \delta_t \tau^t$. *Then,*

$$\left|\delta_2 - \frac{v}{2}\delta_1^2\right| \leq \max\{2, 2|v-1|\} = \left\{ \begin{array}{cc} 2 & \text{if } 0 \leq v \leq 2, \\ 2|v-1|, & \text{elsewhere} \end{array} \right\} \tag{6}$$

and

$$|\delta_t| \leq 2, \text{ for } t \geq 1. \tag{7}$$

Lemma 3 ([26]). *Let* $h \in \mathcal{P}$ *and* $h(\tau) = 1 + \sum_{t=1}^{\infty} \delta_t \tau^t$. *Then,*

$$\left|\delta_2 - v\delta_1^2\right| \leq 2\max\{1, |2v-1|\}, \; v \in \mathbb{C}$$

and the result is sharp for

$$h(\tau) = \frac{1+\tau^2}{1-\tau^2} \text{ and } h(\tau) = \frac{1+\tau}{1-\tau}.$$

Lemma 4 ([27]). *Let the function g given by*

$$g(\tau) = \sum_{t=1}^{\infty} b_t \tau^t$$

be convex in $\mathcal{U}$ *and*

$$f(\tau) = \sum_{t=1}^{\infty} c_t \tau^t$$

be analytic in $\mathcal{U}$. *If*

$$f(\tau) \prec g(\tau),$$

then

$$|c_t| < |b_1|, \quad t \geq 1.$$

In this section, for the recently described classes of multivalent functions, we obtain sharp coefficients estimates of the Taylor series, Fekete–Szegő problems, and coefficient inequalities.

3. Main Results

The Taylor–Maclaurin coefficients for the functions $\mu \in S_l^*(R,T,b)$:

Theorem 1. *Let $\mu \in \mathcal{S}_l^{*,q,b}(R,T)$ be of the form (4), $-1 \leq T < R \leq 1$. Then,*

$$\begin{aligned} |a_{l+1}| &\leq \frac{\widetilde{[l]}_q |b|(R-T)|\vartheta|}{2\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}, \\ |a_{l+2}| &\leq \frac{b\widetilde{[l]}_q (R-T)|\vartheta|^2}{4\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)} \left(5 - T + \frac{\widetilde{[l]}_q b(R-T)}{\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}\right). \end{aligned}$$

The result is sharp for the functions given by (17).

Proof. Let $\mu \in \mathcal{S}_l^{*,q,b}(R,T)$. Then,

$$1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q}\frac{\tau\widetilde{\partial}_q(\tau)}{\mu(\tau)} - 1\right) \prec \overline{p}(R,T;\tau), \tag{8}$$

where

$$\overline{p}(R,T,\tau) = \frac{2R\vartheta^2\tau^2 + (R-1)\vartheta\tau + 2}{2T\vartheta^2\tau^2 + (T-1)\vartheta\tau + 2}.$$

For the Schwarz function s with

$$s(0) = 0 \text{ and } |s(\tau)| < 1,$$

and apply the definition of subordination, such that

$$1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q}\frac{\tau\widetilde{\partial}_q\mu(\tau)}{\mu(\tau)} - 1\right) = \overline{p}(R,T;s(\tau)). \tag{9}$$

Let

$$\begin{aligned} s(\tau) &= \frac{h(\tau)-1}{h(\tau)+1} \\ &= \frac{\delta_1\tau + \delta_2\tau^2 + \delta_3\tau^3 + \ldots}{2 + \delta_1\tau + \delta_2\tau^2 + \ldots} \\ &= \frac{1}{2}\delta_1\tau + \frac{1}{2}\left(\delta_2 - \frac{1}{2}\delta_1^2\right)\tau^2 + \frac{1}{2}\left(\delta_3 - \delta_1\delta_2 + \frac{1}{4}\delta_1^3\right)\tau^3 + \cdots. \end{aligned} \tag{10}$$

Since $\overline{p}(R,T;\tau) = 1 + \sum_{t=1}^{\infty}\overline{Q}_t\tau^t$, then

$$\begin{aligned} &\overline{p}(R,T;s(\tau)) \\ = \; & 1 + \overline{Q}_1\left\{\frac{1}{2}\delta_1\tau + \frac{1}{2}\left(\delta_2 - \frac{1}{2}\delta_1^2\right)\tau^2\ldots\right\} + \overline{Q}_2\left\{\frac{1}{2}\delta_1\tau + \frac{1}{2}\left(\delta_2 - \frac{1}{2}\delta_1^2\right)\tau^2\ldots\right\} + \ldots \\ = \; & 1 + \frac{\overline{Q}_1\delta_1}{2}\tau + \left(\frac{1}{2}\left(\delta_2 - \frac{1}{2}\delta_1^2\right)\overline{Q}_1 + \frac{\overline{Q}_2\delta_1^2}{4}\right)\tau^2 + \ldots. \end{aligned} \tag{11}$$

Also consider the function

$$\overline{p}(R,T;\tau) = \frac{2R\vartheta^2\tau^2 + (R-1)\vartheta\tau + 2}{2T\vartheta^2\tau^2 + (T-1)\vartheta\tau + 2}.$$

Let $\vartheta\tau = \alpha_0$. Then,

$$\begin{aligned}\overline{p}(R,T,\tau) &= \frac{2R\alpha_0^2 + (R-1)\alpha_0 + 2}{2T\alpha_0^2 + (T-1)\alpha_0 + 2} \\ &= \frac{R\alpha_0^2 + \frac{(R-1)}{2}\alpha_0 + 1}{T\alpha_0^2 + \frac{(T-1)}{2}\alpha_0 + 1} \\ &= \left(R\alpha_0^2 + \frac{(R-1)}{2}\alpha_0 + 1\right)\left[1 + \frac{1}{2}(1-T)\alpha_0 + \left(\frac{T^2 - 6T + 1}{4}\right)\alpha_0^2 + \ldots\right] \\ &= 1 + \frac{1}{2}(R-T)\alpha_0 + \frac{1}{4}(R-T)(5-T)\alpha_0^2 + \ldots.\end{aligned}$$

This implies that

$$\overline{p}(R,T;\tau) = 1 + \frac{1}{2}(R-T)\vartheta\tau + \frac{1}{4}(R-T)(5-T)\vartheta^2\tau^2 + \ldots. \tag{12}$$

From (11), it is clear that

$$\begin{aligned}&\overline{p}(R,T;s(\tau)) \\ &= 1 + \frac{1}{4}(R-T)\vartheta\delta_1\tau + \left(\frac{1}{4}(R-T)\vartheta\left(\delta_2 - \frac{1}{2}\delta_1^2\right) \right. \\ &\quad \left. + \frac{(R-T)(5-T)\vartheta^2\delta_1^2}{16}\tau^2\right) + \ldots.\end{aligned} \tag{13}$$

Since $\mu \in \mathcal{S}_l^*(R,T,b)$, then

$$\begin{aligned}&1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q}\frac{\tau\widetilde{\partial}_q\mu(\tau)}{\mu(\tau)} - 1\right) \\ &= 1 + \frac{1}{b\widetilde{[l]}_q}\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)a_{l+1}\tau + \left(\frac{1}{b\widetilde{[l]}_q}\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)a_{l+2} \right. \\ &\quad \left. - \left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)a_{l+1}^2\tau^2\right) + \ldots.\end{aligned} \tag{14}$$

It is straightforward to demonstrate through (9) and by the comparison of the coefficients from (13) and (14) that we obtain

$$a_{l+1} = \frac{b\widetilde{[l]}_q(R-T)\vartheta\delta_1}{4\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}. \tag{15}$$

Taking the modulus, we have

$$|a_{l+1}| \leq \frac{\widetilde{[l]}_q|b|(R-T)\vartheta}{2\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}.$$

Then, comparing the coefficients in (13) and (14), we have

$$\frac{\left(\widetilde{[l+2]_q}-[\widetilde{l}]_q\right)}{b[\widetilde{l}]_q}a_{l+2}$$
$$= \frac{1}{4}(R-T)\vartheta\left(\delta_2-\frac{1}{2}\delta_1^2\right)+\frac{(R-T)(5-T)\vartheta^2\delta_1^2}{16}+\frac{\left(\widetilde{[l+1]_q}-[\widetilde{l}]_q\right)}{[\widetilde{l}]_q b}a_{l+1}^2$$

and

$$a_{l+2}=\frac{b[\widetilde{l}]_q(R-T)\vartheta}{4\left(\widetilde{[l+2]_q}-[\widetilde{l}]_q\right)}\left[\delta_2-\frac{1}{2}\left\{1-\frac{\vartheta}{2}\left(5-T+\frac{b[\widetilde{l}]_q(R-T)}{\widetilde{[l+1]_q}-[\widetilde{l}]_q}\right)\right\}\right],$$

$$|a_{l+2}|=\frac{b[\widetilde{l}]_q(R-T)\vartheta}{4\left(\widetilde{[l+2]_q}-[\widetilde{l}]_q\right)}\left|\delta_2-\frac{v}{2}\delta_1^2\right|, \tag{16}$$

where

$$v=1-\frac{\vartheta}{2}\left(5-T+\frac{b[\widetilde{l}]_q(R-T)}{\left(\widetilde{[l+1]_q}-[\widetilde{l}]_q\right)}\right).$$

When $R>T$ implies $v>2$. Thus, application of Lemma 2 yields the desired conclusion.

Extremal function

$$\mu_*(\tau) = \tau^l+\frac{b[\widetilde{l}]_q(R-T)\vartheta}{2\left(\widetilde{[l+1]_q}-[\widetilde{l}]_q\right)}\tau^{l+1}$$
$$+\frac{b[\widetilde{l}]_q(R-T)\vartheta^2}{4\left(\widetilde{[l+2]_q}-[\widetilde{l}]_q\right)}\left(5-T+\frac{b[\widetilde{l}]_q(R-T)}{\widetilde{[l+1]_q}-[\widetilde{l}]_q}\right)\tau^{l+2}+\ldots. \tag{17}$$

Then, it is clear that

$$1+\frac{1}{b}\left(\frac{1}{[\widetilde{l}]_q}\frac{\tau\widetilde{\partial}_q\mu_*(\tau)}{\mu_*(\tau)}-1\right)=\overline{p}(R,T,\tau),$$

where the series of $\overline{p}(R,T,\tau)$ is given by (12). Hence, $\mu_*\in\mathcal{S}_l^{*,q,b}(R,T)$. □

Using Theorem 1 and setting $q\to 1^-$, $b=1$, and $l=1$ yields the known result that was demonstrated in [28].

Corollary 1 ([28]). *Let $\mu\in\mathcal{S}^*(R,T)$ and μ defined in (1), $-1\le T<R\le 1$. Then,*

$$|a_2| \le \frac{(R-T)|\vartheta|}{2},$$
$$|a_3| \le \frac{(R-T)|\vartheta|^2}{8}\{R-2T+5\}.$$

The result is sharp.

Theorem 2. *Let $\mu \in \mathcal{S}_l^{*,q,b}(R,T)$ and μ defined in (4). Then,*

$$\begin{aligned} &\left|a_{l+2} - \sigma a_{l+1}^2\right| \\ \leq \quad & \frac{\widetilde{[l]}_q |b| (R-T)|\vartheta|}{4\left(\widetilde{[l+2]}_q - 1\right)} \times \max\left\{2, \left|\vartheta\left(-\frac{(R-T)\widetilde{[l]}_q b}{\widetilde{[l+1]}_q - 1} + (T-5) \right.\right.\right. \\ & \left.\left.\left. + \frac{\left(\widetilde{[l+2]}_q - 1\right)\widetilde{[l]}_q b(R-T)}{\left(\widetilde{[l+1]}_q - 1\right)^2}\sigma\right)\right|\right\}. \end{aligned}$$

This result is sharp.

Proof. Since $\mu \in \mathcal{S}_l^{*,q,b}(R,T)$,

$$1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q}\left(\frac{\tau \widetilde{\partial}_q \mu(\tau)}{\mu(\tau)}\right) - 1\right) = \overline{p}(R,T;s(\tau)),\ \tau \in \mathcal{U},$$

where s is the Schwarz function such that $s(0)$ and $|s(\tau)| < 1$ in $\mathcal{U}$. Thus,

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q}\left(\frac{\tau \widetilde{\partial}_q \mu(\tau)}{\mu(\tau)}\right) - 1\right) &= h(\tau), \\ 1 + \frac{1}{b\widetilde{[l]}_q}\left(\frac{\tau \widetilde{\partial}_q \mu(\tau)}{\mu(\tau)}\right) &= \frac{1}{b} + \left(1 + \delta_1 \tau + \delta_2 \tau^2 + \ldots\right), \\ \tau \widetilde{\partial}_q \mu(\tau) &= \widetilde{[l]}_q b \mu(\tau)\left(\frac{1}{b} + \delta_1 \tau + \delta_2 \tau^2 + \ldots\right) \end{aligned}$$

and following the simple calculations, we have

$$\begin{aligned} & \widetilde{[l]}_q \tau^l + \widetilde{[l+1]}_q a_{l+1} \tau^{l+1} + \widetilde{[l+2]}_q a_{l+2} \tau^{l+2} + \ldots \\ = \quad & \widetilde{[l]}_q b\left\{\tau^l + a_{l+1}\tau^{l+1} + a_{l+2}\tau^{l+2} + \ldots\right\}\left(\frac{1}{b} + \delta_1 \tau + \delta_2 \tau^2 + \ldots\right) \\ = \quad & \widetilde{[l]}_q \left\{\tau^l + a_{l+1}\tau^{l+1} + a_{l+2}\tau^{l+2} + \ldots\right\}\left(1 + b\delta_1 \tau + b\delta_2 \tau^2 + \ldots\right). \end{aligned}$$

Comparing the coefficients of both sides, we obtain

$$a_{l+1} = \frac{\widetilde{[l]}_q b \delta_1}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}, \quad a_{l+2} = \frac{\widetilde{[l]}_q b}{\widetilde{[l+2]}_q - \widetilde{[l]}_q}(\delta_1 a_{l+1} + \delta_2).$$

This implies that

$$\begin{aligned} \left|a_{l+2} - \sigma a_{l+1}^2\right| &= \frac{\widetilde{[l]}_q |b|}{\widetilde{[l+2]}_q - \widetilde{[l]}_q}\left|\delta_2 + \frac{\widetilde{[l]}_q b}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}\left(1 - \frac{\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}\sigma\right)\delta_1^2\right| \\ &= \frac{\widetilde{[l]}_q |b|}{\widetilde{[l+2]}_q - \widetilde{[l]}_q}\left|\delta_2 - v\delta_1^2\right|, \end{aligned}$$

where

$$v = \left(\frac{\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}\sigma - 1 \right) \frac{\widetilde{[l]}_q b}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}.$$

Application of Lemma 1, (part iv) for $v = \left(\frac{\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}\sigma - 1 \right) \frac{\widetilde{[l]}_q b}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}$, we obtain the required result. The equality

$$\begin{aligned} \left|a_{l+2} - \sigma a_{l+1}^2\right| &= \frac{\widetilde{[l]}_q |b| (R-T) |\vartheta|^2}{4\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)} \left| \frac{(R-T)\widetilde{[l]}_q b}{\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)} - T + 5 \right. \\ &\left. - \frac{\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)\widetilde{[l]}_q b (R-T)}{\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)^2}\sigma \right| \end{aligned}$$

holds for μ_* given in (17). Consider $\mu_0 : \mathcal{U} \to \mathbb{C}$ defined as follows:

$$\mu_0(\tau) = \tau^l + \frac{\widetilde{[l]}_q b \vartheta (R-T)}{2\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}\tau^{l+2} + \ldots, \tag{18}$$

Hence,

$$1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q}\left(\frac{\tau \widetilde{\partial}_q \mu_0(\tau)}{\mu_0(\tau)} \right) - 1 \right) = \overline{p}\left(R, T; \tau^2\right)$$

where $\overline{p}(R, T; \tau)$ is defined in (12). This demonstrates $\mu_0 \in \mathcal{S}_l^{*,q,b}(R, T)$. Hence, the equality

$$\left|a_{l+2} - \sigma a_{l+1}^2\right| = \frac{\widetilde{[l]}_q |b| (R-T) |\vartheta|}{2\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}$$

holds for the function μ_0 given in (18). □

Setting $q \to 1^-$, $b = 1$, and $l = 1$ in Theorem 1, we obtain the following result.

Corollary 2 ([28]). *Let $\mu \in \mathcal{S}^*(R, T)$ and of the form (1). Then,*

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{(R-T)|\vartheta|}{8} \max\{2, |\vartheta(-(R - 2T + 5) + 2(R-T)\sigma)|\}.$$

This result is sharp.

Theorem 3. *For function $\mu \in \mathcal{A}_l$, μ is defined in (4). If $\mu \in \mathcal{S}_l^{*,q,b}(R, T)$. Then*

$$|a_{l+1}| \leq \frac{\widetilde{[l]}_q |b \overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q},$$

$$|a_{l+t}| \leq \frac{\widetilde{[l]}_q |b| |\overline{Q}_1|}{\left(\widetilde{[t+l]}_q - \widetilde{[l]}_q\right)} \prod_{k=2}^{t} \left(1 + \frac{\widetilde{[l]}_q |b \overline{Q}_1|}{\widetilde{[k-1+l]}_q - \widetilde{[l]}_q} \right), \quad l \in \mathbb{N},\ t \geq 2, \tag{19}$$

where $|\overline{Q}_1|$ is given by (22).

Proof. Suppose $\mu \in \mathcal{S}_l^{*,q,b}(R,T)$ and the function $S(\tau)$ is defined by

$$S(\tau) = 1 + \frac{1}{b}\left(\frac{1}{\widetilde{[l]}_q}\left(\frac{\tau\widetilde{\partial}_q\mu(\tau)}{\mu(\tau)}\right) - 1\right). \tag{20}$$

Then, by Definition 6,

$$S(\tau) \prec \overline{p}(R,T;\tau),$$

where $b \in \mathbb{C}\backslash\{0\}$ and $\overline{p}(R,T;\tau)$ is defined by (3). Hence, by Lemma 4, we obtain

$$\left|\frac{S^{(m)}(0)}{m!}\right| = |\delta_m| \le |\overline{Q}_1|, \quad m \in \mathbb{N}, \tag{21}$$

where

$$S(\tau) = 1 + \delta_1\tau + \delta_2\tau^2 + \ldots$$

and by (5), we have

$$|\overline{Q}_1| = \left|(R-T)\frac{\vartheta}{2}\right|. \tag{22}$$

Also from (20), we find

$$\tau\widetilde{\partial}_q\mu(\tau) = \widetilde{[l]}_q\{b[q(\tau)-1]+1\}\mu(\tau). \tag{23}$$

As $a_l = 1$, from equation (23), we have

$$\begin{aligned} &\left(\widetilde{[t+l]}_q - \widetilde{[l]}_q\right)a_{l+t} \\ = \;& \widetilde{[l]}_q b\{\delta_t + \delta_{t-1}a_{l+1} + \ldots + \delta_1 a_{l+t-1}\} \\ = \;& b\widetilde{[l]}_q \sum_{i=1}^{t} \delta_i a_{l+t-i}. \end{aligned} \tag{24}$$

Equations (21) and (24) yield

$$\begin{aligned} \left(\widetilde{[t+l]}_q - \widetilde{[l]}_q\right)|a_{l+t}| &\le \widetilde{[l]}_q|b||\overline{Q}_1|\sum_{i=1}^{t}|a_{l+t-i}|, \quad l,t \in \mathbb{N}, \\ |a_{l+t}| &\le \frac{\widetilde{[l]}_q|b||\overline{Q}_1|}{\widetilde{[t+l]}_q - \widetilde{[l]}_q}\sum_{i=1}^{t}|a_{l+t-i}|, \quad l,t \in \mathbb{N}. \end{aligned}$$

For $t = 1,2,3$, we have

$$|a_{l+1}| \le \frac{\widetilde{[l]}_q|b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q},$$

$$|a_{l+2}| \le \frac{\widetilde{[l]}_q|b\overline{Q}_1|}{\widetilde{[2+l]}_q - \widetilde{[l]}_q}(1 + |a_{l+1}|),$$

$$\le \frac{\widetilde{[l]}_q|b\overline{Q}_1|}{\widetilde{[2+l]}_q - \widetilde{[l]}_q}\left(1 + \frac{\widetilde{[l]}_q|b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q}\right)$$

and

$$|a_{l+3}| \le \frac{\widetilde{[l]}_q|b\overline{Q}_1|}{\widetilde{[3+l]}_q - \widetilde{[l]}_q}(1 + |a_{l+1}| + |a_{l+2}|)$$

$$|a_{l+3}| \leq \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[3+l]}_q - \widetilde{[l]}_q}\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q} + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[2+l]}_q - \widetilde{[l]}_q}\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q}\right)\right)$$

$$|a_{l+3}| \leq \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[3+l]}_q - \widetilde{[l]}}\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q}\right)\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[2+l]}_q - \widetilde{[l]}_q}\right),$$

respectively. Let us assume that (19) is true for $l + t \leq l + i$, that is

$$|a_{l+i}| \leq \frac{\widetilde{[l]}_q |b||\overline{Q}_1|}{\left(\widetilde{[t+l]}_q - \widetilde{[l]}_q\right)} \prod_{k=2}^{i}\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[k-1+l]}_q - \widetilde{[l]}_q}\right).$$

Consider

$$\begin{aligned}
|a_{l+i+1}| &\leq \frac{\widetilde{[l]}_q |b||\overline{Q}_1|}{\left(\widetilde{[t+1+l]}_q - \widetilde{[l]}_q\right)} \\
&\quad \times (1 + |a_{l+1}| + |a_{l+2}| + \ldots + |a_{l+3}|) \\
&\leq \frac{\widetilde{[l]}_q |b||\overline{Q}_1|}{\left(\widetilde{[t+1+l]}_q - \widetilde{[l]}_q\right)} \\
&\quad \left[\begin{array}{c} 1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q} + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[2+l]}_q - \widetilde{[l]}_q}\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q}\right) \\ + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[3+l]}_q - \widetilde{[l]}}\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q}\right)\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[2+l]}_q - \widetilde{[l]}_q}\right) \\ + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[1+l]}_q - \widetilde{[l]}_q} \prod_{k=2}^{i}\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[k-1+l]}_q - \widetilde{[l]}_q}\right) \end{array}\right] \\
&= \frac{\widetilde{[l]}_q |b||\overline{Q}_1|}{\left(\widetilde{[t+1+l]}_q - \widetilde{[l]}_q\right)} \prod_{k=1}^{i+1}\left(\left(1 + \frac{\widetilde{[l]}_q |b\overline{Q}_1|}{\widetilde{[k-1+l]}_q - \widetilde{[l]}_q}\right)\right).
\end{aligned}$$

The proof is therefore obviously finished. □

Theorem 4. *Let $\mu \in \mathcal{K}_l^{q,b}(R,T)$ with $-1 \leq T < R \leq 1$. Then,*

$$\begin{aligned}
|a_{l+1}| &\leq \frac{b\widetilde{[l]}_q^2 (R-T)|\vartheta|}{2\widetilde{[l+1]}_q\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)} \\
|a_{l+2}| &\leq \frac{b^2\widetilde{[l]}_q (R-T)|\vartheta|}{4\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}\left(5 - T + \frac{\widetilde{[l]}_q b(R-T)}{\widetilde{[l+1]}_q - \widetilde{[l]}_q}\right).
\end{aligned}$$

These findings are sharp.

Proof. Let $\mu \in \mathcal{K}_l^{q,b}(R,T)$, then

$$1 - \frac{1}{b} + \frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu(\tau)\right)}{\widetilde{[l]}_q\widetilde{\partial}_q\mu(\tau)}\right) \prec \overline{p}(R,T;\tau), \tag{25}$$

where

$$\overline{p}(R,T;\tau) = \frac{2R\vartheta^2\tau^2 + (R-1)\vartheta\tau + 2}{2T\vartheta^2\tau^2 + (T-1)\vartheta\tau + 2},$$

such that

$$1-\frac{1}{b}+\frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu(\tau)\right)}{\widetilde{[l]}_q\widetilde{\partial}_q\mu(\tau)}\right)=\overline{p}(R,T;s(\tau)). \tag{26}$$

Since $\mu \in \mathcal{K}_l^{q,b}(R,T)$, then

$$\begin{aligned}
& 1-\frac{1}{b}+\frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu(\tau)\right)}{\widetilde{[l]}_q\widetilde{\partial}_q\mu(\tau)}\right) \\
= \ & 1+\frac{\widetilde{[l+1]}_q}{b\widetilde{[l]}_q^2}\left(\widetilde{[l+1]}_q-\widetilde{[l]}_q\right)a_{l+1}\tau+ \\
& \frac{1}{b\widetilde{[l]}_q^2}\left\{\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)a_{l+2}-\right. \\
& \left.\widetilde{[l+1]}_q^2\left(\frac{\widetilde{[l+1]}_q-\widetilde{[l]}_q}{\widetilde{[l]}_q}\right)a_{l+1}^2\tau^2\right\}+\ldots.
\end{aligned} \tag{27}$$

It is simple to show that by utilizing (26) and comparing the coefficients from (13) and (27), we obtain

$$a_{l+1}=\frac{b\widetilde{[l]}_q^2(R-T)\vartheta\delta_1}{4\widetilde{[l+1]}_q\left(\widetilde{[l+1]}_q-\widetilde{[l]}_q\right)},$$

that is

$$|a_{l+1}|\le\frac{b\widetilde{[l]}_q^2(R-T)\vartheta}{2\widetilde{[l+1]}_q\left(\widetilde{[l+1]}_q-\widetilde{[l]}_q\right)}.$$

Upon comparing the coefficients in (13) and (27), we have

$$\begin{aligned}
& \frac{\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)}{b\widetilde{[l]}_q^2}a_{l+2} \\
= \ & \frac{1}{4}(R-T)\vartheta\left(\delta_2-\frac{1}{2}\delta_1^2\right)+\frac{(R-T)(5-T)\vartheta^2\delta_1^2}{16} \\
& +\widetilde{[l+1]}_q^2\left(\frac{\widetilde{[l+1]}_q-\widetilde{[l]}_q}{b\widetilde{[l]}_q^3}\right)a_{l+1}^2,
\end{aligned}$$

$$|a_{l+2}|=\frac{b\widetilde{[l]}_q^2(R-T)\vartheta}{4\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)}\left|\delta_2-\frac{v}{2}\delta_1^2\right|,$$

where

$$v=1-\frac{\vartheta}{2}\left(5-T+\frac{b\widetilde{[l]}_q(R-T)}{\widetilde{[l+1]}_q-\widetilde{[l]}_q}\right).$$

When $R>T$ implies $v>2$. Thus, application of Lemma 2 yields the desired conclusion.

The extremal function is

$$\begin{aligned}
&\mu_*(\tau)\\
=\ &\tau^l + \frac{b\widetilde{[l]}_q^2(R-T)\vartheta}{2\widetilde{[l+1]}_q\left(\widetilde{[l+1]}_q-\widetilde{[l]}_q\right)}\tau^{l+1} + \frac{b\widetilde{[l]}_q^2(R-T)\vartheta^2}{4\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)}\\
&\times\left(5-T+\frac{b\widetilde{[l]}_q(R-T)}{\widetilde{[l+1]}_q-\widetilde{[l]}_q}\right)\tau^{l+2}+\ldots. \qquad (28)
\end{aligned}$$

Then, it is clear that

$$1-\frac{1}{b}+\frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu_*(\tau)\right)}{\widetilde{[l]}_q\widetilde{\partial}_q\mu_*(\tau)}\right)=\overline{p}(R,T,\tau),$$

where the series of $\overline{p}(R,T,\tau)$ is given by (12). Hence, $\mu_*\in\mathcal{K}_l^{q,b}(R,T)$. □

Theorem 5. *Let $\mu\in\mathcal{K}_l^{q,b}(R,T)$ be of the form (4). Then,*

$$\begin{aligned}
&\left|a_{l+2}-\sigma a_{l+1}^2\right|\\
\leq\ &\frac{\widetilde{[l]}_q^2|b|(R-T)|\vartheta|}{4\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)\widetilde{[l+2]}_q}\\
&\times\max\left\{2,\left|\vartheta\left(-\frac{(R-T)\widetilde{[l]}_q b}{\left(\widetilde{[l+1}_q-\widetilde{[l]}_q\right)}+T-5\right.\right.\right.\\
&\left.\left.\left.+\frac{b\widetilde{[l]}_q\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)(R-T)}{\widetilde{[l+1]}_q\left(\widetilde{[l+1}_q-\widetilde{[l]}_q\right)^2}\sigma\right)\right|\right\}.
\end{aligned}$$

This result is sharp.

Proof. Since $\mu\in\mathcal{K}_l(R,T,b)$, we have

$$1-\frac{1}{b}+\frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu(\tau)\right)}{\widetilde{[l]}_q\widetilde{\partial}_q\mu(\tau)}\right)=\overline{p}(R,T;s(\tau)),\ \tau\in\mathcal{U},$$

with $s(0)=0$ and $|s(\tau)|<1$ in $\mathcal{U}$. Therefore,

$$\begin{aligned}
1-\frac{1}{b}+\frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu(\tau)\right)}{\widetilde{[l]}_q\widetilde{\partial}_q\mu(\tau)}\right) &= h(\tau),\\
\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu(\tau)\right)}{b\widetilde{[l]}_q\widetilde{\partial}_q\mu(\tau)} &= (1-b)+b\left(1+h_1\tau+h_2\tau^2+\ldots\right)
\end{aligned}$$

and after some simple calculation, we have

$$\begin{aligned}
&\widetilde{[l]}_q\widetilde{[l]}_q\tau^{l-1} + (\widetilde{[l+1]}_q^2 a_{l+1}\tau^l \\
&+\widetilde{[l+2]}_q^2 a_{l+2}\tau^{l+1} + \ldots \\
=\ &\widetilde{[l]}_q\left\{\widetilde{[l]}_q\tau^{l-1} + \widetilde{[l+1]}_q a_{l+1}\tau^l + \widetilde{[l+2]}_q a_{l+2}\tau^{l+1} + \ldots\right\} \\
&\times\left(1 + bh_1\tau + bh_2\tau^2 + \ldots\right)
\end{aligned}$$

$$\begin{aligned}
=\ &\widetilde{[l]}_q^2\tau^{l-1} + \left(\widetilde{[l]}_q\widetilde{[l+1]}_q a_{l+1} + \widetilde{[l]}_q^2 bh_1\right)\tau^l \\
&+\left\{\widetilde{[l]}_q\widetilde{[l+2]}_q a_{l+2} + \widetilde{[l]}_q\widetilde{[l+1]}_q bh_1 a_{l+1} + \widetilde{[l]}_q^2 bh_2\right\}\tau^{l+1}.
\end{aligned}$$

Comparing the coefficients of both sides, we obtain

$$a_{l+1} = \frac{\widetilde{[l]}_q^2 bh_1}{\widetilde{[l+1]}_q\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}$$

and

$$a_{l+2} = \frac{\widetilde{[l]}_q b}{\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}\left(\widetilde{[l+1]}_q h_1 a_{l+1} + \widetilde{[l]}_q h_2\right).$$

This implies that

$$\begin{aligned}
&\left|a_{l+2} - \sigma a_{l+1}^2\right| \\
=\ &\left|\frac{\widetilde{[l]}_q^2|b|}{\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)\widetilde{[l+2]}_q}\times\right. \\
&\left.\left(h_2 + \frac{\widetilde{[l]}_q b}{\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}\left(1 - \frac{\widetilde{[l+2]}_q\widetilde{[l]}_q\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}{\widetilde{[l+1]}_q\left(\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)\right)}\sigma\right)h_1^2\right)\right| \\
=\ &\frac{\widetilde{[l]}_q^2|b|}{\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)\widetilde{[l+2]}_q}\left|h_2 - vh_1^2\right|,
\end{aligned}$$

where

$$v = \left(\frac{\widetilde{[l+2]}_q\widetilde{[l]}_q\left(\widetilde{[l+2]}_q - \widetilde{[l]}_q\right)}{\widetilde{[l+1]}_q\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}\sigma - 1\right)\frac{\widetilde{[l]}_q b}{\left(\widetilde{[l+1]}_q - \widetilde{[l]}_q\right)}.$$

Implementation of Lemma 1 part (iv) for v gives the required result. The equality

$$\begin{aligned}&\left|a_{l+2}-\sigma a_{l+1}^2\right|\\ =\ &\frac{\widetilde{[l]}_q^2|b|(R-T)|\vartheta|}{4\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)\widetilde{[l+2]}_q}\times\left|\frac{(R-T)\widetilde{[l]}_q b}{\left(\widetilde{[l+1}_q-\widetilde{[l]}_q\right)}-T+5\right.\\ &\left.-\frac{b\widetilde{[l]}_q\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)(R-T)}{\widetilde{[l+1]}_q\left(\widetilde{[l+1}_q-\widetilde{[l]}_q\right)^2}\sigma\right|\end{aligned}$$

holds for μ_* given in (28). Assume $\mu_0:\mathcal{U}\to\mathbb{C}$ is defined as follows:

$$\mu_0(\tau)=\tau^l+\frac{\widetilde{[l]}_q^2|b|(R-T)}{2\widetilde{[l+2]}_q\left(\widetilde{[l+2]}_q-\widetilde{[l]}_q\right)}\tau^{l+2}+\ldots,$$

where $\overline{p}(R,T;\tau)$ is defined in (12). Hence,

$$1-\frac{1}{b}+\frac{1}{b}\left(\frac{\widetilde{\partial}_q\left(\tau\widetilde{\partial}_q\mu_0(\tau)\right)}{\widetilde{[l]}_q\widetilde{\partial}_q\mu_0(\tau)}\right)=\overline{p}\left(R,T;\tau^2\right).$$

□

Theorem 6. *Let $\mu\in\mathcal{A}_l$ be given by (4). If $\mu\in\mathcal{K}_l(R,T,b)$, then*

$$|a_{l+1}|\leq\frac{|b|\widetilde{[l]}_q^2|\overline{Q}_1|}{\widetilde{[1+l]}_q\left(\widetilde{[1+l]}_q-\widetilde{[l]}_q\right)},$$

$$|a_{l+t}|\leq\frac{|b|\widetilde{[l]}_q^2|\overline{Q}_1|}{\widetilde{[t+l]}_q\left(\widetilde{[t+l]}_q-\widetilde{[l]}_q\right)}\prod_{k=2}^{t}\left(1+\frac{|b|\widetilde{[l]}_q^2|\overline{Q}_1|}{\widetilde{[k-1+l]}_q\left(\widetilde{[k-1+l]}_q-\widetilde{[l]}_q\right)}\right),$$

where $\left|\overline{Q}_1\right|$ is given by (22) and $l\in\mathbb{N}$, $t\geq2$.

Proof. Using the same method as in Theorem 3, we can derive Theorem 6. □

4. Conclusions

In this article, we applied the concept of symmetric q-calculus and the cardioid domain to establish two novel subfamilies, namely, multivalent q-starlike functions and multivalent q-convex functions. We investigated sharp coefficient bounds, Fekete–Szegö functional, coefficient inequalities for the function belonging to newly defined subclasses of multivalent q-starlike functions and q-convex functions. In addition, the study showed how the results are extended and improved by the use of the parameters, including some recently published findings.

Researchers may create many new classes of multivalent functions and employ a variety of ordinary differential and q-analogous of symmetric difference and integral operators in their work. Many new classes may be discovered by putting this article's suggestions into practice. By using the concept of this article, it is possible to generalize the classes described and examined in [29] and investigate them with more relevant findings, including initial coefficient estimates, Toeplitz matrices, Fekete–Szego problems, and Hankel determinants.

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