SOME INFINITE SUMS DERIVED BY USING FRACTIONAL CALCULUS OF LOGARITHMIC FUNCTIONS

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DMS-693-IR December 1994

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Abstract

In the remarkably vast literature on fractional calculus, there are many systematic (and historical) accounts of its applications in a number of areas including (for example) ordinary and partial differential equations, special functions, and summation of series. The object of the present note is to examine rather closely some of the most recent contributions by K. Nishimoto [2] on the use of fractional calculus of logarithmic functions in deriving numerous interesting infinite sums. Some generalizations and relevant connections with certain familiar results in the theory of the Gaussian hypergeometric function are also given.

Key words: Fractional calculus, ordinary and partial differential equations, special functions, summation of series, logarithmic functions, Gaussian hypergeometric function, fractional differintegrals, generalized hypergeometric functions, binomial expansion, mathematical induction, augmentation of parameters, Laplace and inverse Laplace transforms.

1991 *Mathematics Subject Classification.* Primary 26A33, 33C05; Secondary 33C20.

1. Introduction and Preliminaries

In a recent paper in this Journal, Nishimoto [2] made use of the fractional calculus (developed by him in a series of earlier papers and books) in computing the fractional differintegrals (of order $\alpha \in \mathbb{R}$) of such logarithmic functions as

$$\log\left(\frac{z+c}{z}\right), \quad \log\left(\frac{z \pm n}{z \pm n - m - 1}\right), \quad \text{and} \quad \log\left(\frac{z+c}{z-c}\right),$$

where c is a constant and

$$m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad (\mathbb{N} := \{1, 2, 3, \cdots\}).$$

More interestingly, from each of these fractional differintegrals of logarithmic functions, he deduced some remarkable infinite sums. Making use of the Pochhammer symbol $(\lambda)_k = \Gamma(\lambda + k)/\Gamma(k)$, we recall here all five of these infinite sums in the following (*slightly modified*) forms (*cf.* Nishimoto [2, p. 17, Theorem 1(ii); p. 18, Theorem 2(ii); p. 19, Corollary 1(ii); p. 19, Theorem 3(ii); p. 20, Theorem 4]):

$$\sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!} \left(\frac{c}{z+c}\right)^k = \left(\frac{z+c}{z}\right)^{\alpha} - 1 \tag{1}$$

$$(|c| < |z+c|; \quad z \in \mathbb{C}; \quad \alpha \neq 0, -1, -2, \cdots);$$

$$\sum_{k=1}^{\infty} \sum_{r=0}^{m} \frac{(\alpha)_k}{k!} (z+n-r)^{-\alpha-k} = (z+n-m-1)^{-\alpha} - (z+n)^{-\alpha} \tag{2}$$

$$(|z+n-r| > 1 \quad (r = 0, 1, 2, \cdots, m; \quad n \in \mathbb{N}_0); \quad z \in \mathbb{C}; \quad \alpha \neq 0, -1, -2, \cdots);$$

$$\sum_{k=1}^{\infty} \sum_{r=0}^{m} \frac{(\alpha)_k}{k!} (z-n-r)^{-\alpha-k} = (z-n-m-1)^{-\alpha} - (z-n)^{-\alpha} \tag{3}$$

$$(|z-n-r| > 1 \quad (r = 0, 1, 2, \cdots, m; \quad n \in \mathbb{N}_0); \quad z \in \mathbb{C}; \quad \alpha \neq 0, -1, -2, \cdots);$$

$$2\alpha z^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha+1)_{2k}}{(2k+1)!} \left(\frac{c}{z}\right)^{2k+1} = (z-c)^{-\alpha} - (z+c)^{-\alpha} \tag{4}$$

$$(|c| < |z|; \quad z \in \mathbb{C}; \quad \alpha \neq 0, -1, -2, \cdots);$$

$$\sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!} (z+1)^{-\alpha-k} + \sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!} z^{-\alpha-k}$$

$$= 2\alpha \sum_{k=0}^{\infty} \frac{(\alpha+1)_{2k}}{(2k+1)!} z^{-\alpha-2k-1} \tag{5}$$

$$(\min\{|z|, |z+1|\} > 1; \quad z \in \mathbb{C}; \quad \alpha \neq 0, -1, -2, \cdots),$$

where c is a constant in *each* case.

In the present note we shall examine each of the infinite sums (1) to (5) rather closely. In particular, we shall point out relevant connections of these infinite sums with certain familiar results in the theory of the Gaussian hypergeometric function ${}_2F_1(\cdots)$, where ${}_pF_q(\cdots)$ denotes, as usual, a generalized hypergeometric function with p numerator and q denominator parameters. Our alternative (*direct*) approach to the infinite sums (1) to (5), and also to a natural generalization of the infinite sum (5) given in Section 3, *without* using fractional calculus, would show (among other things) that the aforementioned restrictions on the parameter α can be waived fairly simply.

2. Connections with Hypergeometric Functions

First of all, since $z \in \mathbb{C}$, the infinite sums (2) and (3) are essentially the same result. Obviously, (2) with z replaced by $z - 2n$ $(n \in \mathbb{N}_0)$ yields the infinite sum (3), and (3) with z replaced by $z + 2n$ $(n \in \mathbb{N}_0)$ would yield the infinite sum (2). Thus it would suffice to examine *only* the infinite sums (1), (2) or (3), (4), and (5).

In the theory of hypergeometric functions (or, more precisely, the Gaussian hypergeometric function), it is fairly well-known that

$${}_2F_1(a, c; c; z) = {}_2F_1(c, a; c; z) = {}_1F_0(a; —; z)$$

$$:= \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = (1-z)^{-a} \tag{6}$$

$$(|z| < 1; \quad z \in \mathbb{C}; \quad a \in \mathbb{C}),$$

so that, obviously,

$$\sum_{k=1}^{\infty} \frac{(a)_k}{k!} z^k = (1-z)^{-a} - 1 \tag{7}$$

$$(|z| < 1; \quad z \in \mathbb{C}; \quad a \in \mathbb{C}),$$

and that (*cf.*, *e.g.*, Abramowitz and Stegun [1, p. 556, Entry 15.1.10]; see also Prudnikov *et al.* [3, p. 461, Entry 7.3.1.107])

$$ {}_2F_1\left(a, a+\frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{(1+z)^{1-2a} - (1-z)^{1-2a}}{2(1-2a)\,z} \tag{8}$$

$$\left(|z| < 1; \quad z \in \mathbb{C}\setminus\{0\}; \quad a \in \mathbb{C}\setminus\{\tfrac{1}{2}\}\right),$$

where the restrictions $z \neq 0$ *and* $a \neq \frac{1}{2}$ are explicitly stated merely to avoid a zero in the denominator on the right-hand side of (8).

The infinite sum (1), for an *unrestricted* parameter α, is an immediate consequence of the well-known result (7).

In order to derive the infinite sum (2) or (3), *without* using fractional calculus, we observe, again by applying the well-known result (7), that

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{r=0}^{m} \frac{(\alpha)_k}{k!}(z \pm n - r)^{-\alpha-k} &= \sum_{r=0}^{m}\sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!}(z \pm n - r)^{-\alpha-k} \\
&= \sum_{r=0}^{m}(z \pm n - r)^{-\alpha}\left\{\left(1 - \frac{1}{z \pm n - r}\right)^{-\alpha} - 1\right\} \\
&= \sum_{r=0}^{m}\left\{(z \pm n - r - 1)^{-\alpha} - (z \pm n - r)^{-\alpha}\right\} \\
&= (z \pm n - m - 1)^{-\alpha} - (z \pm n)^{-\alpha},\end{aligned}$$

which evidently proves (2) as well as (3) for an *unrestricted* parameter α.

Since

$$(\lambda)_{2k} = 2^{2k}\left(\frac{\lambda}{2}\right)_k\left(\frac{\lambda+1}{2}\right)_k \qquad (\lambda \in \mathbb{C}; \ k \in \mathbb{N}_0), \tag{9}$$

the first member of (4) can be rewritten fairly readily in the hypergeometric form:

$$2\alpha c z^{-\alpha-1}\ {}_2F_1\left(\frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \frac{3}{2}; \frac{c^2}{z^2}\right),$$

which, in view of the known reduction formula (8) with, of course,

$$a = \frac{\alpha+1}{2} \quad \text{and} \quad z \to \frac{c}{z},$$

yields the right-hand side of the infinite sum (4) for an *unrestricted* parameter α.

Finally, we turn to the infinite sum (5). By virtue of the well-known result (7), the first member of (5) is

$$\begin{aligned}(z+1)^{-\alpha}&\left\{\left(1-\frac{1}{z+1}\right)^{-\alpha}-1\right\}+z^{-\alpha}\left\{\left(1-\frac{1}{z}\right)^{-\alpha}-1\right\}\\ &=\left\{z^{-\alpha}-(z+1)^{-\alpha}\right\}+\left\{(z-1)^{-\alpha}-z^{-\alpha}\right\}\\ &=(z-1)^{-\alpha}-(z+1)^{-\alpha},\end{aligned}\tag{10}$$

where we have required only that

$$\min\{|z|,\ |z+1|\}>1 \qquad (z\in\mathbb{C}),$$

the parameter α being *unrestricted*. On the other hand, in view of the identity (9), the right-hand side of (5) can also be rewritten in the hypergeometric form:

$$2\alpha z^{-\alpha-1}\ {}_2F_1\left(\frac{\alpha+1}{2},\ \frac{\alpha+2}{2};\ \frac{3}{2};\ \frac{1}{z^2}\right),$$

which, by means of (8) with

$$a=\frac{\alpha+1}{2} \quad \text{and} \quad z\to\frac{1}{z},$$

yields the last member of (10). This evidently completes our derivation of (5) for an *unrestricted* parameter α.

3. Concluding Remarks and Generalizations

In the preceding section, we have observed that each of the infinite sums (1) to (5) is a rather straightforward consequence of one or both of the familiar results (7) and (8). Our alternative (*direct*) derivation of each of these infinite sums did *not* make use of fractional calculus. Furthermore, we showed that each of the infinite sums (1) to (5) holds true for an *unrestricted* parameter α, thus waiving the obviously unnecessary constraint

$$\alpha\neq 0,-1,-2,\cdots,$$

which was required in the *fractional-calculus* derivation by Nishimoto [2].

Next we turn to the infinite sum (5) which we deduced in Section 2 by applying *both* (7) and (8). Making use of the binomial expansion (6), we can easily rewrite (5) in the form:

$$\sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\alpha)_{k+\ell}}{k!\,\ell!} z^{-k}(-z)^{-\ell} + \sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!} z^{-k}$$

$$= 2\alpha \sum_{k=0}^{\infty} \frac{(\alpha+1)_{2k}}{(2k+1)!} z^{-2k-1} \tag{11}$$

$$(\min\{|z|,\ |z+1|\} > 1; \quad z \in \mathbb{C}; \quad \alpha \in \mathbb{C}).$$

A closer look at the infinite sum (11) would immediately suggest the existence of a natural generalization of (5) in the form:

$$\sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty} \frac{(\alpha_1)_{k+\ell}\cdots(\alpha_p)_{k+\ell}}{(\beta_1)_{k+\ell}\cdots(\beta_q)_{k+\ell}} \frac{z^{-k}(-z)^{-\ell}}{k!\,\ell!} + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdots(\beta_q)_k} \frac{z^{-k}}{k!}$$

$$= \frac{2\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q} \sum_{k=0}^{\infty} \frac{(\alpha_1+1)_{2k}\cdots(\alpha_p+1)_{2k}}{(\beta_1+1)_{2k}\cdots(\beta_q+1)_{2k}} \frac{z^{-2k-1}}{(2k+1)!} \tag{12}$$

$$\big(\min\{|z|,\ |z+1|\} > 1; \quad z \in \mathbb{C}; \quad \alpha_j \in \mathbb{C} \quad (j = 1,\cdots,p);$$

$$\beta_j \in \mathbb{C} \setminus \{0,-1,-2,\cdots\} \quad (j = 1,\cdots,q)\big),$$

which can indeed be proven, by the principle of mathematical induction on the nonnegative integers p and q, using augmentation of parameters by means of the Laplace and inverse Laplace transforms. The details may be omitted.

The general result (12), which corresponds to (11) [and hence also to the infinite sum (5)] when $p-1=q=0$, can also be stated easily in terms of generalized hypergeometric functions. The details may be left as an exercise for the interested reader.

Acknowledgments

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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