

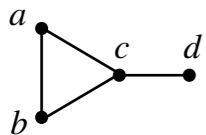
Homomorphism Densities of Graphs in Kernels

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Graphs

A *graph* is a set of vertices V paired with a set of edges E .



$$V = \{a, b, c, d\}$$

$$E = \{ab, ac, bc, cd\}$$

A *graph homomorphism* from a graph H to a graph G is a map f from $V(H)$ to $V(G)$ which preserves edges:

$$uv \in E(H) \Rightarrow f(u)f(v) \in E(G)$$

The number of homomorphisms from H to G is denoted $hom(H, G)$. If $A(G)$ is the adjacency matrix for G , we have:

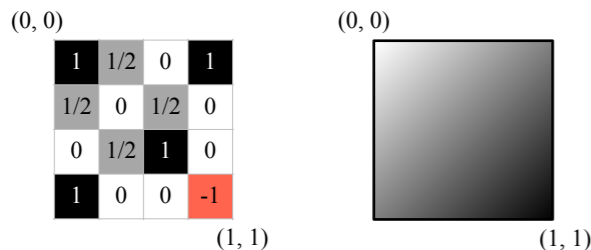
$$hom(H, G) = \sum_{f \in Map(H, G)} \prod_{\{v_i, v_j\} \in E(H)} A(G)(f(v_i), f(v_j)).$$

The *homomorphism density* of H in G is the proportion of homomorphisms in the set of maps from H to G :

$$t(H, G) := \frac{hom(H, G)}{|V(G)|^{|V(H)|}}$$

Kernels

A *kernel* is a symmetric function $W : [0,1]^2 \rightarrow \mathbb{R}$ which is bounded and measurable:



For a graph H and a kernel W , we define the *homomorphism density of H in W* as:

$$t(H, W) := \int_{[0,1]^{|V(H)|}} \prod_{\{u,v\} \in E(H)} W(x_u, x_v) \prod_{u \in V(H)} dx_u.$$

Comparing Graphs and Kernels

Given an adjacency matrix $A(G)$, we may obtain a kernel W_G by superimposing $A(G)$ on top of $[0,1]^2$. In fact, we will find:

$$t(H, G) = t(H, W_G).$$

In this way, a kernel can be interpreted as an abstraction of an adjacency matrix for a graph.

Previous Decomposition Results

Lemma 1. Suppose a graph H is the disjoint union of H_1 and H_2 . Then for a kernel W :

$$t(H, W) = t(H_1, W) \cdot t(H_2, W).$$

- Follows from the definition of $t(H, W)$

Behague et al [1] provide the following decomposition for homomorphism densities of graphs in kernels:

Lemma 2. Suppose H is a graph, W is a kernel, and $p \in \mathbb{R}$. Let $U := W - p$. Then:

$$t(H, W) = \sum_{E \subseteq E(H)} p^{|E(H)| - |E|} t(H[E], U).$$

- Follows from substituting and expanding the integrand

Behague et al [1] prove Theorems 1 and 2 using Lemma 2. Theorem 1 was initially proved by Goodman [2] with a different method. We denote the path on n edges as P_n .

Theorem 1. Suppose $W_1 + W_2 = 1$ for kernels W_1, W_2 . Then:

$$t(C_3, W_1) + t(C_3, W_2) = \sum_{i=1}^2 [t(P_1, W_i)^3 + \frac{3}{2}(t(P_2, W_i) - t(P_1, W_i)^2)].$$

Theorem 2. Suppose $W_1 + W_2 = 1$ for kernels W_1, W_2 . Then:

$$t(C_5, W_1) + t(C_5, W_2) = \sum_{i=1}^2 [t(P_1, W_i)^5 + 5t(P_1, W_i)t(P_4, W_i) - 5t(P_1, W_i)^2t(P_3, W_i)].$$

Our Decomposition Results

Theorem 3. Suppose $W_1 + W_2 = 1$ for kernels W_1, W_2 . Then

$$t(C_{2k+1}, W_1) + t(C_{2k+1}, W_2)$$

can be expressed in terms of sums and products of the simpler densities

$$\{t(P_l, W_i) : l \leq 2k + 1, i \in \{1, 2\}\}.$$

- Let $U := W_1 - t(P_1, W_1)$ and apply Lemma 2 to each term
- $E = E(C_{2k+1})$ terms cancel by parity
- Proper subgraphs of cycles are disjoint unions of paths
- Result follows from Theorem 4

A *partition* of an integer n is an unordered sum of positive integers equalling n . The set of all partitions of n is $P(n)$.

Theorem 4. Let W_1 be a kernel, $x_i := t(P_i, W_1)$, and $U := W_1 - x_1$. For all $n \geq 1$, we have:

$$t(P_n, U) = \sum_{(a_1, \dots, a_k) \in P(n)} \left(\sum_{j=1}^k \binom{r_1 - j + 1}{j, r_2, \dots, r_n, 2r_1 - 2j - k + 1} \right) (-1)^{r_1 - j} \prod_{i=1}^k x_{a_i}$$

where $r_j := |\{i \in [k] : a_i = j\}|$.

Inequalities Involving Densities

A *graphon* is a kernel W where for all $(x, y) \in [0,1]^2$, we have $0 \leq W(x, y) \leq 1$.

For a graph G and a graphon W , we have:

- $t(G, W) \geq 0$
- $1 - W$ is also a graphon
- $t(G, W) + t(G, 1 - W) \geq 0$
- $t(G, 1/2) = (1/2)^{|E(G)|}$

For a pair of graphs (G, H) , we define:

$$c(G, H) := \sup\{\alpha \in \mathbb{R} : t(G, W) + t(G, 1 - W) \geq \alpha(t(H, W) + t(H, 1 - W)) \text{ for every graphon } W\}.$$

- $c(G, H) \geq 0$
- Any graphon W gives an upper bound for $c(G, H)$
- The $1/2$ graphon gives us $0 \leq c(G, H) \leq (1/2)^{|E(G)| - |E(H)|}$

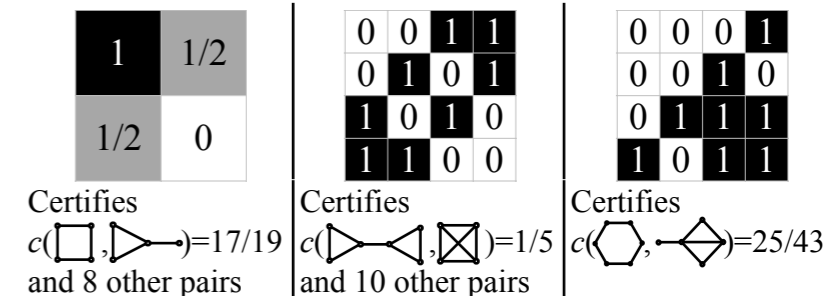
Finding Optimal Graphons

Using computer assistance and external tools in flag algebra, we found lower bounds $l(G, H)$ of $c(G, H)$ for a list of 81 pairs of small graphs (G, H) such that:

- $c(G, H)$ was not known
- $l(G, H)$ was rational

Using a brute-force search over certain types of graphons, we found the following graphons which give upper bounds $u(G, H) = l(G, H)$ for some of the (G, H) pairs.

Theorem 5.



In total, 21 of the 81 bounds have been shown to be sharp.

References

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- A. W. Goodman. "On Sets of Acquaintances and Strangers at any Party". In: *The American Mathematical Monthly* 66.9 (1959), pp. 778-783. ISSN: 00029890, 19300972. URL: <http://www.jstor.org/stable/2310464>.

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