ON A THEOREM OF KOBORI

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Abstract

Let $\mathcal{S}$ and $\mathcal{S}^*$ denote the well-known classes of normalized analytic functions which are, respectively, univalent and starlike in $|z| < 1$. A theorem of Kobori states that, for a function

$$g(z) = z + \sum_{k=2}^{\infty} a_k \, z^k \in \mathcal{S}^*,$$

the sequence of the nth sections

$$S_n(z) = z + \sum_{k=2}^{n} \frac{1}{k} \, a_k \, z^k$$

must be starlike in $|z| < \frac{1}{2}$. Recently, Silverman [10], Gruenberg *et al.* [4], and Rønning [8] extended the above result. In this note, we first improve the results of Ilieff [5], Ruscheweyh [9], and Silverman [10]. We then give a remarkable simple proof of a result of Gruenberg *et al.* [4] and Rønning [8] that $S_4(z)$ is starlike in $|z| < \frac{1}{2}$ for $f \in \mathcal{S}$.

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1. Introduction

Let $\mathcal{A}$ denote the set of analytic functions f in the *open* unit disk D normalized by

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1.$$

Also let $\mathcal{S}$ be the set of univalent functions in $\mathcal{A}$ (*cf.*, *e.g.*, [2] and [11]). Denote by D_r the set $\{z : z \in \mathbb{C} \text{ and } |z| < r\}$. A function $f \in \mathcal{A}$ is said to be convex (starlike) in D_r if it is univalent in D_r with $f(D_r)$ convex (starlike with respect to the origin). By $\mathcal{K}$ and $\mathcal{S}^*$ we denote the subclasses of functions in $\mathcal{S}$ which are, respectively, convex and starlike in D. For

$$f(z) = \sum_{k=1}^{\infty} a_k\, z^k \in \mathcal{A} \qquad (a_1 := 1),$$

let

$$S_n(z, f) = \sum_{k=1}^{n} a_k\, z^k \qquad (n \in \mathbb{N}; a_1 := 1),$$

be the nth section of $f(z)$. In 1934, Kobori [6] proved that, for $f \in \mathcal{K}$ all sections $S_n(z, f)$ are starlike in $D_{\frac{1}{2}}$. Since then many papers (see, *e.g.*, [1], [4], [5], and [7] to [10]) have appeared concerning the section $S_n(z, f)$. Ilieff [5], Ruscheweyh [9], Bernardi [1], and Silverman [10] determined the disks D_{r_n} in which $S_n(z, f)$ is univalent, close-to-convex, and starlike, respectively. On the other hand, the well-known relationship

$$g \in \mathcal{S}^* \iff h(z) = \int_0^z \frac{g(t)}{t}\, dt \in \mathcal{K}$$

furnishes us with an alternative statement of Kobori's theorem: *If $g \in \mathcal{S}^*$, then $S_n(z, h)$ is starlike in $D_{\frac{1}{2}}$ for all n.*

Recently, Gruenberg *et al.* [4] and Rønning [8] extended Kobori's theorem by showing that, if $g \in \mathcal{S}$, then

$$\Re\{S_n'(z, h)\} > 0 \quad \text{in} \quad D_{\frac{1}{2}} \quad \text{for all} \quad n,$$

and $S_n(z, h)$ is starlike in $D_{\frac{1}{2}}$ for positive integers $n \leq 4$ and $n \geq 6$. In this note, we first improve the aforementioned result of Ilieff [5], Ruscheweyh [9], and Silverman [10]. We then give a simple proof of the assertion that $S_4(z, h)$ is starlike in $D_{\frac{1}{2}}$.

2. Radius of Starlikeness Depending on n

Let $f \in \mathcal{K}$. Ilieff [5] gave bounds depending on n for the radius of univalence of $S_n(z, f)$. Let r_n denote the positive root of the equation:

$$1 - (1+n)r^n - nr^{n+1} = 0 \qquad (n \in \mathbb{N}). \tag{1}$$

Ruscheweyh [9] improved the result of Ilieff [5] to the following form:

Theorem 1. *If $f \in \mathcal{K}$, then $S_n(z, f)$ is univalent in D_{r_n}, and maps this circle onto a close-to-convex domain. Furthermore, for an even positive integer n, r_n cannot be replaced by any larger number with respect to $\mathcal{K}$.*

Subsequently, Bernardi [1] proved

Theorem 2. *If $f \in \mathcal{K}$, then $S_n(z, f)$ is starlike in D_{r_n}. This result is sharp for each even positive integer n for the function $f(z) = z/(1-z)$.*

Recently, Silverman [10] proved that, if $f \in \mathcal{K}$, then $S_n(z, f)$ is starlike in $|z| < \{1/(2n)\}^{1/n}$. In view of Theorem 2, the above-mentioned results of Kobori [6], Ruscheweyh [9], and Silverman [10] can be extended to the following form:

Theorem 3. *If $f \in \mathcal{K}$, then $S_n(z, f)$ is starlike in D_{r_n}, where r_n denotes the positive root of Equation* (1) *constrained further by*

$$\frac{1}{2} \leq (2n)^{-1/n} \leq r_n \leq n^{-1/(n-1)} \tag{2}$$

and

$$r_n = 1 - \frac{1}{n}\log(2n) + \frac{1}{2n^2}\log(n)\log(4en) + o\left(\frac{\log(n)}{n^2}\right). \tag{3}$$

Furthermore, for an even positive integer n, r_n cannot be replaced by any larger number.

In fact, it follows from Theorem 2 that $S_n(z, f)$ is starlike in D_{r_n}. For the number r_n defined above, Ruscheweyh [9, Theorem 2] showed that $(2n)^{-1/n} \leq r_n \leq n^{-1/(n-1)}$ and (3) hold true. Since the sequence $(2n)^{-1/n}$ increases with n, the assertion of Theorem 3 follows easily.

3. The Starlikeness of $S_n(z,h)$ for $g \in \mathcal{S}$

Let

$$g(z) = z + \sum_{n=2}^{\infty} a_n\, z^n \in \mathcal{S}$$

and

$$h(z) = \int_0^z \frac{g(t)}{t}\, dt.$$

It was shown by Lin [7] that $S_2(z,h)$ and $S_3(z,h)$ are starlike in $D_{\frac{1}{2}}$. More recently, Gruenberg *et al.* [4] proved that $\Re\{S_n'(z,h)\} > 0$ in $D_{\frac{1}{2}}$, and hence that $S_n(z,h)$ is univalent in $D_{\frac{1}{2}}$ for all n. Rønning [8] has further proved the following result:

Theorem 4. *Let $g \in \mathcal{S}$. Then $S_n(z,h)$ is starlike in $D_{\frac{1}{2}}$ for positive integers $n \le 4$ and $n \ge 6$.*

It should be pointed out that Theorem 4 provides a verification of a certain multiplier conjecture for univalent functions in a special case (*cf.* [3] and [4]). Rønning [8] obtained this result for a little larger set. However, his proof concerning the special case $n = 4$ is difficult and rather lengthy. Here we shall give a shorter (and simpler) proof of Theorem 4 in the special case $n = 4$.

Proof of Theorem 4 in the Special Case $n = 4$. Let

$$a_2 = x + iy, \quad a_3 = u + iv, \quad \text{and} \quad a_4 = s + it. \tag{4}$$

We need to prove that

$$\Re\left\{\frac{zS_4'(z,h)}{S_4(z,h)}\right\} = \left|\frac{z}{S_4(z,h)}\right|^2 \quad \Re\left\{S_4'(z,h)\left(\overline{\frac{S_4(z,h)}{z}}\right)\right\} > 0 \qquad (z \in D_{\frac{1}{2}}). \tag{5}$$

Since the classes $\mathcal{S}$ and $\mathcal{S}^*$ are preserved under rotations, it is sufficient to prove (5) for the point $z = r = \frac{1}{2}$.

Now

$$1536\,\Re\left\{S_4'\left(\frac{1}{2},h\right)\,\overline{S_4\left(\frac{1}{2},h\right)}\right\} = G(x,y,u,v,s,t)$$

$$= 768 + 576x + 96(x^2+y^2) + 256u + 80(xu+yv) + 16(u^2+v^2)$$

$$+ 3(s^2+t^2) + 120s + 36(xs+yt) + 14(us+vt).$$

It is well-known that

$$\left\{g\left(\frac{1}{z^2}\right)\right\}^{-\frac{1}{2}} = z + \sum_{n=1}^{\infty} \frac{b_{2n-1}}{z^{2n-1}} \in \Sigma$$

with

$$\begin{aligned} b_1 &= -\frac{1}{2}\, a_2, \qquad b_3 = \frac{3}{8}\, a_2^2 - \frac{1}{2}\, a_3, \\ b_5 &= \frac{3}{4}\, a_2 a_3 - \frac{5}{16}\, a_2^3 - \frac{1}{2}\, a_4. \qquad\qquad (cf.\ [2,\ \text{p. } 132]). \end{aligned} \tag{6}$$

Applying the area theorem to $\{g(1/z^2)\}^{-\frac{1}{2}}$, we obtain

$$|b_1|^2 + 3|b_3|^2 + 5|b_5|^2 \le 1. \tag{7}$$

Substituting from (4) and (6) into (7), we conclude that

$$\begin{aligned} G_1(x,y,u,v,s,t) &= 64(x^2+y^2) + 108(x^2+y^2)^2 + 125(x^2+y^2)^3 + 192(u^2+v^2) \\ &\quad - \left[288 + 600(x^2+y^2)\right](ux^2 - uy^2 + 2xyv) + 320(s^2+t^2) \\ &\quad + 720(x^2+y^2)(u^2+v^2) + 400s(x^3 - 3xy^2) - 960s(xu - yv) \\ &\quad + 400t(3x^2y - y^3) - 960t(xv + yu) - 256 \le 0. \end{aligned} \tag{8}$$

In order to prove (5), it is sufficient to prove that $G + G_1 \ge 0$. We write

$$\begin{aligned} G + G_1 = &\left\{512 + 576x + 160(x^2+y^2) + 108(x^2+y^2)^2 + 125(x^2+y^2)^3 + 256u \right. \\ &+ 208(u^2+v^2) + 720(x^2+y^2)(u^2+v^2) + 80(xu + yv) \\ &\left. - \left[288 + 600(x^2+y^2)\right](ux^2 - uy^2 + 2xyv)\right\} + 323(s^2+t^2) \\ &+ 2s\left\{200(x^3 - 3xy^2) - 480(xu - yv) + 18x + 7u + 60\right\} \\ &+ 2t\left\{200(3x^2y - y^3) - 480(xv + yu) + 18y + 7v\right\} \\ = &\ G_2(x,y,u,v) + 323(s^2+t^2) + 2s\, G_3(x,y,u,v) + 2t\, G_4(x,y,u,v). \end{aligned}$$

Since

$$
\begin{aligned}
&323(s^2+t^2)+2s\,G_3(x,y,u,v)+2t\,G_4(x,y,u,v)\\
&\geq -\frac{1}{323}\left(G_3^2+G_4^2\right)\\
&= -\frac{1}{323}\left\{40000(x^2+y^2)^3-192000(x^2+y^2)(ux^2-uy^2+2xyv)\right.\\
&\qquad +7200(x^4-y^4)+230400(x^2+y^2)(u^2+v^2)+324(x^2+y^2)+49(u^2+v^2)\\
&\qquad +2800\big[u(x^3-3xy^2)+v(3x^2y-y^3)\big]-17280u(x^2+y^2)-6720x(u^2+v^2)\\
&\qquad +24000(x^3-3xy^2)-57600(xu-yv)\\
&\qquad \left.+252(xu+yv)+2160x+840u+3600\right\}\\
&= G_5(x,y,u,v),
\end{aligned}
\tag{9}
$$

we have

$$
\begin{aligned}
&G+G_1\geq G_2+G_5\\
&\quad=\left\{\frac{161776}{323}+\frac{183888}{323}\,x+\frac{51356}{323}\,(x^2+y^2)-\frac{24000}{323}\,(x^3-3xy^2)\right.\\
&\qquad\left.+\frac{27684}{323}\,x^4+216x^2y^2+\frac{42084}{323}\,y^4+\frac{375}{323}\,(x^2+y^2)^3\right\}\\
&\qquad+(u^2+v^2)\left\{\frac{67135}{323}+\frac{6720}{323}\,x+\frac{2160}{323}\,(x^2+y^2)\right\}-2u\left\{\frac{900}{323}\,(x^4-y^4)\right.\\
&\qquad\left.+\frac{1400}{323}\,(x^3-3xy^2)+\frac{37872}{323}\,x^2-\frac{55152}{323}\,y^2-\frac{41594}{323}\,x-\frac{40924}{323}\right\}\\
&\qquad-2v\left\{\frac{1800}{323}\,xy(x^2+y^2)+\frac{1400}{323}\,(3x^2y-y^3)+288xy+\frac{16006}{323}\,y\right\}\\
&\quad=G_6(x,y)+(u^2+v^2)\,G_7(x,y)-2u\,G_8(x,y)-2v\,G_9(x,y)\\
&\quad\geq\frac{1}{G_7}\left\{G_6G_7-(G_8)^2-(G_9)^2\right\},
\end{aligned}
\tag{10}
$$

where we have used the fact that $G_7>0$.

But

$$
\begin{aligned}
G_6G_7 &- (G_8)^2 - (G_9)^2 \\
&= \left\{\frac{45955}{323}(x^2+y^2) + \frac{318960}{323}x\right\}(x^2+y^2)^2 + \frac{1746292}{323}x^4 + \frac{1330344}{323}x^2y^2 \\
&\quad - \frac{21892}{17}y^4 + \frac{7418432}{323}x^3 - \frac{7224768}{323}xy^2 + \frac{19822400}{323}x^2 \\
&\quad - \frac{3012544}{323}y^2 + \frac{31046656}{323}x + \frac{1496832}{17}.
\end{aligned}
\tag{11}
$$

In view of the inequality:

$$
\begin{aligned}
&\left\{\frac{45955}{323}(x^2+y^2) + \frac{318960}{323}x\right\}(x^2+y^2)^2 \\
&\qquad \geq -\frac{454100}{323}(x^4 + 2x^2y^2 + y^4),
\end{aligned}
\tag{12}
$$

Equation (11) reduces to the following form:

$$
\begin{aligned}
G_6G_7 - (G_8)^2 - (G_9)^2 &\geq -\frac{45792}{17}y^4 + \left(-\frac{3012544}{323} - \frac{7224768}{323}x + \frac{24832}{19}x^2\right)y^2 \\
&\quad + \frac{1496832}{17} + \frac{31046656}{323}x + \frac{19822400}{323}x^2 \\
&\quad + \frac{7418432}{323}x^3 + \frac{1292192}{323}x^4 \\
&= G_{10}(x, y^2).
\end{aligned}
\tag{13}
$$

Setting $y^2 = w$ and writing

$$
G_{10}(x, w) = -\frac{45792}{17}w^2 + G_{11}(x)\,w + G_{12}(x),
$$

we find that, for each $x \in [-2, 2]$, the graph of $G_{10}(x, w)$ as a function of w is a parabola opening downward. Thus the inequality $G_{10} > 0$ needs to be verified only at the endpoints of the interval $0 \leq w \leq 4 - x^2$, that is,

$$
G_{10}(x, w) \geq \min\left\{G_{10}(x, 0),\ \ G_{10}(x, 4 - x^2)\right\}. \tag{14}
$$

But

$$\begin{aligned} G_{10}(x,0) &= G_{12}(x) \\ &= \frac{1292192}{323}x^2\left(x+\frac{115913}{40381}\right)^2 + \frac{288}{323}\cdot\frac{1286465209}{40381}x^2 \\ &\qquad + \frac{31046656}{323}x + \frac{1492192}{17} \\ &\geq 6736 \end{aligned} \tag{15}$$

and

$$\begin{aligned} G_{10}(x,\,4-x^2) &= x^2\left(\frac{14643200}{323}x + \frac{31483904}{323}\right) + \frac{2147584}{323}x + \frac{2468864}{323} \\ &\geq \frac{2197504}{323}x^2 + \frac{2147584}{323}x + \frac{2468864}{323} \\ &= \left(\frac{2197504}{323}x + \frac{8389}{17168}\right)^2 + \frac{2468864}{323} - \frac{2197504}{323}\left(\frac{8389}{17168}\right)^2 \\ &\geq 6019. \end{aligned} \tag{16}$$

Hence (14) shows that $G_{10} > 0$ and the desired result follows at once from (10) and (13). The proof of Theorem 4 in the special case $n = 4$ is thus completed.

In conclusion, we remark that the elementary area theorem provides enough information to prove Theorem 4 in the case $n = 4$. Unfortunately, along similar lines, we are unable to prove Theorem 4 in the case $n = 5$. We do indeed conjecture that Theorem 4 is also true in the case $n = 5$.

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