

CERTAIN SUBCLASSES OF STARLIKE FUNCTIONS. II<sup>†</sup>

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## ABSTRACT

Recently, the authors introduced two interesting subclasses  $\mathcal{F}^*(\alpha, \beta, \gamma)$  and  $\mathcal{C}(\alpha, \beta, \gamma)$  of starlike functions in the unit disk  $\mathcal{U}$ , and proved various coefficient inequalities and distortion theorems for functions belonging to these subclasses. The object of the present sequel is to apply some of these coefficient inequalities in order to deduce substantially more general distortion theorems involving fractional derivatives and fractional integrals of functions belonging to the classes  $\mathcal{F}^*(\alpha, \beta, \gamma)$  and  $\mathcal{C}(\alpha, \beta, \gamma)$ . Relevant connections of the results presented here with those of the earlier paper are also indicated.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $\mathcal{A}$  denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disk  $\mathcal{U} = \{z: |z| < 1\}$ . Further, let  $\mathcal{S}$  denote the class of all functions in  $\mathcal{A}$  which are univalent in the unit disk  $\mathcal{U}$ . Then a function  $f(z)$  in  $\mathcal{S}$  is said to be starlike of order  $\alpha$  if and only if (cf. [1], [9], and [7])

$$(1.2) \quad \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U})$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ). We denote by  $\mathcal{S}^*(\alpha)$  the subclass of  $\mathcal{S}$  consisting of all starlike functions of order  $\alpha$ . Throughout this paper, it should be

understood that functions such as  $zf'(z)/f(z)$ , which have removable singularities at  $z = 0$ , have had these singularities removed in statements like (1.2).

A function  $f(z)$  belonging to the class  $\mathcal{S}$  is said to be convex of order  $\alpha$  if and only if

$$(1.3) \quad \operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathcal{U})$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ). Let us denote by  $\mathcal{K}(\alpha)$  the class of all functions in  $\mathcal{S}$  which are convex of order  $\alpha$ . We note that  $f(z) \in \mathcal{K}(\alpha)$  if and only if  $zf'(z) \in \mathcal{S}^*(\alpha)$ , and that

$$(1.4) \quad \mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^*, \mathcal{K}(\alpha) \subseteq \mathcal{K}(0) \equiv \mathcal{K}, \text{ and } \mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}$$

for  $0 \leq \alpha < 1$ .

The classes  $\mathcal{S}^*(\alpha)$  and  $\mathcal{K}(\alpha)$  were first introduced by Robertson [9], and were studied subsequently by Schild [11], MacGregor [5], Pinchuk [8], Jack [4], and others (cf., e.g., [1] and [7]).

Let  $\mathcal{T}$  denote the subclass of  $\mathcal{A}$  consisting of functions of the form

$$(1.5) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

We denote by  $\mathcal{T}^*(\alpha)$  and  $\mathcal{L}(\alpha)$ , respectively, the classes obtained by taking the intersections of  $\mathcal{S}^*(\alpha)$  and  $\mathcal{K}(\alpha)$  with  $\mathcal{T}$ , that is,

$$(1.6) \quad \mathcal{T}^*(\alpha) = \mathcal{S}^*(\alpha) \cap \mathcal{T} \text{ and } \mathcal{L}(\alpha) = \mathcal{K}(\alpha) \cap \mathcal{T}.$$

The classes  $\mathcal{T}^*(\alpha)$  and  $\mathcal{L}(\alpha)$  were introduced by Silverman [12].

Define the function  $g(z)$  by

$$(1.7) \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0).$$

Then a function  $f(z)$  belonging to the class  $\mathcal{F}$  is said to be in the class  $\mathcal{F}^*(\alpha, \beta, \gamma)$  if

$$(1.8) \quad \left| \left( \frac{zf'(z)}{g(z)} - 1 \right) / \left( \frac{zf'(z)}{g(z)} + (1-2\beta) \right) \right| < \gamma \quad (z \in \mathcal{U})$$

for  $g(z) \in \mathcal{F}^*(\alpha)$ , where  $0 \leq \beta < 1$  and  $0 < \gamma \leq 1$ . Further, we say that the function  $f(z)$  is in the class  $\mathcal{L}(\alpha, \beta, \gamma)$  if it belongs to the class  $\mathcal{F}$  and satisfies the condition (1.8) for  $g(z) \in \mathcal{L}(\alpha)$ ,  $0 \leq \beta < 1$  and  $0 < \gamma \leq 1$ .

The classes  $\mathcal{F}^*(\alpha, \beta, \gamma)$  and  $\mathcal{L}(\alpha, \beta, \gamma)$  were introduced by Srivastava and Owa [14].

We note that when the function  $f(z)$  belonging to the class  $\mathcal{F}$  is univalent, the classes  $\mathcal{F}^*(\alpha, \beta, \gamma)$  and  $\mathcal{L}(\alpha, \beta, \gamma)$  evidently become the subclasses of the class  $\mathcal{K}(\alpha, \beta)$  which was studied recently by Gupta [3].

Finally, among many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) given in the literature (cf., e.g., [2, Chapter 13], [10], [13, p. 28 et seq.], and [15]), we choose to recall here the following definitions which were used earlier by Owa [6] (and, more recently, by Owa and Srivastava [7]):

DEFINITION 1. The fractional integral of order  $\lambda$  is defined, for a function  $f(z)$ , by

$$(1.9) \quad D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta,$$

where  $\lambda > 0$ ,  $f(z)$  is an analytic function in a simply-connected region of the  $z$ -plane containing the origin, and the multiplicity of  $(z-\zeta)^{\lambda-1}$  is removed by requiring  $\log(z-\zeta)$  to be real when  $z - \zeta > 0$ .

DEFINITION 2. The fractional derivative of order  $\lambda$  is defined, for a function  $f(z)$ , by

$$(1.10) \quad D_z^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta,$$

where  $0 \leq \lambda < 1$ ,  $f(z)$  is an analytic function in a simply-connected region of the  $z$ -plane containing the origin, and the multiplicity of  $(z-\zeta)^{-\lambda}$  is removed as in Definition 1 above.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order  $n + \lambda$  is defined by

$$(1.11) \quad D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^\lambda f(z),$$

where  $0 \leq \lambda < 1$  and  $n \in \mathcal{N}_0 = \{0, 1, 2, \dots\}$ .

In the present sequel to our recent paper [14] we give various distortion theorems involving fractional derivatives and fractional integrals of functions belonging to the classes  $\mathcal{F}^*(\alpha, \beta, \gamma)$  and  $\mathcal{C}(\alpha, \beta, \gamma)$ . Our derivations of these distortion theorems are based upon the following lemmas:

LEMMA 1 (Srivastava and Owa [14, Theorem 1]). Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{F}^*(\alpha, \beta, \gamma)$ . Then

$$(1.12) \quad \sum_{n=2}^{\infty} \left[ (1+\gamma)^n a_n - \frac{(1-\alpha)(1-\gamma+2\beta\gamma)}{n-\alpha} \right] \leq 2\gamma(1-\beta).$$

LEMMA 2 (Srivastava and Owa [14, Theorem 2]). Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{C}(\alpha, \beta, \gamma)$ . Then

$$(1.13) \quad \sum_{n=2}^{\infty} \left[ (1+\gamma)^n a_n - \frac{(1-\alpha)(1-\gamma+2\beta\gamma)}{n(n-\alpha)} \right] \leq 2\gamma(1-\beta).$$

## 2. DISTORTION THEOREMS INVOLVING OPERATORS OF FRACTIONAL CALCULUS

We begin by proving

THEOREM 1. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{F}^*(\alpha, \beta, \gamma)$ . Then

$$(2.1) \quad \left| D_z^{-\lambda} f(z) \right| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)(2+\lambda)} |z| \right\}$$

and

$$(2.2) \quad \left| D_z^{-\lambda} f(z) \right| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)(2+\lambda)} |z| \right\}$$

for  $\lambda > 0$  and  $z \in \mathcal{U}$ . The results (2.1) and (2.2) are sharp.

PROOF. Define the function  $F(z)$  by

$$(2.3) \quad \begin{aligned} F(z) &= \Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n z^n \end{aligned}$$

for  $\lambda > 0$ . Since the function

$$(2.4) \quad \phi(n) = \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} \quad (n \geq 2)$$

is decreasing in  $n$  ( $n \geq 2$ ), we have

$$(2.5) \quad 0 < \phi(n) \leq \phi(2) = \frac{2}{2+\lambda} \quad (n \geq 2).$$

Furthermore, it follows from Lemma 1 that

$$(2.6) \quad \sum_{n=2}^{\infty} a_n \leq \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{2(2-\alpha)(1+\gamma)}.$$

With the aid of (2.5) and (2.6), we observe that

$$(2.7) \quad \begin{aligned} |F(z)| &\leq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n \\ &\leq |z| - \left(\frac{2}{2+\lambda}\right) |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| - \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)(2+\lambda)} |z|^2, \end{aligned}$$

which gives (2.1), and that

$$(2.8) \quad \begin{aligned} |F(z)| &\leq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n \\ &\leq |z| + \left(\frac{2}{2+\lambda}\right) |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)(2+\lambda)} |z|^2 \end{aligned}$$

which implies (2.2).

Finally, we can easily see that the results (2.1) and (2.2) are sharp for the function  $f(z)$  given by

$$(2.9) \quad D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)(2+\lambda)} z \right\}.$$

COROLLARY 1. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{F}^*(\alpha, \beta, \gamma)$ . Then  $D_z^{-\lambda} f(z)$  is included in a disk with its center at the origin and radius  $r_0$  given by

$$(2.10) \quad r_0 = \frac{1}{\Gamma(2+\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)(2+\lambda)} \right\}.$$

Following the lines of proof of Theorem 1, and using Lemma 2 in place of Lemma 1, we obtain

THEOREM 2. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{C}(\alpha, \beta, \gamma)$ . Then

$$(2.11) \quad \left| D_z^{-\lambda} f(z) \right| \cong \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)(2+\lambda)} |z| \right\}$$

and

$$(2.12) \quad \left| D_z^{-\lambda} f(z) \right| \cong \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)(2+\lambda)} |z| \right\}$$

for  $\lambda > 0$  and  $z \in \mathcal{U}$ . The results (2.11) and (2.12) are sharp for the function  $f(z)$  defined by

$$(2.13) \quad D_z^{-\lambda} f(z) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)(2+\lambda)} z \right\}.$$

COROLLARY 2. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{C}(\alpha, \beta, \gamma)$ . Then  $D_z^{-\lambda} f(z)$  is included in a disk with its center at the origin and radius  $r_1$  given by

$$(2.14) \quad r_1 = \frac{1}{\Gamma(2+\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)(2+\lambda)} \right\}.$$

Applying Theorem 1, we next prove

THEOREM 3. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{F}^*(\alpha, \beta, \gamma)$ . Then

$$(2.15) \quad \left| D_z^{1-\lambda} f(z) \right| \leq \frac{|z|^\lambda}{\Gamma(2+\lambda)} \left\{ (1+\lambda) + \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)} |z| \right\}$$

for  $\lambda > 0$  and  $z \in \mathcal{U}$ . The result (2.15) is sharp.

PROOF. Note that Lemma 1 implies

$$(2.16) \quad \sum_{n=2}^{\infty} n a_n \leq \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)}.$$

By using (2.5) and (2.16), we have

$$(2.17) \quad \begin{aligned} |F'(z)| &\leq 1 + |z| \sum_{n=2}^{\infty} \frac{n \Gamma(n+1) \Gamma(2+\lambda)}{\Gamma(n+1+\lambda)} a_n \\ &\leq 1 + \left( \frac{2}{2+\lambda} \right) |z| \sum_{n=2}^{\infty} n a_n \\ &\leq 1 + \frac{2\{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)\}}{(2-\alpha)(1+\gamma)(2+\lambda)} |z|. \end{aligned}$$

where  $F(z)$  is defined by (2.3).

Now the assertion (2.15) of Theorem 3 follows if we combine (2.17) with the assertion (2.2) of Theorem 1. Further, by taking the function  $f(z)$  given by (2.9), we can easily see that the result (2.15) is sharp.

For the functions  $f(z)$  belonging to the class  $\mathcal{C}(\alpha, \beta, \gamma)$ , we similarly have

**THEOREM 4.** Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{C}(\alpha, \beta, \gamma)$ . Then

$$(2.18) \quad \left| D_z^{1-\lambda} f(z) \right| \cong \frac{|z|^\lambda}{\Gamma(2+\lambda)} \left\{ (1+\lambda) + \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)} |z| \right\}$$

for  $\lambda > 0$  and  $z \in \mathcal{U}$ . The result (2.18) is sharp for the function  $f(z)$  given by (2.13).

Finally, we prove the following distortion theorems for fractional derivatives of functions belonging to the classes  $\mathcal{F}^*(\alpha, \beta, \gamma)$  and  $\mathcal{C}(\alpha, \beta, \gamma)$ .

**THEOREM 5.** Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{F}^*(\alpha, \beta, \gamma)$ . Then

$$(2.19) \quad \left| D_z^\lambda f(z) \right| \cong \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)} |z| \right\}$$

and

$$(2.20) \quad \left| D_z^\lambda f(z) \right| \cong \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)} |z| \right\}$$

for  $0 \leq \lambda < 1$  and  $z \in \mathcal{U}$ . The results (2.19) and (2.20) are sharp.

PROOF. Define the function  $G(z)$  by

$$(2.21) \quad \begin{aligned} G(z) &= \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n \end{aligned}$$

for  $0 \leq \lambda < 1$ . We note that

$$(2.22) \quad 1 \leq \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} < n$$

for  $0 \leq \lambda < 1$  and  $n \geq 2$ . Thus it follows from (2.16) and (2.22) that

$$(2.23) \quad \begin{aligned} |G(z)| &\geq |z| - |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} n a_n \\ &\geq |z| - \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)} |z|^2, \end{aligned}$$

which implies (2.19), and that

$$(2.24) \quad \begin{aligned} |G(z)| &\leq |z| + |z|^2 \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n \\ &\leq |z| + |z|^2 \sum_{n=2}^{\infty} n a_n \\ &\leq |z| + \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)} |z|^2, \end{aligned}$$

which proves (2.20).

Finally, by taking the function  $f(z)$  defined by

$$(2.25) \quad D_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\gamma)(1+\gamma)} z \right\},$$

the results (2.19) and (2.20) are easily seen to be sharp.

COROLLARY 3. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{F}^*(\alpha, \beta, \gamma)$ . Then  $D_z^\lambda f(z)$  is included in a disk with its center at the origin and radius  $r_2$  given by

$$(2.26) \quad r_2 = \frac{1}{\Gamma(2-\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(1-\beta)}{(2-\alpha)(1+\gamma)} \right\}.$$

THEOREM 6. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{C}(\alpha, \beta, \gamma)$ . Then

$$(2.27) \quad \left| D_z^\lambda f(z) \right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)} |z| \right\}$$

and

$$(2.28) \quad \left| D_z^\lambda f(z) \right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)} |z| \right\}$$

for  $0 \leq \lambda < 1$  and  $z \in \mathcal{U}$ . The results (2.27) and (2.28) are sharp for the function  $f(z)$  given by

$$(2.29) \quad D_z^\lambda f(z) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)} \left\{ 1 - \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)} z \right\}.$$

COROLLARY 4. Let the function  $f(z)$  defined by (1.5) be in the class  $\mathcal{C}(\alpha, \beta, \gamma)$ . Then  $D_z^\lambda f(z)$  is included in a disk with its center at the origin

and radius  $r_3$  given by

$$(2.30) \quad r_3 = \frac{1}{\Gamma(2-\lambda)} \left\{ 1 + \frac{(1-\alpha)(1+\gamma) + 2\gamma(3-\alpha)(1-\beta)}{2(2-\alpha)(1+\gamma)} \right\}.$$

### 3. CONCLUDING REMARKS

From Definition 2 we immediately have

$$(3.1) \quad D_z^0 f(z) = f(z),$$

which, in view of Definition 3, yields

$$(3.2) \quad D_z^{n+0} f(z) = \frac{d^n}{dz^n} D_z^0 f(z) = f^{(n)}(z) \quad (n \in \mathcal{N}_0).$$

It follows from (3.1) and (3.2) that

$$(3.3) \quad \lim_{\lambda \rightarrow 0} D_z^{-\lambda} f(z) = f(z) \quad \text{and} \quad \lim_{\lambda \rightarrow 0} D_z^{1-\lambda} f(z) = f'(z).$$

Thus, by letting  $\lambda \rightarrow 0$  in our theorems, we shall obtain the corresponding results given in our earlier paper [14].

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