Homographic solutions of the quasihomogeneous N-body problem

by

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B.Sc., State University of Moldova, 2008

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ABSTRACT

We consider the N-body problem given by quasihomogeneous force functions of the form $\frac{C_1}{r^a} + \frac{C_2}{r^b}$ (C_1, C_2, a, b constants and $0 < a \leq b$) and address the fundamentals of homographic solutions. Generalizing techniques of the classical N-body problem, we prove necessary and sufficient conditions for a homographic solution to be either homothetic, or relative equilibrium. We further prove an analogue of the Lagrange-Pizzetti theorem based on our techniques. We also study the central configurations for quasihomogeneous force functions and settle the classification and properties of simultaneous and extraneous central configurations. In the last part of the thesis, we combine these findings with the Lagrange-Pizzetti theorem to show the link between homographic solutions and central configurations, to prove the existence of homographic solutions and to give algorithms for their construction.

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DEDICATION

To young researchers of Celestial Mechanics

The advancement in any science happens by passing the torch of discoveries from one generation to another, to shed light on newer and newer grounds.

Chapter 1

Introduction

1.1 Introduction

Historycally, the main motivation for the N-body problem was to understand the motion of celestial bodies like the Sun and the eight planets around it, attracting each other with the Newtonian force of gravitation. From the physical point of view, the problem could be stated as follows: *Given only the initial positions in space and the initial velocities of a group of celestial bodies, determine the position of each particle at every moment in time, or describe their motion.*

Besides celestial bodies, there are many other systems in which the particles interact with attracting forces of laws different from the inverse square of the distance. For example, the *van der Waals* group of forces in chemistry, which describe the interaction for various dipole molecules, are given by the general formula[1] $F = \frac{C_1}{r^s} + \frac{C_2}{r^t}$, where the exponents s, t are some positive integers and $C_1 \cdot C_2 \neq 0$. The *Lennard-Jones* force function[2] describes well the interaction of neutral particles in quantum chemistry and is given by $U = const\left(\frac{C_1}{r^{12}} - \frac{C_2}{r^6}\right)$.
In Celestial Mechanics itself, the *Manev* force function $U = \frac{C_1}{r} + \frac{C_2}{r^b}$, where b is a positive integer, was extensively used especially in the form: $F = \frac{C_1}{r^2} + \frac{C_2}{r^3}$ (an inverse-cubic perturbation to the Newtonian force itself). It can give a good explanation for the theory of the Moon and the perihelion advance of inner planets (see [2]). It also represents an approximation to the Einsteinian formula that describes the advance of perihelion of Mercury, an effect that cannot be predicted using the Newtonian force

[1] The formulas for forces and force functions in this introduction are given for the interaction of only 2 bodies.

[2] Given a force function U, the corresponding force F is $F = \frac{\partial U}{\partial r}$, as it will be seen in Section 2.1.

alone ([11]).

All these examples motivate the treatment of general force functions of the form: $U = \frac{C_1}{r^a} + \frac{C_2}{r^b},\ b \geq a > 0$, a topic introduced in 1993 by Diacu [5] as the *quasihomogeneous N-body problem.* Most of the work in this area has been focused on collisions and central configurations[3] (as can be seen in [6], [5], [10], [7], [2] and others). In this thesis, we focus on *homographic solutions*, in which the configuration formed by the N bodies remains similar to itself in time.
These types of solutions represent one of the main application of central configurations, for which the force that acts on each of the N bodies is proportional to the position vector of that body. These homographic solutions belong to the very few types of orbits for which there exists a complete theory for the Newtonian potential. Since they were only marginally treated in the context of quasihomogeneous potentials, our ambition is to write here the basic foundation of quasihomogeneous homographic solutions, trying to parallel the results obtained in the theory of the classical N-body problem.

1.2 Our Claims

For the quasihomogeneous N-body problem with $\{a \neq 2 \text{ and } b \neq 2\}$, and also for the quasihomogeneous 3-body problem with $\{a \neq 2 \text{ and } b > a\}$ or $\{b \neq 2 \text{ and } a < b\}$, we make the following claims:

Claim 1. A homographic solution is:

1. homothetic, if and only if it's total angular momentum is zero;
2. a relative equilibrium, if and only if it is planar and rotates with a constant, non-zero angular velocity.

Claim 2. A solution is homographic if and only if the configuration of the N bodies forms equivalent central configurations during the time of its existence.

Claim 3. Prove the existence of homographic solutions and present how to construct them.

In addition, for the quasihomogeneous N-body problem, with no restrictions on $a,\ b$:

[3] The definition for central configuration will be given in 4.1.

Claim 4. A central configuration r is a simultaneous central configuration if and only if at least one of the following holds:

1. $\forall c > 0,\ cr$ is a simultaneous central configuration;

2. $\exists c > 0,\ c \neq 1$, such that cr is a central configuration.

The Importance of Our Claims. The first three claims show that, with the exception of forces inversely proportional to the cube of the distance ($a = b = 2$), the results for quasihomogeneous forces closely imitate those of Newtonian one.

On the necessary and sufficient conditions Claim 1 provides, the whole theory of homographic solutions is based. According to Wintner [15], Lagrange considered it to be the main part of his theory for the homographic solutions of the 3-body problem.

Claim 2 proves the strong link between these solutions and central configurations. This allows us to use all the results concerning central configurations, obtained in the research of the Newtonian N-body problem, to the variety of quasihomogeneous potentials.

Claim 3, through its existence proof, provides the mathematical soundness to all the research concerning these homographic solution. It is also the most practical, since using it we can explicitly construct and analyze homographic solutions of any type, as well as decide whether or not an observed experimental solution is a particular homographic solution.

Finally, besides being an inevitable ingredient for obtaining Claim 3, Claim 4 has theoretical importance in itself due to the popular research involving central configurations, not necessarily related to homographic solutions (see [12]). In particular, its consequences complete Diacu's findings concerning central configurations in [6] and proves R. Jones conjectures [7].

1.3 Plan

Chapter 1 contains the motivation of the work and a statement of the claims which will be proved in this thesis, followed by an overview of the structure of the thesis itself.

Chapter 2 defines the concept of quasihomogeneous force function and derives the equations of the quasihomogeneous N-body problem. The ten first integrals and their important consequences are obtained: the Lagrange-Jacobi identity and the basic properties of rectilinear, collinear, planar and flat solutions.

Chapter 3 introduces the homographic solutions (general, homothetic and relative equilibrium) and the connection of those orbits with the spatial classification of solutions (flat/planar), ending with some necessary and sufficient conditions for a homographic solution to be either homothetic or a relative equilibrium (Claim 1).

Chapter 4 is where the two kinds of central configurations (simultaneous and extraneous) are described. Using their properties, Claim 4 is proved.

Next, we show that for a homographic solution, the configurations always form equivalent central configurations (Claim 2). Finally, an exact procedure of constructing any homographic solution is provided, and the existence of homographic solutions is proved (Claim 3).

Chapter 6 contains a description of the main results and claims of the thesis. Additionally, it hints at possible future development of the theory of homographic solutions in the context of quasihomogeneous potentials.

Chapter 2

The first integrals of the quasihomogeneous N-body problem

2.1 The quasihomogeneous N-body problem

Consider a system of N bodies, each of them mathematically represented by points, in which their mass is concentrated (such a model approximates a collection of bodies, whose distance from one another is much bigger than the maximum of their size). It is to be understood that the system of N bodies is not subject to any exterior influence.

For the i-th particle in the system, let m_i be its mass and r_i be the *position vector*, in some coordinate system $(x,\ y,\ z,\ O)$. As introduced by Diacu, [5], define the *quasihomogeneous force function* as $U:\ \{(r_1,\ r_2,\ ...,\ r_N) \in R^{3N} |\ r_i \neq r_j,\ \forall i, j = 1..N\} \to R^+$, given by:

$$\begin{cases} U = V + W, \\ V = \sum_{1 \le j < i \le N} \frac{m_j m_i}{|r_j - r_i|^a}, \\ W = \sum_{1 \le j < i \le N} \frac{k m_j m_i}{|r_j - r_i|^b}, \\ k > 0,\ 0 < a \le b. \end{cases} \tag{2.1.1}$$

(As we see, the name "quasihomogeneous" is justified by the homogeneous nature of the force functions V and W in U.) For $a = b = 1$, the force function above becomes the classical/Newtonian force function: $G \sum_{1 \le j < i \le N} m_j m_i \frac{1}{|r_j - r_i|}$.

The *force* that acts on the i-th body is defined as the vector function $F_i = \frac{\partial U}{\partial r_i} =$

U_{r_i}. For example, the component of the force due to V can be computed as follows[1]: $\frac{\partial V}{\partial r_i} = \frac{\partial}{\partial r_i}\left(\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}\frac{m_j m_i}{|r_j-r_i|^a}\right) = \sum_{j=1,j\neq i}^{j=N}\frac{\partial}{\partial(r_j-r_i)}\left(\frac{m_j m_i}{[(r_j-r_i)^2]^{\frac{a}{2}}}\right)\frac{\partial(r_j-r_i)}{\partial r_i}$; now the differentiations can be continued as: $\frac{\partial}{\partial(r_j-r_i)^2}\left(\frac{1}{[(r_j-r_i)^2]^{\frac{a}{2}}}\right)\frac{\partial(r_j-r_i)^2}{\partial(r_j-r_i)}\frac{\partial(r_j-r_i)}{\partial r_i}$, where $\frac{\partial(r_j-r_i)^2}{\partial(r_j-r_i)} = (i\frac{\partial}{\partial(x_j-x_i)} + j\frac{\partial}{\partial(y_j-y_i)} + k\frac{\partial}{\partial(z_j-z_i)})[(x_j-x_i)^2+(y_j-y_i)^2+(z_j-z_i)^2]$. Finally, we obtain $\frac{\partial V}{\partial r_i} = \sum_{j=1,\,j\neq i}^{N} m_j m_i(r_j-r_i)\frac{a}{|r_i-r_j|^{a+2}}$; after similar computations for W, we get the expression for the force:

$$U_{r_i} = \sum_{j=1,\,j\neq i}^{N} m_j m_i(r_j-r_i)\left[\frac{a}{|r_i-r_j|^{a+2}} + \frac{k\cdot b}{|r_i-r_j|^{b+2}}\right]. \qquad (2.1.2)$$

Let us first notice that, since $a,\ b,\ k$ are positive constants, the force is oriented *from* the body on which it acts; i.e., it is an attractive force. For $a = b = 1$, the force becomes $F_i = const\sum_{j=1,\,j\neq i}^{N}\frac{m_j m_i(r_j-r_i)}{|r_i-r_j|^3}$. This is the Newtonian force of gravitational attraction, discovered by Issac Newton in 17th century, observing the motion of planets in our solar system, and published in his *Principia* (see [3]).

The general quasihomogeneous N-body problem is an *initial value problem* for ordinary differential equations, which is obtained by using Newton's second law of motion $m_i r_i'' = U_{r_i}$ (throughout the paper, " $'$ " denotes differentiation with respect to time):

Problem. *Given initial values for the positions $r_i(0)$ and velocities $r_i'(0)$ of N particles $(i = 1, ..., N)$, with $r_i(0) \neq r_j(0)$ for all mutually distinct i and j, find or describe the solution of the second order system:*

$$m_i r_i'' = \sum_{j=1,\,j\neq i}^{N} m_j m_i\left[\frac{a}{|r_i-r_j|^{a+2}} + \frac{k\cdot b}{|r_i-r_j|^{b+2}}\right](r_j-r_i), \qquad (i = 1, ..., N). \quad (2.1.3)$$

(These equations are also called the *equations of motion* for the system of bodies.) Based on standard results of the theory of ordinary differential equations, there always exists a unique solution to this initial value problem, on a maximum interval $[0,\,t^*)$. If $t^* = \infty$ then we say that the solution exists for all time, otherwise t^* is called a *singularity* of the solution.

[1] The coefficient $\frac{1}{2}$ is set because of the double appearance of the term $m_j m_i\frac{1}{|r_j-r_i|^a}$.

2.2 The ten first integrals

First integrals, or conservation integrals, are functions that remain constant along any given solution of the system, the constant depending on the initial condition of the solution. In other words, integrals provide relations between the position and velocity vectors, so that each of the integrals, if it is algebraic with respect to the components of r_i and r_i', allows the reduction of the system's dimension by one dimension. The classical N-body problem has 10 independent algebraic first integrals, which we obtain below for quasihomogeneous problem as well. In 1887 H. Bruns [1] proved, for the classical force, that there are no other algebraic first integrals. The reader can consult the book by Pollard [9] for an alternative description of the classical integrals.

2.2.1 Conservation of momentum

With $p_i = m_i r_i'$ being called the *momentum* of the particle i, $p = \sum_{i=1}^{N} p_i$ will be the total momentum of the system. Let us add upon i the equations in (2.1.3), to get:

$$\begin{aligned} p' &= \sum_{i=1}^{i=N} p_i' = \sum_{i=1}^{i=N} \sum_{j=1,\, j\neq i}^{N} m_j m_i [\frac{a}{|r_i - r_j|^{a+2}} + \frac{k \cdot b}{|r_i - r_j|^{b+2}}](r_j - r_i) = 0, \\ \sum_{i=1}^{N} m_i r_i' &= const = p_o, \end{aligned}$$

because every time the term with $(r_j - r_i)$ occurs, the term with $(r_i - r_j)$ will also occur and they will cancel each other. As $p = const = p_o$, in physics, this represents the law of *conservation of the momentum* and it gives us the first three integrals. Integrating this equation, we get:

$$\sum_{i=1}^{i=N} m_i r_i = p_0 t + v_0, \tag{2.2.1}$$

where the constant vector $v_0 = \sum_{i=1}^{i=N} m_i r_{i0}$ provides the next three first integrals. But the magnitude $\frac{\sum_{i=1}^{i=N} m_i r_i}{M}$, where $M = \sum_{i=1}^{N} m_i$, is known as the position vector for the center of mass (see [9], page 39); therefore, equation (2.2.1) implies that the center of mass moves in a straight line, with a constant velocity vector.

In order to understand easier all other important features of the motion in the system, we will study the positions of the N bodies with respect to the center of mass;

that is, we set the origin of the coordinate system at the center of mass:

$$\sum_{i=1}^{N} m_i r_i = 0. \tag{2.2.2}$$

This equation is often called the *integral of the center of mass.*

2.2.2 Conservation of energy

Writing the equations of motions as $m_i r_i'' = \frac{\partial U}{\partial r_i}$, if we dot-multiply them by $\frac{dr_i}{dt}$ and sum over all i, to get:

$$\frac{dU}{dt} = \sum_i m_i r_i'' r_i' = \sum_i m_i \frac{d}{dt}\left[\frac{1}{2}(r_i')^2\right].$$

But $r_i' = v_i$ is the velocity vector of the particle i, while $\frac{1}{2}\sum_i m_i v_i^2 = T$ is called the *kinetic energy* of the system, so we get

$$\begin{aligned} U' &= T', \\ T - U &= h. \end{aligned} \tag{2.2.3}$$

The magnitude $-U$ is called the *potential energy;* therefore, the last equation is the law of *conservation of energy.* The scalar h is the energy constant.

2.2.3 Conservation of the angular momentum

The *angular momentum* of a particle i is defined to be the cross product of its position vector and momentum: $r_i \times p_i = m_i r_i \times r_i'$. Let us cross multiply each side of system (2.1.3) by r_i and sum over i. Then

$$\sum_i m_i r_i \times r_i'' = \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} m_j m_i [\frac{a}{|r_i - r_j|^{a+2}} + \frac{k \cdot b}{|r_i - r_j|^{b+2}}](r_i \times r_j),$$

because $r_i \times r_i = 0$. At the same time, in the above double sum, the terms $r_i \times r_j$ cancel the terms $r_j \times r_i$, so that the right side is equal to zero. Integration yields:

$$C = \sum_i m_i r_i \times r_i', \tag{2.2.4}$$

$$C = \sum_i m_i \begin{pmatrix} y_i z_i' - z_i y_i' \\ z_i x_i' - x_i z_i' \\ x_i y_i' - y_i x_i' \end{pmatrix}.$$

The constant vector C is the total angular momentum of the system and it forms the last three first integrals.

2.3 Consequences of the first integrals

2.3.1 Consequences of the energy integral: Lagrange-Jacobi identity

Lagrange-Jacobi identity for quasihomogeneous potentials was also obtained in [10]. This relation depends on the explicit form of the quasihomogeneous force function, therefore it is not identical to the classical results.

We define the *moment of inertia* I of the system as:

$$I = \sum_{i=1}^{N} m_i r_i^2. \tag{2.3.1}$$

Differentiating it twice with respect to time, we get:

$$I'' = 2\sum_i m_i v_i \cdot v_i + 2\sum_i r_i m_i r_i'', \tag{2.3.2}$$

which, using equation (2.1.3), is equivalent to $I'' = 4T + 2\sum_i \sum_{j=1,\, j\neq i}^{N} m_j m_i [\frac{a}{|r_i - r_j|^{a+2}} + \frac{k \cdot b}{|r_i - r_j|^{b+2}}](r_j r_i - r_i^2)$. We have: $r_j r_i - r_i^2 = \frac{1}{2}[r_j^2 - r_i^2 - (r_j - r_i)^2]$ and

$\sum_i \sum_{j=1,\, j\neq i}^{N} m_j m_i \frac{r_j^2}{|r_i - r_j|^{a+2}} = \sum_j \sum_{i=1,\, i\neq j}^{N} m_j m_i \frac{r_j^2}{|r_i - r_j|^{a+2}} = \sum_i \sum_{j=1,\, j\neq i}^{N} m_j m_i \frac{r_i^2}{|r_i - r_j|^{a+2}}$,
therefore $I'' - 4T = -\sum_i \sum_{j=1,\, j\neq i}^{N} m_j m_i [\frac{a}{|r_i - r_j|^{a+2}} + \frac{k \cdot b}{|r_i - r_j|^{b+2}}]|r_j - r_i|^2$. With (2.1.1) and (2.2.3), we get the Lagrange-Jacobi identity:

$$I'' = 4h + 2(2-a)V + 2(2-b)W\,. \tag{2.3.3}$$

Additionally, notice that (2.3.2) is equivalent to $I'' = 4T + 2\sum_i r_i \frac{\partial U}{\partial r_i}$; comparing this to the above identity, after the use of energy integral, we find:

$$\sum_i r_i \frac{\partial U}{\partial r_i} = -aV - bW. \tag{2.3.4}$$

This last relation (important for a later section) can also be obtained by noticing that V, W are homogeneous functions of degree $-a$ and $-b$, respectively. In the configuration space $r = (r_1, r_2, ..., r_N)$, $\sum_i r_i \frac{\partial V}{\partial r_i} = r\frac{\partial V}{\partial r}$, therefore, according to Euler's theorem for homogeneous functions, $r\frac{\partial V}{\partial r} = -aV$. With similar reasoning for W, formula (2.3.4) follows.

2.3.2 Consequences of the integrals of the center of mass and of the total angular momentum integrals

The next definitions and consequences, unlike those above, do not depend on the form of the force function; therefore their proof is identical to those for the classical force function (and a version of them can be found, for example, in [15]). However, for the convenience of the reader, we will present them here, with more details. These results will be used in the subsequent sections of the thesis.

The central concepts of the paper, like homographic solutions, depend on some rotation matrices. We want to introduce a special matrix-function, $F(t)$, that will allow us to reduce by one the order of the derivative of the rotation matrix Ω and, as we will further see, to ultimately reduce the discussion to some algebraic equations.

Lemma 1. *Any real-valued rotation 3-matrix $\Omega(t)$, of class C^2, determines a unique skew-symmetric 3-matrix $F(t)$, given by*

$$F(t) = \Omega^{-1}\Omega'. \tag{2.3.5}$$

Conversely, given a particular constant rotation 3-matrix $\Omega(0)$, any continuous skew-symmetric 3-matrix $F(t)$ determines a unique rotation 3-matrix $\Omega(t)$, which satisfies the same formula above.

Proof. Ω is an orthogonal matrix, so $\Omega^T = \Omega^{-1} \implies \Omega^T\Omega = E \implies (\Omega^T\Omega)' = 0$. Using the fact that for two matrix-functions $A(t)$ and $B(t)$ for which the product can be defined, we can apply the product rule $(AB)' = A'B + AB'$, we obtains:

$$\Omega^{-1}\Omega' = -(\Omega^{-1}\Omega')^T.$$

This implies that the rotational matrix Ω defines a unique skew-symmetric matrix-function $F(t)$:

$$F(t) = \Omega^{-1}\Omega' = \begin{pmatrix} 0 & -f_3 & f_2 \\ f_3 & 0 & -f_1 \\ -f_2 & f_1 & 0 \end{pmatrix}. \tag{2.3.6}$$

On other hand, given a particular skew-symmetric matrix $F(t)$ and a constant matrix $\Omega(0)$, the equation $F(t) = \Omega^{-1}\Omega' \iff \Omega'(t) = \Omega(t)F(t)$ is a homogeneous, linear differential equation for which the I.V.P. always has a unique solution $\Omega = \Omega(t)$, given by:

$$\Omega(t) = \Omega(0) \cdot \exp\left(\int_0^t F(\tau)d\tau\right). \tag{2.3.7}$$

The integral of a skew-symmetric matrix is always a skew-symmetric matrix. Using the familiar properties of the exponential of a matrix M: $\exp(M^T) = (\exp M)^T$ and $\exp(-M) = (\exp M)^{-1}$, for M skew-symmetric, we get that $(\exp M)^{-1} = (\exp M)^T$ and thus Ω is orthogonal. Also, because $\det(\exp M) = \exp(trace(M))$, $\det(\Omega) = +1$. Thus, $\Omega(t)$ in (2.3.7) is indeed a rotation matrix. □

We can find the derivative of the auxiliary matrix $F(t)$ to be $F' = \Omega^{-1}\Omega'' + (\Omega F)^T\Omega'$and so:

$$F' = \Omega^{-1}\Omega'' - F^2, \ where \tag{2.3.8}$$

$$F^2 = \begin{pmatrix} -f_2^2 - f_3^2 & f_1 f_2 & f_1 f_3 \\ f_2 f_1 & -f_3^2 - f_1^2 & f_2 f_3 \\ f_3 f_1 & f_3 f_2 & -f_1^2 - f_2^2 \end{pmatrix}. \tag{2.3.9}$$

If the rotation is around the z-axis, in the positive sense, then it should be given by

$$\Omega(t) = \begin{pmatrix} \cos w & -\sin w & 0 \\ \sin w & \cos w & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.3.10}$$

where $w = w(t)$. But, if we perform the simple computations in (2.3.5), this determines:

$$F(t) = \begin{pmatrix} 0 & -f_3 & 0 \\ f_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; f_3(t) = w'(t), \tag{2.3.11}$$

while Lemma 1 indicates that the last form of $F(t)$ is not only necessary, but also sufficient for $\Omega(t)$ to have the form of a rotation around the z-axis (2.3.10).

Based on the existence of a constant total angular momentum vector, naturally arrises the notion of a plane perpendicular to that vector. In the next definitions, by *solution* we consider the solution of the quasihomogeneous N-body problem $r(t) = (r_1(t), r_2(t), ..., r_N(t))$.

Definition 1. The *invariable plane* for a solution with non-zero total angular momentum, $C \neq 0$, is defined as the plane through the center of mass and perpendicular to the vector $C = (C_x, C_y, C_z)$:

$$C_x x + C_y y + C_z z = 0. \tag{2.3.12}$$

The invariable plane is not only constant, but also invariable in every inertial barycentric coordinate system.

Many important N-body systems have all the motion in a single plane; for example, such is (in approximation) the solar system of planets.

Definition 2. A solution $r(t)$ of the $N-$body problem is called *planar* if there exists a fixed plane π_0 such that $r_i(t) \in \pi_0, \forall i, \; \forall t \in I$, where I is the interval of existence for the solution.

But if the plane π is not necessarily fixed, then the solution $r(t)$ is called *flat*: $\forall t \in I, \; \exists \pi = \pi(t)$ such that $r_i(t) \in \pi(t)$.

Depending on the initial conditions, there exist not only planar or flat solutions, but also flat and non-planar solutions; for instance, if $N = 3$, the configuration is always flat, but the initial velocities can be easily chosen such that the plane containing them will change its position in space.

Lemma 2. *If the solution is planar (with π_0) and has a total angular momentum ($C \neq 0$), then the fixed plane π_0 coincides with the invariable plane.*

Proof. For a planar solution, π_0 will contain the center of mass; choose the coordinate system such that $\pi_0 = (x, y)$. Because $z_i(t) = 0, \; \forall t$, from (2.2.4) it follows that $C \perp (x, y)$. According to Definition 1, we finished the proof. □

Next we introduce some relations useful for any flat solutions.

According to Definition 2, at any moment of time, there is a plane that contains all N bodies. Let the position in those planes be described by the $\bar{r}_i = (\xi_i,\, \eta_i,\, \zeta_i)^T$ of the system (ξ, η, ζ), centered at the same origin O as the system $(x,\, y,\, z)$. The flat solution can be thought of as a motion inside the plane plus some rotation (the same for all bodies) around the origin due to the change in the position of the plane that contains them. We can write:

$$r_i = \Omega \bar{r}_i = \Omega \begin{pmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{pmatrix}, \; \zeta_i = 0, \; \forall i = 1, .., N. \tag{2.3.13}$$

Denote:

$$I^{\xi\xi} = \sum_i m_i \xi_i^2, \; I^{\eta\eta} = \sum_i m_i \eta_i^2, \; I^{\xi\eta} = \sum_i m_i \xi_i \eta_i. \tag{2.3.14}$$

For any $m{+}m$ scalars $a_i,\, b_i$ one has: $\begin{vmatrix} \sum_{i=1}^m a_i^2 & \sum_{i=1}^m a_i b_i \\ \sum_{i=1}^m a_i b_i & \sum_{i=1}^m b_i^2 \end{vmatrix} = \frac{1}{2} \sum_{i=1}^m \sum_{k=1}^m \begin{vmatrix} a_i & b_i \\ a_k & b_k \end{vmatrix}^2$ and the factor $\frac{1}{2}$ in front means that we exclude repetitions in the sum. Also, since the rotation does not change the distance from the origin, $r_i^2 = \xi_i^2 + \eta_i^2$, and with (2.3.1) we have:

$$\begin{vmatrix} I^{\xi\xi} & I^{\xi\eta} \\ I^{\xi\eta} & I^{\eta\eta} \end{vmatrix} = \sum_{1 \le i < k \le N}^{m} m_i m_k \begin{vmatrix} \xi_i & \xi_k \\ \eta_i & \eta_k \end{vmatrix}^2, \; I^{\xi\xi} + I^{\eta\eta} = I. \tag{2.3.15}$$

Because the angular momentum vector is constant and invariable with respect to the coordinate system, we choose the orientation of the z-axis of the system $(x,\, y,\, z)$ to be in the direction of C:

$$C_x = 0, \; C_y = 0, \; C_z = |C|, \tag{2.3.16}$$

so that $\Omega^{-1} C = \Omega^{-1}(0,\, 0,\, |C|)^T$. (For the case of zero angular momentum, the above relation holds trivially). In preparation for the proof of the next lemma, we now need to show that, with $f_1,\, f_2,\, f_3$ from the definition of the auxiliary matrix (2.3.6), the following holds:

$$\Omega^{-1}\begin{pmatrix} 0 \\ 0 \\ |C| \end{pmatrix} = \begin{pmatrix} f_1 I^{\eta\eta} - f_2 I^{\xi\eta} \\ f_2 I^{\xi\xi} - f_1 I^{\xi\eta} \\ \sum_i m_i(\xi_i\eta_i' - \eta_i\xi_i') + f_3(I^{\xi\xi} + I^{\eta\eta}) \end{pmatrix}. \tag{2.3.17}$$

First, $\bar{r_i} \times \bar{r_i}' = [0,\, 0,\, (\xi_i\eta_i' - \eta_i\xi_i')]$ and $\bar{r_i} \times [(f_1,\, f_2,\, f_3) \times \bar{r_i}] = [f_1\eta_i^2 - f_2\xi_i\eta_i,\; f_2\xi_i^2 - f_1\xi_i\eta_i,\; f_3(\xi_i^2 + \eta_i^2)]$. Therefore, we can show that the right-hand side of (2.3.17) is the vector: $\sum_i m_i\bar{r_i} \times \bar{r_i}' + \sum_i m_i\bar{r_i} \times [(f_1,\, f_2,\, f_3) \times \bar{r_i}]$. So, because from (2.2.4), $\Omega^{-1}C = \Omega^{-1}\sum_i m_i(r_i \times r_i')$, in order to prove (2.3.17), we need to show that $\Omega^{-1}(r_i \times r_i') = \bar{r_i} \times \bar{r_i}' + \bar{r_i} \times [(f_1,\, f_2,\, f_3) \times \bar{r_i}]$. But, from (2.3.13), after taking the derivative with respect to time and using $F = \Omega^{-1}\Omega'$, we find that $\Omega^{-1}r_i' = F\bar{r_i} + \bar{r_i}' = (f_1,\, f_2,\, f_3) \times \bar{r_i} + \bar{r_i}'$. Since $\Omega^{-1}(r_i \times r_i') = \Omega^{-1}r_i \times \Omega^{-1}r_i' = \bar{r_i} \times \Omega^{-1}r_i'$, formula (2.3.17) becomes true.

Lemma 3. *If the solution is flat and does not have an invariable plane ($C = 0$), then it is planar.*

Proof. (2.3.17) implies that $\begin{cases} f_1 I^{\eta\eta} - f_2 I^{\xi\eta} = 0 \\ f_1 I^{\xi\eta} - f_2 I^{\xi\xi} = 0 \end{cases}$ and, since from (2.3.1) $I > 0$ (it does not make physical sense to have more than one body at the origin), not both of $I^{\xi\xi}$, $I^{\eta\eta}$ can be zero. One can divide one of the equation in the system by that I^{ii} which is not zero and substitute one of $f_1,\ f_2$ in the second equation, to find that either $f_1 = f_2 = 0$, or $(I^{\xi\eta})^2 - I^{\xi\xi}I^{\eta\eta} = 0$.

In the first case, the situation corresponds to (2.3.10). From (2.3.13) it follows that the third coordinate of vectors r_i remains the same as of vectors $\bar{r_i}$, during the rotation around z-axis. Thus $z_i = 0$ for all times and this means that the motion takes place in the (x, y) plane; so the solution is planar.

In the second case, formula (2.3.15) implies that for all distinct $i,\ k = 1, ..., N$, $\xi_i\eta_k - \xi_k\eta_i = 0$ for all times. But this means that $(\xi_i,\, \eta_i,\, 0) \times (\xi_k,\, \eta_k,\, 0) = 0$ and therefore all N vectors $\bar{r_i}$ are collinear with the origin. Hence, the coordinate system $(\xi,\, \eta,\, \zeta)$ can be chosen such that all N bodies are on the ξ axis for all time. Then $\eta_i = 0$, $I^{\eta\eta} = 0$, $I^{\xi\eta} = 0$, $\xi_i\eta_i' - \eta_i\xi_i' = 0$ and $I^{\xi\xi} = I$ and (2.3.17), with $C = 0$, reduces to $s_2 I = 0 = s_3 I$. Because $I > 0$, it follows that $s_2 = s_3 = 0$. Similar computations as in relations (2.3.10), (2.3.11) show that this corresponds to a rotation around the x-axis; thus, the motion takes place in a plane parallel to the $(z,\, y)$ plane for all times, so the solution is again planar. □

Lemma 4. *If the solution has an invariable plane ($C \neq 0$) and, at a moment t^*, the N masses lie on a line, then at that moment, the N masses lie in the invariable plane.*

Proof. The hypothesis of the lemma implies that for all $i, \ k = 1, ..., N$ the body i is collinear with body k and with the origin; i.e., $r_i \times r_k = 0$, and therefore $(r_i \times r_k) \cdot r_i' = 0 \implies (r_i \times r_i') \cdot r_k = 0$. Summing over all i gives that $C \cdot r_k = 0, \ \forall k = 1..N$; Definition 1 shows that all particles must lie in the invariable plane. □

Simpler than planar or flat solutions are the solutions in which all bodies lie on a line.

Definition 3. A solution $r(t)$ of the $N-$body problem is called *rectilinear* if there exists a fixed line Λ_0 such that $r_i(t) \in \Lambda_0, \ \forall t \in I$, where I is the interval of existence of the solution.

But if the line Λ is not necessarily fixed, then the solution $r(t)$ is called *collinear*: $\forall t \in I, \ \exists \Lambda = \Lambda(t)$ such that $r_i(t) \in \Lambda(t)$.

Lemma 5. *Every collinear solution is planar.*

Proof. If $C \neq 0$ then Lemma 4, for which a collinear solution is just a particular case, implies that the solution is planar. If, on the other hand, $C = 0$, then we merely apply Lemma 3, because a collinear solution is necessarily flat. □

The next Lemma provides an early example of homographic solutions (which will be described in detail in Chapter 3):

Lemma 6. *If a collinear solution is not rectilinear, then the geometrical configuration formed by the N masses remains similar to itself when t varies (i.e., the solution is homographic). The size of the configuration is independent of time if and only if so is the angular velocity of the rotating line $\Lambda(t)$, which contains the collinear configuration.*

Proof. By Lemma 5 above, the solution is planar; choose the plane of motion to be the $(x, \, y)$ plane of the coordinate system. In this plane, consider another coordinate system (possibly not inertial) $(\xi, \, \eta)$ such that the line $\Lambda(t)$, which contains all bodies according to Definition 3, coincides with the ξ-axis; so, $\eta_i = 0, \ \forall i$. The ξ-axis, together with all bodies, will be rotating with a certain angular velocity $w'(t)$ around the z-axis, so that (2.3.13) holds, with $\Omega(t)$ given by (2.3.10). We want to prove that $\xi_i(t) = s(t)\xi_i(0)$, for some real function $s = s(t)$ that does not depend on i.

Differentiation of (2.3.13) gives that $r_i'' = \Omega''\bar{r}_i + 2\Omega'\bar{r}_i' + \Omega\bar{r}_i''$; using (2.3.6) and (2.3.8), we easily find that $\Omega^{-1}r_i'' = \bar{r}_i'' + 2F\bar{r}_i' + (F' + F^2)\bar{r}_i$. Substitution of (2.3.11), and (2.3.9), with $\zeta_i = 0$, give that:

$$\Omega^{-1}r_i'' = \begin{pmatrix} \xi_i'' - 2w'\eta_i' - w'^2\xi_i - w''\eta_i \\ \eta_i'' + 2w'\xi_i' - w'^2\eta_i + w''\xi_i \\ 0 \end{pmatrix}.$$

Now, because all bodies are on the ξ-axis for all time, the η-component of the forces that act on body i should be zero; therefore, the absolute value of the projection of the acceleration on the η-axis should be zero. The second line in the above formula implies $2w'\xi_i' + w''\xi_i = 0$, $\forall i$. Because the solution is not rectilinear, $w' \neq 0$; also $\xi_i \neq 0$ at least for $N-1$ values of i (we can have at most 1 body in the origin, at moment t). Dividing the previous relation by $w'\xi_i$ and integrating from 0 to t, we get: $2\ln|\xi_i|\,|_0^t = -\ln|w'(t)|\,|_0^t$, $\left(\frac{\xi_i(t)}{\xi_i(0)}\right)^2 = \left|\frac{w'(0)}{w'(t)}\right|$, therefore we can write $\xi_i(t) = s(t)\cdot\xi_i(0)$, where $s(t) \neq 0$ only depends on $w(t)$, at least for $N-1$ bodies. Using the center of mass equation (2.2.2), it follows that for an Nth body situated in the center of mass at a particular moment t^*, $\xi_N(t^*) = \frac{s(t^*)\sum_{i=1}^{N-1}\xi_i(0)}{m_N} = 0$. This shows that either $\sum_{i=1}^{N-1}\xi_i(0)$ and, thus, $\xi_N(0) = 0$, or $s(t^*) = 0$; therefore, in either case formula $\xi_i(t) = s(t)\cdot\xi_i(0)$ is satisfied even for the troublesome Nth body and thus the first part of the lemma is proved.

To prove the second part, substitute $s(t)\cdot\xi_i(0)$ for $\xi_i(t)$, and $\eta_i = 0$ into (2.3.13), using (2.3.10), to get $r_i(t) = [s(t)\cos w\xi_{i0},\ s(t)\sin w\xi_{i0},\ 0]$; then substitute this into the definition (2.2.4) to get: $C_x = C_y = 0$ and $C_z = w's(t)^2\sum_i m_i\xi_{i0}^2$. This shows that the plane of motion is the invariable plane and that

$$|C| = |w'|s(t)^2\sum_i \xi_{i0}^2 > 0. \tag{2.3.18}$$

Because $C = const$ and $s \neq 0$, this shows that $s(t) = const \iff w'(t) = const_2$; together with $r_i(t) = s\Omega(t)\bar{r}_{i0}$, this proves the second part of the lemma as well. □

Finally, we can give a necessary and sufficient criterion for a collinear solution to be rectilinear:

Lemma 7. *A collinear solution does not have an invariable plane if and only if it is rectilinear.*

Proof. If the solution is rectilinear, from Definition 3 it follows there is a line Λ_0 that contains all $r_i(t)$ together with the origin, for all time. This also implies that their velocities $r_i'(t)$ will also be on the same line, so that $r_i \times r_i' = 0$ and $C = \sum_i m_i r_i \times r_i' = 0$. The converse is thus proved; for the direct statement, notice that if the collinear solution is not rectilinear then formula (2.3.18) in the above proof shows that $C \neq 0$.

□

Chapter summary. We have introduced the notion of quasihomogeneous force functions and the equations that describe the evolution of a system of N bodies: the quasihomogeneous N-body problem. We also obtained the ten first integrals and, based on them, we showed some important consequences, such as the Lagrange-Jacobi identity (used in Section 4.1) and the basic properties of rectilinear, collinear, planar and flat solutions (used in Section 4.2).

The next chapter introduces homographic solutions; as we shall see, one of the main ingredients for all proofs regarding properties of those solutions will be Lemma 1 and its relations proved above.

Chapter 3

Homographic solutions

3.1 Introduction to homographic solutions

Homographic solutions are among the very few types of known explicit solutions of the general N-body problem. In their treatment, we were inspired by the corresponding theory for Newtonian case, as presented by A. Wintner [15].

Definition 4. A solution $r(t)$ is called *homographic* if the configuration formed by $r_i(t),\ i = 1, ..., N$, remains similar to itself during the time interval I of existence of that solution.

That is, there exist a scalar function $s(t) : I \to R^+$ $(s(t) > 0,\ \forall t)$ and a matrix function (rotational matrix[1]) $\Omega(t) : I \to R^9$, such that:

$$\begin{aligned} r_i(t) &= s(t) \cdot \Omega(t) \cdot r_i(t_0), \quad i = 1, ..., N \\ s(t_0) &= 1,\ \Omega(t_0) = E, \end{aligned} \tag{3.1.1}$$

where E denotes the identity matrix.

The translation is also a similarity transformation, but we exclude it since the center of gravity is fixed at the origin. Thus, in a homographic solution, the configuration can only rotate and/or dilate. Considering the two possibilities separately, we get two independent types of homographic solutions:

Definition 5. A homographic solution $r(t)$ is called *homothetic* if $\forall t \in I,\ \Omega(t) = E\ \&\ s(t) \neq const$, i.e.:

$$r_i(t) = s(t) r_i(t_0),\ i = 1, ..., N. \tag{3.1.2}$$

[1] $\Omega(t) \in SO(3)$, that is, it is special $(\det \Omega = +1)$ and orthogonal $(\Omega^T \Omega = E)$.

Definition 6. A homographic solution $r(t)$ is called a *relative equilibrium* if $\forall t \in I$, $s(t) = const$ & $\Omega(t) \neq const$, i.e.:

$$r_i(t) = \Omega(t) r_i(t_0),\ i = 1, ..., N. \tag{3.1.3}$$

Thus, in a homothetic solution the configuration is dilating without rotation, while in a relative equilibrium, the configuration is rotating without dilating.

There are deep facts connecting the homographic character of the solutions and their flatness: *If the homographic solution is not flat, then it is homothetic; if the homographic solution is flat, then it is planar.* For the Newtonian force function, the former is due to Pizzetti, and the latter to Lagrange.

Using them, one can show that, for the Newtonian case, *a homographic solution is: homothetic, if and only if it's total angular momentum is zero; a relative equilibrium, if and only if it is planar and rotates with a constant, non-zero angular velocity* (Lagrange-Pizzetti).

The extension of these facts for the quasihomogeneous case will be done in the next sections, while here we will prepare some of the needed ingredients.

In a homographic solution, the rotation does not change the mutual distances between bodies, therefore, with the substitution of position vectors in the homographic form (3.1.1), the quasihomogeneous force function

$$U = V + W = \sum_{1 \leq j < i \leq N} m_j m_k \left(\frac{1}{|r_j - r_i|^a} + \frac{k}{|r_j - r_i|^b} \right),\ k > 0,\ 0 < a \leq b,$$

takes the form

$$U = \frac{V_0}{s^a(t)} + \frac{W_0}{s^b(t)}, \tag{3.1.4}$$

where $V_0 = V(r_0)$ and $W_0 = W(r_0)$. Also, the force acting on the i-th body, $\frac{\partial U(t)}{\partial r_i} = U_{r_i} = \sum_{j \neq i} m_j m_i (r_j - r_i) [\frac{a}{|r_i - r_j|^{a+2}} + \frac{k \cdot b}{|r_i - r_j|^{b+2}}]$, becomes:

$$m_i r_i'' = \quad \Omega m_i \sum_{j \neq i} m_j (r_{j0} - r_{i0}) [\frac{a}{s^{a+1} |r_{i0} - r_{j0}|^{a+2}} + \frac{k \cdot b}{s^{b+1} |r_{i0} - r_{j0}|^{b+2}}]. \tag{3.1.5}$$

At the initial moment, the accelerations due to $V(t)$ and $W(t)$ are :

$$\frac{V_{r_i 0}}{m_i} = \sum_{j \neq i} m_j (r_{j0} - r_{i0}) \frac{a}{|r_{i0} - r_{j0}|^{a+2}},\ \frac{W_{r_i 0}}{m_i} = \sum m_j (r_{j0} - r_{i0}) \frac{k \cdot b}{|r_{i0} - r_{j0}|^{b+2}}. \tag{3.1.6}$$

Notice that these two vectors need not be parallel.

Using the new notations, one can write the equations of motions as:

$$s^{b+1}\Omega^{-1}r_i'' = (s^{b-a}\frac{V_{r_i0}}{m_i} + \frac{W_{r_i0}}{m_i}). \tag{3.1.7}$$

From Definition 4, for the position vectors we can write: $r_i = \Omega s r_{i0}$ and so $r_i' = \Omega' s r_{i0} + \Omega s' r_{i0}$, $r_i'' = \Omega s'' E r_{i0} + 2\Omega' s' r_{i0} + \Omega'' s r_{i0}$. Multiplying with Ω^{-1} the last equation and using the relations: $F(t) = \Omega^{-1}\Omega'$, $F' = \Omega^{-1}\Omega'' - F^2$ obtained in Section 2.3.2, we obtain

$$\Omega^{-1}r_i' = (s'E + sF)r_{i0} \tag{3.1.8}$$

and then $\Omega^{-1}r_i'' = [s''E + 2s'F + s(F' + F^2)]r_{i0}$.

By substituting the last expression into the left-hand side of (3.1.7) we get the *homographic form of equations of motion* as:

$$K(t) \cdot r_{i0} = s^{b-a}\frac{V_{r_i0}}{m_i} + \frac{W_{r_i0}}{m_i}, \tag{3.1.9}$$

where the 3-matrix $K(t)$ is defined by $K(t) = s^{b+1}[s''E + 2s'F + s(F' + F^2)]$ and thus has the form:

$$K(t)=s^{b+1}\begin{pmatrix} s''+s(-f_2^2-f_3^2) & 2s'(-f_3)+s(-f_3'+f_1f_2) & 2s'f_2+s(f_2'+f_1f_3) \\ 2s'f_3+s(f_3'+f_1f_2) & s''+s(-f_1^2-f_3^2) & 2s'(-f_1)+s(-f_1'+f_3f_2) \\ 2s'(-f_2)+s(-f_2'+f_1f_3) & 2s'f_1+s(f_1'+f_3f_2) & s''+s(-f_2^2-f_1^2) \end{pmatrix}. \tag{3.1.10}$$

For both theorems to follow, a crucial step will be to prove that the rotation takes place around a fixed axis; the next lemma (for which a sketch of the proof can also be found in [15]) gives a necessary and sufficient criterion for that.

Lemma 8. *A rotation given by $\Omega(t)$ is one about a fixed axis if and only if there exists a constant orthogonal matrix P, such that all elements of the third row of the matrix $P^{-1}F(t)P$ are equal to zero for all times, where $F(t) = \Omega^{-1}\Omega'$.*

Proof. The direct part of the lemma is trivial, since for (2.3.10) and (2.3.11) one can take P to be the identity matrix.

For the converse part, let us first show that if a rotation $\Omega = \Omega(t)$ determines the skew-symmetric matrix $F = F(t)$, then the rotation $P^{-1}\Omega P$ determines the corresponding $P^{-1}FP$, where $P^{-1} = P^T = const$. Let $\bar{\Omega} = P^{-1}\Omega P$ and $\bar{F} = P^{-1}FP$; we easily check that $\bar{\Omega}$ and $\bar{F}$ are indeed a rotation and, respectively, a skew-symmetric matrix. But $\bar{\Omega}^T\bar{\Omega}' = (P^T\Omega^T P)(P^{-1}\Omega' P) = P^{-1}FP$ and Lemma 1 confirms the claim.

Now, if $P^{-1}FP$ vanishes in the third row and because it is skew-symmetric, it will have the form (2.3.11) and then the uniquely determined $\bar{\Omega} = P^{-1}\Omega P$ rotation will have the form (2.3.10), that is, it will be a rotation around the z-axis. But then, setting the equation for finding the axis of rotation u for Ω: $\Omega u = u$, we get: $\bar{\Omega}P^{-1}u = P^{-1}u$, which is in turn the equation for the axis of rotation of $\bar{\Omega}$. Thus we obtain: $u = Pz = const$; that is, the rotation given by Ω is indeed around a fixed axis. □

Corollary 1. *If $F(t) = F_0 = const$, then the rotation takes place around a fixed axis.*

Proof. A theorem by Wintner and Murnaghan ([14], p. 340-341) guarantees that for a constant, real 3-matrix F_0, one can always find a constant, real, orthogonal matrix P, such that $P^{-1}F_0P$ will have upper triangular form. Because F_0 is skew-symmetric, $P^{-1}F_0P$ will be so as well and will have zero entries in the third row. Lemma 8 completes the proof. □

3.2 Extension of Lagrange's Theorem

The goal of this section is to extend Lagrange's theorem to the quasihomogenous case. The force functions where none of a and b is equal to 2 will be treated in the first subsection. For the case when $a = b = 2$, the theorem does not hold even for $N = 3$ bodies: A. Wintner [15] built a solution in which the three bodies start from a configuration in the (x, y) plane and rotate around the x-axis.

For the situation in which only one of a and b is equal to 2 and $N = 3$, we managed to prove that the extension applies as well; this is discussed in the second subsection.

3.2.1 Quasihomogeneous force function with $\{a \neq 2$ and $b \neq 2\}$.

In this subsection we will prove the most general extension of Lagrange's theorem to quasihomogeneous force functions:

Theorem 1. *For the quasihomogeneous N-body problem with $a \neq 2$ and $b \neq 2$, if a homographic solution is flat, then it is planar.*

The theorem holds for the collinear case, and without any restrictions on a, b, as shown in Lemma 5 (Section 2.3.2). Thus, we now suppose that the solution is homographic and flat, but not collinear. We plan to prove that the homographic solution is such that, if there exist any rotation, then it should take place around a fixed axis; then we show that the only allowed position of this axis will be the one perpendicular to the plane of the initial configuration.

Choose the plane in which all r_{i0} lie to be the $(x,\, y)$ plane, thus $z_{i0} = 0,\ i = 1, ..., N$ and $F^z_{i0} = 0$. We begin with a lemma that will simplify the flat configuration.

Lemma 9. *For a flat, non-collinear solution of an $N-$body problem, there exists an initial moment t_0, a real number $e \neq 1$, and a pair of bodies $\{1,\, 2\}$, such that*

$$\begin{cases} x_{20} = e \cdot x_{10}, \qquad x_{10} \neq 0 \\ y_{20} = y_{10} \neq 0. \end{cases} \tag{3.2.1}$$

Also there exists a pair of bodies $\{\alpha,\, \beta\}$ such that:

$$\det(\bar{F}_{\alpha 0},\, \bar{F}_{\beta 0}) \;\; \neq \;\; 0, \tag{3.2.2}$$

where $\bar{F}_{j0},\ j = \alpha, \beta$, are the forces that act upon each body, in which the third component is omitted, at the instant t_0. (The first pair of bodies needs not be identical to the second one).

Proof. Choose a body 1 not situated at the origin (it does not make physical sense to have more than one body at that origin); thus $x_{10} \neq 0$ and $y_{10} \neq 0$. Because the configuration is not collinear at some moment t_0, there should exist at least one more body (index 2) not situated on the line containing the vector r_{10}; in particular, not both coordinates of the second body are equal to those of the first body. We can choose the y-axis to be perpendicular to $(r_{20} - r_{10})$, and thus $y_{10} = y_{20}$, then $x_{20} \neq x_{10}$ and formula (3.2.1) becomes true.

To prove the second assertion, fix the line determined by the vector force of one body ($F_{\alpha 0}$). Because the configuration is not collinear, we can argue that it is possible to find a most distant body (index β) from that line and that $F_{\alpha 0} \nparallel F_{\beta 0}$. This is equivalent to $F_{\alpha 0} \times F_{\beta 0} \neq 0$, which directly implies (3.2.2). $\square$

Denoting the elements of the $K(t)$ matrix by $K_{kj},\ k, j \in \{1, 2, 3\}$, marking the

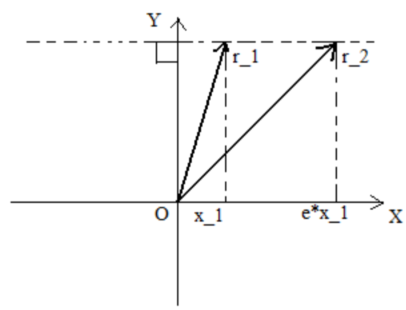

Figure 3.2.1: Two arbitrary bodies satisfying relations (3.2.1).

components of $\frac{V_{r_i0}}{m_i}$, $\frac{W_{r_i0}}{m_i}$ with a k exponent and applying the equation of motion (3.1.9) to $r_{i0},\ i = 1, 2$, considered in Lemma 9, we get the following system:

$$\begin{cases} K_{k1}(t)x_{10} + K_{k2}(t)y_{10} = s^{b-a}\frac{V^k_{r_10}}{m_1} + \frac{W^k_{r_10}}{m_1} \\ K_{k1}(t)ex_{10} + K_{k2}(t)y_{10} = s^{b-a}\frac{V^k_{r_20}}{m_2} + \frac{W^k_{r_20}}{m_2}. \end{cases}$$

This 6×6 system, when keeping k fixed, has a non-vanishing determinant $\Delta = (1-e)x_{10}y_{10}$; because $z_{i0} = 0$, (3.1.6) implies that $\frac{V_{z_i0}}{m_i} = \frac{W_{z_i0}}{m_i} = 0,\ i = 1, 2$. Solving it, we find:

$$\begin{cases} K_{11} = \frac{s^{b-a}(\frac{V_{x_10}}{m_1} - \frac{V_{x_20}}{m_2}) + \frac{W_{x_10}}{m_1} - \frac{W_{x_20}}{m_2}}{(1-e)x_{10}}, \\ K_{21} = \frac{s^{b-a}(\frac{V_{y_10}}{m_1} - \frac{V_{y_20}}{m_2}) + \frac{W_{y_10}}{m_1} - \frac{W_{y_20}}{m_2}}{(1-e)x_{10}}, \\ K_{12} = \frac{s^{b-a}(\frac{V_{x_20}}{m_2} - e\frac{V_{x_10}}{m_1}) + \frac{W_{x_20}}{m_2} - e\frac{W_{x_10}}{m_1}}{(1-e)y_{10}}, \\ K_{22} = \frac{s^{b-a}(\frac{V_{y_20}}{m_2} - e\frac{V_{y_10}}{m_1}) + \frac{W_{y_20}}{m_2} - e\frac{W_{y_10}}{m_1}}{(1-e)y_{10}}, \\ K_{31} = K_{32} = 0. \end{cases} \tag{3.2.3}$$

With A, B, C, D constants determined by initial conditions, we get:

$$\begin{cases} K_{12} + K_{21} = s^{b-a}\left(\frac{\frac{V_{x_20}}{m_2} - e\frac{V_{x_10}}{m_1}}{(1-e)y_{10}} + \frac{\frac{V_{y_10}}{m_1} - \frac{V_{y_20}}{m_2}}{(1-e)x_{10}}\right) + \frac{\frac{W_{x_20}}{m_2} - e\frac{W_{x_10}}{m_1}}{(1-e)y_{10}} + \frac{\frac{W_{y_10}}{m_1} - \frac{W_{y_20}}{m_2}}{(1-e)x_{10}} = s^{b-a}2A + 2B, \\ K_{11} - K_{22} = s^{b-a}\left(\frac{\frac{V_{x_10}}{m_1} - \frac{V_{x_20}}{m_2}}{(1-e)x_{10}} - \frac{\frac{V_{y_20}}{m_2} - e\frac{V_{y_10}}{m_1}}{(1-e)y_{10}}\right) + \frac{\frac{W_{x_10}}{m_1} - \frac{W_{x_20}}{m_2}}{(1-e)x_{10}} - \frac{\frac{W_{y_20}}{m_2} - e\frac{W_{y_10}}{m_1}}{(1-e)y_{10}} = s^{b-a}C + D. \end{cases} \tag{3.2.4}$$

Introducing these relations and the fifth equation of (3.2.3) into the definition (3.1.10) of the matrix $K(t)$, we get:

$$\begin{cases} s^{b+2}f_1f_2 = s^{b-a}A + B, \\ s^{b+2}(f_1^2 - f_2^2) = s^{b-a}C + D \end{cases} \tag{3.2.5}$$

and

$$\begin{cases} -2s'f_2 + s(-f_2' + f_3f_1) = 0, \\ 2s'f_1 + s(f_1' + f_2f_3) = 0. \end{cases} \tag{3.2.6}$$

From system (3.2.5) we find that $f_1(t)$ and $f_2(t)$ are given by the following expressions (where we retained only those solutions who satisfy $f_1^2 \geq 0,\ f_2^2 \geq 0$ and denoted by σ_i the sign of f_i):

$$\begin{cases} f_1 = \sigma_1 \frac{s^{\frac{-b-2}{2}}}{\sqrt{2}} \sqrt{\sqrt{(s^{b-a}C+D)^2 + 4(s^{b-a}A+B)^2} + (s^{b-a}C+D)}, \\ f_2 = \sigma_2 \frac{s^{\frac{-b-2}{2}}}{\sqrt{2}} \sqrt{\sqrt{(s^{b-a}C+D)^2 + 4(s^{b-a}A+B)^2} - (s^{b-a}C+D)}. \end{cases} \tag{3.2.7}$$

To shorten the expressions, let us denote: $H = \sqrt{(s^{b-a}C+D)^2 + 4(s^{b-a}A+B)^2}$, $I = (s^{b-a}A+B)$, $J = (s^{b-a}C+D)$.

Next we find the derivatives of f_1 and f_2, substitute all results in (3.2.6), and after long computations, the equations (3.2.6) become:

$$s^{\frac{-b-2}{2}} \frac{-s'\sigma_2\{(2-b)(H-J)H + (b-a)s^{b-a}[-C(H-J)+4AI]\} + 4s\sigma_1 f_3\sqrt{I^2}H}{2\sqrt{2}\sqrt{H-J}H} = 0,$$
$$s^{\frac{-b-2}{2}} \frac{s'\sigma_1\{(2-b)(H+J)H + (b-a)s^{b-a}[C(H+J)+4AI]\} + 4s\sigma_2 f_3\sqrt{I^2}H}{2\sqrt{2}\sqrt{H+J}H} = 0.$$

Since $s > 0$, the above equations imply:

$$-s'\sigma_2\{(H-J)[(2-b)H-(b-a)s^{b-a}C] + 4(b-a)s^{b-a}AI\} + 4s\sigma_1 f_3\sqrt{I^2}H = 0, \tag{3.2.8}$$
$$s'\sigma_1\{(H+J)[(2-b)H+(b-a)s^{b-a}C] + 4(b-a)s^{b-a}AI\} + 4s\sigma_2 f_3\sqrt{I^2}H = 0. \tag{3.2.9}$$

We will continue the discussion in cases, as to whether or not some of f_1 or f_2 is zero.

Case 1. If $f_1 = f_2 = 0$, then (2.3.10) and (2.3.11) show that the rotation (if any) is about the z-axis for all time. But initially, as we chose in the beginning of the proof, all bodies are in the (x, y) plane, therefore they keep moving in that plane for all time and the solution is planar. This can happen, say, when $A = B = C = D = 0$, for all values of a and b.

Case 2. If $f_1 \neq 0$ but $f_2 = 0$, then from the first of (3.2.6), $f_3 = 0$. Also, (3.2.7) implies that $I = 0$ and $H = J \neq 0$. Introducing these in (3.2.9), we get that $s'[(2-b)J + (b-a)s^{b-a}C] = 0$. One possibility is $s = const$; to find the others, let us consider two subcases:

1. In the homogeneous case ($a = b$), the above equation requires $b = 2$.
2. Let $a < b$. After equating to zero the coefficients of the non-constant func-

tion s^{b-a} in the above relation, we get the following system:

$$\begin{cases} C(2-a) = 0, \\ D(2-b) = 0. \end{cases}$$

Since $C = D = 0$ is excluded by $J \neq 0$, it follows that either $a = 2$ and $D = 0$, or $b = 2$ and $C = 0$.
Thus, if $a \neq 2$ and $b \neq 2$, the only allowed possibility is $s = const$.

Case 3. If $f_2 \neq 0$ but $f_1 = 0$, then the treatment is similar and with identical conclusions: $f_3 = 0,\ I = 0,\ H = -J \neq 0$ and, if $s \neq const$, then either $a = b = 2$ or $a < b$. In the last situation, either $a = 2$ and $D = 0$, $C \neq 0$, or $b = 2$ and $C = 0$, $D \neq 0$.

Case 4. If $f_1 \neq 0$ and $f_2 \neq 0$, then (3.2.7) gives that $I \neq 0$ and $H \neq 0$. Let us add or subtract the (3.2.8) and (3.2.9) equations, according to: $\sigma_1 = -/+\sigma_2$; in any case, we get: $s'(-\sigma_2)\{(H-J)[(2-b)H-(b-a)s^{b-a}C]+8(b-a)s^{b-a}AI+(H+J)[(2-b)H+(b-a)s^{b-a}C]\} = 0$. So, $s = const$ (when (3.2.8) gives that $f_3 = 0$) or $(2-b)H^2+(b-a)s^{b-a}CJ+4(b-a)s^{b-a}AI = 0$. We again treat two subcases:

1. For $a = b$, we find $b = 2$.
2. For $a < b$, expanding the products and assuming $s \neq const$, one gets the system:

$$\begin{cases} (2-a)(C^2+4A^2) = 0, \\ (4-b-a)(CD+4AB) = 0, \\ (2-b)(D^2+4B^2) = 0. \end{cases}$$

   Since $(C^2+4A^2) = (D^2+4B^2) = 0$ is excluded by $I \neq 0$, it follows that either $a = 2$ and $(D^2+4B^2) = 0,\ A \neq 0$, or $b = 2$ and $(C^2+4A^2) = 0,\ B \neq 0$.

But in each situation encountered in subcases 1 and 2 above, equations (3.2.7) give that $f_1 = C_1 s^{-2}$ and $f_2 = C_2 s^{-2}$, for some non-zero constants C_1, C_2. Substitution into the second of (3.2.6) shows that $f_3 = 0$ again.
Also, notice that for $a \neq 2$ and $b \neq 2$ it follows that $s = const$.

Thus, in the cases where at least one of f_1, f_2 is not zero, we got that $f_3 = 0$.

Because $F \neq (0)$, (2.3.5) shows that $\Omega \neq const \implies \Omega^{-1} \neq const$. In each of discussed cases, $F(t)$ can be put in the form $F(t) = f(t) \cdot F_0$. The proof of Corollary 1 tells us that there exists a constant matrix P such that the matrix $P^{-1}F_0P$ vanishes in the third row, therefore $P^{-1}FP$ also vanishes in the third row. Then Lemma 8 grants that the rotation is about a fixed axis, its position being yet to be determined. For that, consider the equation for finding the axis of rotation $\Omega u = u$, $u = (u_x, u_y, u_z)$, which after differentiation implies $\Omega' u = 0$. Using this in $F = \Omega^{-1}\Omega'$, we get $Fu = 0$; thus, with the form of F given by (2.3.6) and $f_3 = 0$, we obtain:

$$\begin{cases} f_2 u_z = 0, \\ -f_1 u_z = 0, \\ -f_2 u_x + f_1 u_y = 0. \end{cases}$$

Since not both f_1 and f_2 vanish, this clearly shows that the axis of rotation is fixed in the (x, y) plane.

Taking the case when none of a, b is equal to two, we obtained that $s = const$. Thus, each body moves on circles of constant position, centered on the axis of rotation.

This is physically impossible, because we can identify a body moving in a most remote plane from the origin and thus the resultant force should have a component that will force the body to move off that plane. To show this mathematically, in view of $F = const$, equation (3.1.10) reduces to $K(t) = F^2$ and because $f_3 = 0$, from (2.3.8), $\det \bar{K} = 0$, where $\bar{K}$ denotes that we omitted the third line and column in $K(t)$. On the other hand, write (3.1.9) as the matrix equation $\left(\frac{\bar{V}_{r_1 0}}{m_1} + \frac{\bar{W}_{r_1 0}}{m_1}, \frac{\bar{V}_{r_2 0}}{m_2} + \frac{\bar{W}_{r_2 0}}{m_2} \right) = \bar{K}(t)(\bar{r}_{10}, \bar{r}_{20})$, taking the second pair of bodies in Lemma 9. Use (3.2.2) and $\bar{F}_{i0} = \bar{V}_{r_i 0} + \bar{W}_{r_i 0}$ to get $\det \left(\frac{\bar{V}_{r_1 0}}{m_1} + \frac{\bar{W}_{r_1 0}}{m_1}, \frac{\bar{V}_{r_2 0}}{m_2} + \frac{\bar{W}_{r_2 0}}{m_2} \right) \neq 0$; this would imply that both $\bar{K}(t)$ and $(\bar{r}_{10}, \bar{r}_{20})$ have a non-vanishing determinant.

We reached a direct contradiction and therefore the situation: $s = const$ and none of a, b equal to zero, does not hold.

We are forced to conclude that, except when $a = 2$ or $b = 2$, only the case with $f_1 = f_2 = 0$ is allowed to exist, and thus Theorem 1 is proved.

3.2.2 Quasihomogeneous force function with $a \neq 2$ or $b \neq 2$, $N = 3$.

The goal of the subsection is to prove the following theorem:

Theorem 2. *For the quasihomogeneous 3-body problem with $a = 2$ and $b > 2$, or $b = 2$ and $a < 2$, if a homographic solution is flat, then it is planar.*

We will use all the above results in this section that do not make explicit use of the degres a, b and will try to reach a contradiction when assuming that there is some rotation not around the z-axis. Since in the Case 1 above, the theorem holds trivially, the analysis will be continued in the cases where: at least one of f_1, f_2 is different of zero, thus $f_3 = 0$ and the axis of rotation is in the (x, y) plane. As shown in the last part of the previous proof, the situation with $s = const$ leads to a contradiction, no matter what a, b are. Therefore, we are left to analyze the situation with: $s \neq const$ and only one of a, b is equal to two. First, Cases 2 and 3 will be treated, then Case 4.

3.2.2.1 Only one of f_1, f_2 is equal to zero.

The matrix $K(t)$ from equation (3.1.10) for this case becomes:

$$K(t) = s^{b+1} \begin{pmatrix} s'' + s(-f_2^2) & 0 & 0 \\ 0 & s'' + s(-f_1^2) & 0 \\ 0 & 0 & s'' + s(-f_2^2 - f_1^2) \end{pmatrix}.$$

Comparisson with (3.2.3) gives that $K_{12} = K_{21} = 0$, or:

$$\begin{cases} \frac{V_{x_2 0}}{m_2} = e\frac{V_{x_1 0}}{m_1}, \ \frac{W_{x_2 0}}{m_2} = e\frac{W_{x_1 0}}{m_1} \\ \frac{V_{y_1 0}}{m_1} = \frac{V_{y_2 0}}{m_2}, \ \frac{W_{y_1 0}}{m_1} = \frac{W_{y_2 0}}{m_2} \end{cases},$$

which when used in the second of (3.2.4) for $K_{11} - K_{22}$ shows that:

$$s^{b-a}\left(\frac{V_{x_1 0}}{m_1 x_{10}} - \frac{V_{y_1 0}}{m_1 y_{10}}\right) + \frac{W_{x_1 0}}{m_1 x_{10}} - \frac{W_{y_1 0}}{m_1 y_{10}} = s^{b-a}C + D.$$

But, as shown in Cases 2 and 3, either $a = 2$ and $D = 0$, $C \neq 0$, or $b = 2$ and $C = 0$, $D \neq 0$, which means that one of the following two systems should hold (but not both):

$$\begin{cases} \frac{W_{x_1 0}}{m_1 x_{10}} - \frac{W_{y_1 0}}{m_1 y_{10}} = 0, \\ \frac{V_{x_1 0}}{m_1 x_{10}} - \frac{V_{y_1 0}}{m_1 y_{10}} \neq 0, \end{cases} ; \quad \begin{cases} \frac{W_{x_1 0}}{m_1 x_{10}} - \frac{W_{y_1 0}}{m_1 y_{10}} \neq 0, \\ \frac{V_{x_1 0}}{m_1 x_{10}} - \frac{V_{y_1 0}}{m_1 y_{10}} = 0. \end{cases} \tag{3.2.10}$$

We will prove that actually none of the above systems can hold, and this will mean that the cases discused in this subsubsection lead to a contradiction. For this, we compute directly the corresponding accelerations. The x-component due to force function W has the form: $\frac{W_{x_i0}}{m_i} = k \cdot b\sum_{j=1,\, j\neq i}^{3} m_j(x_{j0} - x_{i0})\frac{1}{l_{ji}^{b+2}}$ (we denoted by $l_{ji} = |r_{j0} - r_{i0}|$ the distance between bodies j, i at the initial moment of time); the component due to force function V will have the same form, just with the exponent $b+2$ replaced by $a+2$ and the coefficient kb replaced with a.

From Lemma 9, $x_{20} = ex_{10},\ e \neq 1,\ y_{20} = y_{10}$, while the integral of the center of mass $\sum_{j=1}^{3} r_{j0} = 0$ gives that $m_3x_{30} = -x_{10}(m_1 + em_2)$ and $m_3y_{30} = -y_{10}(m_1 + m_2)$.

Using these, $\frac{W_{x_10}}{m_1x_{10}} = \frac{kb}{x_{10}}\left[\frac{m_2x_{10}(e-1)}{l_{12}^{b+2}} + \frac{m_3(x_{30}-x_{10})}{l_{13}^{b+2}}\right] = kb\left[\frac{m_2(e-1)}{l_{12}^{b+2}} - \frac{m_1+em_2+m_3}{l_{13}^{b+2}}\right]$ and $\frac{V_{x_10}}{m_1x_{10}} = a\left[\frac{m_2(e-1)}{l_{12}^{a+2}} - \frac{m_1+em_2+m_3}{l_{13}^{a+2}}\right]$. In a similar manner, we obtain that $\frac{W_{y_10}}{m_1y_{10}} = -kb\frac{m_1+m_2+m_3}{l_{13}^{b+2}}$ and $\frac{V_{y_10}}{m_1y_{10}} = -a\frac{m_1+m_2+m_3}{l_{13}^{a+2}}$. Thus, the sought differences in (3.2.10) can be put in the form:

$$\begin{cases} \frac{W_{x_10}}{m_1x_{10}} - \frac{W_{y_10}}{m_1y_{10}} = kb\frac{(1-e)m_2}{l_{13}^{b+2}}\left[1 - \left(\frac{l_{13}}{l_{12}}\right)^{b+2}\right] \\ \frac{V_{x_10}}{m_1x_{10}} - \frac{V_{y_10}}{m_1y_{10}} = a\frac{(1-e)m_2}{l_{13}^{a+2}}\left[1 - \left(\frac{l_{13}}{l_{12}}\right)^{a+2}\right] \end{cases}. \tag{3.2.11}$$

Obviously, if one of them is equal to zero, then (because $e \neq 1$) $\frac{l_{13}}{l_{12}} = 1$ and then the other difference is also equal to zero:

$$\frac{W_{x_10}}{m_1x_{10}} - \frac{W_{y_10}}{m_1y_{10}} = 0 \iff \frac{V_{x_10}}{m_1x_{10}} - \frac{V_{y_10}}{m_1y_{10}} = 0. \tag{3.2.12}$$

In conclusion, our claim is proved, equation (3.2.10) is contradicted and the case analyzed in this subsubsection is not allowed to exist.

3.2.2.2 None of f_1, f_2 is equal to zero.

Here the matrix $K(t)$ becomes:

$$K(t) = s^{b+1}\begin{pmatrix} s'' + s(-f_2^2) & sf_1f_2 & 0 \\ sf_1f_2 & s'' + s(-f_1^2) & 0 \\ 0 & 0 & s'' + s(-f_2^2 - f_1^2) \end{pmatrix}.$$

The comparison with equations (3.2.3) gives that $K_{12} = K_{21} = s^{b-a}A + B \neq 0$, or:

$$\begin{cases} \frac{\frac{V_{x_2 0}}{m_2} - e\frac{V_{x_1 0}}{m_1}}{(1-e)y_{10}} = A = \frac{\frac{V_{y_1 0}}{m_1} - \frac{V_{y_2 0}}{m_2}}{(1-e)x_{10}} \\ \frac{\frac{W_{x_2 0}}{m_2} - e\frac{W_{x_1 0}}{m_1}}{(1-e)y_{10}} = B = \frac{\frac{W_{y_1 0}}{m_1} - \frac{W_{y_2 0}}{m_2}}{(1-e)x_{10}} \end{cases}, \tag{3.2.13}$$

and the second of equation (3.2.4) give that

$$\begin{cases} \frac{\frac{V_{x_1 0}}{m_1} - \frac{V_{x_2 0}}{m_2}}{(1-e)x_{10}} - \frac{\frac{V_{y_2 0}}{m_2} - e\frac{V_{y_1 0}}{m_1}}{(1-e)y_{10}} = C \\ \frac{\frac{W_{x_1 0}}{m_1} - \frac{W_{x_2 0}}{m_2}}{(1-e)x_{10}} - \frac{\frac{W_{y_2 0}}{m_2} - e\frac{W_{y_1 0}}{m_1}}{(1-e)y_{10}} = D \end{cases}. \tag{3.2.14}$$

As we found at the end of Case 4, only one of the following systems should hold:

$$\begin{cases} a = 2 \\ D = B = 0 \\ A \neq 0 \end{cases}, \quad \begin{cases} b = 2 \\ C = A = 0 \\ B \neq 0 \end{cases}; \tag{3.2.15}$$

let us investigate them separately.

In the case $a = 2$, systems (3.2.13) and (3.2.14) yield the following conditions:

$$\begin{cases} \frac{V_{x_2 0}}{y_{10}m_2} - e\frac{V_{x_1 0}}{y_{10}m_1} = \frac{V_{y_1 0}}{x_{10}m_1} - \frac{V_{y_2 0}}{x_{10}m_2} \\ \frac{V_{x_2 0}}{m_2} \neq e\frac{V_{x_1 0}}{m_1} \;\&\; \frac{V_{y_1 0}}{m_1} \neq \frac{V_{y_2 0}}{m_2}, \\ \frac{W_{x_2 0}}{m_2 x_{10}} = e\frac{W_{x_1 0}}{m_1 x_{10}}, \\ \frac{W_{y_1 0}}{m_1 y_{10}} = \frac{W_{y_2 0}}{m_2 y_{20}}, \\ \frac{W_{x_1 0}}{m_1 x_{10}} - \frac{W_{y_1 0}}{m_1 y_{10}} = 0. \end{cases} \tag{3.2.16}$$

We can continue the proof if we could show that at least one pair of initial position vectors $r_{j0},\ j \in \{1, 2, 3\}$ form an angle greater than 90^0.

Mathematically, let, as previously, r_{10} be the position vector of the body not situated at the origin and let r_{II0} and r_{III0} be the other two position vectors. The integral of the center of mass states that:

$$m_{III}r_{III0} = -(m_1 r_{10} + m_{II}r_{II0}). \tag{3.2.17}$$

None of r_{II0} and r_{III0} can be collinear with r_{10}, otherwise the previous equation

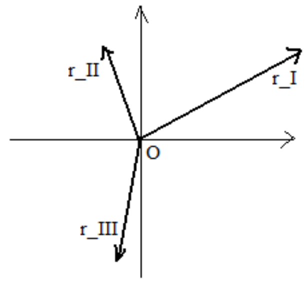

Figure 3.2.2: A possible position for the three vectors in (3.2.17).

requires that the third one will be collinear as well, but we excluded the collinear configurations. Using the scalar product, we find $\cos\angle(r_{10},\, r_{II0}) \sim r_{10}\cdot r_{II0} = x_{10}x_{II0} + y_{10}y_{II0}$, $\cos\angle(r_{10},\, r_{III0}) \sim -\left(x_{10}\frac{x_{10}m_1+x_{II0}m_{II}}{m_{III}} + y_{10}\frac{y_{10}m_1+y_{II0}m_{II}}{m_{III}}\right) = -[(\frac{m_1}{m_{II}})(x_{10}^2 + y_{10}^2) + \frac{m_{II}}{m_{iii}}(x_{10}x_{II0} + y_{10}y_{II0})]$, where we used (3.2.17) in coordinates.
When $\cos\angle(r_{10},\, r_{II0}) \geq 0$, $\cos\angle(r_{10},\, r_{III0}) < 0$, because $x_{10}^2 + y_{10}^2 > 0$.
We see that in any case, we can denote by r_{20} that vector of $\{r_{III0},\, r_{II0}\}$ such that

$$\angle(r_{10},\, r_{20}) > 90^0. \tag{3.2.18}$$

Take the y-axis perpendicular to $r_{10} - r_{20}$, which will make this pair of vectors satisfy all conditions of Lemma 9.

A first consequence is $e \neq 0$, otherwise r_{20} is along the y- axis and the triangle based on r_{10} and r_{20} is right, which is a contradiction to (3.2.18). Returning to (3.2.16), substituting the fifth and then the third equations into the forth equation, we get:

$$\frac{W_{y_20}}{m_2y_{20}} = \frac{W_{x_20}}{m_2x_{20}}. \tag{3.2.19}$$

The relation (3.2.12) of the previous subsubsection is not specific to body 1 alone; similar computations (now allowed by $e \neq 0$) show that:

$$\begin{cases} \frac{W_{x_20}}{m_2x_{20}} - \frac{W_{y_20}}{m_2y_{20}} = kb\frac{\frac{(e-1)}{e}m_1}{l_{23}^{b+2}}\left[1-\left(\frac{l_{23}}{l_{12}}\right)^{b+2}\right] \\ \frac{V_{x_20}}{m_2x_{20}} - \frac{V_{y_20}}{m_2y_{20}} = a\frac{\frac{(e-1)}{e}m_1}{l_{23}^{a+2}}\left[1-\left(\frac{l_{23}}{l_{12}}\right)^{a+2}\right] \end{cases}, \tag{3.2.20}$$

and therefore:

$$\frac{W_{x_20}}{m_2x_{20}} - \frac{W_{y_20}}{m_2y_{20}} = 0 \iff \frac{V_{x_20}}{m_2x_{20}} - \frac{V_{y_20}}{m_2y_{20}} = 0. \tag{3.2.21}$$

Using (3.2.12) and (3.2.21), we get that the fifth equation of (3.2.16) and equation (3.2.19) imply:

$$\frac{V_{x_1 0}}{m_1 x_{10}} - \frac{V_{y_1 0}}{m_1 y_{10}} = 0 \,\&\, \frac{V_{x_2 0}}{m_2 x_{20}} - \frac{V_{y_2 0}}{m_2 y_{20}} = 0. \tag{3.2.22}$$

Finally, we substitute these expressions into the right-hand side of the first equation in (3.2.16) and get:

$$\frac{1}{y_{10}} \left(\frac{V_{x_2 0}}{m_2} - e\frac{V_{x_1 0}}{m_1} \right) \left(1 + \frac{y_{10}^2}{e \cdot x_{10}^2} \right) = 0.$$

But if $1+\frac{y_{10}^2}{e \cdot x_{10}^2} = 0$, or $y_{10}^2 = -e \cdot x_{10}^2$, then $r_{10}^2 = y_{10}^2 + x_{10}^2 = (1-e)x_{10}^2$, $r_{20}^2 = -e(1-e)x_{10}^2$ and $r_{10}^2 + r_{20}^2 = (1-e)^2 x_{10}^2$, while $(r_{10} - r_{20})^2 = (x_{10} - x_{20})^2 = (1-e)^2 x_{10}^2$. That is, this corresponds exactly to a right triangle formed by r_{10} and r_{20} – a situation that is excluded by relation (3.2.18).

Therefore, we are forced to conclude that $\frac{V_{x_2 0}}{m_2} - e\frac{V_{x_1 0}}{m_1} = 0$, which is a direct contradiction of the second equation in (3.2.16).

As this ended the treatment of the case $a = 2$, we now move to the second one, $b = 2$ and the second system of (3.2.15), the treatment of which will be similar.

In this case, systems (3.2.13) and (3.2.14) yield the following conditions:

$$\begin{cases} \frac{W_{x_2 0}}{y_{10} m_2} - e\frac{W_{x_1 0}}{y_{10} m_1} = \frac{W_{y_1 0}}{x_{10} m_1} - \frac{W_{y_2 0}}{x_{10} m_2} \\ \frac{W_{x_2 0}}{m_2} \neq e\frac{W_{x_1 0}}{m_1} \,\&\, \frac{W_{y_1 0}}{m_1} \neq \frac{W_{y_2 0}}{m_2}, \\ \frac{V_{x_2 0}}{x_{10} m_2} = e\frac{V_{x_1 0}}{x_{10} m_1}, \\ \frac{V_{y_1 0}}{y_{10} m_1} = \frac{V_{y_2 0}}{y_{20} m_2}, \\ \frac{V_{x_1 0}}{x_{10} m_1} - \frac{V_{y_1 0}}{y_{10} m_1} = 0. \end{cases} \tag{3.2.23}$$

Substituting the fifth and then the third equations (with the use of $e \neq 0$) into the forth equation, we get:

$$\frac{V_{y_2 0}}{m_2 y_{20}} = \frac{V_{x_2 0}}{m_2 x_{20}}. \tag{3.2.24}$$

With (3.2.12) and (3.2.21), we find that the fifth equation of (3.2.23) and equation (3.2.24) imply:

$$\frac{W_{x_1 0}}{m_1 x_{10}} - \frac{W_{y_1 0}}{m_1 y_{10}} = 0 \,\&\, \frac{W_{x_2 0}}{m_2 x_{20}} - \frac{W_{y_2 0}}{m_2 y_{20}} = 0. \tag{3.2.25}$$

Now we substitute these expressions into the right-hand side of the first equation in (3.2.16) and get:

$$\frac{1}{y_{10}}\left(\frac{W_{x_20}}{m_2}-e\frac{W_{x_10}}{m_1}\right)\left(1+\frac{y_{10}^2}{e\cdot x_{10}^2}\right)=0.$$

Again, since $1+\frac{y_{10}^2}{e\cdot x_{10}^2}=0$ is interdicted by the fact that the triangle based on r_{10} and r_{20} is not right, this relation is a direct contradiction of the second equation in (3.2.23).

To summarize, both $a = 2$ and $b = 2$ situations of the current subsubsection (when none of $f_1,\ f_2$ is equal to zero) lead to contradictions.

Together with the contradiction obtained in the previous subsubsection, it means that the only possible situation remains Case 1, when $f_1 = f_2 = 0$ and the rotation takes place around the z-axis, perpendicular to the initial plane of the configuration. Thus, the configuration formed by the 3-bodies remains planar and Theorem 2 is proved.

3.3 Extension of Pizzetti's Theorem

The goal of this section is to extend Pizzetti's theorem to quasihomogenous potentials.

Theorem 3. *For the quasihomogeneous N-body problem with $a \neq 2$ and $b \neq 2$, if the homographic solution is not flat, then it is homothetic.*

If $N = 3$, then the solution is always flat. So, throughout this section, we consider $N > 3$.

Obviously, we need to prove that there is no rotation whatsover, that is, $\Omega(t) = \Omega(t_0) = E$. This will be done in two large steps: first, we will show that the rotation, if any, should take place around a fixed axis; second, for that rotation, represented by a particular angular velocity function $w' = w'(t)$, we obtain $w'(t) = 0$.

Using the proof of Corollary 1, if we could prove that

$$F(t) = f(t)\cdot F_0, \tag{3.3.1}$$

where $f(t)$ is a scalar function and F_0 a constant skew-symmetric 3-matrix whose entries are $f_{i0},\ i = 1, 2, 3$, then $P^{-1}FP$ would also have zero entries on the third row. Therefore, with Lemma 8, the above formula would be sufficient for proving that the rotation takes place around a fixed axis.

Before beginning the proof of (3.3.1), we will use non-flatness to simplify the expression of $K(t)$.

Lemma 10. *For a non-flat solution of a quasihomogeneous $N-$body problem, there exists an initial moment t_0 and two triplets of bodies, such that:*

$$\det(r_{10}, r_{20}, r_{30}) \neq 0, \quad (3.3.2)$$

$$\det(F_{40}, F_{50}, F_{60}) \neq 0, \quad (3.3.3)$$

where r_{i0}, $i = 1, 2, 3$, are the position vectors of those three bodies and F_{j0}, $j = 4, 5, 6$, are the total forces that act upon the other three bodies, at the instant t_0.(The triplet (1,2,3) needs not be identical to the triplet (4,5,6)).

Proof. Because the solution is not flat, there should be a moment $t = t_0$ such that the configuration is not flat at that moment and $N > 3$.

To prove the first assertion, choose a body with $i = 1$, not situated at the center of gravity O (it does not make any physical sense to have more than one body at that center). The configuration cannot be collinear at $t = t_0$, therefore we can find a body, $i = 2$, not situated along the vector r_{10}. Consider another body (non-flatness allows that) off the plane determined by vectors r_{10} and r_{20} and the claim is proved.

For the second claim, not all forces will be parallel with any particular plane, for one can pick a most distant body from that plane and argue that the force acting on it should have a component perpendicular to that plane. In particular, not all forces can be parallel with one another. Therefore, we can find two bodies, with $i = 1$ and $i = 2$, such that $F_{10} \nparallel F_{20}$, and consider the plane P_F built on these vectors (by translating the F_{20} vector till its origin coincides with that of F_{10}), and passing through the body 1. Because the solution is not flat, there should exist bodies above this plane, and below it (otherwise the vector F_{10} would have components perpendicular to that plane). Now consider a most distant body with $i = 3$ from that plane, in the upper half-space. With the same argument as above, the force acting on this body cannot be parallel with the plane P_F, which also shows it is not identical to $i = 2$. Thus the three forces are not co-planar, for which formula (3.3.3) is satisfied. □

Equation of motion (3.1.9) can be written as the matrix equation $(s^{b-a}\frac{V_{r_10}}{m_1} +$

$\frac{W_{r_10}}{m_1},\ s^{b-a}\frac{V_{r_20}}{m_2}+\frac{W_{r_20}}{m_2},\ s^{b-a}\frac{V_{r_30}}{m_3}+\frac{W_{r_30}}{m_3}) = K(t)(r_{10}, r_{20}, r_{30})$, and using (3.3.2) we get:

$$K(t) = (s^{b-a}\frac{V_{r_10}}{m_1}+\frac{W_{r_10}}{m_1},\ s^{b-a}\frac{V_{r_20}}{m_2}+\frac{W_{r_20}}{m_2},\ s^{b-a}\frac{V_{r_30}}{m_3}+\frac{W_{r_30}}{m_3})(r_{10}, r_{20}, r_{30})^{-1} = s^{b-a}D_1+D_2, \tag{3.3.4}$$

where D_1 and D_2 are constant 3-matrices.

We will also need the following two quantities, produced with $K(t) = s^{b+1}[s''E + 2s'F + s(F' + F^2)]$ using simple matrix operations: $\frac{1}{2}(K + K^T) = s^{b+1}(s''E + sF^2)$, $\frac{1}{2}(K - K^T) = s^{b+1}(sF' + 2s'F)$. With the simplified expression for K, these relations become:

$$s^{b+1}(s''E + sF^2) = s^{b-a}D_3 + D_4, \tag{3.3.5}$$

$$s^{b+1}(sF' + 2s'F) = s^{b-a}D_5 + D_6, \tag{3.3.6}$$

where $D_i,\ i = 3, 4, 5, 6$, are other constant 3-matrices.

The diagonal and the non-diagonal elements of the matrix equation (3.3.5) imply, respectively, the following two systems:

$$\begin{cases} s^{b+2}(f_1^2 - f_2^2) = s^{b-a}A + B \\ s^{b+2}(f_2^2 - f_3^2) = s^{b-a}C + D \end{cases} \tag{3.3.7}$$

and

$$\begin{cases} s^{b+2}f_1f_2 = s^{b-a}G + H \\ s^{b+2}f_1f_3 = s^{b-a}I + J \\ s^{b+2}f_2f_3 = s^{b-a}K + L \end{cases}, \tag{3.3.8}$$

where the 10 constants come from the matrices $D_i,\ i = 1,\ 2,\ 3,\ 4,\ 5,\ 6$.

These systems will turn out to be sufficient for proving (3.3.1), which is equivalent to

$$f_i(t) = f(t) \cdot f_{i0}. \tag{3.3.9}$$

It is easy to check that if $s(t) = const$, then all of f_i will be some constants and the assertion proved. Thus, we will assume, in all cases needed for proving (3.3.9), that $s(t) \neq const$.

First. If at least two of $f_i(t)$ are equal to zero, then the above formula is trivially satisfied.

Second. Suppose that only one of f_i is zero, say $f_1 = 0$, (because of the symmetry of the equations, the treatment for the other two possibilities is identical). The systems (3.3.7) and (3.3.8) become:

$$\begin{cases} s^{b+2}f_2^2 = s^{b-a}(-A) - B \;\; > 0 \\ s^{b+2}f_3^2 = s^{b-a}(-A-C) - B - D > 0 \\ s^{b+2}f_2 f_3 = s^{b-a}K + L \;\; \neq 0. \end{cases} \tag{3.3.10}$$

With a notation for convenience: $A + C = M$, $B + D = N$ and σ_i standing for the sign $(+/-)$ of f_i, we find that the non-zero functions are given by:

$$f_2(t) = \sigma_2 \sqrt{-\frac{s^{b-a}A + B}{s^{b+2}}} \neq 0, \;\; f_3(t) = \sigma_3 \sqrt{-\frac{s^{b-a}M + N}{s^{b+2}}} \neq 0. \tag{3.3.11}$$

When $a = b$, (3.3.9) is immediately satisfied. Let thus $a < b$. To get the relation between the constants, take the product of the first two equations in (3.3.10) and compare to the third, to obtain $(s^{b-a}A + B)(s^{b-a}M + N) = (s^{b-a}K + L)^2$. After equating the coefficients of the function $s^{b-a}(t) > 0$ with same degree, we have $(AN + BM)^2 = 4AMBN$ which implies:

$$AN = BM. \tag{3.3.12}$$

All subcases can be split in two: $N = 0$ and $N \neq 0$. The first case, together with $f_3 \neq 0$, implies that $M \neq 0$ & $B = 0$, and therefore $A \neq 0$. Then (3.3.11) shows that both f_i are proportional to $f(t) = \sqrt{\frac{s^{b-a}}{s^{b+2}}}$. In the second case, when $N \neq 0$, $A = \frac{BM}{N}$ and so (3.3.11) gives $f_2(t) = \frac{\sigma_2}{\sigma_3}\sqrt{\frac{B}{N}} \cdot f_3(t)$. Thus, in both cases (3.3.9) is satisfied.

Third. Now we will deal with the most general case, when none of f_i is equal to zero. Equations (3.3.8) give:

$$s^{b+2}f_1 = \frac{s^{b-a}G + H}{f_2}, \;\; f_3 = \frac{s^{b-a}I + J}{s^{b-a}G + H} f_2; \tag{3.3.13}$$

therefore:

$$s^{b+2}f_1^2 = \frac{(s^{b-a}G + H)(s^{b-a}I + J)}{(s^{b-a}K + L)}, \tag{3.3.14}$$

$$s^{b+2}f_2^2 = \frac{(s^{b-a}K + L)(s^{b-a}G + H)}{(s^{b-a}I + J)}, \tag{3.3.15}$$

$$s^{b+2}f_3^2 = \frac{(s^{b-a}I + J)(s^{b-a}K + L)}{(s^{b-a}G + H)}. \tag{3.3.16}$$

Clearly, when $a = b$, (3.3.9) is easily satisfied. Let $a < b$. Now we take the differences between the above equations of indices 1&2 and 2&3 and introduce the results in (3.3.7), after this we expand all the products from brackets and get the following equations:

$$\begin{gathered}
s^{3(b-a)}AIK + s^{2(b-a)}[A(IL+JK)+BIK] + s^{b-a}[AJL+B(IL+JK)] + BJL = \\
= s^{3(b-a)}G(I^2-K^2) + s^{2(b-a)}[2G(IJ-KL)+H(I^2-K^2)]+ \\
+s^{b-a}[G(J^2-L^2)+2H(IJ-KL)] + H(J^2-L^2); \\
s^{3(b-a)}CIG + s^{2(b-a)}[C(IH+GJ)+DIG] + s^{b-a}[CJH+D(IH+GJ)] + DJH = \\
= s^{3(b-a)}K(G^2-I^2) + s^{2(b-a)}[2K(GH-IJ)+L(G^2-I^2)]+ \\
+s^{b-a}[K(H^2-J^2)+2L(GH-IJ)] + L(H^2-J^2).
\end{gathered}$$

Again, because these relations are to be satisfied for a continuous range of times t, we need to equate the coefficients of the terms with same degree of s and so get the following system:

$$\begin{cases}
AIK = G(I^2-K^2) \\
A(IL+JK)+BIK = 2G(IJ-KL)+H(I^2-K^2) \\
AJL+B(IL+JK) = G(J^2-L^2)+2H(IJ-KL) \\
BJL = H(J^2-L^2);
\end{cases} \tag{3.3.17}$$

$$\begin{cases}
CIG = K(G^2-I^2) \\
C(IH+GJ)+DIG = 2K(GH-IJ)+L(G^2-I^2) \\
CJH+D(IH+GJ) = K(H^2-J^2)+2L(GH-IJ) \\
DJH = L(H^2-J^2).
\end{cases} \tag{3.3.18}$$

The relation we want to prove $f_i(t) = f(t) \cdot f_{i0}$ is equivalent to showing that $f_3 = pf_2$ and $f_1 = qf_2$, where p and q are some non-zero constants. The second of (3.3.13) then requires that $s^{b-a}(I-pG) = pH-J$, while the first of (3.3.13) combined with (3.3.15) requires that $s^{b-a}(I-qK) = qL-J$. Because these relations are to be satisfied for all time, we need to prove that there exist some non-zero constants $p,\ q$ such that:

$$I = pG,\ J = pH,\ I = qK,\ J = qL, \tag{3.3.19}$$

or, in case $G,\ I,\ K$ are all different of zero, one needs to prove that:

$$\frac{H}{G}=\frac{J}{I}=\frac{L}{K}. \tag{3.3.20}$$

We will first work on excluding the cases when some or all of those 6 constants in the equation above are equal to zero.

Case 1. Assume that at least one of $G,\ I,\ K$ constants is zero: let it be $I=0$ (the other cases are have identical conclusions, due to the symmetry of systems (3.3.7) and (3.3.8)). Then the first equation in system (3.3.17) will require that at least another constant from $\{G,\ I,\ K\}$ should also be equal to zero. Let then also $G=0$.

1. The situation when $G=I=K=0$ is trivial: the equations (3.3.14)-(3.3.16) show that the functions will have the form: $f_i(t)=f(t)\cdot f_{i0}$.
2. If $G=I=0$ and $K\neq 0$, then the system (3.3.17), after substitutions gives that $HJ^2=0$ and, with (3.3.14) it follows that $f_1=0$, which is a contradiction with the initial assumption that none of f_i is zero.

Case 2. Assume now that all of $G,\ I,\ K$ are different of zero, but one of H, J, L is equal to zero: let $J=0$. The forth equation in (3.3.17) then requires that at least one more from among H, J, L is equal to zero. Let then $H=0$ as well.

1. The situation: $H=J=L=0$ is again trivial since the equations (3.3.14)-(3.3.16) show that in this case the functions will have the required form: $f_i(t)=f(t)\cdot f_{i0}$.
2. If $H=J=0$ but $L\neq 0$, then the system (3.3.17), after substitutions, gives that $GI^2=0$, which is a contradiction.

Case 3. Thus, we can now assume that none of the constants $G,\ I,\ K, H,\ J,\ L$ is equal to zero. This allows us to eliminate the constants $A,\ B,\ C,\ D$ from the systems (3.3.17) and(3.3.18). Next, we found that a shorter way to prove (3.3.20) is to make the notations: $\frac{H}{G}=P,\ \frac{J}{I}=Q,\ \frac{L}{K}=R$, and then relation to prove becomes: $P=Q=R\,(\neq 0)$. After these eliminations and notations, the systems (3.3.17) and (3.3.18) transform to:

$$\begin{cases} RI^2 - QK^2 + P(Q\frac{I^2}{R} - R\frac{K^2}{Q}) = QI^2 - RK^2 + P(I^2 - K^2) \\ QR(I^2 - K^2) - P(\frac{R^2K^2}{Q} - \frac{QI^2}{R}) = Q^2I^2 - R^2K^2 + P(I^2Q - K^2R) \\ -PI^2 + QG^2 + R(\frac{PG^2}{Q} - \frac{QI^2}{P}) = G^2P - I^2Q + R(G^2 - I^2) \\ QP(G^2 - I^2) + R(\frac{G^2P^2}{Q} - \frac{Q^2I^2}{P}) = P^2G^2 - Q^2I^2 + R(G^2P - I^2Q). \end{cases} \tag{3.3.21}$$

The first two equations in this system are equivalent to:

$$\begin{cases} I^2(\frac{R-P}{R})(R - Q) = K^2(\frac{P-Q}{Q})(R - Q) \\ I^2(R - Q)(R - P) = K^2(R - Q)(P - Q)\frac{R^2}{Q^2}. \end{cases} \tag{3.3.22}$$

All situations can be split as follows:

1. $R \neq Q$.
   (a) $R = P$. The first of (3.3.22) requires $R = Q$ - a contradiction.
   (b) $R \neq P$. Equating I^2 from both of (3.3.22) one again gets $R = Q$, thus impossible.
2. $R = Q$. The third equation in (3.3.21) simplify to: $\frac{P^2+Q^2}{P} = 2Q \implies P = Q$ and this proves that $\frac{H}{G} = \frac{J}{I} = \frac{L}{K}$ is satisfied. Therefore, $f_i(t) = f(t) \cdot f_{i0}$ is also proved in the most general case.

By now, we exhausted all possible types of $f_i(t)$ and proved that the required relation $f_i(t) = f(t) \cdot f_{i0}$, which is equivalent to $F(t) = f(t) \cdot F_0$, is always satisfied. As we showed in the beginning of the section, this means that we have proved that the rotation takes place around a fixed axis, for all time. The rotation axis can be chosen now to be the z-axis and so, the rotation matrix is given by (2.3.10) and the auxiliary matrix by (2.3.11). Then the second equation in (3.3.7) gives

$$s^{b+2}(w'(t))^2 = s^{b-a}(-C) - D. \tag{3.3.23}$$

From (2.3.10) and (3.1.1) in Definition (4) of a homographic solution, we get:

$$r_i = \{s(x_{i0}\cos w - y_{i0}\sin w),\ s(x_{i0}\sin w + y_{i0}\cos w),\ sz_{i0}\}.$$

The projection of the integral of the total angular momentum $\sum_{i=1}^{N} r_i \times r_i' = C$ on the z-axis is $C_z = \sum_i m_i(x_i y_i' - y_i x_i')$ and after simple computations:

$$C_z = s^2 w' \sum_i m_i (x_{i0}^2 + y_{i0}^2). \tag{3.3.24}$$

Suppose that $w' \neq 0$, which implies Ω is not a constant.

If $\sum_i m_i(x_{i0}^2 + y_{i0}^2) = 0$, then that $x_{i0} = y_{i0} = 0$ and from the above representation for r_i, it follows that all bodies are situated on the z-axis, which contradicts the fact that our solution is not flat. Therefore, $\sum_i m_i(x_{i0}^2 + y_{i0}^2) \neq 0$ and we can divide (3.3.24) by $\sum_i m_i(x_{i0}^2 + y_{i0}^2)$ to get: $s^2 w' = \mathit{const} \neq 0$.

Introducing this in (3.3.23), we get:

$$s^{b-2} + C_1 s^{b-a} = C_2,$$

for some constants C_1 and C_2. Except when $b = 2$ or $a = 2$, this relation implies that $s = \mathit{const} = 1$. This means that each body is rotating on fixed planes (in fact, on fixed circles), parallel with one another and perpendicular to the z-axis. And this is physically impossible, since we can find a most remote from the origin plane of rotation and argue that all other bodies will create a resultant force not contained in that plane of rotation, and thus determining the body to move off that plane. Mathematically, equation $K(t) = s^{b+1}[s''E + 2s'F + s(F' + F^2)]$ reduces to $K(t) = F' + F^2$ and because $f_1 = f_2 = 0$, $\det K = 0$. On the other hand, write the equation of motion (3.1.9) as the matrix equation $(\frac{V_{r_4 0}}{m_4} + \frac{W_{r_4 0}}{m_4}, \frac{V_{r_5 0}}{m_5} + \frac{W_{r_5 0}}{m_5}, \frac{V_{r_6 0}}{m_6} + \frac{W_{r_6 0}}{m_6}) = K(t)(r_{40}, r_{50}, r_{60})$, taking the second triplet of bodies in Lemma 10, and use $F_{i0} = V_{r_i 0} + W_{r_i 0}$. This would imply that both $K(t)$ and (r_{40}, r_{50}, r_{60}) have a non-vanishing determinant and we arrived to a direct contradiction.

We are led to conclude that $w'(t) = 0$ and $\Omega(t) = \mathit{const} = E$, thus the theorem is proved, unless $b = 2$ or $a = 2$.

3.4 Counterexamples to the Lagrange and Pizzetti theorems for $a = b = 2$

In the first subsection to follow, we point out some of the difficulties in extending the above two theorems to a higher number of bodies, in the intermediate range of a and

b values (that is, for $a \neq 2$ *or* for $b \neq 2$). In the next two subsections we'll show that if the degree of the force functions are: $a = b = 2$ (which corresponds to the cubic force of attraction), then even for $N = 3$ and $N = 4$ one can construct counterexamples to Theorems 1 and 3, respectively. For the case $N = 3$, we use the idea of Wintner [15] and fill in all the necessary details.

3.4.1 The difficulty of extending Lagrange's theorem to $N = 4$ and $a \neq 2$ or $b \neq 2$

As we showed that Theorem 2 holds, we may wonder whether such an extension can be proved or disproved for a higher number of bodies. The main difficulty encountered for extending was that for $N = 4$ bodies in an arbitrary configuration, the corresponding expression of formula (3.2.11) is:

$$\frac{W_{x_1 0}}{m_1 x_{10}} - \frac{W_{y_1 0}}{m_1 y_{10}} = \frac{kbm_2(1-e)}{l_{13}^{b+2}} \left[1 - \frac{l_{13}^{b+2}}{l_{12}^{b+2}} - \frac{m_4}{m_2(e-1)} \left(\frac{y_{40}}{y_{10}} - \frac{x_{40}}{x_{10}} \right) \left(1 - (\frac{l_{13}}{l_{14}})^{b+2} \right) \right].$$

No direct conclusion similar to (3.2.12) can easily be made. On the other hand, this could suggest that this theorem does not hold for all configuration 4 bodies, and we tried to find a counterexample in which some special initial conditions could lead to a solution that was flat, but rotated around the x-axis, thus non-planar. In simple configurations, like the four bodies at vertexes of a square, or the fourth body situated at the center of mass, the solution failed to rotate, thus remaining planar. We then focused on the arrangement depicted in Fig. 3.4.1, in which l_{12} was arbitrary, while $y_{10},\ y_{30},\ y_{40}$ were variables to be determined from the condition that the given configuration leads to the rotation around the the x-axis, when $a = 2$ and $b > 2$ [2].

Even for small values of b (like, $b = 3$) the 3 equations for establishing the existence of the appropriate solution $\{y_{10},\ y_{30},\ y_{40}\}$ were of a degree higher than 7, which made the analytical proof of the existence of the solution extremely hard, if not impossible. Subsequently, extensive computer assisted numerical simulations showed that the counterexample failed: there where no initial values of $y_{10},\ y_{30},\ y_{40}$ to make the solution non-planar.

The proof of Theorem 3 shows that the corresponding analysis for $N = 4$ and $a = 2,\ b > 2$ or $b = 2,\ a < 2$ would be even harder. Moreover, the next development of the theory in this paper (see the next Corollary based on Theorems 3, 1, 2) requires

[2] The same approach was tried for the similar situation with $b = 2$ and $a < 2$.

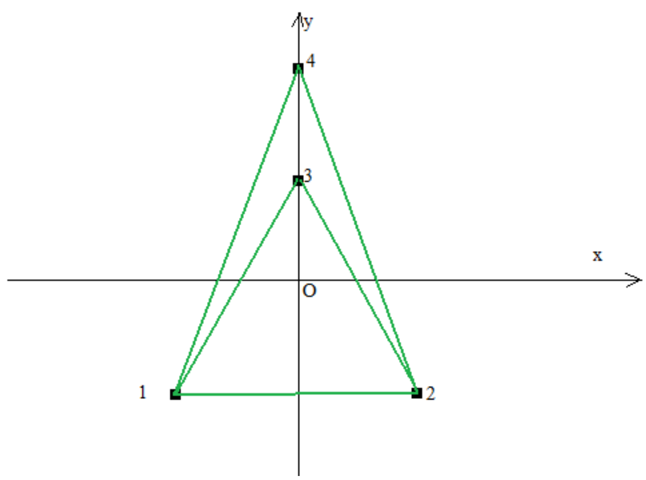

Figure 3.4.1: The configuration we tried for the failed counterexample.

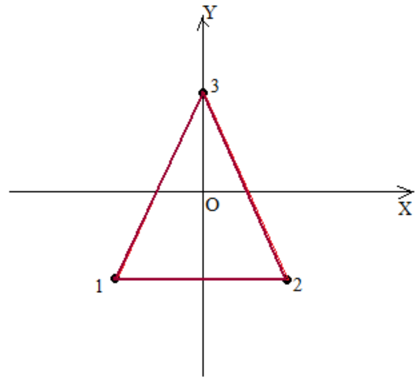

Figure 3.4.2: The isosceles configuration of equations (3.4.1).

that we first prove the flat theorem, then the non-flat one.

All this being said, the only remaining potentials for which we will make a firm conclusion is $a = b = 2$.

3.4.2 A counterexample to Theorem 1, $a = b = 2$

Let $N = 3$; we want to find such initial conditions (coordinates $x_{i0},\ y_{i0},\ z_{i0}$ and velocities $(x_i')_0,\ (y_i')_0,\ (z_i')_0$, where $i = 1,\ 2,\ 3$) that will result in the solution to the equations of motion be a flat (since $N = 3$) and homographic, but rotating around the x-axis solution (thus non-planar).

Choose the initial positions and masses of the 3 bodies in the $(x,\ y)$-plane as follows:

$$\begin{cases} m_2 = m_1, \\ x_{20} = -x_{10} = \frac{l_{12}}{2} > 0 = x_{30}, \\ y_{20} = y_{10} =< 0 < y_{30}, \\ 2m_1 y_{10} = -m_3 y_{30} \\ z_{j0} = 0,\ \forall j = 1,\ 2,\ 3. \end{cases} \tag{3.4.1}$$

Let us notice that the integrals of the center of mass,

$$\sum_{j=1}^{3} m_j x_{j0} = 0,\ \sum_{j=1}^{3} m_j y_{j0} = 0, \tag{3.4.2}$$

are satisfied.

Because $a = b = 2$, let $G = 2(k+1)$; thus $U = V + W = (k+1)V = \frac{G}{2}\sum_{i<j}\frac{m_i m_j}{|l_{ij}|^2}$ and the (general) equations of motions are:

$$\begin{cases} m_i x_i'' = U_{x_i} = m_i G \sum m_j \frac{(x_j - x_i)}{|l_{ji}|^4} \\ m_i y_i'' = U_{y_i} = m_i G \sum m_j \frac{(y_j - y_i)}{|l_{ji}|^4} \\ m_i z_i'' = U_{z_i} = m_i G \sum m_j \frac{(z_j - z_i)}{|l_{ji}|^4} \end{cases}, \quad i = 1, 2, 3. \tag{3.4.3}$$

From the symmetrical arrangement of the three bodies it follows that the following are guaranteed:

$$\begin{cases} U_{x_2 0} = -U_{x_1 0} < 0 = U_{x_3 0}, \\ U_{y_2 0} = U_{y_1 0} > 0, \\ U_{z_i 0} = 0, \ \forall i. \end{cases} \tag{3.4.4}$$

Let us notice that adding upon i each of the equations in (3.4.3) (and using the above symmetry equations), we get a more exact expression for the integral of the center of masses, *which, however, do not impose any additional conditions on the coordinates of the 3 bodies*:

$$\sum_i U_{x_i} = 0, \ \sum_i U_{y_i} = 0. \tag{3.4.5}$$

To prove that the solution will be homographic, i.e. $r_i = s(t)\Omega(t)r_{i0}$, but non-planar, we want the rotation to be around the x-axis, in the positive direction:

$$\begin{cases} \Omega(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos w & -\sin w \\ 0 & \sin w & \cos w \end{pmatrix}, \\ w'(t) > 0, \ w(0) = 0 \\ s(t) > 0, \ s(0) = 1, \\ s'(t) \neq 0. \end{cases} \tag{3.4.6}$$

In coordinate form, the solution should satisfy:

$$x_i = x_{i0} s; \quad y_i = y_{i0} s \cos w; \quad z_i = y_{i0} s \sin w. \tag{3.4.7}$$

Let $X = \frac{U_{x_1 0}}{m_1 x_{10}}$, $Y = \frac{U_{y_1 0}}{m_1 y_{10}}$. Then, using expressions (3.4.1), (3.4.2), (3.4.5)(3.4.4), the formula:

$$U_{x_i0} = X \cdot m_i x_{i0},\ U_{y_i0} = Y \cdot m_i y_{i0} \tag{3.4.8}$$

will hold not only for the first body, but for all $i = 1, 2, 3$.

To get all the conditions that are to be satisfied by the initial conditions, we substitute (3.4.7) into the equations of motion (3.4.3). For instance, the x-component of the force due to the force function W, that acts on the i-th body becomes:

$$W_{x_i} = m_i G \sum_{j=1,\, j\neq i}^{3} m_j \frac{s(x_{j0} - x_{i0})}{s^{b+2}|l_{ji}|^4} = s^{-3} U_{x_i0}. \tag{3.4.9}$$

After all substitutions, making use of (3.4.8), we get:

$$\begin{aligned} s'' &= s^{-3}X, && (3.4.10)\\ s''\cos w - 2s'w'\sin w - s(w')^2\cos w - sw''\sin w &= s^{-3}Y\cos w, && (3.4.11)\\ s''\sin w + 2s'w'\cos w - s(w')^2\sin w + sw''\cos w &= s^{-3}Y\sin w. && (3.4.12) \end{aligned}$$

Multiply (3.4.11) by $-\sin w$ and (3.4.12) by $\cos w$ and add them to get $2s'w'+sw'' = 0$. Next multiply (3.4.11) by $\cos w$ and (3.4.12) by $\sin w$ and add them to get $s''-s(w')^2 = s^{-3}Y$. Thus, the equations of motion conditions are

$$\begin{cases} s'' = Xs^{-3} < 0, \\ 2s'w' + sw'' = (s^2w')' = 0, \\ s'' - s(w')^2 = Ys^{-3}. \end{cases} \tag{3.4.13}$$

Next we notice that the three equations are to be satisfied by only two functions ($s(t)$ and $w(t)$); to ensure the existence of solutions, we will eliminate one of them. After substitution of the first equation for s'' into the third, we get that the third equation can be replaced by $s^2w'(t) = (X-Y)^{\frac{1}{2}}$, if we require $X - Y > 0$. But, using the computations carried out in Section 3.2.2, formula (3.2.11), replacing $e = -1,\ k \to \frac{G}{2}$, we get: $\frac{U_{x_10}}{m_1x_{10}} - \frac{U_{y_10}}{m_1y_{10}} = G\frac{2m_1}{l_{13}^4}\left\{1 - \left(\frac{l_{13}}{l_{12}}\right)^4\right\}$. Thus, $X - Y$ is a positive constant if we choose the initial position such that $l_{13} = l_{23} < l_{12}$, and then the second equation in (3.4.13) is eliminated by the other two. System (3.4.13) becomes equivalent to the following conditions of the the real functions $s(t)$ and $w(t)$:

$$\begin{cases} s'' = Xs^{-3}, \\ s^2 w'(t) = (X-Y)^{\frac{1}{2}} . \end{cases}$$

With the substitution $s' = p(s)$, the first equation transforms to $\frac{dp}{ds}p(s) = Xs^{-3}$, which has the solution $p^2 = (s')^2 = -Xs^{-2} + const$. Choose $s'_{t=0} = (-X)^{\frac{1}{2}}$, which makes $s' = \frac{(-X)^{\frac{1}{2}}}{s}$ and, after integration, $s^2(t) = 2(-X)^{\frac{1}{2}}t + const_2$. Because $s(0) = 1$ and $s(t) > 0$, we have: $const_2 = 1$ and $s(t) = [1 + 2(-X)^{\frac{1}{2}}t]^{\frac{1}{2}}$.

Replacing that expression into the second equation of the system above, we get $\frac{dw}{dt} = \frac{(X-Y)^{\frac{1}{2}}}{1+2(-X)^{\frac{1}{2}}t} \neq 0$ and after integration, $w(t) = \frac{(X-Y)^{\frac{1}{2}}}{2(-X)^{\frac{1}{2}}} \ln \frac{1+2(-X)^{\frac{1}{2}}t}{2(-X)^{\frac{1}{2}}} + const_3$. Since $w(0) = 0$, $const_3 = \frac{(X-Y)^{\frac{1}{2}}}{2(-X)^{\frac{1}{2}}} \ln[2(-X)^{\frac{1}{2}}]$ and $w(t) = \frac{(X-Y)^{\frac{1}{2}}}{2(-X)^{\frac{1}{2}}} \ln[1 + 2(-X)^{\frac{1}{2}}t]$.

In conclusion, the above mentioned initial positions ((3.4.1) and $l_{13} = l_{23} < l_{12}$) and the initial velocities determined from (3.4.7) with the real functions $s(t)$, $w(t)$ that satisfy conditions (3.4.6), means that the expressions (3.4.7) satisfy the equations of motions; therefore, the found solution is flat ($N = 3$), homographic but non-planar ($w' \neq 0 \implies \Omega(t) \neq const$).

3.4.3 A counterexample to Theorem 3, $a = b = 2$

Here we let $N = 4$; we want to find such initial conditions that will result in the solution to the equations of motion be non-flat, but rotating around the x-axis (thus not homothetic).

We choose the initial positions and masses of the 4 bodies in the (x, y)-plane as follows (see Fig. 3.4.3):

$$\begin{cases} m_2 = m_1, \\ m_3 = m_4, \\ x_{20} = -x_{10} = \frac{l_{12}}{2} > 0 = x_{30} = x_{40}, \\ y_{20} = y_{10} =< 0 < y_{30} = y_{40}, \\ 2m_1 y_{10} = -2m_3 y_{30} \\ z_{10} = z_{20} = 0 < z_{40} = -z_{30}. \end{cases} \tag{3.4.14}$$

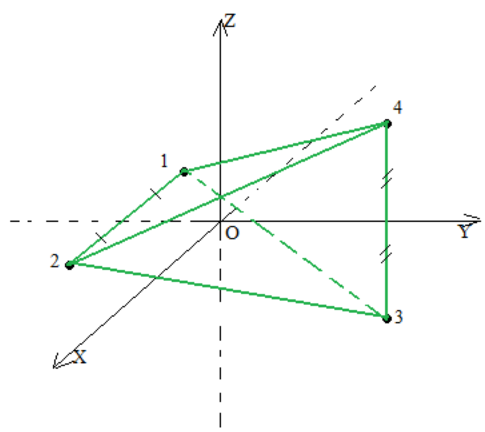

Figure 3.4.3: The tetrahedron configuration of conditions (3.4.14).

The integrals center of mass,

$$\sum_{j=1}^{4} m_j x_{j0} = 0,\ \sum_{j=1}^{4} m_j y_{j0} = 0,\ \sum_{j=1}^{4} m_j z_{j0} = 0 \tag{3.4.15}$$

are satisfied.

The (general) equations of motions are (3.4.3), unchanged.

From the symmetry it follows that the following are guaranteed:

$$\begin{cases} U_{x_2 0} = -U_{x_1 0} < 0 = U_{x_3 0} = U_{x_4 0}, \\ U_{y_2 0} = U_{y_1 0} > 0 > U_{y_3 0} = U_{y_4 0}, \\ U_{z_1 0} = U_{z_2 0} = 0 < U_{z_3 0} = -U_{z_4 0}. \end{cases} \tag{3.4.16}$$

The equations of the center of mass (3.4.5) remain the same, for the same reason. To prove that the solution will be not homothetic, we want the rotation to be around the x-axis, in the positive direction, i.e. we require the same system (3.4.6). In coordinates form, the solution should satisfy:

$$x_i = x_{i0}s;\quad y_i = s(y_{i0}\cos w - z_{i0}\sin w);\quad z_i = s(y_{i0}\sin w + z_{i0}\cos w). \tag{3.4.17}$$

Notice that since, initially, all bodies are not in one and the same plane, they will remain in different planes forever, that is, *the homographic solution will be non-flat.*

We make the notations: $X = \frac{U_{x_1 0}}{m_1 x_{10}} < 0,\ Y = \frac{U_{y_1 0}}{m_1 y_{10}} < 0,\ Z = \frac{U_{z_3 0}}{m_1 z_{30}} < 0$. Then, using expressions (3.4.14), (3.4.15), (3.4.5)(3.4.16), the formula:

$$U_{x_1 0} = X \cdot m_i x_{i0},\ U_{y_i 0} = Y \cdot m_i y_{i0},\ U_{z_i 0} = Z \cdot m_i z_{i0} \tag{3.4.18}$$

will hold for all $i = 1, 2, 3, 4$.

After all substitutions of (3.4.17) into the equations of motion (3.4.3), making use of (3.4.18), we get:

$$s'' = s^{-3}X \tag{3.4.19}$$

$$\begin{aligned} y_{i0}[s''\cos w - 2s'w'\sin w - s(w')^2\cos w - sw''\sin w] - z_{i0}[s''\sin w + \\ +2s'w'\cos w - s(w')^2\sin w + sw''\cos w] = s^{-3}(Yy_{i0}\cos w - Zz_{i0}\sin w) \end{aligned} \tag{3.4.20}$$

$$\begin{aligned} y_{i0}[s''\sin w + 2s'w'\cos w - s(w')^2\sin w + sw''\cos w] + z_{i0}[s''\cos w - \\ -2s'w'\sin w - s(w')^2\cos w - sw''\sin w] = s^{-3}(Yy_{i0}\sin w + Zz_{i0}\cos w). \end{aligned} \tag{3.4.21}$$

Multiplying (3.4.20) by $-\sin w$ and (3.4.21) by $\cos w$ and adding them, we obtain $y_{i0}(2s'w' + sw'') + z_{i0}[s'' - s(w')^2] = z_{i0}Zs^{-3}$. Next multiply (3.4.20) by $\cos w$ and (3.4.21) by $\sin w$ and add them to get $y_{i0}[s'' - s(w')^2] - z_{i0}[2s'w' + sw'') = s^{-3}Y$. For $i = 1, 2$, these two equations are equivalent to the last two equations in (3.4.13). If we can make $Z = Y$, then the above equations (3.4.19), (3.4.20), (3.4.21) will be implied by the system (3.4.13) for any $i = 1, 2, 3, 4$.

We can easily compute that $Y = -2G\frac{m_3}{l_{13}^4}\left(\frac{m_1}{m_3} + 1\right)$, $Z = -2G\frac{m_3}{l_{13}^4}\left(\frac{m_1}{m_3} + \left(\frac{l_{13}}{l_{34}}\right)^4\right)$, therefore the condition $Z = Y$ is satisfied if we put $l_{13} = l_{34}$ in the initial positions.

Next, we repeat the same analysis of (3.4.13), and with the condition $X > Y$ equivalent to $l_{13} < l_{12}$, the same solutions $s(t)$ and $w(t)$ are obtained.

In summary, the above mentioned initial positions ((3.4.14) and $l_{13} = l_{34} = l_{23} < l_{12}$) and the initial velocities determined from (3.4.17) with the real functions $s(t)$, $w(t)$ that satisfy conditions (3.4.6), means that the expressions (3.4.17) satisfy the equations of motions. Therefore, the found solution is non-flat ($N = 4$), homographic but not homothetic ($w' \neq 0 \implies \Omega(t) \neq const$).

3.5 Extension of Lagrange-Pizzetti Theorem

Here we will prove important consequences of the theorems from the previous two sections (partly announced in Section 3.1). Also, we will find the expressions for the conservation integrals for homographic solutions, and gather the relations needed for the next chapter.

Theorems 1 and 2 tell us that a non-planar homographic solution cannot be flat; together with Theorem 3, we have the following:

Corollary 2. *For a quasihomogeneous N-body problem with $a \neq 2$ and $b \neq 2$, and also for a quasihomogeneous 3-body problem with $a \neq 2$ or $b \neq 2$, if a homographic solution is not planar, then it is homothetic.*

Let us first find the formulas for kinetic energy and total angular momentum for planar homographic solution. Namely, in this case, the z- coordinate can be chosen to be zero and therefore, the rotation matrix $\Omega(t)$ and the corresponding F-matrix are given by:

$$\Omega(t) = \begin{pmatrix} \cos w & -\sin w & 0 \\ \sin w & \cos w & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; F(t) = \begin{pmatrix} 0 & -w'(t) & 0 \\ w'(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5.1}$$

where $w'(t)$ is the angular velocity of the rotating configuration. From formula (3.1.8) of Section (3.1), by multiplying it with $\Omega(t)$, we get: $r_i'(t) = (s'\Omega + s\Omega F)r_{i0} = \{(s\cos w)'x_{i0} - (s\sin w)'y_{i0},\ (s\sin w)'x_{i0} + (s\cos w)'y_{i0},\ 0\}$. Thus $(r_i')^2 = \{[(s\cos w)']^2 + [(s\sin w)']^2\}x_{i0}^2 + \{[(s\sin w)']^2 + [(s\cos w)']^2\}y_{i0}^2$, and, after simple calculations, $(r_i')^2 = (s'^2 + s^2 w'^2)r_{i0}^2$.

We also consider the moment of inertia $I = \sum_i m_i r_i^2$, which for a homographic solution becomes

$$I = \sum_i m_i (s r_{i0})^2 = s^2 I_0, \tag{3.5.2}$$

because the rotation does not change the distance from the origin $|r_i|$.

Therefore, we find that the kinetic energy $T = \frac{1}{2}\sum_i m_i (r_i')^2$ takes the form:

$$T = \frac{1}{2}(s'^2 + s^2 w'^2) I_0. \tag{3.5.3}$$

Next, using the definition (3.1.1) of homographic solutions as $r_i = s\Omega r_{i0}$, we find that:

$$r_i \times r_i' = \begin{pmatrix} s\cos w \cdot x_{i0} - s\sin w \cdot y_{i0} \\ s\sin w \cdot x_{i0} + s\cos w \cdot y_{i0} \\ 0 \end{pmatrix} \times \begin{pmatrix} (s\cos w)'x_{i0} - (s\sin w)'y_{i0} \\ (s\sin w)'x_{i0} + (s\cos w)'y_{i0} \\ 0 \end{pmatrix},$$

and after reductions, it simplifies to $r_i \times r_i' = (0,\ 0,\ w'(s r_{i0})^2)$.

Using all these formulas, the absolute value of the total angular momentum $C = \sum_i m_i r_i \times r_i'$ becomes:

$$|C| = \left| \sum_i m_i w' (s r_{i0})^2 \right| = w' s^2 I_0. \tag{3.5.4}$$

(Since $C = (0,\ 0,\ w'\sum_i m_i (s r_{i0})^2)$ is an integration constant, it follows that w' cannot change the sign during the motion; therefore we can choose, from the beginning, the orientation of the coordinate system such that the configuration rotates in the counterclockwise direction, i.e. $w' \geq 0$, and thus (3.5.4) makes sense.)

Now we are ready to state and prove the extension of Lagrange-Pizzetti theorem:

Theorem 4. *For a quasihomogeneous N-body problem with $a \neq 2$ and $b \neq 2$, and also for a quasihomogeneous 3-body problem with $a \neq 2$ or $b \neq 2$, the following statements hold:*
(I) A homographic solution is homothetic if and only if $C = 0$ (i.e., iff the motion has no invariable plane);
(II) A homographic solution is a relative equilibrium if and only if it is planar and rotates with a constant, non-zero, angular velocity.

Proof. Let us discuss separately planar and non-planar cases.

The planar case. Since $s > 0$ by definition, equation (3.5.4) implies that $C = 0 \iff w' = 0$. According to the definition (3.1.2), $w' = 0$ tells us that the solution is homothetic, so part (I) is proved. Next, from (3.1.3), for a relative equilibrium, $s = const > 0$ and $w' \neq 0$; thus, formula (3.5.4) shows that $w' = const$. Conversely, if $w' = const \neq 0$, then the same formula implies that $s = const$; hence, (II) is also proved.

The non-planar case. According to Corollary 2, the homographic solution is homothetic. This, on the one hand, excludes/completes the (II) part of the theorem; on the other hand, it completes the converse part of (I) (our solution is homothetic, even when $C = 0$). We are left to prove the direct part of (I). But, for a homothetic solution, $C = \sum_i m_i r_i \times r_i' = \sum m_i s r_{i0} \times s' r_{i0} = 0$, which completes the entire proof. □

Some consequences of this theorem concern collinear solutions.

First, for a collinear solution to be a relative equilibrium, in view part (II), it should not be rectilinear (but this condition is not sufficient).

Second, let a solution be collinear and not rectilinear; using Lemma 7 of Section 2.3.2, we see that $C \neq 0$. Then from Lemma 6 it follows that the solution is homographic, but, with part (I) of the current theorem, it is not homothetic. On the other hand, a homothetic and collinear solution (thus, with part (I), $C = 0$) is identical to a homographic and rectilinear one (using Lemma 7 again).

In what follows, we prepare the relations that, apart from being self-relevant, will also be used at the end of the next chapter.

Using the above Corollary, it follows that the kinetic energy in the non-planar case is $T = \frac{1}{2}\sum_i m_i(s' r_{i0})^2 = \frac{1}{2}s'^2 I_0$. Thus, formula (3.5.3) holds for the non-planar case as well, if we consider $w' = 0$. For the same reason, formula (3.5.4) holds for non-planar solutions, with $w' = 0$, because using part (I) of Theorem 4 we see that $C = 0$ in this case.

The expression of the force function (3.1.4) holds both for planar, and non-planar situations; using (3.1.4) and (3.5.3), we can write the energy integral for homographic solutions $h = T - U$ as:

$$h = \frac{1}{2}(s'^2 + s^2 w'^2) I_0 - V_0 s^{-a} - W_0 s^{-b}. \tag{3.5.5}$$

In Section 2.3.1, we obtained that the *Lagrange-Jacobi* identity for the quasihomogeneous force function is $I'' = 4h + 2(2-a)V + 2(2-b)W$. Substituting (3.1.4) and (3.5.2) for $V,\ W,\ I$, the homographic version of that relation becomes:

$$(ss'' + s'^2) I_0 - (2-a) V_0 s^{-a} - (2-b) W_0 s^{-b} = 2h. \tag{3.5.6}$$

Finally, using the general formula (3.1.10) for $K(t)$, and expressions (3.5.1), for the planar case we get:

$$K(t) = s^{b+1} \begin{pmatrix} s'' - sw'^2 & -2s'w' - sw' & 0 \\ 2s'w' + sw'' & s'' - sw'^2 & 0 \\ 0 & 0 & s'' \end{pmatrix}, \tag{3.5.7}$$

while for the non-planar case (where, according to Corollary 2, $\Omega = const$, $F = const$) we have:

$$K(t) = s^{b+1} \begin{pmatrix} s'' & 0 & 0 \\ 0 & s'' & 0 \\ 0 & 0 & s'' \end{pmatrix}.$$

Hence, we again see that the planar relation (3.5.7) can be applied for non-planar solutions as well, with the same convention: $w' = 0$.

Chapter summary. We have introduced the equations that define and classify the homographic solutions (general, homothetic and relative equilibria). Then, in the ensuing three sections, we proved the tight connection of those types with the spatial classification of solutions (flat/planar), culminating with some necessary and sufficient conditions for a homographic solution to be either homothetic or a relative equilibrium (Theorem 4).

But we haven't addressed the question of *existence* and *construction* of such solutions. The second section of Chapter 4 is devoted to these issues, and it will largely use the consequences presented in Section 3.5. As we shall see, the construction of homographic solutions is based on the notion of *central configuration*, with which we begin the next chapter.

Chapter 4

Central configurations and the homographic solutions

4.1 Central Configurations

To definine central configurations (denoted shortly by CC) of a system of N bodies, with fixed masses m_i, we will follow A. Wintner [15]:

Definition 7. The configuration r, formed by the N position vectors r_i of the particles in the system, is said to be a *central configuration* (CC), if:

$$\exists l = const \ : \ \forall i \in \{1, .., N\}, \ F_i = \frac{\partial U}{\partial r_i} = l m_i r_i. \tag{4.1.1}$$

In vectorial form,

$$\exists l = const \ : \ M^{-1}\frac{\partial U}{\partial r} - lr = 0, \tag{4.1.2}$$

where M is the diagonal matrix with entries x_1, y_1, z_1, ..., x_N, y_N, z_N.

Multiply (4.1.1) by r_i and sum over i to get: $\sum_i r_i \frac{\partial U}{\partial r_i} = l \sum_i m_i r_i^2 = lI$. But, using expression (2.3.4) we have:

$$l = \frac{-(aV + bW)}{I} < 0. \tag{4.1.3}$$

Another type of configurations for the quasihomogeneous force functions, introduced by Diacu *et al.* in [6], occurs when considering the component force functions V and W separately:

Definition 8. The configuration r is called a *simultaneous central configuration* (SCC) if r is a CC for each of the force functions V and W, i.e.:

$$\begin{cases} \exists l_v = const: & M^{-1}\frac{\partial V}{\partial r} - l_v r = 0 \ \& \\ \exists l_w = const: & M^{-1}\frac{\partial W}{\partial r} - l_w r = 0. \end{cases} \tag{4.1.4}$$

Using the same method as for the force function U, we can find that:

$$l_v = -\frac{aV}{I} < 0, \; l_w = -\frac{bW}{I} < 0. \tag{4.1.5}$$

Note that if r is a CC for V or for W, then (4.1.3) holds with the respective modifications.

As an example, in the 3-body problem, consider the three masses on a straight line, with m_2 symmetrically situated between the other two and $m_1 = m_3$. This configuration is simultaneous, which shows that in general, the class of SCC is not empty (and of CC is not empty as well).

There are simple connections between the two types of configurations, reflected by the next two propositions.

Proposition 1. *The configuration r is a SCC if and only if r is a CC for U and r is a CC for at least V or W.*

Proof. The direct statement follows easily by adding together the equations (4.1.4); r will be a CC for U with $l = l_v + l_w$. For the converse statement, substitute the equation for l_v in the relation (4.1.2) of CC for U, to get $M^{-1}\frac{\partial W}{\partial r} + l_v r - lr = 0$. From this we see that r will also be a CC for W (with $l_w = l - l_v$) and, by the definition of SCC, a SCC as well. □

Proposition 2. *The configuration r is a SCC if and only if $aV\frac{\partial W}{\partial r} = bW\frac{\partial V}{\partial r}$ and r is a CC for at least one of the potentials: U, W or V.*

Proof. For a SCC, the condition of a CC for W can be written as $\frac{\partial W}{\partial r} = l_w Mr = \frac{l_w}{l_v} l_v Mr = \frac{l_w}{l_v}\frac{\partial V}{\partial r}$, and using (4.1.5) the direct part is proved.

Now let r be a CC for U (when it is a CC for W or for V, the treatment is even simpler) and for this configuration, let $\frac{\partial W}{\partial r} = \frac{bW}{aV}\frac{\partial V}{\partial r}$. Then, substituting this last relation for $\frac{\partial W}{\partial r}$ in (4.1.2), we get: $\frac{\partial V}{\partial r} = \frac{1}{\frac{bW}{aV}+1} lMr$. Using (4.1.3) and (4.1.5), it follows that r is a CC for V as well. Applying Proposition 1, it follows that r is a SCC and the converse statement is proved. □

(A trivial case where this proposition applies is the case of homogeneous force functions: when $a = b$, for all i, $\frac{\partial W}{\partial r_i} = k\frac{\partial V}{\partial r_i}$, where $k > 0$ is the constant of proportionality of W: $U = V + W = \sum_{1\leq j<i\leq n} m_j m_i \left(\frac{1}{r_{ji}^a} + \frac{k}{r_{ji}^b}\right)$.)

Intuitively, if we rotate the entire system of N bodies that form a CC, the new configuration will still be central. To see this mathematically, we write the condition of a CC (4.1.1) in the form:

$$\sum_{j\neq i} m_j(r_j - r_i)[\frac{a}{|r_i - r_j|^{a+2}} + \frac{k \cdot b}{|r_i - r_j|^{b+2}}] - lr_i = 0,$$

for some constant l and for all $i \in \{1, .., N\}$. Now take any rotation matrix $A \in SO(3)$, which changes each position vector r_i into Ar_i, and multiply both sides of the above relation by A. Because distances between particles remain unchanged after rotation, we get: $\sum_{j\neq i} m_i m_j (Ar_j - Ar_i)[\frac{a}{|Ar_i - Ar_j|^{a+2}} + \frac{k\cdot b}{|Ar_i - Ar_j|^{b+2}}] - m_i l A r_i = 0$, which shows that the new configuration is central, with the same constant l.

For the classical force this is also true under the action of homotheties (proportional dilating or contracting). Because the shape of the configuration remains unchanged under such transformations, we are led to the idea of equivalent central configurations and equivalence classes:

Definition 9. Two CC-s, r and r^*, are called *equivalent* if they can be made identical via a rotation and/or a homothety. That is, there exists a real rotation matrix R and a constant $c > 0$, such that:

$$r^* = c \cdot R \cdot r. \tag{4.1.6}$$

All CC-s that are equivalent to one another form a *(equivalence) class* of CC-s.

For Manev-type force functions (quasihomogeneous force functions with $a = 1$), Diacu et. all [6] concluded that homotheties of CC-s are not always CC-s.
In this section, we will investigate the general question: *Given a CC r, is $c \cdot r$ also a CC, for some $c > 0$? If so, then under what conditions?*

In connection with this question, R. Jones [7] introduced the following subtype of CC-s:

Definition 10. A CC r is called an *extraneous central configuration* (denote it by ECC) if there exists a positive constant c, $c \neq 1$, such that cr is not a CC.

Let {CC}, {SCC} and {ECC} denote, respectively, the sets of all possible: CC-s, SCC-s and ECC-s. R. Jones made the following conjectures:

Conjecture 1. (R. Jones) *For a quasihomogeneous force function:*

1. *{SCC} ={CC}\{ECC}.*

2. *If r is a ECC, then for all $c > 0,\ c \neq 1,\ cr$ is not a CC.*

Let us first show that for a homogeneous potential of any degree $-a,\ a > 0$, the homotheties of a CC are again CC-s. The idea we follow was used by R. Moeckel [8] to prove the property for the Newtonian case.

Lemma 11. *If r is a CC for V, then for any constant $c > 0$, the configuration $c \cdot r$ is also a CC for V.*

Proof. We need to show the existence of a suitable negative constant l_v^* such that equation (4.1.2) be satisfied by this constant and the configuration $c \cdot r$; i.e. we need to prove that $\exists l_v^* < 0$, such that:

$$\begin{cases} M^{-1}\frac{\partial V(cr)}{\partial(cr)} - l_v^* cr = 0, \\ l_v^* = -\frac{aV(cr)}{I(cr)}. \end{cases} \tag{4.1.7}$$

But $\frac{\partial V(cr)}{\partial r} = c\frac{\partial V(cr)}{\partial(cr)}$ and $\frac{\partial}{\partial r}V(cr) = \frac{\partial}{\partial r}\sum_{i<j}\frac{1}{|cr_i - cr_j|^a} = \frac{1}{c^a}\frac{\partial}{\partial r}V(r)$, thus $M^{-1}\frac{\partial V(cr)}{\partial(cr)} - l_v^* cr = \frac{1}{c^{a+1}}(M^{-1}\frac{\partial V(r)}{\partial r} - l_v^* c^{a+2} r)$. If one takes $l_v^* = l_v c^{-a-2} < 0$ then, in view of first equation in (4.1.4), the last expression becomes zero and formula (4.1.7) becomes true. (Additionally, we can easily check that the second equation of the system is also satisfied then.) □

Obviously, for a homogeneous force function, {ECC} is empty.

We are ready to prove the main result of this section:

Theorem 5. *Let r be a CC for the quasihomogeneous force function U. The following statements are equivalent:*

1. *r is a SCC;*

2. *$\forall c > 0,\ cr$ is a SCC;*

3. *$\exists c > 0,\ c \neq 1$, such that cr is a CC.*

Proof. If r is a SSC for U, then r is a CC both for V and W. Using Lemma 11, cr is a CC both for V and W and the implication (1) $\implies$ (2) of the theorem follows.

The second implication is obvious and therefore we are left to prove: (3) $\implies$ (1). The hypothesis of the third statement implies that $\exists l^* = const: \ M^{-1}\frac{\partial U(cr)}{\partial (cr)} - l^* cr = 0$. Using $U = V + W$ and $\frac{\partial}{\partial r}V(cr) = \frac{1}{c^a}\frac{\partial}{\partial r}V(r)$, (similarly for W), we get:

$$\exists l^* = const \ : \ \frac{1}{c^{a+1}}[M^{-1}(\frac{\partial V(r)}{\partial r} + c^{a-b}\frac{\partial W(r)}{\partial r}) - l^* c^{a+2} r] = 0.$$

One the other hand, r is a CC for U, so we can substitute $M^{-1}\frac{\partial V}{\partial r}$ from (4.1.2) into the above relation to obtain:

$$\frac{c^{a-b}}{c^{a+1}}[M^{-1}\frac{\partial W(r)}{\partial r} - (\frac{l^* c^{a+2} - l}{c^{a-b} - 1})r] = 0,$$

for some constants $l, \ l^*, \ c, \ c \neq 1$. But this means there exists a constant $l_w = \frac{l^* c^{a+2} - l}{c^{a-b} - 1}$ such that the second equation in (4.1.4) is satisfied, and so r is a CC for W (detailed computations show that $l_w = \frac{-bW}{I}$, as expected). Proposition 1 helps us complete the proof. □

As a consequence, once a configuration is a ECC, then for all $c > 0, \ c \neq 1$, cr is not a CC. Another conclusion follows from comparing Theorem 5 with the definition of a ECC:

Corollary 3. *The set of CC-s , $\{CC\}$, consists of two disjoint subsets: $\{SCC\}$ (invariant both under rotation, and under homothety) and $\{ECC\}$ (only invariant under rotation).*

Thus, Jones's Conjectures 1 are proved.

In [6] Diacu *et al.* prove that in the collinear 3-body problem, most of the CC-s are not simultaneous; this implies that, in general, the {ECC} set is not empty and suggests that {SCC} is a poor set.

An example of non-trivial central configurations for quasihomogeneous force functions are the equilateral triangle configurations for the three body problem with Manev potentials ($a = 1$). It has been shown that there exists exactly 2 classes of such configurations, correspondig to the two orientations of a triangle in a plane, [6].

4.2 Homographic solutions and central configurations

If in Chapter 3 we discussed the main properties and types of homographic solutions, in this section we will address the question of existence of these solutions. Also, we will show how homographic solutions can actually be constructed.

The following theorem[1] shows the connection between homographic solutions and central configurations:

Theorem 6. *Consider the quasihomogeneous N-body problem with $a \neq 2$ and $b \neq 2$, or the quasihomogeneous 3-body problem with $a \neq 2$ or $b \neq 2$. If a solution is homographic, then during its time of existence:*

1. *in case of homogeneous potentials ($a = b$), the configuration forms equivalent central configurations;*

2. *in case of non-homogeneous potentials ($b>a$), the configuration forms equivalent simultaneous central configurations if the solution is not a relative equilibrium, and equivalent central configurations if the solution is a relative equilibrium.*

Conversely, if the solution forms equivalent central configurations while evolving in time, then that solution is homographic.

Proof. We first aim to show that, in the conditions stated in the theorem, if a solution is homographic, then the initial configuration r_0 should be a CC. Subsequently, we will show that the same will be true for the configuration $r(t)$ at any moment in time.

As in Section 3.5 of part I, if the solution is planar, then we orient the coordinate system so that $z_i = 0,\ \forall i$, and the angular velocity of the rotating configuration is positive $w'(t) \geq 0$. Then, with the convention $w'(t) = 0$ for a non-planar solution (see Corollary 2), all relations specific for the planar case, described in detail in Section 3.5, will hold for a general homographic solution.

Since $I_0 > 0,\ V_0 > 0,\ W_0 > 0$, we can denote:

$$p_0 = \frac{V_0}{I_0} > 0,\ q_0 = \frac{W_0}{I_0} > 0,\ h_0 = \frac{h}{I_0} > 0,\ C_0 = \frac{C}{I_0}. \tag{4.2.1}$$

[1] For the 3-body problem, with classical potential, this result is due to Lagrange.

With them, the relations for the energy integral (3.5.5), the total angular momentum (3.5.4) and the Lagrange-Jacobi identity (3.5.6) become, respectively:

$$\begin{aligned} h_0 &= \frac{1}{2}(s'^2 + s^2 w'^2) - p_0 s^{-a} - q_0 s^{-b}, && (4.2.2)\\ |C_0| &= w' s^2, && (4.2.3)\\ 2h_0 &= (ss'' + s'^2) - (2-a)p_0 s^{-a} - (2-b)q_0 s^{-b}. && (4.2.4) \end{aligned}$$

We can equate s'^2 from (4.2.2) and (4.2.4); also, we notice that, from (4.2.3), $\frac{(s^2w')'}{s} = 2s'w' + sw'' = 0$. Altogether, we obtain:

$$\begin{cases} s'' - sw'^2 = -\frac{ap_0}{s^{a+1}} - \frac{bq_0}{s^{b+1}}, \\ 2s'w' + sw'' = 0. \end{cases} \quad (4.2.5)$$

We are going to use this information in the equations of motion, which for homographic solutions were deduced in Section 3.1, equation (3.1.9), to be: $K(t) \cdot r_{i0} = s^{b-a}\frac{V_{r_i 0}}{m_i} + \frac{W_{r_i 0}}{m_i}$, $i \in \{1, ..., N\}$.

For the planar case, substituting (4.2.5) into $K(t)$ given by (3.5.7), we get that the matrix $K(t)$ becomes:

$$K(t) = \begin{pmatrix} -s^{b-a}ap_0 - bq_0 & 0 & 0 \\ 0 & -s^{b-a}ap_0 - bq_0 & 0 \\ 0 & 0 & -s^{b-a}ap_0 - bq_0 + s^{b+2}w'^2 \end{pmatrix}. \quad (4.2.6)$$

The same substitution into $K(t)$ for the non-planar case, with $w' = 0$, gives that:

$$K(t) = \begin{pmatrix} -s^{b-a}ap_0 - bq_0 & 0 & 0 \\ 0 & -s^{b-a}ap_0 - bq_0 & 0 \\ 0 & 0 & -s^{b-a}ap_0 - bq_0 \end{pmatrix}.$$

Knowing that for the planar case, $z_{i0} = 0$, $V_{z_i 0} = W_{z_i 0} = 0$, we find that for any homographic solution, $K(t)r_{i0} = (-s^{b-a}ap_0 - bq_0) \cdot r_{i0}$ and the equations of motion can be written as:

$$(-s^{b-a}ap_0 - bq_0) \cdot r_{i0} = s^{b-a}\frac{V_{r_i 0}}{m_i} + \frac{W_{r_i 0}}{m_i},\ i \in \{1, ..., N\}. \quad (4.2.7)$$

Now we need to divide the treatment in two cases, as to whether or not $s^{b-a} =$

const.

1. Let $b = a$ (homogeneous force function) or $s = const = 1$ (relative equilibrium solution). With $U_{r_i0} = V_{r_i0} + W_{r_i0}$, equation (4.2.7) becomes $U_{r_i0} = (-ap_0 - bq_0)m_i r_{i0}$. But, in view of the definition (4.1.1), this means that the initial configuration given by r_{i0} is a CC, with $l = (-ap_0 - bq_0) = -\frac{aV_0+bW_0}{I_0} < 0$.

As shown in Section 4.1, the rotation of a CC is a CC of the same class. Also, because in a homographic motion the successive configurations are formed by composing rotations (which preserve the class of a CC) and homotheties, then according to Definition (9) and Lemma (11), it follows that $r_i(t)$ will form a CC of the same class as r_{i0}, for all times of the solution t.

2. If $b > a$ (purely quasihomogeneous case) and $s \neq const$ (the homographic solution is not a relative equilibrium), then (4.2.7) can be written as: $s^{b-a}\left(\frac{V_{r_i0}}{m_i} + ap_0 r_{i0}\right) + \left(\frac{W_{r_i0}}{m_i} + bq_0 r_{i0}\right) = 0$. Because it has to be satisfied for all times t , that relation implies the following system:

$$\begin{cases} V_{r_i0} = \left(\frac{\partial V}{\partial r_i}\right)_{t=0} = -ap_0 m_i r_{i0}, \\ W_{r_i0} = \left(\frac{\partial W}{\partial r_i}\right)_{t=0} = -bq_0 m_i r_{i0}. \end{cases}$$

But according to definition 8, this system implies that the initial configuration r_{i0} is a simultaneous central configuration, with constants $l_v = -ap_0 = -\frac{aV_0}{I_0}$, $l_w = -q_0 = -\frac{bW_0}{I_0}$.

Moreover, using Theorem 5 of Section 4.1, it follows that during the time of the solution, $r_i(t)$ form simultaneous central configurations, of the same class as those formed by r_{i0} (applying again Definition 9).

Thus, the proof of the first part of Theorem 6 is finished.

The second (converse) part is rather straightforward. Namely, if $r(t)$ is a solution of the N-body problem that forms central configurations of a particular class, then, according to Definition (9) and Definition (4), $r(t)$ will be necessarily homographic. □

Diacu *et al.,* using totally different methods, proved in [6] that, for the Manev-type N-body problem, if a solution is homothetic, then it forms simultaneous central configurations for all time. Thus, this can be seen as a particular case of Theorem 6.

As follows from the definition (4) of a homographic solution, from the arguments of Section 3.5 and of the beginning of the above proof, any homographic solution

$r_i(t),\ i = 1, ..., N$, can be represented as:

$$\begin{cases} r_i(t) = s(t)\Omega(t)r_{i0}, \\ \Omega(t) = \begin{pmatrix} \cos w & -\sin w & 0 \\ \sin w & \cos w & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{cases} \tag{4.2.8}$$

with $s,\ w$ satisfying:

$$\begin{cases} s(t) > 0,\ s(0) = 1 \\ w(0) = 0,\ w'(0) \geq 0, \\ \begin{cases} w'(t) \geq 0, & r(t) - planar, \\ w'(t) = 0, & r(t) - non-planar. \end{cases} \end{cases} \tag{4.2.9}$$

The next theorem concerns the construction of any homographic solution.

Theorem 7. *Consider the quasihomogeneous N-body problem with $a \neq 2$ and $b \neq 2$, or the quasihomogeneous 3-body problem with $a \neq 2$ or $b \neq 2$. Let $r_i(t),\ i = 1, ..., N$, be some vector-functions defined by (4.2.8), where $s(t),\ w(t)$ are some real functions and $r_{i0},\ i = 1, ..., N$, are some constant position vectors. Then:*

1. *for homogeneous force functions ($a = b$), $r_i(t)$ represent a homographic solution if and only if r_{i0} form a central configuration and $s(t),\ w(t)$ satisfy (4.2.5), (4.2.9);*

2. *for non-homogeneous force functions ($b>a$), $r_i(t)$ represent a relative equilibrium solution if and only if r_{i0} form a central configuration, $s(t) = 1$ and $w(t)$ satisfies (4.2.5), (4.2.9);*

3. *for non-homogeneous force functions ($b>a$), $r_i(t)$ represent a homographic solution which is not a relative equilibrium if and only if r_{i0} form a simultaneous central configuration, $s(t) \neq const$ and $s(t),\ w(t)$ satisfy (4.2.5), (4.2.9);*

where $p_0,\ q_0$ are given by (4.2.1).

Proof. The direct part of all three situations follows immediately from the previous theorem and it's proof.

To prove the converse, we need to show that the functions $r_i(t)$ satisfy the general equations of motion $m_i r_i'' = V_{r_i} + W_{r_i}$, or

$$\Omega^{-1} m_i r_i'' = \Omega^{-1} U_{r_i}. \tag{4.2.10}$$

We analyze the above 3 cases separately, beginning with the third one, the most general:

- For $a < b$, $s \neq const$, because r_{i0} form a SCC, formulas (4.1.1), (4.1.5) imply that $V_{r_i0} = -\frac{aV_0}{I_0} m_i r_{i0}$, $W_{r_i0} = -\frac{bW_0}{I_0} m_i r_{i0}$. We can write $\Omega^{-1} U_{r_i} = \frac{V_{r_i0}}{s^{a+1}} + \frac{W_{r_i0}}{s^{b+1}}$ (see Section 3.1, formula (3.1.7)) and, using (4.2.8), $(r_i)'' = (s\Omega)'' r_{i0}$. Substituting all these into (4.2.10), the relation to be proved becomes: $s^{b+1}\Omega^{-1}(s\Omega)'' r_{i0} = (-s^{b-a} \cdot a\frac{V_0}{I_0} - b\frac{W_0}{I_0}) r_{i0}$.
  Now, with direct differentiations, $s^{b+1}\Omega^{-1}(s\Omega)'' = s^{b+1}[s''E + 2s'\Omega^{-1}\Omega' + s\Omega^{-1}\Omega'']$, which is, according to Lemma 1 and equations 2.3.5, 2.3.8, equal to $K(t)$ in (3.1.10). Using (3.5.1) we find that

$$K(t) = s^{b+1} \begin{pmatrix} s'' - sw'^2 & -2s'w' - sw' & 0 \\ 2s'w' + sw'' & s'' - sw'^2 & 0 \\ 0 & 0 & s'' \end{pmatrix}. \tag{4.2.11}$$

  But because (4.2.5) are satisfied by s, w, this matrix transforms exactly to (4.2.6). With the same reasoning as in the proof of Theorem 6, using the convention about the sign of w', we get that $s^{b+1}\Omega^{-1}(s\Omega)'' r_{i0} = K(t) r_{i0} = (-s^{b-a} a p_0 - b q_0) \cdot r_{i0}$. Thus, with p_0, q_0 given by (4.2.1), the equation of motion is proved.

- When $a = b$, $W = kV$ and because r_{i0} form a CC, Proposition 2 tells us that r_{i0} are SCC-s, and from here on the treatment is almost identical to the above case. The relation to be proved becomes: $s^{b+1}\Omega^{-1}(s\Omega)'' r_{i0} = -b(\frac{V_0}{I_0} + \frac{W_0}{I_0}) r_{i0}$, while $s^{b+1}\Omega^{-1}(s\Omega)'' r_{i0}$ turns out to exactly equate $(-s^{b-a} a p_0 - b q_0) \cdot r_{i0}$. Thus, with $a = b$ and p_0, q_0 given by (4.2.1), the equations of motion are proved.

- If $b > a$, but $s = const = 1$, the treatment is even simpler. The vectors r_{i0} forming a CC, we have $U_{r_i0} = \frac{-(aV_0 + bW_0)}{I} m_i r_{i0}$; with (3.1.7), $\Omega^{-1} U_{r_i} = \frac{V_{r_i0}}{1^{a+1}} + \frac{W_{r_i0}}{1^{b+1}} = U_{r_i0}$. Consequently, the relation to prove becomes $\Omega^{-1}\Omega'' r_{i0} = (-a\frac{V_0}{I_0} -$

$b\frac{W_0}{I_0})r_{i0}$. After substituting $s = 1$ in the derivations of the first case, we find that $\Omega^{-1}\Omega'' r_{i0} = (-ap_0 - bq_0) \cdot r_{i0}$ and this case is proved as well.

□

In Section 4.1 we showed that the set of simultaneous central configurations is not empty, in general. According to this last theorem, to prove the existence of homographic solutions for any degree of the homogeneous force function, it is thus sufficient to prove the existence of solutions of the system (4.2.5), with the initial conditions given in (4.2.9):

$$\begin{cases} s'' - sw'^2 = -\frac{ap_0}{s^{a+1}} - \frac{bq_0}{s^{b+1}}, \\ 2s'w' + sw'' = 0, \\ s(0) = 1,\ \ w(0) = 0, \\ s'(t) \in R,\ \ w'(t) \geq 0. \end{cases} \tag{4.2.12}$$

While it is possible to invoke directly classical results in the theory of differential equations, we can first show a direct path for solving this system. Because $\frac{(s^2w')'}{s} = 2s'w' + sw''$, the second equation above is equivalent to the first integral $s^2w' = const = w'(0)$. Also we notice that $[\frac{1}{2}s'^2 + \frac{1}{2}s^2w'^2 - p_0s^{-a} - q_0s^{-b}]' = s's'' - ss'w'^2 + sw'(2s'w' + sw'') + ap_0s^{-a-1}s' + bq_0s^{-b-1}s'$ and, using the latter, and then the former equation in (4.2.12), we get $[\frac{1}{2}s'^2 + \frac{1}{2}s^2w'^2 - p_0s^{-a} - q_0s^{-b}]' = s'[s'' - sw'^2 + \frac{ap_0}{s^{a+1}} + \frac{bq_0}{s^{b+1}}] = 0$.

If $s' = 0$, then using (4.2.12), $s(t) = 1$, while $w(t) = t \cdot w'(0)$. That is, the simplest homographic solutions are the relative equilibria, when one imposes on the bodies arranged in a central configuration a constant angular velocity $w' = w'(0)$ (which can be computed from the first equation in (4.2.12) to be $\sqrt{\frac{aV_0 + bW_0}{I_0}}$). As an example, such relative equilibria can be constructed using the equilateral central configuration for the Manev problem, mentioned at the end of Section 4.1. For a discussion about the stability of such periodic solutions see [13], [4].

When $s' \neq 0$, the first equation in (4.2.12) is equivalent to $\frac{1}{2}s'^2 + \frac{1}{2}s^2w'^2 - p_0s^{-a} - q_0s^{-b} = const = \frac{1}{2}(s'(0)^2 + w'(0)^2 - p_0 - q_0$. The first two equations in (4.2.12) become equivalent to the system:

$$\begin{cases} s^2w' = C_1, \\ \frac{1}{2}s'^2 + \frac{1}{2}s^2w'^2 - p_0s^{-a} - q_0s^{-b} = C_2, \end{cases}$$

where $C_1,\ C_2$ are constants[2].

Substituting $s^2 w'^2 = \frac{C_1}{s^2}$ into the second equation, we get the initial value problem: $\frac{ds}{dt} = \pm\sqrt{2h_0 + 2(\frac{p_0}{s^a} + \frac{q_0}{s^b}) - \frac{C_1^2}{s^2}}, \quad s(0) = 1$. For the majority of values of a, b, this equation cannot be integrated in terms of elementary functions, but standard results of ordinary differential equations assure that there is a unique solution to it, $s(t)$, on some interval $t \in [0,\, t_0^*)$. Since this solution is continuously differentiable, it follows that there will exist a unique solution to $\frac{dw}{dt} = \frac{|C_0|}{[s(t)]^2}, \quad w(0) = 0$ as well, and the existence of homographic solutions is now proved.

Chapter summary. We introduced central configurations and proved several properties about simultaneous and extraneous central configurations. We showed that these types are mutually exclusive and determined by the possibility for a homothety of a central configuration to be again a central configuration. Using these results, we showed that homographic solutions evolve by forming equivalent central configurations. We also introduced a procedure for constructing any homographic solution (Theorem 7), which helped us prove the existence of these solutions.

[2]Because a homographic solution always satisfies (4.2.2) and (4.2.3), for a solution pair $s(t),\ w(t)$, the constants $C_1,\ C_2$ will be, respectivelly, $|C_0|$ and h_0.

Chapter 5

Conclusions

In this work we have addressed the basics of the theory of homographic solutions for the N-body problem given by quasihomogeneous force functions.

After writing the equations for the quasihomogeneous N-body problem and enlisting the main consequences of its first ten integrals, we derived the homographic form of the equations of motion. Separating the cases of flat and non-flat solutions, we proved that for a quasihomogeneous problem with $a \neq 2$ and $b \neq 2$, if a homographic solution is flat, then it is necessarily planar (this property also holds for $a \neq 2$ *or* $b \neq 2$, when $N = 3$), while if it is non-flat, then it should be homothetic. The main consequence of this result was that if a homographic solution is not planar, then it is homothetic. This fact and and the simplified relations of the first integrals written for planar and non-planar solutions, allowed us to prove the extension of the Lagrange-Pizzetti theorem for quasihomogeneous case: (i) a homographic solution is homothetic if and only if $C = 0$ (i.e., when it has no invariable plane); (ii) a homographic solution is a relative equilibrium if and only if it is planar and rotates with a constant, non-zero, angular velocity.

Next, the main properties of the two types of central configurations (simultaneous and extraneous) have been determined. We showed that these types are mutually exclusive and determined by the possibility for a homothety of a central configuration to be again a central configuration, a property that only holds for simultaneous central configurations. In our opinion, working exclusively in the space of configuration vectors r is the reason of the relative easyness with which these results were obtained.

Joining the Lagrange-Pizzetti theorem and the associated results we proved about central configurations, we showed that for quasihomogeneous force functions, like for the Newtonian case, a solution is homographic if and only if the bodies form central configurations of the same type (equivalent) for all time. Finally, the last theorem we proved provides an exact way of constructing any desired homographic solution, starting with a given central configuration and two real functions as solutions for a specific system of differential equations (functions which play the role of the angle of rotation and, respectively, the dilatation coefficient for the configuration of the homographic solution). Also, showing that the corresponding initial value problem always has solutions, it was easy to prove the existence of homographic solutions.

All the results concerning homographic solutions were obtained for quasihomogeneous force functions with degrees $a \neq 2$ and $b \neq 2$, for any $N \geq 2$, and for $a \neq 2$ and $b \geq a$, or for $b \neq 2$ and $a \leq b$, if $N = 3$. For $a = 2$ and $b = 2$, we showed that even in the case of 3 bodies, the theorems of Lagrange and Pizzetti do not hold.

The cases that remain unsolved correspond to force functions with $a < 2 = b$ or $a = 2 < b$, for $N \geq 4$. The long and unsuccesful search for a counterexample of Lagrange's theorem encourages us to state (at least for $N = 4$) the following:

Conjecture 2. *For the quasihomogeneous* 4*-body problem with* $a < 2 = b$ *or* $a = 2 < b$*, if a homographic solution is flat, then it is planar.*

Before ending this work, we would like to point out that the research on homographic solutions does not end here. An enumeration of all possible homographic solutions should be caried out, based, on one hand, on the value of the constants of integration and, on the other hand, on the variety of central configurations known for different values of N. And this will possibly constitute the object of our future research.

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